New quasi-symmetric designs by the Kramer-Mesner method[☆]

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Abstract

A t-(v, k, λ) design is quasi-symmetric if there are only two block intersection sizes. We adapt the Kramer-Mesner construction method for designs with prescribed automorphism groups to the quasi-symmetric case. Using the adapted method, we find many new quasi-symmetric 2-$(28, 12, 11)$ and 2-$(36, 16, 12)$ designs, establish the existence of quasi-symmetric 2-$(56, 16, 18)$ designs, and find three new unitals 2-$(217, 7, 1)$ of non-prime power order.

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1. Introduction

A t-(v, k, λ) *design* is a set of v points together with a collection of k-element subsets called blocks such that every t-subset of points is contained in exactly λ blocks. The design is *quasi-symmetric* if any two blocks intersect either in x or in y points, for non-negative integers $x < y$. We refer to the monograph [27] and the survey [26] for the main results and definitions about quasi-symmetric designs, and to [2] for designs in general.

It is known that there are no quasi-symmetric designs for $t \geq 5$, and for $t = 4$ the only examples are the 4-$(23, 7, 1)$ design and its complement. Apart from this 4-design and its residual with parameters 3-$(22, 7, 4)$, it has been conjectured that the only quasi-symmetric designs for $t = 3$ are the Hadamard 3-designs, designs with parameters 3-$((\lambda + 1)(\lambda^2 + 5\lambda + 5), (\lambda + 1)(\lambda + 2), \lambda)$ (which exist for $\lambda = 1$ while existence is unknown for $\lambda \geq 2$), a hypothetical 3-$(496, 40, 3)$ design, and the complements of these designs.

The classification of feasible parameters for quasi-symmetric 2-designs is a difficult problem and there are many triples (v, k, λ) for which existence is unde-

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cided (from now on we omit the parameter t if it is equal 2). Therefore it makes sense to use computational methods for the construction of designs. A celebrated technique for the construction of designs with prescribed automorphism groups is the Kramer-Mesner method [17]; see also [16, Chapter 9.2].

The purpose of this paper is to adapt the Kramer-Mesner method to quasi-symmetric designs. A number of enhancements can be made under the assumption that there are only two block intersection sizes. Using this enhanced method, we significantly increase the number of known quasi-symmetric $(28, 12, 11)$ and $(36, 16, 12)$ designs. We find quasi-symmetric $(56, 16, 18)$ designs, which had previously been unknown. We also find three new $(217, 7, 1)$ designs; one such design had been known. These four $(217, 7, 1)$ designs are the only known unitals of non-prime power order.

The layout of our paper is as follows. In section 2 we describe the adaptations to the Kramer-Mesner method for quasi-symmetric designs. We give a detailed account of software used for the various steps of the computation. We also present an idea how the method can be applied when the group is given as a permutation group on the blocks of the quasi-symmetric design, instead on the points as usual.

In sections 3 to 5 we describe constructions of new quasi-symmetric designs. The results rely heavily on computer calculations and it is impossible to present all details. Instead, we give intermediate results in the proofs of the proposition and theorems, such as the number of orbits, the number of solutions of the Kramer-Mesner system, the number of non-isomorphic designs and orders of their full automorphism groups. This enables independent verification of the steps of our computations.

In the final section 6 we give an updated table of known quasi-symmetric designs. An on-line version of the table available at

`https://web.math.pmf.unizg.hr/~krcko/results/quasisym.html`

contains links to the actual designs, again enabling verification of our results.

2. The Kramer-Mesner method for quasi-symmetric designs

Let G be a group of permutations of a v-element set, say $S = \{1, \ldots, v\}$. To find all t-(v, k, λ) designs with G as an automorphism group, we need the orbits $\mathcal{T}_1, \ldots, \mathcal{T}_m$ of t-element subsets of S, and the orbits $\mathcal{K}_1, \ldots, \mathcal{K}_n$ of k-element subsets of S induced by the action of G. Let a_{ij} be the number of subsets from $\mathcal{K}_j$ containing a given subset $T \in \mathcal{T}_i$. Clearly a_{ij} does not depend on the choice of T. The $m \times n$ matrix $A = [a_{ij}]$ is the *Kramer-Mesner* matrix. Designs correspond to 0-1 solutions of the linear system $A \cdot x = \lambda\, j_m$, where j_m is the all-1 vector of length m.

Finding 0-1 solutions of a system of linear equations is a known NP-complete problem. To make the computation feasible, the number of variables n has to be kept sufficiently small. Since we are looking for quasi-symmetric designs, we can limit the search to orbits $\mathcal{K}_j$ such that $|K_1 \cap K_2|$ is either x or y, for any two sets $K_1, K_2 \in \mathcal{K}_j$. We shall call such $\mathcal{K}_j$ the *good orbits* and assume that all other orbits have been left out, thereby reducing n. Furthermore, two orbits $\mathcal{K}_i$ and $\mathcal{K}_j$ are called *compatible* if $|K_1 \cap K_2|$ is either x or y, for any $K_1 \in \mathcal{K}_i$, $K_2 \in \mathcal{K}_j$. The $n \times n$ matrix $C = [c_{ij}]$ with $c_{ij} = 1$ if $\mathcal{K}_i$ and $\mathcal{K}_j$ are compatible, and $c_{ij} = 0$ otherwise, will be called the *compatibility matrix*.

Algorithms based on lattice basis reduction have been used to solve the Kramer-Mesner system $A \cdot x = \lambda\, j_m$; see [28] and the references therein. In [18], a simple backtracking solver was used. It can be made quite efficient by utilizing the compatibility matrix C. Once an orbit $\mathcal{K}_i$ has been chosen, the search is limited to the compatible orbits $\mathcal{K}_j$, such that $c_{ij} = 1$. Depending on the compatibility matrix, systems with thousands and sometimes even tens of thousands of variables can be solved. Examples will be given in the sequel. We implemented our compatibility matrix solver in the programming language C.

We use GAP [11] to generate the orbits and build the Kramer-Mesner matrix A. We also implemented some time-critical routines for generating the orbits in C. A straightforward approach for finding good orbits is to generate all orbits of k-subsets, and to eliminate the ones with intersection sizes other than x and y. The total number of orbits is often too large for this approach. We use the following trick if the group order $|G|$ exceeds the number of blocks of the design b. Clearly we need only orbits of size at most b, and every such orbit has a non-trivial stabilizer $H \leq G$. We build these short orbits by this algorithm:

1: **for** every subgroup H of index $[G : H] \leq b$ up to conjugation **do**
2:     find the orbits of H on S (the "seeds")
3:     combine the seeds into k-subsets of S
4:     take representatives of the k-subsets under the action of G
5: **end for**

It suffices to loop over the non-conjugate subgroups $H \leq G$ because we need only one representative of each k-subset under the action of G. If $H_1, H_2 \leq G$ are conjugate (i.e. $H_2 = \alpha^{-1} H_1 \alpha$ for some $\alpha \in G$), then H_1 fixes the k-subset K if and only if H_2 fixes K^α. We use the GAP command **ConjugacyClasses-Subgroups** for the loop in line 1 of the algorithm, and the commands **Orbits** and **OrbitRepresentatives** for lines 2 and 4. For line 3 we use our own procedure written in C. The algorithm will be used in the proofs of Theorems 4.1 and 5.1.

As the final step of the computation, the constructed designs need to be checked for isomorphism. Different solutions of the Kramer-Mesner system may give isomorphic designs. We use the program nauty [22] for isomorphism testing and to compute the full automorphism groups of the designs. To analyze the groups an for all other group-related computations, such as finding subgroups, we use GAP [11].

Suppose we are looking for 2-designs with parameters (v, b, r, k, λ) and an automorphism group G. Here r is the number of blocks through a point. The group G is usually given as a permutation group on the v points, but a permutation representation on the b blocks is sometimes more readily available. For example, the block graph of unknown quasi-symmetric designs may be a known strongly regular graph, and we can take G to be an automorphism group of the graph. In this case we apply the following approach, which we shall call the *dual Kramer-Mesner method*. We compute orbits of 2-subsets and r-subsets of the set $S = \{1, \ldots, b\}$. An orbit of r-subsets is *good* if its members pairwise intersect in λ points, and two orbits are *compatible* if members of one always intersect members of the other in λ points. The compatibility matrix and the Kramer-Mesner system are defined as before, except that the 2-orbits now correspond to block intersections and thus have to be covered by the r-orbits either x or y times. Solutions of the system represent duals of the sought-after quasi-symmetric designs. An example will be presented in section 4.

3. $(28, 12, 11)$ and $(36, 16, 12)$ designs

The classic quasi-symmetric $(28, 12, 11)$ design with $x = 4$, $y = 6$, and $(36, 16, 12)$ design with $x = 6$, $y = 8$ are obtained as the derived and the residual design of the symplectic symmetric $(64, 28, 12)$ design, respectively [15]. The symplectic group $Sp(6, 2)$ of order 1451520 acts on these designs. The designs have the symmetric difference property (SDP): the symmetric difference of any two blocks (of the quasi-symmetric designs) or any three blocks (of the symmetric design) is a block or a block complement. Up to isomorphism, there are four symmetric $(64, 28, 12)$ SDP designs, and four quasi-symmetric $(28, 12, 11)$ and $(36, 16, 12)$ SDP designs [13]. Full automorphism groups of the quasi-symmetric SDP designs are of orders 1451520, 10752, 1920, and 672 [24].

In [10], quasi-symmetric $(28, 12, 11)$ designs with an automorphism of order 7 without fixed points and blocks were classified; there are exactly 246 such designs. By embedding them as derived designs in symmetric designs, the authors found 8784 non-isomorphic symmetric $(64, 28, 12)$ designs and quasi-symmetric $(36, 16, 12)$ designs. The enumeration of quasi-symmetric $(28, 12, 11)$ designs with automorphisms of order 7 was performed with the help of tactical decomposition matrices [9]. It was shown that nine such matrices correspond to quasi-symmetric designs and the program BDX was used to convert them to incidence matrices of the designs [19].

To repeat the classification by the Kramer-Mesner method, one would generate the $\binom{28}{12}/7 = 3067740$ orbits of 12-element subsets of $S = \{1, \ldots, 28\}$. Among them are 187572 good orbits, with intersection numbers $x = 4$, $y = 6$. The corresponding Kramer-Mesner system is too large for our solver. It was shown in [18] how tactical decomposition matrices can be used to reduce the size of the system. The nine tactical decomposition matrices from [9] give systems with number of columns ranging from 24696 to 116620. We could handle the smallest system by our compatibility matrix solver. In the other cases it was more efficient to write a backtracking program for "indexing" the tactical decomposition matrices, i.e. transforming them directly to incidence matrices of quasi-symmetric designs, proceeding column-by-column. The number of non-isomorphic designs obtained in each case agrees with the results of [9] and [10].

To make the Kramer-Mesner approach feasible, we take a larger group. The full automorphism group $Sp(6, 2)$ of the classic $(28, 12, 11)$ design possesses a subgroup G isomorphic to the dihedral group of order 12, generated by the permutations

$$\alpha = (1,2,3,4,5,6)(7,8,9,10,11,12)(13,14,15,16,17,18)(19,20,21,22,23,24)(25,26,27),$$
$$\beta = (1,6)(2,5)(3,4)(7,11)(8,10)(13,17)(14,16)(19,23)(20,22)(25,27).$$

Proposition 3.1. *Up to isomorphism there are* 13656 *quasi-symmetric* $(28,12,11)$ *designs with* $x = 4$*,* $y = 6$ *and* $G = \langle\alpha,\beta\rangle$ *as an automorphism group.*

Proof. There are 47 orbits of 2-subsets and 2543568 orbits of 12-subsets of $S = \{1,\ldots,28\}$ under the action of G. Among them are 1097 good orbits, with intersection numbers $x = 4$, $y = 6$. We set up the 47×1097 Kramer-Mesner system and solved it using our compatibility matrix solver. The total number of solutions is 654336. By applying nauty [22] we found that they give rise to 13656 non-isomorphic quasi-symmetric $(28,12,11)$ designs, including the classic design, the SDP design with full automorphism group of order 10752, and two more designs with automorphisms of order 7.

For $(36,16,12)$ designs, we take a group of order 24 isomorphic to the symmetric group S_4. The full automorphism group $Sp(6,2)$ of the classic $(36,16,12)$ design possesses such a subgroup G generated by

$$\alpha = (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)(25,26,27)(28,29,30)(31,32,33),$$
$$\beta = (1,4)(2,7)(5,9)(6,11)(8,10)(13,16)(14,19)(15,21)(17,22)(18,24)(20,23)(26,27)(29,30)(32,34).$$

Proposition 3.2. *Up to isomorphism there are* 35572 *quasi-symmetric* $(36,16,12)$ *designs with* $x = 6$*,* $y = 8$ *and* $G = \langle\alpha,\beta\rangle$ *as an automorphism group.*

Proof. The group G induces 50 orbits on the 2-subsets and 304774697 orbits on the 16-subsets of $S = \{1,\ldots,36\}$. Among them are 1300 good orbits, with intersection numbers $x = 6$, $y = 8$. The 50×1300 Kramer-Mesner system has 886528 solutions respecting the compatibility matrix. They give rise to 35572 non-isomorphic quasi-symmetric $(36,16,12)$ designs, including the classic design and the SDP design with full automorphism group of order 1920.

The preceding propositions are examples how to increase the number of know designs by the Kramer-Mesner method. Take a known design, compute its full automorphism group, take a subgroup and find all designs with the subgroup as an automorphism group. Besides the known design, one may get other, non-isomorphic designs. We did this for many groups operating on quasi-symmetric $(28,12,11)$ and $(36,16,12)$ designs.

A direct construction of quasi-symmetric designs based on Hadamard matrices and mutually orthogonal Latin squares (MOLS) was described in [4] and [21]. From a Hadamard matrix of order 8 and either two or three MOLS of order 8, many non-isomorphic quasi-symmetric $(28, 12, 11)$ and $(36, 16, 12)$ designs can be constructed. Most of these designs have trivial automorphism groups.

By summarizing all designs found by the Kramer-Mesner method and by the construction from [4, 21], and eliminating isomorphic copies by nauty [22], we can give lower bounds on the number of quasi-symmetric $(28, 12, 11)$ and $(36, 16, 12)$ designs.

Theorem 3.3. *There are at least* 58891 *quasi-symmetric* $(28, 12, 11)$ *designs and at least* 522079 *quasi-symmetric* $(36, 16, 12)$ *designs up to isomorphism.*

The distribution of the designs by order of full automorphism group is given in Table 1.

$\vert \mathrm{Aut} \vert$	#28	#36	$\vert \mathrm{Aut} \vert$	#28	#36	$\vert \mathrm{Aut} \vert$	#28	#36
1451520	1	1	224	8	16	36	33	0
10752	1	3	192	652	7000	32	1299	0
4608	3	3	168	2	23	28	12	0
2304	0	4	160	564	5628	24	360	216596
1920	4	4	144	12	32	21	95	0
1536	13	61	128	4745	5084	20	26	70
1344	4	8	120	17	17	18	7	0
768	18	248	96	26039	5858	14	50	0
672	8	16	84	15	72	12	12908	5
640	1	3	80	372	3744	10	28	0
576	12	12	72	11	31	7	47	0
512	14	60	64	110	0	4	0	14
384	102	300	60	8	0	3	172	466
360	1	1	54	0	1	2	62	1080
320	4	12	48	1224	36114	1	9554	237018
288	10	14	42	3	0			
256	258	2456	40	2	4			

Table 1: The distribution of designs from Theorem 3.3 by order of full automorphism group.

4. $(56, 16, 18)$ designs

Let $\mathcal{W}$ be the Witt 5-$(24, 8, 1)$ design and $\mathcal{D} = \mathrm{der}(\mathrm{der}(\mathrm{res}(\mathcal{W})))$. Here res denotes the residual, and der the derived design with respect to a point. The design $\mathcal{D}$ is quasi-symmetric with parameters $(21, 6, 4)$, $x = 0$, $y = 2$. The full automorphism group $\mathrm{Aut}\,\mathcal{D}$ of order 40320 is a split extension $M_{21}.\mathbb{Z}_2$ by the Mathieu group M_{21}. It acts as a permutation group on the 56 blocks of the design $\mathcal{D}$. We will denote this permutation group of degree 56 by G. It has a subgroup $H \cong (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2).A_5$ of order 960, generated by the permutations

$$\alpha = (1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20)(21,22,23,24,25)(26,27,28,29,30)$$
$$(31,32,33,34,35)(36,37,38,39,40)(41,42,43,44,45)(46,47,48,49,50)(51,52,53,54,55),$$
$$\beta = (1,6,8)(2,21,26)(3,32,34)(4,11,5)(7,15,22)(9,16,13)(10,29,17)(12,33,30)(14,19,31)$$
$$(18,23,35)(24,28,36)(25,37,39)(27,38,40)(42,51,49)(43,52,45)(44,46,47)(48,54,53)$$
$$(50,56,55).$$

Theorem 4.1. *There are three quasi-symmetric* $(56,16,18)$ *designs with* $x=4$, $y=8$ *and* $H=\langle\alpha,\beta\rangle$ *as an automorphism group. The first one has* H *as its full automorphism group, the second one has a split extension* $H.\mathbb{Z}_2$ *of order* 1920*, and the third one has* G *as its full automorphism group.*

PROOF. The action of H induces 7 orbits on the 2-subsets of $S=\{1,\ldots,56\}$. Using the algorithm described in section 2, we found 40 orbits of 16-subsets of size at most $b=231$ with intersection numbers $x=4$, $y=8$. The 7×40 Kramer-Mesner system has 5 solutions respecting the compatibility matrix, giving rise to 3 non-isomorphic quasi-symmetric designs. The full automorphism groups were computed with nauty [22] and analyzed with GAP [11].

Quasi-symmetric designs with parameters $(56,16,18)$ had previously been unknown. The binary codes spanned by the incidence vectors of their blocks are self-orthogonal. The first two designs of Theorem 4.1 span codes of dimension 23 and minimum distance 8, while the third design spans a code of dimension 19 and minimum distance 16. The codes were analyzed with the GAP package GUAVA [8]. The first two codes are far from optimal, but the third code has minimum distance equal to the best known binary linear code of length 56 and dimension 19 [12].

The block graph of the three designs of Theorem 4.1 is the Cameron graph, a strongly regular graph with parameters $SRG(231,30,9,3)$ [5]. The full automorphism group of the Cameron graph is a permutation group G on 231 vertices, isomorphic to the split extension $M_{22}.\mathbb{Z}_2$ of order 887040. We can use it to construct the three $(56,16,18)$ designs by the dual Kramer-Mesner method, as explained in section 2. The group G has three subgroups of order 960 up to conjugation. We compute orbits of 66-subsets ($r=66$) of the set $S=\{1,\ldots,231\}$ of size at most $v=56$, within which the subsets intersect in $\lambda=18$ elements. Two orbits are compatible if subsets from one intersect subsets from the other in $\lambda=18$ elements. The Kramer-Mesner system has rows labeled by the orbits of 2-subset, columns labeled by the good orbits of the 66-subsets, and every 2-orbit has to be covered $x=4$ or $y=8$ times. One subgroup of order 960 leads to a 63×168 Kramer-Mesner system with 96 solutions respecting the compatibility matrix, giving the second and the third quasi-symmetric designs of Theorem 4.1. The second subgroup gives a 63×60 Kramer-Mesner system with 18 solutions and the first and third design of Theorem 4.1. Finally, the third subgroup gives a 60×24 Kramer-Mesner system without solutions.

5. $(217, 7, 1)$ designs

Designs with parameters $t = 2$ and $\lambda = 1$ (Steiner 2-designs) are automatically quasi-symmetric with intersection numbers $x = 0$, $y = 1$. An important family are the *unitals* with parameters $(q^3 + 1, q + 1, 1)$. For any prime power q there exists a projective plane of order q^2 with a unitary polarity, and the set of absolute points and non-absolute lines forms a unital of order q. The only known unital with q not a prime power is a cyclic $(217, 7, 1)$ design discovered independently by Mathon [20] and Bagchi and Bagchi [1].

In [18], unitals of order $q = 4$ with a non-abelian automorphism group of order 39 were classified. Tactical decomposition matrices were used to reduce the size of the Kramer-Mesner system. For Steiner 2-designs the good k-orbits are easily recognized as the ones covering every 2-orbit at most once. The corresponding column of the Kramer-Mesner matrix A has only 0 and 1 entries. Similarly, two compatible k-orbits do not cover the same 2-orbit, i.e. the corresponding columns of A are "disjoint". The backtracking solver from [18] detects when partial solutions violate this condition and stops trying to extend them. Therefore the compatibility matrix approach does not give any advantage in the case of Steiner 2-designs.

However, we can now handle the much larger $q = 6$ case thanks to the algorithm for short orbits from section 2, albeit only with rather large groups. The full automorphism group of the $(217, 7, 1)$ design of Mathon, Bagchi and Bagchi is a semidirect product $\mathbb{Z}_{217}.\mathbb{Z}_{30}$ of order 6510. It has a subgroup $G \cong \mathbb{Z}_{217}.\mathbb{Z}_6$ of order 1302 generated by the permutations

$$\alpha = (1, 2, 3, \ldots, 217),$$

$$\begin{aligned}\beta = \ &(2, 38, 68, 93, 150, 89)(3, 75, 135, 185, 82, 177)(4, 112, 202, 60, 14, 48)(5, 149, 52, 152, 163, 136)\\ &(6, 186, 119, 27, 95, 7)(8, 43, 36, 211, 176, 183)(9, 80, 103, 86, 108, 54)(10, 117, 170, 178, 40, 142)\\ &(11, 154, 20, 53, 189, 13)(12, 191, 87, 145, 121, 101)(15, 85, 71, 204, 134, 148)(16, 122, 138, 79, 66, 19)\\ &(17, 159, 205, 171, 215, 107)(18, 196, 55, 46, 147, 195)(21, 90, 39, 105, 160, 25)(22, 127, 106, 197, 92, 113)\\ &(23, 164, 173, 72, 24, 201)(26, 58, 157, 131, 37, 31)(28, 132, 74, 98, 118, 207)(29, 169, 141, 190, 50, 78)\\ &(30, 206, 208, 65, 199, 166)(32, 63, 125)(33, 100, 192, 124, 212, 213)(34, 137, 42, 216, 144, 84)\\ &(35, 174, 109, 91, 76, 172)(41, 179, 77, 209, 102, 49)(44, 73, 61, 51, 115, 96)(45, 110, 128, 143, 47, 184)\\ &(56, 83, 214, 70, 167, 67)(57, 120, 64, 162, 99, 155)(59, 194, 198, 129, 180, 114)(62, 88, 182, 188, 193, 161)\\ &(69, 130, 217, 181, 151, 126)(81, 140, 153, 200, 203, 97)(94, 187, 156)(104, 123, 175, 146, 158, 168)\\ &(111, 165, 210, 139, 116, 133).\end{aligned}$$

Theorem 5.1. *There are four $(217, 7, 1)$ designs with $G = \langle \alpha, \beta \rangle$ as an automorphism group. One of them is the unital of Mathon, Bagchi and Bagchi, and the remaining ones have G as their full automorphism group.*

Proof. The action of G induces 21 orbits on the 2-subsets and 1141 orbits of size at most $b = 1116$ with intersection numbers $x = 0$, $y = 1$ on the 7-subsets of $S = \{1, \ldots, 217\}$. The Kramer-Mesner system has 96 solutions, giving rise to 4 non-isomorphic designs.

5.1. The new unitals as relative difference families

Although the designs of Theorem 5.1 were found by the Kramer-Mesner method, they are more easily described by difference families. Also, in this way it becomes apparent how closely related to each other they are. Let G be an additively written group of order v with a subgroup H. A (v, H, k, λ) relative difference family (RDF) is a collection $\{B_1, \ldots, B_t\}$ of k-subsets of G such that the equation $x - y = g$ has exactly λ solution pairs $(x, y) \in \cup_{i=1}^{t} B_i \times B_i$ for $g \in G \setminus H$, and no solution pairs for $g \in H$. If $|H| = k$ and $\lambda = 1$, the development $\{B_i + g \mid g \in G, i = 1, \ldots, t\}$ of the RDF together with the right cosets of H as blocks forms a Steiner 2-$(v, k, 1)$ design with G as a point-regular automorphism group. See [7] for more details and algebraic constructions of RDFs leading, amongst others, to the known $(217, 7, 1)$ design.

By identifying $i \in \{0, \ldots, 216\}$ with α^i, we can write the designs of Theorem 5.1 as RDFs in $\mathbb{Z}_{217}$ relative to the subgroup $H = \langle 31 \rangle$ of order 7. Let $B_1 = \{0, 1, 37, 67, 88, 92, 149\}$, $B_2 = 8B_1$, $B_3 = 11B_1$, $B_4 = 15B_1$, and $B_5 = 29B_1$ (multiplication is modulo 217). Then $\{B_1, B_2, B_3, B_4, B_5\}$ is a RDF giving rise to the design of Mathon, Bagchi and Bagchi, while $\{B_1, B_2, B_3, B_4, -B_5\}$, $\{B_1, B_2, B_3, -B_4, -B_5\}$, and $\{B_1, B_2, -B_3, B_4, -B_5\}$ are RDFs giving rise to the other three designs. Thus, the three new unitals could more easily be obtained by changing signs of base blocks, as explained in [6]. In the terminology of [6], the RDFs generating the designs of Theorem 5.1 are *similar*. Notice that the automorphism β corresponds to the multiplier 37 of order 6. The first RDF also has 11 as a multiplier of order 30, while the other three RDFs only have 37 and its powers as multipliers.

6. An updated existence table

Neumaier [23] distinguished four types of quasi-symmetric 2-designs: multiples of symmetric designs, strongly resolvable designs, Steiner 2-designs with $v > k^2$, and residuals of biplanes. All other quasi-symmetric 2-designs are called *exceptional* and a table of feasible parameters with $2k \leq v \leq 40$ is given in [23].

An updated and extended table of exceptional quasi-symmetric 2-designs with $2k \leq v \leq 70$ appears in Shrikhande's survey [26]. It contains information on existence and in most cases on the number of non-isomorphic designs. Tables of feasible parameters with information about existence also appear in [25], grouped according to the associated strongly regular graph. These tables include Steiner 2-designs along with exceptional quasi-symmetric designs.

In Table 2 we include the parameters from [26] for which quasi-symmetric designs are known to exist and two more rows: exceptional $(78, 36, 30)$ designs with $x = 15$, $y = 18$ and unitals $(217, 7, 1)$. We give new estimates on the number of non-isomorphic quasi-symmetric designs in the column "Nqsd". Where no reference is given, see [26].

Quasi-symmetric $(64, 24, 46)$ designs with $x = 8$, $y = 12$ were first constructed in [3] as part of an infinite series. Recently Jungnickel and Tonchev [14] proved that the number of quasi-symmetric designs in this series grows exponentially and established the lower bound for $(64, 24, 46)$ designs given in Table 2.

Designs with parameters $(66, 30, 29)$, $x = 12$, $y = 15$ and $(78, 36, 30)$, $x = 15$, $y = 18$ are also part of an infinite series [4, 21]. From a Hadamard matrix of

v	k	λ	r	b	x	y	Nqsd	Reference
21	6	4	16	56	0	2	1	
21	7	12	40	120	1	3	1	
22	6	5	21	77	0	2	1	
22	7	16	56	176	1	3	1	
23	7	21	77	253	1	3	1	
28	12	11	27	63	4	6	≥ 58891	Thm 3.3
31	7	7	35	155	1	3	5	
36	16	12	28	63	6	8	≥ 522079	Thm 3.3
45	9	8	44	220	1	3	1	
49	9	6	36	196	1	3	≥ 44	
56	16	6	22	77	4	6	≥ 2	
56	16	18	66	231	4	8	≥ 3	Thm 4.1
63	15	35	155	651	3	7	≥ 1	
64	24	46	126	336	8	12	≥ 28826	[14]
66	30	29	65	143	12	15	≥ 10000	Prop 6.1
78	36	30	66	143	15	18	≥ 10000	Prop 6.1
217	7	1	36	1116	0	1	≥ 4	Thm 5.1

Table 2: Existence of quasi-symmetric designs.

order $2u$ (where u is an even integer) and either $u-2$ or $u-1$ MOLS of order $2u$, quasi-symmetric designs with parameters $(2u^2-u^2-u, u^2-u-1)$, $x = u(u-2)/2$, $y = u(u-1)/2$ and $(2u^2+u, u^2, u^2-u)$, $x = u(u-1)/2$, $y = u^2/2$ can be constructed. Ingredients for the construction always exist when u is a power of two; the $u = 4$ case was discussed in section 3. The only other instance when enough MOLS are known is $u = 6$, giving quasi-symmetric $(66, 30, 29)$ and $(78, 36, 30)$ designs. The construction allows arbitrary choices and many non-isomorphic designs can be obtained even from a single Hadamard matrix and set of MOLS. This was already shown in section 3 for $u = 4$, and here we give rough estimates for $u = 6$.

Proposition 6.1. *There are at least* 10000 *non-isomorphic quasi-symmetric* $(66, 30, 29)$ *designs with* $x = 12$, $y = 15$ *and at least as many* $(78, 36, 30)$ *designs with* $x = 15$, $y = 18$.

Nauty [22] was used to establish that the constructed designs are not isomorphic. All of them have trivial automorphism groups. It seems that the number of designs obtained by the construction from [4] and [21] also grows exponentially with u.

An on-line version of Table 2 is available on the web page mentioned in the introduction. It contains links to incidence matrices of the newly constructed designs, and most other known quasi-symmetric designs from Table 2.

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