$\rightsquigarrow \sum_{n=0}^{\infty} a_n x^n \ldots r = $ rad. kon. $\rightsquigarrow f : \langle -r, r\rangle \to \mathbb{R}$; $f \in C^\infty(\langle -r, r\rangle)$

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot x^{n-1} \cdot n$$

$\rightsquigarrow f \in C^\infty(\langle -r, r\rangle) \rightsquigarrow f(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow$ Taylor. red.

$\rightsquigarrow \ln(1+x) : \langle -1, 1\rangle \to \mathbb{R}$

$\rightsquigarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n \rightsquigarrow r = 1 \rightsquigarrow \ln(1+x) \quad \forall x \in \langle -1, 1\rangle$

Q: $x = 1$? $\rightsquigarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \rightsquigarrow$ konvergira.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \stackrel{?}{=} \ln(1+1) = \ln 2 \; ?$$

Teorem 6.15. (Abelov teorem) Ako red $\sum_{n=0}^{\infty} a_n$ konvergira i ima sumu s onda je i s def. fja. $f : \langle -1, 1\rangle \to \mathbb{R}$ i

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \forall |x| < 1$$

vrijedi $\lim_{x \to 1^-} f(x) = s$.

Dokaz: Trebamo dokazati

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \quad 0 < 1-x < \delta \Rightarrow |f(x) - s| < \varepsilon .$$

Vrijedi:

$$f(x) \cdot \frac{1}{1-x} = \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{m=0}^{\infty} x^m = \sum_{k=0}^{\infty} (a_0 + a_1 + \dots + a_k) x^k$$

apsolutna konverg.

Teorem o prod. redova

$$= \sum_{k=0}^{\infty} s_k x^k \quad \text{za } |x| < 1 \quad \text{gdje je } s_k = a_0 + \dots + a_k .$$

$$\Rightarrow \begin{cases} f(x) = (1-x) \displaystyle\sum_{n=0}^{\infty} s_n x^n & \text{odnosno} \\ s = s(1-x) \displaystyle\sum_{n=0}^{\infty} x^n & \text{pa oduzimanjem dobivamo} \end{cases}$$

$(*)\quad f(x)-s=(1-x)\sum_{n=0}^{\infty}(s_n-s)x^n \quad$ za $\quad |x|<1.$

Neka je $\varepsilon>0$. Zbog $\lim s_n=s$ $\exists\, n_\varepsilon\in\mathbb{N}$ t.d.
$\forall n\in\mathbb{N}\ (n>n_\varepsilon)\Rightarrow |s_n-s|<\frac{\varepsilon}{2}$. Neka je $\delta=\frac{\varepsilon}{2}\left(\sum_{n=0}^{n_\varepsilon}|s_n-s|\right)^{-1}$.

Desna strana od $(*)$ rastavimo na sumu od 0 do n_ε i od $n_\varepsilon+1$ do ∞. Tada za $0<1-x<\delta$ imamo

$$|f(x)-s|\le(1-x)\sum_{n=0}^{n_\varepsilon}|s_n-s|x^n+(1-x)\sum_{n=n_\varepsilon+1}^{\infty}|s_n-s|x^n$$

$$\le \underbrace{(1-x)}_{\sim\delta}\sum_{n=0}^{n_\varepsilon}|s_n-s|+(1-x)\cdot\frac{\varepsilon}{2}\underbrace{\left(\sum_{n=n_\varepsilon+1}^{\infty}x^n\right)}_{\le\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}}$$

$$\le\underbrace{\delta\cdot\sum_{n=0}^{n_\varepsilon}|s_n-s|}_{=\frac{\varepsilon}{2}}+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \checkmark.$$

$\boxed{\text{d.z.}}$ Dokažite da je f definirana na $\langle -1,1\rangle$ odnosno da red $\sum a_n x^n$ konvergira za $|x|<1$.

VM

Primjer 6.23. Vrijedi $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$.

Taylorov red fj. $f(x) = \ln(1+x)$ $\forall x \in \langle -1, 1 \rangle$

je jednak $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n$ i konvergira (dokažite!) k k $\forall x \in \langle -1, 1]$.

Iz Abelovog teorema (znamo da red $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ konvergira po Leibn. kri.)

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \lim_{x \to 1^-} \ln(1+x) \underset{\text{neprek.}}{=} \ln(2).$$

Korolar 6.3. Ako redovi $\sum a_n$, $\sum b_n$ i $\sum c_n$ konv. k A, B i C i ako je $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $\forall n$, onda je

$$C = A \cdot B$$

Dokaz: Označimo $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ i $h(x) = \sum_{n=0}^{\infty} c_n x^n$ $\forall |x| < 1$.

Vrijedi: $h(x) = f(x) \cdot g(x)$ $\forall x \in \langle -1, 1 \rangle$

Tm. o produktu redova: vrijedi zato što red $\sum a_n x^n$ konvergira apsolutno! (d.z.)

cijeli red $\sum a_n$ možda ne konvergira apsolutno.

Po Abelovom teoremu, $C = \lim_{x \to 1^-} h(x)$, $A = \lim_{x \to 1^-} f(x)$ i $B = \lim_{x \to 1^-} g(x)$

pa po teoremu o jednostrukom limesu vrijedi

$$C = \lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} f(x) \cdot g(x) = \lim_{x \to 1^-} f(x) \cdot \lim_{x \to 1^-} g(x)$$

$$= A \cdot B$$

$\blacksquare$