$$1 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

x, x' ; y, y', ... $\equiv \bmod \ell$

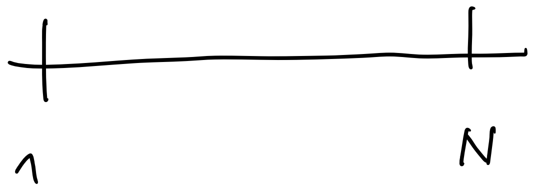
$$\rightsquigarrow (\quad)(\quad) = (\quad)^2 + (\quad)^2 + (\quad)^2 + (\quad)^2$$

5. Aritmetičke funkcije

- $\pi(x) = \#\{p \text{ prost} : p < x\}$ Teorem: $\pi(x) \sim \dfrac{x}{\ln x}$

Problem: Kolika je vjerojatnost da su dva "slučajno" odabrana prirodna broja a i b t.d. $(a,b) = 1$?

N



$$p_N = \frac{\sum_{b=1}^{N} \varphi(b)}{\sum_{b=1}^{N} b}$$

(a,b)
$a \le b$

$$p = \lim_{N \to +\infty} \frac{\sum_{b=1}^{N} \varphi(b)}{\sum_{b=1}^{N} b} = \frac{6}{\pi^2}$$

• $f: \mathbb{N} \to \mathbb{C}$ je multiplikativna ako je $f(1)=1$ i $f(mn)=f(m)f(n)$ $\forall m,n$ t.d. $(m,n)=1$.

• oznaka: $f(x) = O(g(x))$ ako postoji $C>0$ t.d. $|f(x)| \leq C \cdot g(x)$ $\forall x$.

g ↑ pozitivna fja.

$f \rightsquigarrow g$ i $g(n) := \sum_{d \mid n} f(d)$. Pokažimo da je g također multipl.

Neka je $(m,n)=1$.

$$g(mn) = \sum_{d \mid mn} f(d) = \sum_{\substack{d' \mid m \\ d'' \mid n}} f(d'd'') = \sum_{\substack{d' \mid m \\ d'' \mid n}} f(d') f(d'') = \sum_{d' \mid m} f(d') \cdot \sum_{d'' \mid n} f(d'')$$

$$= g(m) \cdot g(n).$$

Def. 5.1. Möbiusova fja $\mu(n), n \in \mathbb{N}$ je definirana sa

$$\mu(n) = \begin{cases} 0, & \text{ako } n \text{ nije kvadratno slobodan.} \\ (-1)^k & \text{ako je } n = p_1 \cdots p_k \text{ i gdje su } p_1 \ldots p_k \text{ različiti prosti brojevi.} \end{cases}$$

$\mu(1) = 1$; $\mu(p) = -1$ za p prost.

$$M(x) = \sum_{n<x} \mu(n)$$

$\uparrow$

Mertensova fja.

$\forall \varepsilon > 0$ vrijedi:

$$M(x) = O(x^{\frac{1}{2}+\varepsilon})$$

onda vrijedi Riemannova slutnja.

$\mu(n) \ldots \quad \ldots \quad O((x \ln(\ln x))^{\frac{1}{2}}) = O(x^{\frac{1}{2}+\varepsilon})$

μ je multiplikativna $\Rightarrow$ $\nu(n) = \sum_{d|n} \mu(d)$ također mult.

Tj. $\nu(1) = 1$ dok je za $n \geq 1$

$$\nu(n) = \nu(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = \nu(p_1^{\alpha_1}) \nu(p_2^{\alpha_2}) \ldots \nu(p_n^{\alpha_n})$$

$$= (\mu(1) + \mu(p_1) + \mu(p_1^2) + \ldots)(\quad) \ldots (\quad)$$

$$= (1 - 1 + \ldots 0) \cdot \ldots = 0$$

Teorem 15.2. (Möbiusova formula inverzije)

Neka je $f : \mathbb{N} \to \mathbb{C}$ proizvoljna fja, te neka je $F(n) = \sum_{d \mid n} f(d)$, $\forall n \in \mathbb{N}$. Tada je

$$f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right).$$

Obratno, ako je $f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right)$, onda je $F(n) = \sum_{d \mid n} f(d)$.

Dokaz: $\Rightarrow$

$$\sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \mu(d) \sum_{d' \mid \frac{n}{d}} f(d') = \sum_{d' \mid n} f(d') \sum_{d \mid \frac{n}{d'}} \mu(d)$$

$d'd \mid n$

$$= \sum_{d' \mid n} f(d')\, \nu\left(\frac{n}{d'}\right) = f(n) \checkmark.$$

Obratno, neka je $f(n) = \sum_{d'|n} \mu\left(\frac{n}{d'}\right) F(d')$. Sada je

$$\sum_{d|n} f(d) = \sum_{d|n} f\left(\frac{n}{d}\right) = \sum_{d|n} \sum_{d'|\frac{n}{d}} \mu\left(\frac{n}{dd'}\right) F(d')$$

$$= \sum_{d'|n} F(d') \sum_{d|\frac{n}{d'}} \mu\left(\frac{n}{d'}/d\right) = \sum_{d'|n} F(d') \cdot \nu\left(\frac{n}{d'}\right) = F(n).$$

Primijenimo li Teorem 5.2. na relaciji $\sum_{d|n} \varphi(d) = n$

dobivamo $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = n \sum_{d|n} \frac{\mu(d)}{d}$.

Def. 5.3. Neka je $n \in \mathbb{N}$. Def. $\tau(n)$ = broj pozitivnih djelitelja od n i $\sigma(n)$ = zbroj svih pozit. djelitelja od n.

Vrijedi: $\tau(n) = \sum_{d \mid n} 1$ i $\sigma(n) = \sum_{d \mid n} d$ pa su funkcije multiplikativne. Budući da je $\tau(p^j) = j+1$ i $\sigma(p^j) = 1 + p + \dots + p^j = \frac{p^{j+1}-1}{p-1}$ slijedi

$$\tau(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = \prod (\alpha_i + 1)$$

$$\sigma(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = \prod \frac{p_i^{\alpha_i+1}-1}{p_i - 1}.$$

Prop. 5.4. 1) $\sigma(n) < n(1+\ln n)$ za $n \geq 2$.

2) $\varphi(n) > \frac{1}{4}\frac{n}{\ln n}$ za $n \geq 2$.

<u>Dokaz</u>: 1) Imamo

$$\sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} \leq n \sum_{d \leq n} \frac{1}{d} < n\left(1 + \int_1^n \frac{1}{x}dx\right) = n(1+\ln n).$$

2) Fja. $f(n) = \frac{\sigma(n)\varphi(n)}{n^2}$ je multiplikativna.

Nadalje, $f(p^j) = \frac{p^{j+1}-1}{p-1} \cdot \frac{p^{j-1}(p-1)}{p^{2j}} = 1 - \frac{1}{p^{j+1}} \geq 1 - \frac{1}{p^2}$, pa je

$$f(n) \geq \prod_{p|n}\left(1 - \frac{1}{p^2}\right) \geq \prod_{m=2}^{\infty}\left(1 - \frac{1}{m^2}\right) = \frac{1\cdot\cancel{3}}{2\cdot\cancel{2}} \cdot \frac{\cancel{2}\cdot\cancel{4}}{\cancel{3}\cdot\cancel{3}} \cdot \frac{\cancel{3}\cdot 5}{\cancel{4}\cdot 4} \cdot \ldots = \frac{1}{2}$$

$\Rightarrow \sigma(n)\cdot\varphi(n) \geq \frac{1}{2}n^2$. Iz (1) slijedi $\sigma(n) < 2n\ln n$

$$\Rightarrow \varphi(n) > \frac{1}{4}\frac{n}{\ln n}.$$