

# A NOTE ON MEDIAL QUASIGROUPS

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ABSTRACT. In this short note we prove two results about medial quasigroups. First, let  $\varphi$  and  $\psi$  be binary operations defined by multiplication, left and right division in a medial quasigroup. Then  $\varphi$  and  $\psi$  are mutually medial, i.e.  $\varphi(\psi(a, b), \psi(c, d)) = \psi(\varphi(a, c), \varphi(b, d))$ . Second, four points  $a, b, c, d$  in an idempotent medial quasigroup form a parallelogram if and only if  $d = (a/b)(b \setminus c)$ .

Let  $(Q, \cdot)$  be a quasigroup. We will denote left and right division in  $Q$  by  $\setminus$  and  $/$ , i.e.  $a \setminus c = b \iff ab = c \iff c/b = a$ . By a *formula* we mean any expression built up from a number of variables using the operations  $\cdot, \setminus$  and  $/$ . More precisely:

- (1) the variables  $x, y, \dots$  are formulae;
- (2) if  $\varphi, \psi$  are formulae, then so are  $\varphi \cdot \psi, \varphi \setminus \psi$  and  $\varphi / \psi$ .

A formula  $\varphi$  containing (at most) two variables gives rise to a new binary operation  $Q \times Q \rightarrow Q$ , which we will also denote by the letter  $\varphi$ .

A quasigroup is *medial* if the identity  $ab \cdot cd = ac \cdot bd$  holds. It is known that  $(Q, \cdot)$  is medial if and only if either (and therefore both) of the quasigroups  $(Q, \setminus)$  and  $(Q, /)$  are medial. Our first result states that binary operations defined by formulae in a medial quasigroup are *mutually medial*:

**Theorem 1.** *Let  $\varphi, \psi$  be the binary operations defined by any two formulae in a medial quasigroup  $Q$ . Then the following identity holds:*

$$\varphi(\psi(a, b), \psi(c, d)) = \psi(\varphi(a, c), \varphi(b, d)).$$

*Proof.* It is easily verified that the operations  $\cdot, \setminus$  and  $/$  are mutually medial, i.e. the identities  $ab \setminus cd = (a \setminus c)(b \setminus d)$ ,  $ab / cd = (a / c)(b / d)$  and  $(a \setminus b) / (c \setminus d) = (a / c) \setminus (b / d)$  hold. For example, if we denote  $x = a \setminus c$  and  $y = b \setminus d$ , then  $ax = c$  and  $by = d$ . Using mediality we get  $cd = ax \cdot by = ab \cdot xy \implies ab \setminus cd = xy = (a \setminus c)(b \setminus d)$ .

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A binary operation defined by a formula  $\varphi$  is mutually medial with multiplication, left and right division:

$$\varphi(ab, cd) = \varphi(a, c) \cdot \varphi(b, d), \quad (1)$$

$$\varphi(a \setminus b, c \setminus d) = \varphi(a, c) \setminus \varphi(b, d), \quad (2)$$

$$\varphi(a / b, c / d) = \varphi(a, c) / \varphi(b, d). \quad (3)$$

This is obvious for  $\varphi(a, b) = a$  and  $\varphi(a, b) = b$ , and follows by induction for more complicated formulae. Supposing the identities are true for  $\varphi_1$  and  $\varphi_2$ , we see that they also hold for  $\varphi = \varphi_1 \cdot \varphi_2$ :

$$\begin{aligned} \varphi(ab, cd) &= \varphi_1(ab, cd) \cdot \varphi_2(ab, cd) = \varphi_1(a, c)\varphi_1(b, d) \cdot \varphi_2(a, c)\varphi_2(b, d) = \\ &= \varphi_1(a, c)\varphi_2(a, c) \cdot \varphi_1(b, d)\varphi_2(b, d) = \varphi(a, c) \cdot \varphi(b, d). \end{aligned}$$

The argument is similar for identities (2), (3) and formulae  $\varphi = \varphi_1 \setminus \varphi_2$ ,  $\varphi = \varphi_1 / \varphi_2$ .

Finally, mutual mediality of  $\varphi$  and  $\psi$  is obtained by induction on  $\psi$ :

$$\begin{aligned} \varphi(\psi(a, b), \psi(c, d)) &= \varphi(\psi_1(a, b)\psi_2(a, b), \psi_1(c, d)\psi_2(c, d)) \stackrel{(1)}{=} \\ &= \varphi(\psi_1(a, b), \psi_1(c, d)) \cdot \varphi(\psi_2(a, b), \psi_2(c, d)) = \\ &= \psi_1(\varphi(a, c), \varphi(b, d)) \cdot \psi_2(\varphi(a, c), \varphi(b, d)) = \\ &= \psi(\varphi(a, c), \varphi(b, d)). \end{aligned}$$

Identity (2) is used if  $\psi = \psi_1 \setminus \psi_2$ , and identity (3) if  $\psi = \psi_1 / \psi_2$ .  $\square$

**Corollary 2.** *If  $(Q, \cdot)$  is a medial quasigroup, then the binary operation defined by a formula  $\varphi$  is also medial.*

Of course,  $(Q, \varphi)$  need not be a quasigroup. Special cases of Theorem 1 and Corollary 2 have been used in [1] and [7]. Identity (1) was proved earlier by Puharev [2].

In [3], some geometric concepts have been introduced in a medial quasigroup  $Q$ . For example, the points  $a, b, c, d \in Q$  are said to form a *parallelogram*, denoted by  $\text{Par}(a, b, c, d)$ , if there are points  $p, q \in Q$  such that  $pa = qb$  and  $pd = qc$ . This quaternary relation satisfies the axioms of *parallelogram space* (for definitions and further references see [8]). In particular, given any three points  $a, b, c \in Q$  there is a unique  $d \in Q$  such that  $\text{Par}(a, b, c, d)$ . In [5], the parallelogram relation in idempotent medial quasigroups (satisfying the additional identity  $aa = a$ ) was characterized in several more direct ways. In even more special quasigroups, explicit formulae for the fourth vertex  $d$  of a parallelogram as a function of  $a, b$  and  $c$  are known; see [1], [4], [6] and [9]. Here we give such a formula valid in a general IM-quasigroup.

**Theorem 3.** *Let  $Q$  be an idempotent medial quasigroup and  $a, b, c, d \in Q$ . Then,  $\text{Par}(a, b, c, d)$  holds if and only if there are  $x, y \in Q$  such that  $xb = a$ ,  $by = c$  and  $xy = d$ .*

*Proof.* Let  $x, y \in Q$  be elements satisfying  $xb = a$ ,  $by = c$  and  $xy = d$ . By taking  $p = a$  and  $q = x$ , we see that  $pa = qb$  and  $pd = xb \cdot xy = x \cdot by = qc$ , i.e.  $\text{Par}(a, b, c, d)$  holds.

Now suppose  $\text{Par}(a, b, c, d)$  holds and denote  $x = a / b$ ,  $y = b \setminus c$ . Then,  $xb = a$  and  $by = c$ . According to [3, Corollary 5], for any  $p \in Q$  there is a unique  $q \in Q$  such that  $pa = qb$  and  $pd = qc$ . Specially, for  $p = a$  we see that  $a = qb \implies q = x$  and  $ad = qc = xc = x \cdot by = xb \cdot xy = a \cdot xy$ . Cancelling  $a$  from the left yields  $xy = d$ .  $\square$

**Corollary 4.** *In an idempotent medial quasigroup,  $\text{Par}(a, b, c, d)$  holds if and only if  $d = (a / b)(b \setminus c)$ .*

Formulae for the fourth vertex of a parallelogram in hexagonal, quadratical, GS and  $G_2$ -quasigroups contain only multiplication. In fact, such formulae follow from Corollary 4 because left and right division can be expressed by multiplication in these four classes of quasigroups. The formulae given in [1], [4] and [9] are shorter because other identities valid in these particular classes of quasigroups were taken into account.

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