Existence of Weak Solutions for Immiscible Compressible Two-Phase Flow in Porous Media by the Concept of Global Pressure

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Outline

Flow Equations

Mass conservation for $\alpha \in \{w, g\}$:

$$\Phi \frac{\partial}{\partial t}(\rho_\alpha(p_\alpha) S_\alpha) + \operatorname{div}(\rho_\alpha(P_\alpha)\mathbf{q}_\alpha) = \mathscr{F}_\alpha,$$

The Darcy-Muscat law for $\alpha \in \{w, g\}$:

$$\mathbf{q}_\alpha = -\frac{kr_\alpha(S_\alpha)}{\mu_\alpha} \mathbb{K}(\nabla P_\alpha - \rho_\alpha(P_\alpha)\mathbf{g}),$$

Capillary law: $P_c(S_g) = P_g - P_w,$

No void space: $S_w + S_g = 1.$

Introduction of the Global Pressure

In total flow $\mathbf{Q}_t = \rho_w(P_w)\mathbf{q}_w + \rho_g(P_g)\mathbf{q}_g$ as a function of S_g, P_g,

$$\mathbf{Q}_t = -\lambda(S_g,P_g)\mathbb{K}\left(\nabla P_g - f_w(S_g,P_g)\nabla P_c(S_g) - \rho(S_g,P_g)\mathbf{g}\right),$$

eliminate saturation gradient (in order to decouple equations in the fractional flow formulation).

- Idea: introduction of a new pressure-like variable that will eliminate ∇S_g term (*Chavent (1976)*, *Antontsev-Monakhov (1978)*)
- Introduce global pressure P, such that $P_g = P_g(S_g,P)$.
- Then $P_w(S_g,P) = P_g(S_g,P) - P_c(S_g)$.
- Find functions $P_g(S_g,P)$ and $\omega(S_g,P)$ that satisfy:

$$\nabla P_g - f_w(S_g,P_g(S_g,P))P_c(S_g)\nabla S_g = \omega(S_g,P)\nabla P$$

Solution:

$$P_g(S_g,P) = P + P_c(0) + \int_0^{S_g} f_w(s,P_g(s,P))P_c'(s)\,ds$$

and:

$$\omega(S_g,P) = \exp\left(-\int_0^{S_g} (\nu_g(s,P)-\nu_w(s,P))\frac{\rho_w(s,P)\rho_g(s,P)\lambda_w(s)\lambda_g(s)P_c'(s)}{(\rho_w(s,P)\lambda_w(s)+\rho_g(s,P)\lambda_g(s))^2}\,ds\right).$$

where

$$\nu_w(S_g,P) = \frac{\rho_w'(P_w(S_g,P))}{\rho_w(P_w(S_g,P))}, \quad \nu_g(S_g,P) = \frac{\rho_g'(P_g(S_g,P))}{\rho_g(P_g(S_g,P))},$$

are fluid compressibilities.

Notation:

$$\begin{aligned}
\rho_\alpha(S_g,P) &= \rho_\alpha(P_\alpha(S_g,P)),\\
\lambda(S_g,P) &= \rho_w(S_g,P)\lambda_w(S_w) + \rho_g(S_g,P)\lambda_g(S_g),\\
f_\alpha(S_g,P) &= \rho_\alpha(S_g,P)\lambda_\alpha(S_\alpha)/\lambda(S_g,P),\ \alpha = w,g
\end{aligned}$$

New Saturation Variable θ

Energy estimates suggest the use of the new saturation variable θ,

$$\theta = \beta(S) = \int_0^S \sqrt{\lambda_g(s)\lambda_w(s)} P_c'(s)\, ds,$$

which is invertible and denote $S_g = \mathscr{S}(\theta)$.
Diffusivity coefficient:

$$A(S_g, P) = \rho_w(S_g, P)\rho_g(S_g, P) \frac{\sqrt{\lambda_w(S_w)\lambda_g(S_g)}}{\lambda(S_g, P)}$$

and rewrite phase mass fluxes as:

$$\rho_w(S_g,P)\mathbf{q}_w = -\Lambda_w(S_g,P)\mathbb{K}\nabla P + A(S_g,P)\mathbb{K}\nabla\theta + \lambda_w(S_g)\rho_w(S_g,P)^2\mathbb{K}\mathbf{g},$$
$$\rho_g(S_g,P)\mathbf{q}_g = -\Lambda_g(S_g,P)\mathbb{K}\nabla P - A(S_g,P)\mathbb{K}\nabla\theta + \lambda_g(S_g)\rho_g(S_g,P)^2\mathbb{K}\mathbf{g},$$

(θ, P) Formulation

$$
\begin{aligned}
\Phi \frac{\partial}{\partial t}(\rho_w(S_g,P)S_w) - \operatorname{div}(\Lambda_w(S_g,P)\mathbb{K}\nabla P) + \operatorname{div}(A(S_g,P)\mathbb{K}\nabla\theta) \\
+ \operatorname{div}(\lambda_w(S_g)\rho_w(S_g,P)^2\mathbb{K}\mathbf{g}) + \rho_w(S_g,P) f_w(S_g,P)F_P = \rho_w(S_g,P)S_w^* F_I,
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\Phi \frac{\partial}{\partial t}(\rho_g(S_g,P)S_g) - \operatorname{div}(\Lambda_g(S_g,P)\mathbb{K}\nabla P) - \operatorname{div}(A(S_g,P)\mathbb{K}\nabla\theta) \\
+ \operatorname{div}(\lambda_g(S_g)\rho_g(S_g,P)^2\mathbb{K}\mathbf{g}) + \rho_g(S_g,P) f_g(S_g,P)F_P = \rho_g(S_g,P)S_g^* F_I,
\end{aligned} \tag{2}
$$

where $S_g = \mathscr{S}(\theta)$, $S_w = 1 - S_g$.

Boundary conditions: Ω bounded, Lipschitz domain, $\partial\Omega = \Gamma_{inj} \cup \Gamma_{imp}$, $Q_T = \Omega \times]0,T[$.

$$\theta = 0, \quad P = 0 \quad \text{on } \Gamma_{inj} \times]0,T[\tag{3}$$

$$\mathbf{q}_w \cdot \mathbf{n} = \mathbf{q}_g \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{imp} \times]0,T[, \tag{4}$$

where $\mathbf{n}$ is outward pointing unit normal on $\partial\Omega$

Initial conditions:

$$\theta(x,0) = \theta_0(x), \quad P(x,0) = p_0(x) \quad \text{in } \Omega. \tag{5}$$

Assumptions

(A.1) The porosity Φ belongs to $L^\infty(\Omega)$, and there exist constants, $\phi_M \geq \phi_m > 0$, such that $0 < \phi_m \leq \Phi(x) \leq \phi_M$ a.e. in Ω.

(A.2) The permeability tensor $\mathbb{K}$ belongs to $(L^\infty(\Omega))^{n\times n}$, and there exist constants $k_M \geq k_m > 0$, such that for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ it holds:

$$k_m|\xi|^2 \leq \mathbb{K}(x)\xi \cdot \xi \leq k_M|\xi|^2.$$

(A.3) Relative mobilities satisfy $\lambda_w, \lambda_g \in C([0,1];\mathbb{R}^+)$, $\lambda_w(S_w = 0) = 0$ and $\lambda_g(S_g = 0) = 0$; λ_j is a non decreasing function of S_j. Moreover, there exist constants $\lambda_M \geq \lambda_m > 0$ such that for all $S_g \in [0,1]$

$$0 < \lambda_m \leq \lambda_w(S_g) + \lambda_g(S_g) \leq \lambda_M.$$

(A.4) There exist constants $p_{c,min} > 0$ and $M > 0$ such that the capillary pressure function $S_g \mapsto P_c(S_g)$, $P_c \in C([0,1[;\mathbb{R}^+) \cap C^1(]0,1[;\mathbb{R}^+)$, for all $S_g \in]0,1[$

$$P_c'(S_g) \geq p_{c,min} > 0,$$

$$P_c(S_g)(1-S_g) + \int_0^1 P_c(s)\,ds + \sqrt{\lambda_g(S_g)\lambda_w(S_g)}P_c'(S_g) \leq M.$$

(A.5) There exist $S^\# \in]0,1[$, $0 < \gamma$ and $M > 0$ such that for all $S \in]0, S^\#]$

$$S^{-\gamma}\lambda_g(S)(P_c(S) - P_c(0)) + S^{2-\gamma}P_c'(S) \leq M,$$

and for all $S \in [S^\#, 1[$

$$(1-S)^{2-\gamma}P_c'(S) \leq M.$$

(A.6) ρ_w and ρ_g are $C^1(\mathbb{R})$ non decreasing functions, and there exist $\rho_m, \rho_M > 0$ such that for all $p \in \mathbb{R}$ it holds

$$\rho_m \leq \rho_w(p), \rho_g(p) \leq \rho_M, \quad 0 < \rho_w'(p), \rho_g'(p) \leq \rho_M.$$

(A.7) $F_I, F_P \in L^2(Q_T)$, $F_I, F_P \geq 0$, and $0 \leq S_w^* \leq 1$ a.e. in Q_T.

(A.8) There exist $0 < \tau < 1$ and $C > 0$ such that for all $S_1, S_2 \in [0,1]$

$$C\left|\int_{S_1}^{S_2} \sqrt{\lambda_g(s)\lambda_w(s)}\, ds\right|^\tau \geq |S_1 - S_2|.$$

(A.9) $S_g^* = 1$.

Main Theorem

Existence of weak solution of (θ, P)-formulation.

$$V = \{\varphi \in H^1(\Omega) \colon \varphi|_{\Gamma_{inj}} = 0\}.$$

Theorem

Let (A.1)-(A.9) hold and assume $(\theta_0, p_0) \in L^2(\Omega) \times L^2(\Omega)$, $0 \le \theta_0 \le \beta(1)$ a.e. in Ω. Then there exists a weak solution (P, θ) of the problem (1), (2), (3), (4), (5), satisfying

$$P \in L^2(0,T;V),\ \theta \in L^2(0,T;V),\ 0 \le \theta \le \beta(1) \text{ a.e. in } Q_T,\ S = \mathscr{S}(\theta),$$
$$\Phi\partial_t(\rho_w(S,P)(1-S)) \in L^2(0,T;V'), \quad \Phi\partial_t(\rho_g(S,P)S) \in L^2(0,T;V').$$

Main Theorem - The Proof

- Introducing the regularized problem.
  - Capillary pressure may be unbounded at $S = 0$, and its derivative may be unbounded in $S = 0, 1$. - Regularize capillary pressure and its derivative
  - Degeneracy of the "diffusivity" term - Add small constant η to this term
- Existence result for the regularized problem
  - Time discretization
  - Uniform estimates with respect to h
  - Passage to the limit as $h \to 0$.
- Passage to the limit as regularization parameter $\eta \to 0$. (compactness lemma (Chavent-Jaffré, Galusinski-Saad).)

Regularization

Regularization of the capillary pressure is taken as:

$$P_c^\eta(S) = P_c(0) + \int_0^S R_\eta(P_c'(s))\,ds, \tag{6}$$

since its derivative is regularized as follows:

$$R_\eta(P_c'(S)) = \begin{cases} 2(1-\frac{S}{\eta})\frac{P_c(\eta)-P_c(0)}{\eta} + (2\frac{S}{\eta}-1)P_c'(\eta) & \text{for } S \leq \eta \\ P_c'(S) & \text{for } \eta \leq S \leq 1-\eta \\ P_c'(1-\eta) & \text{for } 1-\eta \leq S \leq 1, \end{cases} \tag{7}$$

$P_c^\eta(S)$ properties:

- $P_c^\eta(S)$ is bounded, monotone and $C^1([0,1])$ function, for any $\eta > 0$.
- $\frac{d}{dS}P_c^\eta(S) \geq p_{c,min}/2 > 0$.
- $|R_\eta(P_c'(S))| \leq p_{c,max}^\eta < +\infty$, $p_{c,max}^\eta \to \infty$ when $\eta \to 0$.
- $R_\eta(P_c'(S)) \leq P_c'(S)$, for $S \geq \eta$

Regularization

Define:

$$P_g^\eta(S,P) = P + P_c(0) + \int_0^S f_w(s,P) R_\eta(P_c'(s))\,ds, \tag{8}$$

$$P_w^\eta(S,P) = P - \int_0^S f_g(s,P) R_\eta(P_c'(s))\,ds. \tag{9}$$

$$\omega^\eta(S,P) = \exp\left(-\int_0^S (\nu_g(s,P) - \nu_w(s,P)) \frac{\rho_w(s,P)\rho_g(s,P)\lambda_w(s)\lambda_g(s) R_\eta(P_c'(s))}{(\rho_w(s,P)\lambda_w(s) + \rho_g(s,P)\lambda_g(s))^2}\,ds\right).$$

$$\beta^\eta(S) = \int_0^S \sqrt{\lambda_w(s)\lambda_g(s)} R_\eta(P_c'(s))\,ds. \tag{10}$$

Replace

$\rho_\alpha(S,P)$	with	$\rho_\alpha^\eta(S,P) = \rho_\alpha(P_\alpha^\eta(S,P))$
$A(S,P)$	with	$A^\eta(S,P) = \frac{\rho_w(S,P)\rho_g(S,P)}{\lambda(S,P)} \lambda_w(S)\lambda_g(S) R_\eta(P_c'(S)) + \eta$
$\Lambda_\alpha(S,P)$	with	$\Lambda_\alpha^\eta(S,P) = \lambda_\alpha(S)\rho_\alpha(S,P)\omega^\eta(S,P)$

Regularized problem

$$\Phi\partial_t(\rho_w^\eta(S^\eta,P^\eta)(1-S^\eta)) - \operatorname{div}(\Lambda_w^\eta(S^\eta,P^\eta)\mathbb{K}\nabla P^\eta) + \operatorname{div}(A^\eta(S^\eta,P^\eta)\mathbb{K}\nabla S^\eta)$$
$$+\operatorname{div}(\lambda_w(S^\eta)\rho_w^\eta(S^\eta,P^\eta)^2\mathbb{K}\mathbf{g}) + \rho_w^\eta(S^\eta,P^\eta)f_w(S^\eta,P^\eta)F_P = \rho_w^\eta(S^\eta,P^\eta)(1-S^*)F_I$$

$$\Phi\partial_t(\rho_g^\eta(S^\eta,P^\eta)S^\eta) - \operatorname{div}(\Lambda_g^\eta(S^\eta,P^\eta)\mathbb{K}\nabla P^\eta) - \operatorname{div}(A^\eta(S^\eta,P^\eta)\mathbb{K}\nabla S^\eta)$$
$$+\operatorname{div}(\lambda_g(S^\eta)\rho_g^\eta(S^\eta,P^\eta)^2\mathbb{K}\mathbf{g}) + \rho_g^\eta(S^\eta,P^\eta)f_g(S^\eta,P^\eta)F_P = \rho_g^\eta(S^\eta,P^\eta)S^*F_I$$

Theorem

Let (A.1)-(A.8) hold and assume that $(s_0,p_0) \in V \times V$, $0 \le s_0 \le 1$ a.e. in Ω. For all $\eta > 0$ sufficiently small there exists a weak solution (P^η,S^η) of the regularized problem satisfying

$$P^\eta,\ S^\eta \in L^2(0,T;V),\ 0 \le S^\eta \le 1 \text{ a.e. in } Q_T,$$
$$\Phi\partial_t(\rho_w^\eta(S^\eta,P^\eta)(1-S^\eta)), \Phi\partial_t(\rho_g^\eta(S^\eta,P^\eta)S^\eta) \in L^2(0,T;V').$$

Proof of Theorem 2 (Time Discretization)

Divide $[0,T]$ into N subintervals: $h = T/N$,

$$t_n = nh \quad J_n =]t_{n-1}, t_n], \quad \text{for} 1 \leq n \leq N$$

$$\partial^h v(t) = \frac{v(t+h) - v(t)}{h},$$

for $h > 0$. For any Hilbert space define

$$l_h(\mathscr{H}) = \{v \in L^\infty(0,T;\mathscr{H}) : v \text{ is constant in time on each subinterval } J_n \subset [0,T]\}.$$

For $v^h \in l_h(\mathscr{H})$ is set $v^n = (v^h)^n = v^h|_{J_n} \Rightarrow v^h = \sum_{n=1}^N v^n \chi_{]t_{n-1},t_n]}(t)$, $\quad v^h(0) = v^0$.

For h a discrete system is defined with unknowns $P^h, S^h \in l_h(V)$.

Proposition

Assume (A.1)–(A.8), $0 \leq S^* \leq 1$, $0 \leq s_0 \leq 1$ and $p_0, s_0 \in V$. Then there exists a solution $P^h, S^h \in l_h(V)$ of discrete system, such that

$$0 \leq S^h \leq 1 \quad a.e. \text{ in } Q_T.$$

Proof - based on the Schauder fixed point theorem.

Proof of Theorem 2 (Uniform Estimates)

Test functions used (Introduced by Galusinski-Saad)

$$\varphi = G_g(P_g^{\eta,k}) = \int_{P_c(0)}^{P_g^{\eta,k}} \frac{1}{\rho_g(z)} dz, \quad \psi = G_w(P_w^{\eta,k}) = \int_0^{P_w^{\eta,k}} \frac{1}{\rho_w(z)} dz \tag{11}$$

Defining

$$\mathscr{H}^\eta(S,P) = \left[\rho_w(P_w^\eta) G_w(P_w^\eta) - P_w^\eta\right](1-S) + \left[\rho_g(P_g^\eta) G_g(P_g^\eta) - P_g^\eta\right] S + \int_0^S P_c^\eta(z) dz.$$

Basic estimate:

$$\begin{aligned} &\int_\Omega \Phi \mathscr{H}^\eta(S^h,P^h)(T) dx + \int_{Q_T} (|\nabla P^h|^2 + |\nabla \beta^\eta(S^h)|^2) \, dxdt + \eta \int_{Q_T} |\nabla S^h|^2 dxdt \\ &\leq C \int_{Q_T} (|F_I|^2 + |F_P|^2 + 1) dxdt + \int_\Omega \Phi \mathscr{H}^\eta(s^0,p^0) dx, \end{aligned} \tag{12}$$

which gives weak convergences:

$$S^h \rightharpoonup S, \; P^h \rightharpoonup P, \beta^\eta(S^h) \rightharpoonup \beta^\eta(S)$$

Proof of Theorem 2 (Passage to the Limit as $h \to 0$)

Introduce:

$$r_w^k = \rho_w(P_w^\eta(P^k, S^h))(1 - S^k), \quad r_g^k = \rho_g(P_g^\eta(P^k, S^k))S^k,$$

- $r_\alpha^h \to r_\alpha$ strongly in $L^2(Q_T)$ and a.e. in Q_T,
- P^h converges to P a.e. in Q_T, (and weakly), S^h converges to S a.e. in Q_T (and weakly)! This follows from the continuity of the inverse of the mapping:

$$(S, P) \mapsto (\rho_w(P_w^\eta(S, P))(1 - S), \rho_g(P_g^\eta(S, P))).$$

- limit values can be identified: $r_w = \rho_w(P_w^\eta(P, S))(1 - S), \quad r_g = \rho_g(P_g^\eta(P, S))S.$

We have all that is needed to pass to the limit as $h \to 0$ in the discrete system!

Uniform bounds with respect to η

After passing to the limit $h \to 0$ we get

$$\int_\Omega \Phi \mathscr{H}^\eta(S^\eta, P^\eta)(T)dx + \int_{Q_T} (|\nabla P^\eta|^2 + |\nabla \beta^\eta(S^\eta)|^2)\, dxdt + \eta \int_{Q_T} |\nabla S^\eta|^2 dxdt$$
$$\leq C \int_{Q_T} (|F_I|^2 + |F_P|^2 + 1) dxdt + \int_\Omega \Phi \mathscr{H}^\eta(s^0, p^0) dx.$$

It follows:

$$(P^\eta)_\eta \text{ is uniformly bounded in } L^2(0,T;V),$$
$$(\beta^\eta(S^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T;V),$$
$$(\sqrt{\eta}\nabla S^\eta)_\eta \text{ is uniformly bounded in } L^2(Q_T)^d,$$
$$(\Phi \partial_t(\rho_w(P_w^\eta)(1-S^\eta)))_\eta \text{ is uniformly bounded in } L^2(0,T;V'),$$
$$(\Phi \partial_t(\rho_g(P_g^\eta)S^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T;V').$$

Compactness Result in the Degenerate Case

1.

Lemma

For every $c > 0$ and for sufficiently small $\eta_0 > 0$ the following set

$$E_c^{\eta_0} = \{(r_w^\eta = \rho_w(P_w^\eta(S,P))(1-S), r_g^\eta = \rho_g(P_g^\eta(S,P))S) \colon 0 < \eta \le \eta_0,$$
$$\|P\|_{L^2(0,T;V)} \le c, \quad \|\beta^\eta(S)\|_{L^2(0,T;V)} \le c,$$
$$\|\Phi\partial_t r_w^\eta\|_{L^2(0,T;V')} + \|\Phi\partial_t r_g^\eta\|_{L^2(0,T;V')} \le c\}$$

is relatively compact in $L^2(Q_T) \times L^2(Q_T)$.

2. The mapping

$$(S,P) \mapsto (r_w^\eta, r_g^\eta)$$

is a homeomorphism.

Convergences with respect to η

Lemma

Up to subsequences the following convergence results hold for $(\theta^\eta)_\eta$, $\theta^\eta = \beta^\eta(S^\eta)$ *and* $(P^\eta)_\eta$*:*

$$P^\eta \rightharpoonup P \ \text{ weakly in } L^2(0,T;V) \text{ and a.e. in } Q_T, \tag{13}$$

$$\theta^\eta \rightharpoonup \theta \ \text{ weakly in } L^2(0,T;V) \text{ and a.e. in } Q_T, \tag{14}$$

$$S^\eta \to \mathscr{S}(\theta) \ \text{ a.e. in } Q_T, \tag{15}$$

$$\Phi\partial_t(\rho_w(P_w^\eta(S^\eta,P^\eta))(1-S^\eta)) \rightharpoonup \Phi\partial_t(\rho_w(P_w(\mathscr{S}(\theta),P))(1-\mathscr{S}(\theta))) \tag{16}$$
$$\text{weakly in } L^2(0,T;V')$$

$$\Phi\partial_t(\rho_g(P_g^\eta(S^\eta,P^\eta))S^\eta) \rightharpoonup \Phi\partial_t(\rho_g(P_g(\mathscr{S}(\theta),P))\mathscr{S}(\theta)) \ \text{ weakly in } L^2(0,T;V'). \tag{17}$$

Moreover, $0 \le \theta \le \beta(1)$ *a.e. in* Q_T.

We have obtained all convergences needed to pass to the limit as $\eta \to 0$ in the weak formulation of the regularized problem!

Conclusion

The global pressure formulation makes the coupling between the two equations less strong, implying that in the mathematical analysis of the system:

- Less regularization is needed.
- More general data can be used.

Thank you for your attention