Multiscale Discretizations for Flow, Transport, and Mechanics in Porous Media

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Outline

- Motivation
- Mortar mixed finite element (MMFE) methods for multiphase flow problem
- Time splitting for MMFE for multiphase flow and mixed/Godunov methods for diffusion-dispersion and reactive transport
- Numerical experiments
- Extensions to DG and DG-MMFE for flow and Galerkin for elasticity
- Conclusions
- Current and Future Work

Motivation

- Goal is to solve heterogeneous subsurface flow problems: multiphase flow, coupled with reactive transport and geomechanics in a multiscale setting.
- Applications: NAPL remediation, monitoring of nuclear wastes, C02 sequestration saline aquifers.
- Traditional method - uniform grid everywhere, too expensive. Mortars lead to attractive dynamic meshing strategies and multiphysics couplings.
- Cannot avoid if physical domain is irregular! No single smooth map to a regular computational grid exists.

Societal Needs in Relation to Geological Systems

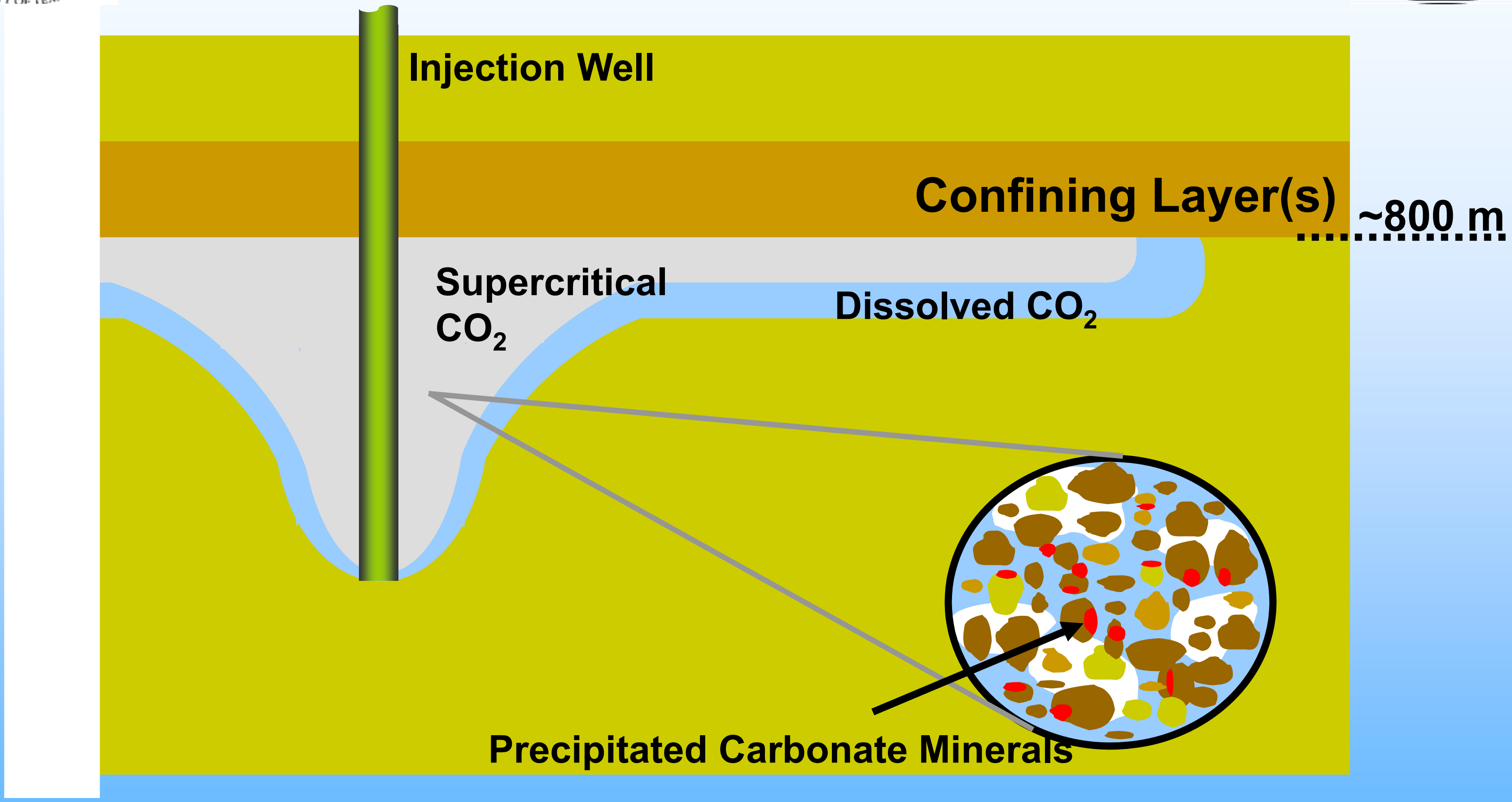
Resources Recovery
- Petroleum and natural gas recovery from conventional/unconventional reservoirs
- *In situ* mining
- Hot dry rock/enhanced geothermal systems
- Potable water supply
- Mining hydrology

Waste Containment/Disposal
- Deep waste injection
- Nuclear waste disposal
- CO_2 sequestration
- Cryogenic storage/petroleum/gas

Underground Construction
- Civil infrastructure
- Underground space
- Secure structures

Site Restoration
- Aquifer remediation
- Acid-rock drainage

CO_2 Injection and Trapping Mechanisms



Stratigraphic Trapping | **Solubility Trapping** | **Hydrodynamic Trapping** | **Mineral Trapping**

Celia et al, 2002

Diffusion-Dispersion

$$\frac{\partial(\varphi_i^* c_{iw})}{\partial t} - \nabla \cdot \mathbf{D}_i^* \nabla c_{iw} = 0$$

Solved fully-implicitly using Expanded MFEM with full-tensor

Introduce $\tilde{\mathbf{z}} = -\nabla c, \quad \mathbf{z} = \mathbf{D}_i^* \tilde{\mathbf{z}}$

Find $\tilde{\mathbf{z}}_{h,iw}^{m+1}|_{\Omega_j} \in \tilde{\mathbf{V}}_{h,j},\ \mathbf{z}_{h,iw}^{m+1}|_{\Omega_j} \in \mathbf{V}_{h,j},\ c_{h,iw}^{m+1}|_{\Omega_j} \in W_{h,j}$ such that:

$$\left(\frac{\varphi_i^{*,m+1} c_{h,iw}^{m+1} - \widehat{T}_i}{\Delta \tau^{m+1}}, w \right)_{\Omega_j} + \left(\nabla \cdot \mathbf{z}_{h,iw}^{m+1}, w \right)_{\Omega_j} = 0, w \in W_{h,j}$$

$$(\tilde{\mathbf{z}}_{h,iw}^{m+1}, \mathbf{v})_{\Omega_j} = (c_{h,iw}^{m+1}, \nabla \cdot \mathbf{v})_{\Omega_j} - \left\langle \mathcal{P}_j c_{h,iw}, \mathbf{v} \cdot \mathbf{n}_j \right\rangle_{\Gamma_j}, \mathbf{v} \in \mathbf{V}_{h,j}$$

$$(\mathbf{z}_{h,iw}^{m+1}, \tilde{\mathbf{v}})_{\Omega_j} = (\mathbf{D}_i^{*,m+1} \tilde{\mathbf{z}}_{h,iw}^{m+1}, \tilde{\mathbf{v}})_{\Omega_j}, \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_{h,i}$$

Simplifying Assumptions

- Flow is independent of transport.
- Inter-phase distribution of species assumed to be ``locally equilibrium'' controlled, instantaneously.
- Ignore adsorption.

Preliminaries

$\bar{\Omega} = \cup_{i=1}^{n_b} \bar{\Omega}_i$: computational domain is decomposed into non-overlapping subdomain blocks

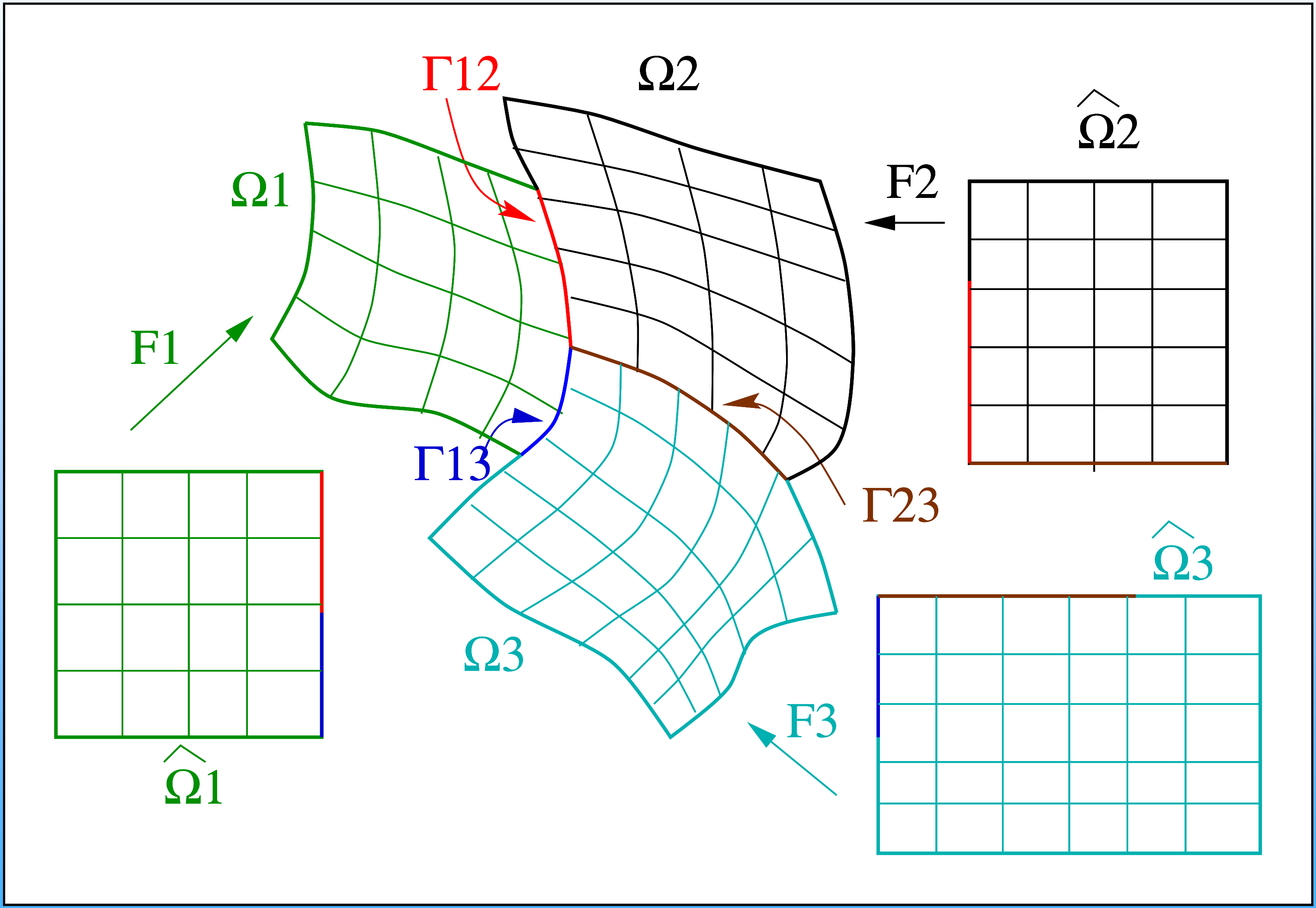
$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j, \quad \Gamma = \cup_{i,j=1}^{n_b} \Gamma_{ij}, \quad \Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \backslash \partial\Omega$$

On each block Ω_i : $\mathcal{T}_{h,i}$ – finite element partition
$\mathbf{V}_{h,i} \times W_{h,i} \subset H(\text{div};\Omega_i) \times L^2(\Omega_i)$ – MFE spaces on $\mathcal{T}_{h,i}$

On each interface $\Gamma_{i,j}$: $\mathcal{T}_{H,i,j}$ – interface finite element grid
$M_{H,i,j} \subset L^2(\Gamma_{i,j})$ – mortar space on $\mathcal{T}_{H,i,j}$

$$\mathbf{V}_h = \bigoplus_{i=1}^{n_b} \mathbf{V}_{h,i}, \qquad W_h = \bigoplus_{i=1}^{n_b} W_{h,i} \qquad M_H = \bigoplus_{1\le i<j\le n_b} M_{H,i,j}$$

Mortar Domain Decomposition



Equations for Multiphase Flow

Mass balance in each sub-domain: $\dfrac{\partial(\varphi\rho_\alpha S_\alpha)}{\partial t}+\nabla\cdot(\rho_\alpha \mathbf{u}_\alpha) = q_\alpha$

Darcy's law (constitutive equation): $\mathbf{u}_\alpha = -\dfrac{k_{r\alpha}(S_\alpha)K}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g \nabla D)$

Saturation constraint: $\sum_\alpha S_\alpha = 1$

Capillary pressure relation: $p_c(S_w) = p_n - p_w$

Continuity: On each interface $\Gamma_{i,j}$ physically meaningful BC applied:

$$p_\alpha|_{\Omega_i} = p_\alpha|_{\Omega_j} \quad [\mathbf{u}_\alpha \cdot \mathbf{n}]_{i,j} \equiv \mathbf{u}_\alpha|_{\Omega_i} \cdot \mathbf{n}_i + \mathbf{u}_\alpha|_{\Omega_j} \cdot \mathbf{n}_j = 0$$

Expanded MMFE Method for Flow

Introduce a pressure gradient term to avoid inverting $k_{r\alpha}$:

$$\tilde{\mathbf{u}}_\alpha = -K/\mu_\alpha(\nabla p_\alpha - \rho_\alpha g \nabla D), \quad \mathbf{u}_\alpha = k_{r\alpha}(S_\alpha)\tilde{\mathbf{u}}_\alpha$$

In a backward Euler multi-block, we seek $\mathbf{u}^n_{\alpha,h}|_{\Omega_i} \in \mathbf{V}_{h,i}$, $\tilde{\mathbf{u}}^n_{\alpha,h}|_{\Omega_i} \in \tilde{\mathbf{V}}_{h,i}$, $p^n_h|_{\Omega_i} \in W_{h,i}$, $S^n_h|_{\Omega_i} \in W_{h,i}$, $p^n_H|_{\Gamma_{i,j}} \in M_{H,i,j}$, $S^n_H|_{\Gamma_{i,j}} \in M_{H,i,j}$ for $1 \le i < j \le n_b$, such that:

$$\left(\frac{\Delta(\varphi\rho_{\alpha,h}S_{\alpha,h})^n}{\Delta t^n}, w\right)_{\Omega_i} + \left(\nabla\cdot\rho^n_{\alpha,h}\mathbf{u}^n_{\alpha,h}, w\right)_{\Omega_i} = (q^n_\alpha, w)_{\Omega_i}, w \in W_{h,i}$$

$$\left(\left(\frac{K}{\mu_{\alpha,h}}\right)^{-1}\tilde{\mathbf{u}}^n_{\alpha,h}, \mathbf{v}\right)_{\Omega_i} = (p^n_{\alpha,h}, \nabla\cdot\mathbf{v})_{\Omega_i} - \left\langle p^n_{\alpha,H}, \mathbf{v}\cdot\mathbf{n}_i\right\rangle_{\Gamma_i} + (\rho^n_{\alpha,h} g\nabla D, \mathbf{v})_{\Omega_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}$$

$$(\mathbf{u}^n_{\alpha,h}, \tilde{\mathbf{v}})_{\Omega_i} = (k^n_{r\alpha,h}\tilde{\mathbf{u}}^n_{\alpha,h}, \tilde{\mathbf{v}})_{\Omega_i}, \quad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_{h,i}$$

$$\left\langle \left[\mathbf{u}^n_{\alpha,h}\cdot\mathbf{n}\right]_{i,j}, \zeta\right\rangle_{\Gamma_{i,j}} = 0, \quad \zeta \in M_{H,i,j}$$

Reduction to an Interface Problem

Let $\mathbf{M}_H = M_H \times M_H$. Define

$$b^n(\psi,\eta) = \sum_{1\le i<j\le n_b} \sum_\alpha \int_{\Gamma_{i,j}} \left[\rho^n_{\alpha,h}\mathbf{u}^n_{\alpha,h}(\psi)\cdot\mathbf{n}\right]_{ij} \eta_\alpha \, ds,$$

where $\psi = (p^n_{w,H}, S^n_{w,H}) \in \mathbf{M}_H$, $\eta = (\eta_w, \eta_w) \in \mathbf{M}_H$

Define the non-linear interface operator $\mathcal{B}^n : \mathbf{M}_H \to \mathbf{M}_H$ by

$$\left\langle \mathcal{B}^n\psi, \eta \right\rangle = b^n(\psi,\eta), \quad \forall \eta \in \mathbf{M}_H$$

Then $(\psi, p^n_{\alpha,h}(\psi), S^n_{\alpha,h}(\psi), \mathbf{u}^n_{\alpha,h}(\psi))$ solves the multiphase flow equations when $\mathcal{B}^n(\psi) = 0$

Interface problem is solved by “inexact Newton-GMRES” scheme

Reactive Species Transport

Mass balance of species i in phase α :

$$\frac{\partial(\varphi c_{i\alpha} S_\alpha)}{\partial t} + \nabla \cdot (c_{i\alpha}\mathbf{u}_\alpha - \varphi S_\alpha \mathbf{D}_{i\alpha} \nabla c_{i\alpha}) = r(c_{i\alpha})$$

$$\mathbf{D}_{i\alpha} \nabla c_{i\alpha} \cdot \mathbf{n} = 0$$

Diffusion-Dispersion tensor $\mathbf{D}_{i\alpha} = \mathbf{D}_{i\alpha}^{\text{diff}} + \mathbf{D}_{i\alpha}^{\text{hyd}}$:

Molecular diffusion: $\mathbf{D}_{i\alpha}^{\text{mol}} = \tau_\alpha d_{\text{m},i\alpha} \mathcal{I}$

Physical dispersion: $\varphi S_\alpha \mathbf{D}_{i\alpha}^{\text{hyd}} = d_{t,\alpha} |\mathbf{u}_\alpha| \mathcal{I} + (d_{l,\alpha} - d_{t,\alpha}) \frac{\mathbf{u}_\alpha \mathbf{u}_\alpha^T}{|\mathbf{u}_\alpha|}$

Source term: $r(c_{i\alpha}) = r_{i\alpha}^I + \varphi S_\alpha r_{i\alpha}^C + q_{i\alpha}$,

where $r_{i\alpha}^I$ is influx/efflux from other phases,

$r_{i\alpha}^C$ is chemical rate of decay

$q_{i\alpha}$ is a source (or sink) term

Phase-Summed Equations

Assume an equilibrium partitioning of species between phases:

$$c_{i\alpha} = \theta_{i\alpha} c_{i\alpha_0}$$

Sum over all phases for a given species:

$$\frac{\partial(\varphi_i^* c_{iw})}{\partial t} + \nabla \cdot (c_{iw}\mathbf{u}_i^* - \mathbf{D}_i^* \nabla c_{iw}) = r_i^*(\mathbf{c}_w)$$

$$\mathbf{D}_{iw} \nabla c_{iw} \cdot \mathbf{n} = 0$$

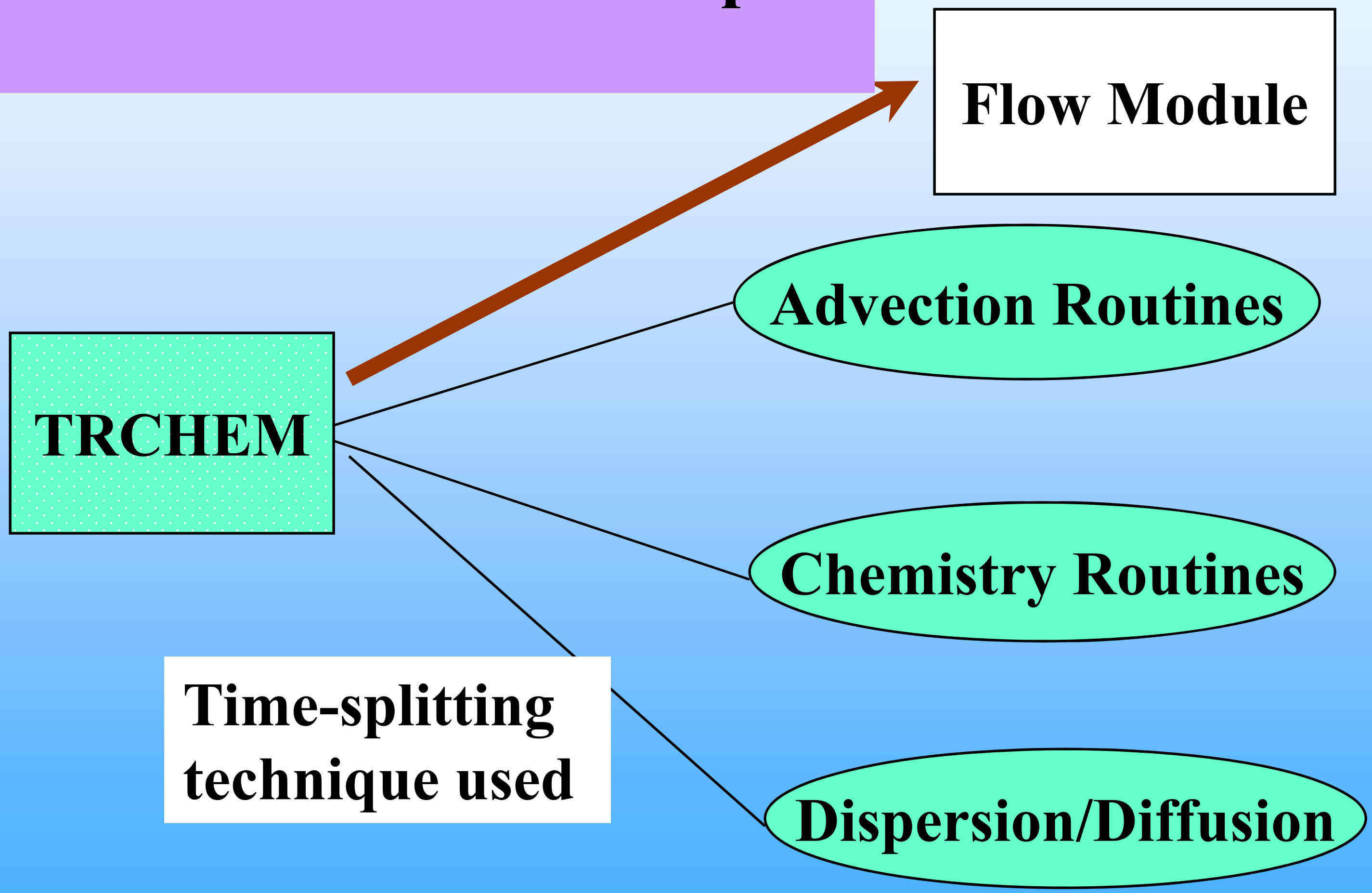
$$\varphi_i^* = \varphi \sum_\alpha \theta_{i\alpha} S_\alpha \qquad u_i^* = \sum_\alpha \theta_{i\alpha} \mathbf{u}_\alpha$$

$$\mathbf{D}_i^* = \varphi \sum_\alpha S_\alpha \theta_{i\alpha} \mathbf{D}_{i\alpha} \qquad r_i^*(\mathbf{c}_w) = \varphi \sum_\alpha r_{i\alpha}^C - r_{iR} + \sum_\alpha q_{i\alpha}$$

Note: $\sum_\alpha r_{i\alpha}^I + r_{iR} = 0,$ where r_{iR} is the influx/efflux of species i into the stationary phase

IPARS-TRCHEM Structure

IPARS TRCHEM =
Flow and Reactive Transport



Advection

$$\left(\frac{\partial \varphi_i^* c_{iw}}{\partial t}, w\right)_{\Omega_j} + (\nabla \cdot (c_{iw}\mathbf{u}_i^*), w)_{\Omega_j} = \left(\sum_\alpha q_{i\alpha}, w\right)_{\Omega_j}, w \in W_j$$

Solved using a Godunov scheme

First order Godunov scheme

Let $T_i^m = \varphi_i^{*,m} c_{h,iw}^m$, solve for $\overline{T}_i$ from

$$\left(\frac{\overline{T}_i - T_i^m}{\Delta\tau^{m+1}}, w\right)_{\Omega_j} + \sum_{E \in \mathcal{T}_{h,j}} \left\langle c_{h,iw}^{m,\text{upw}} \mathbf{u}_{h,i}^{*,m+1/2} \cdot \mathbf{n}_E, w \right\rangle_{\partial E} = \left(\sum_\alpha q_{i\alpha}, w\right)_{\Omega_j}$$

Chemical Reaction

$$\text{Define } \Phi(t) \equiv \text{diag}\{\varphi_i^*(t)\}, \quad \mathbf{T} = \mathbf{T}(t) \equiv \Phi(t)\mathbf{c}_w, \text{ and}$$

$$r_i^{*,C}(\mathbf{T}) \equiv \varphi \sum_\alpha r_{i\alpha}^C(\Phi^{-1}(t)\mathbf{T}) \quad \text{Then} \quad \frac{\partial \mathbf{T}_i}{\partial t} = r_i^{*,C}(\mathbf{T})$$

Solved by explict ODE integration using Runge-Kutta

Second order Runge-Kutta scheme

$$k_{1,i} = \Delta\tau^{m+1}\, r_i^{*,C}(\overline{\mathbf{T}})$$

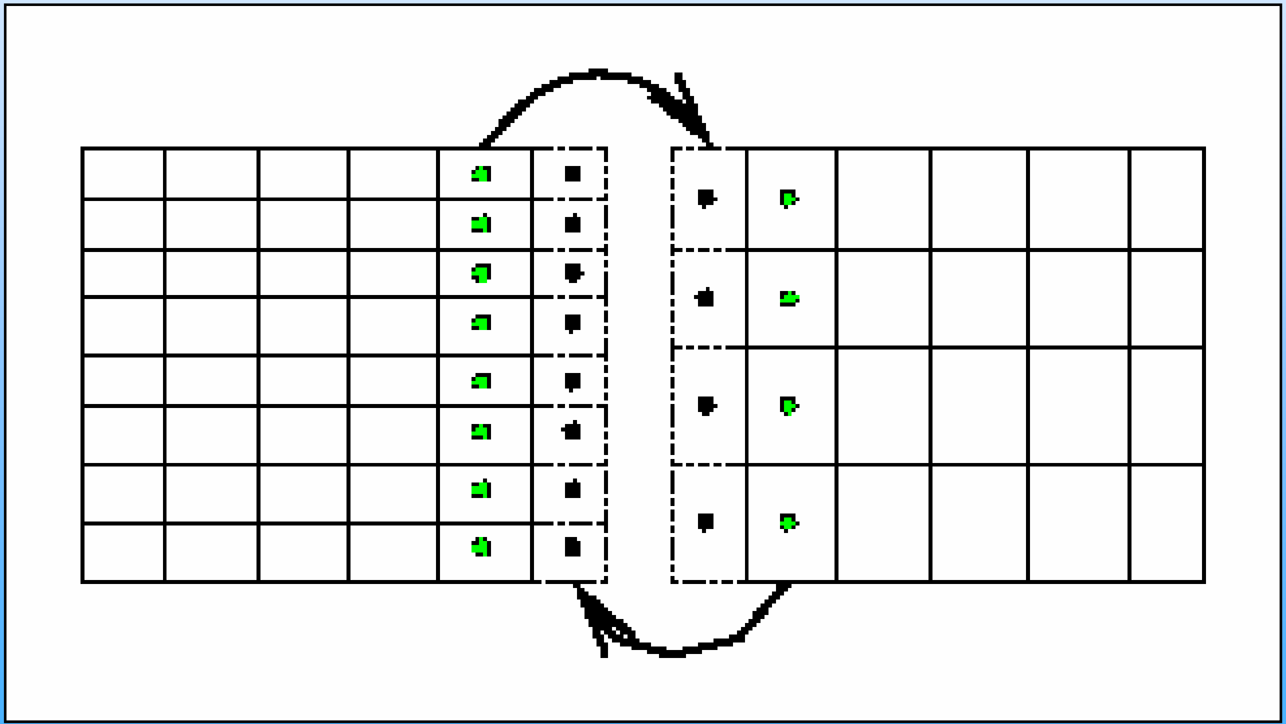
$$k_{2,i} = \Delta\tau^{m+1}\, r_i^{*,C}\left(\overline{\mathbf{T}} + \frac{1}{2}\mathbf{k}_1\right)$$

$$\widehat{T}_i = \overline{T} + k_{2,i}$$

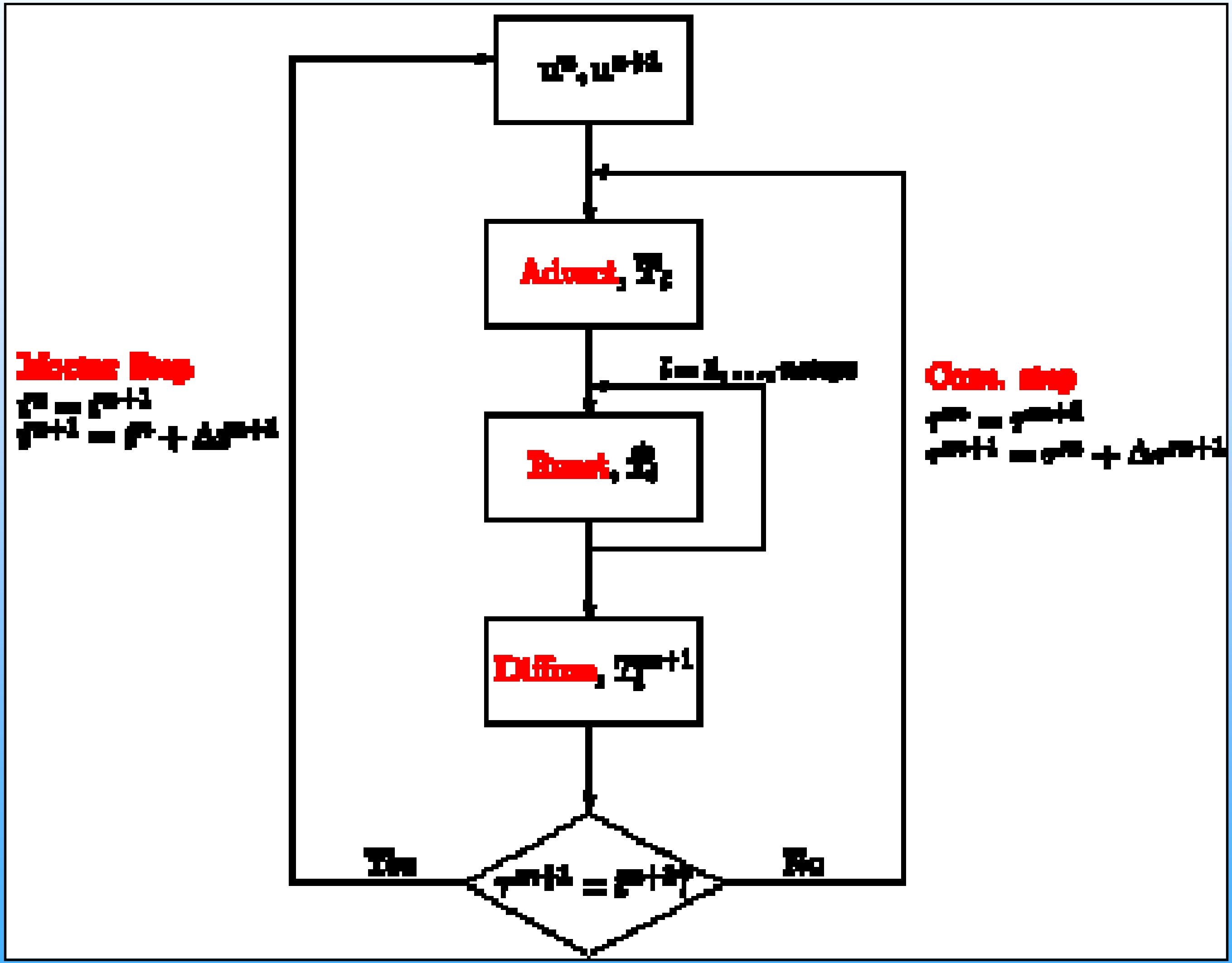
Accounting for Non-matching grids in Transport Problem

$\mathcal{P}_j : L^2(\Gamma_j) \to L^2(\Gamma_k)$ is an L^2-orthogonal proj. s.t. $\forall \phi \in L^2(\Gamma_j)$

$\langle \phi - \mathcal{P}_j \phi, \mathbf{v} \cdot \mathbf{n}_j \rangle_{\Gamma_{k,j}} = 0$, $\forall \mathbf{v} \in \mathbf{V}_{h,i}, \forall k$ such that $\overline{\Omega}_k \cap \overline{\Omega}_j \neq \emptyset$

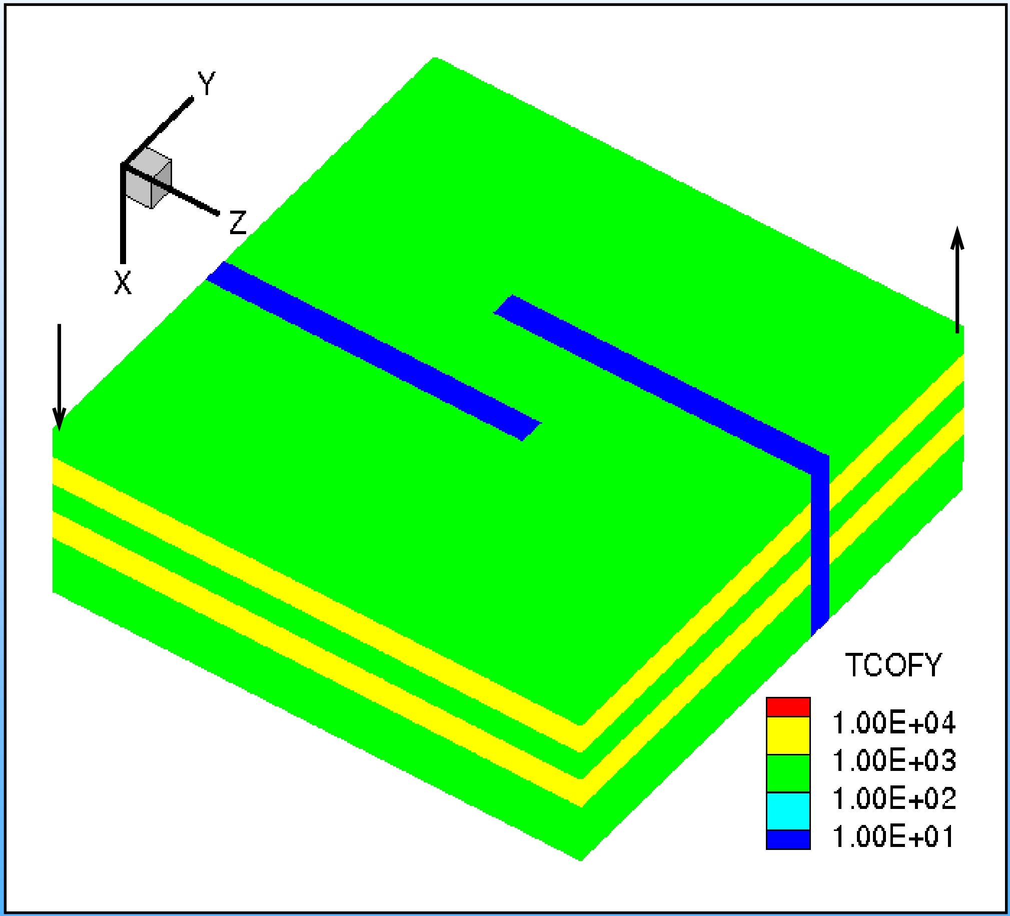


Algorithm

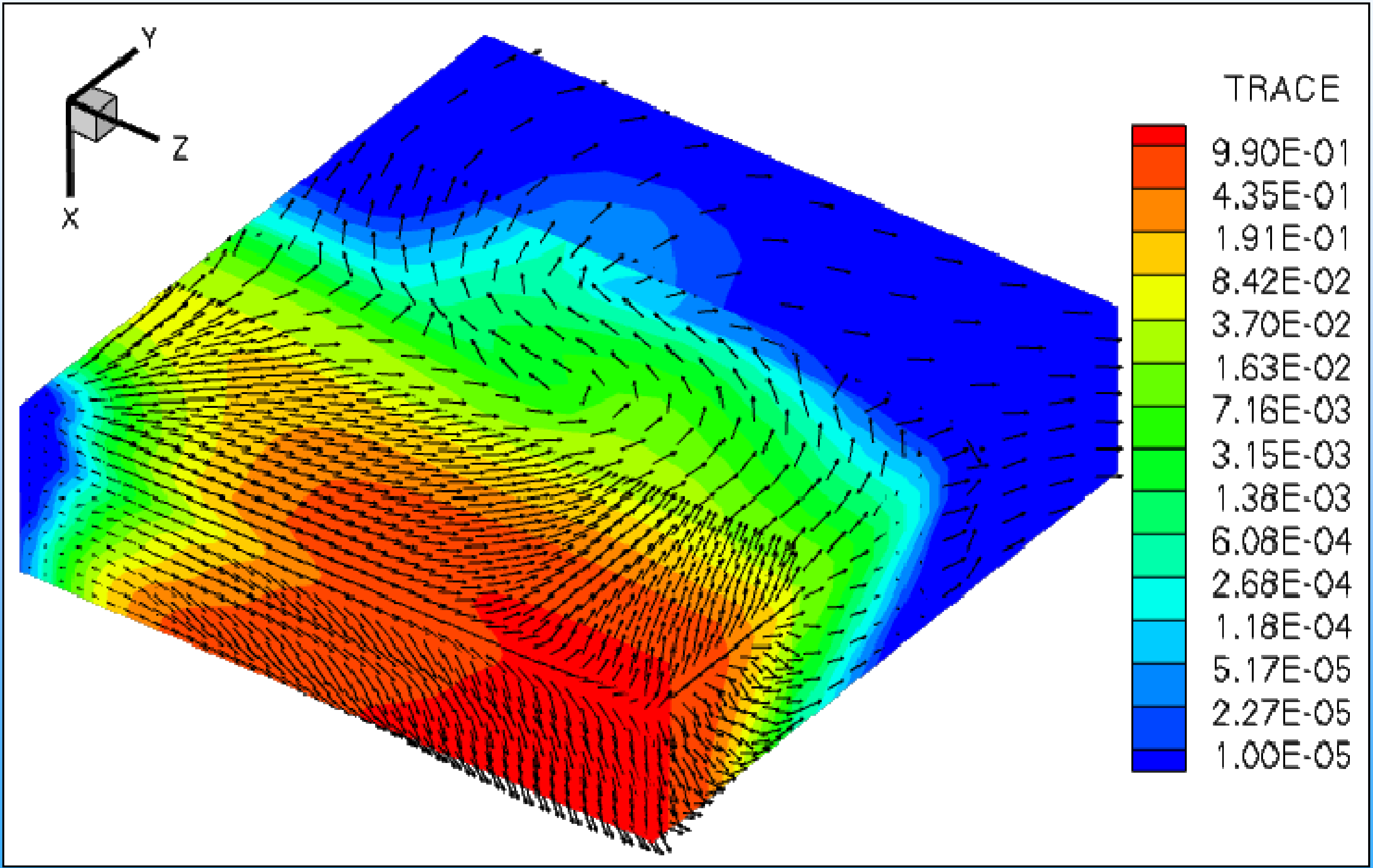


NAPL Remediation

- Bio-remediation of NAPL using microbes
- Advection-Diffusion-Reaction
- Discontinuous permeability field with barriers
- Two flowing phases - quarter-five spot
- External BC: no-flow and zero diffusive flux
- IC: NAPL, microbes occupy $0 < y < 40$ ft and O_2, N_2 occupy $40 < y < 400$ ft.
- Domain: 20 ft x 400 ft x 400 ft
- Reference case: NX=20, NY=40, NZ=40

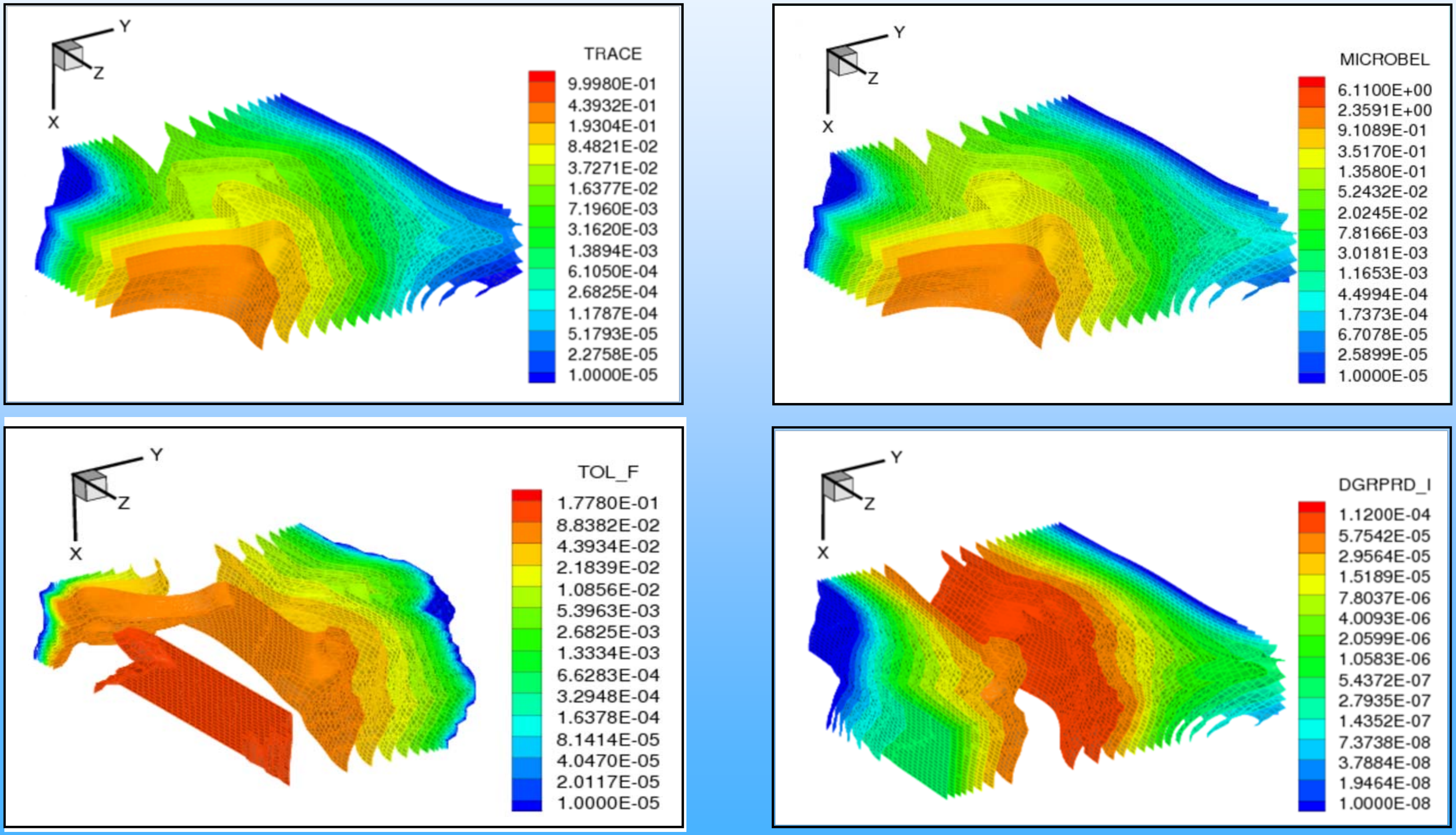


Flow Pattern in Multi-block



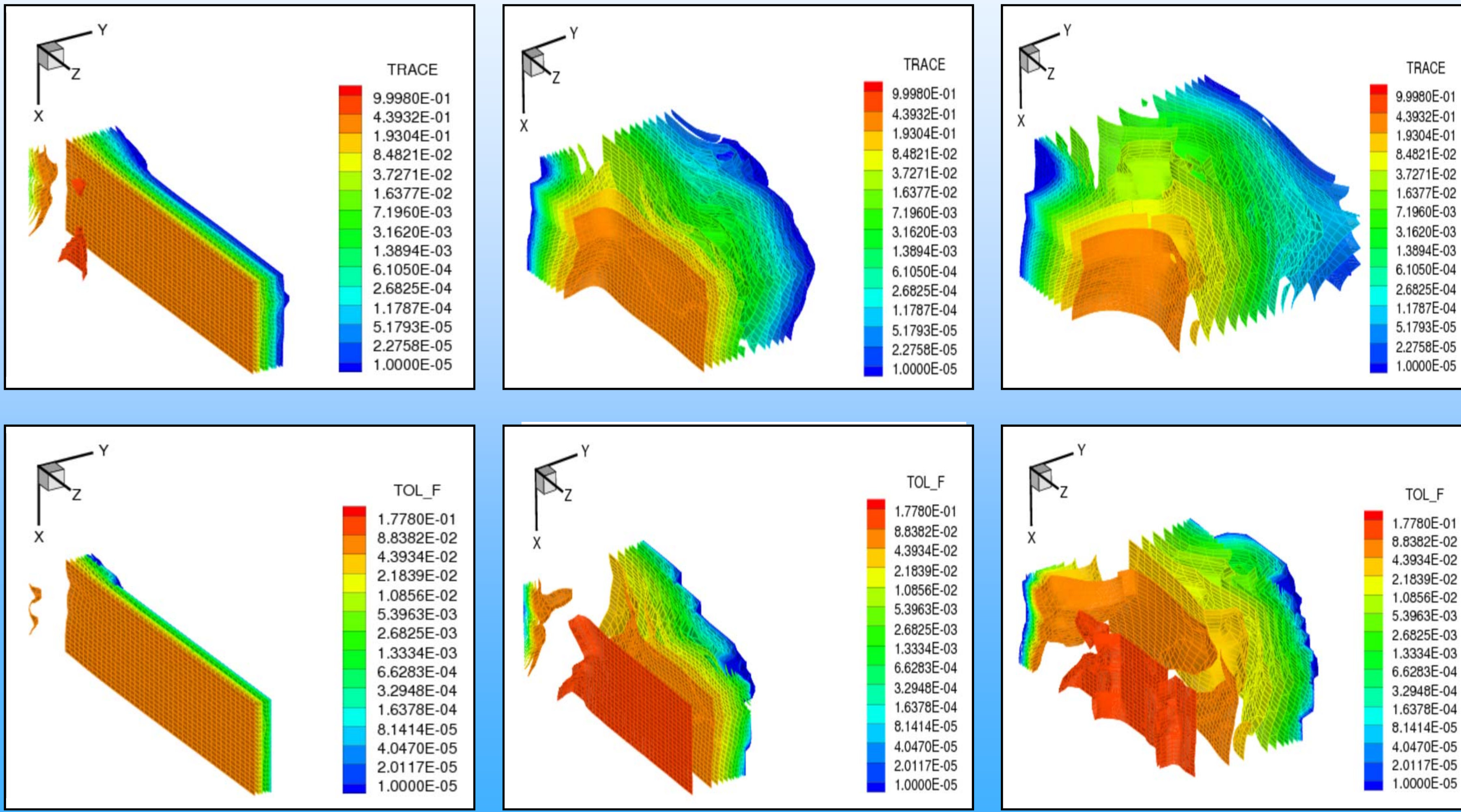
Reference Solution

Concentrations of tracer, NAPL, microbes and bio-degraded product at 100 days



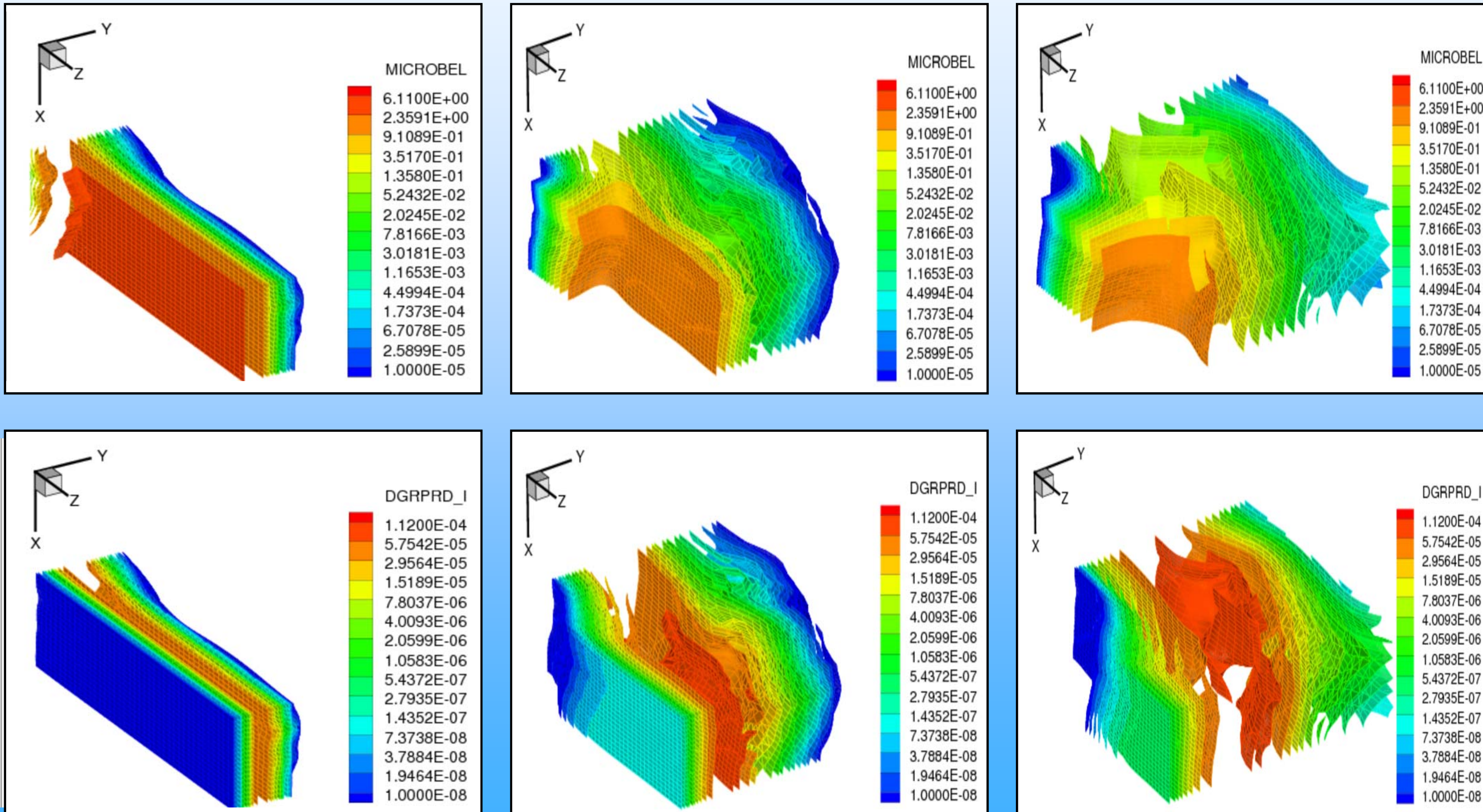
Comparison to Mortar Scheme

Tracer & NAPL concentrations at 5, 50, 100 days



Comparison to Mortar Scheme, cont'd

Microbe & product concentrations at 5, 50, 100 days



Multinumeric Extensions

- DG and Mixed FEM can be combined for treating flow using mortar spaces
- DG is applicable for both flow and transport on non-matching grids
- Examples for single-phase slightly compressible flow follow

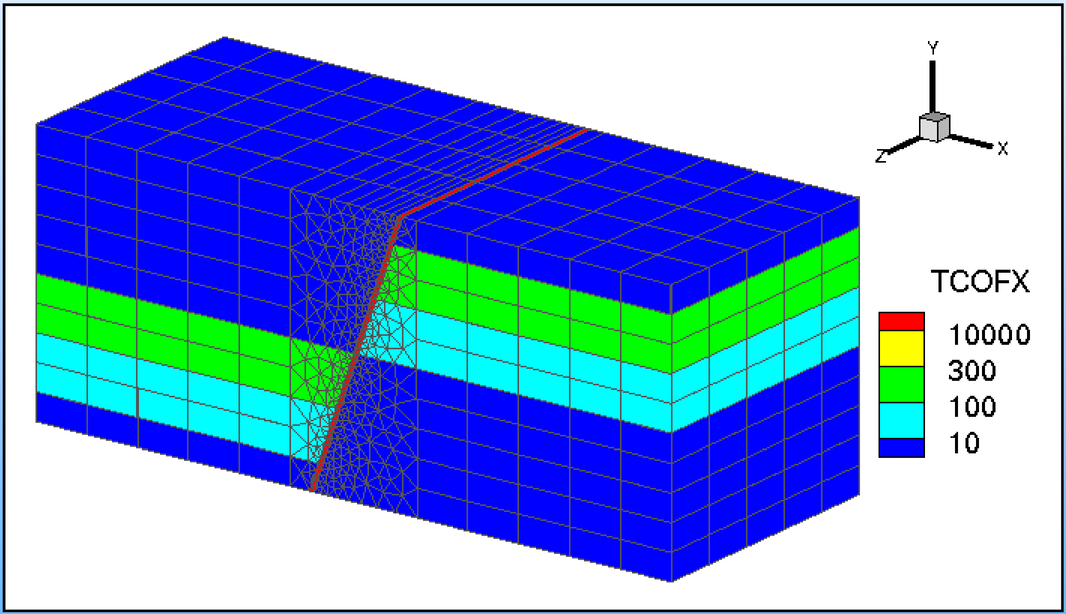
DG-MFEM, 3 blocks with a fault

- 250 x 100 x 100 ft
- 2 ft wide fault: 10000 mD, ϕ=0.01
- 4 geological layers: 10, 100, 300, 10 mD, ϕ=0.2
- BC:

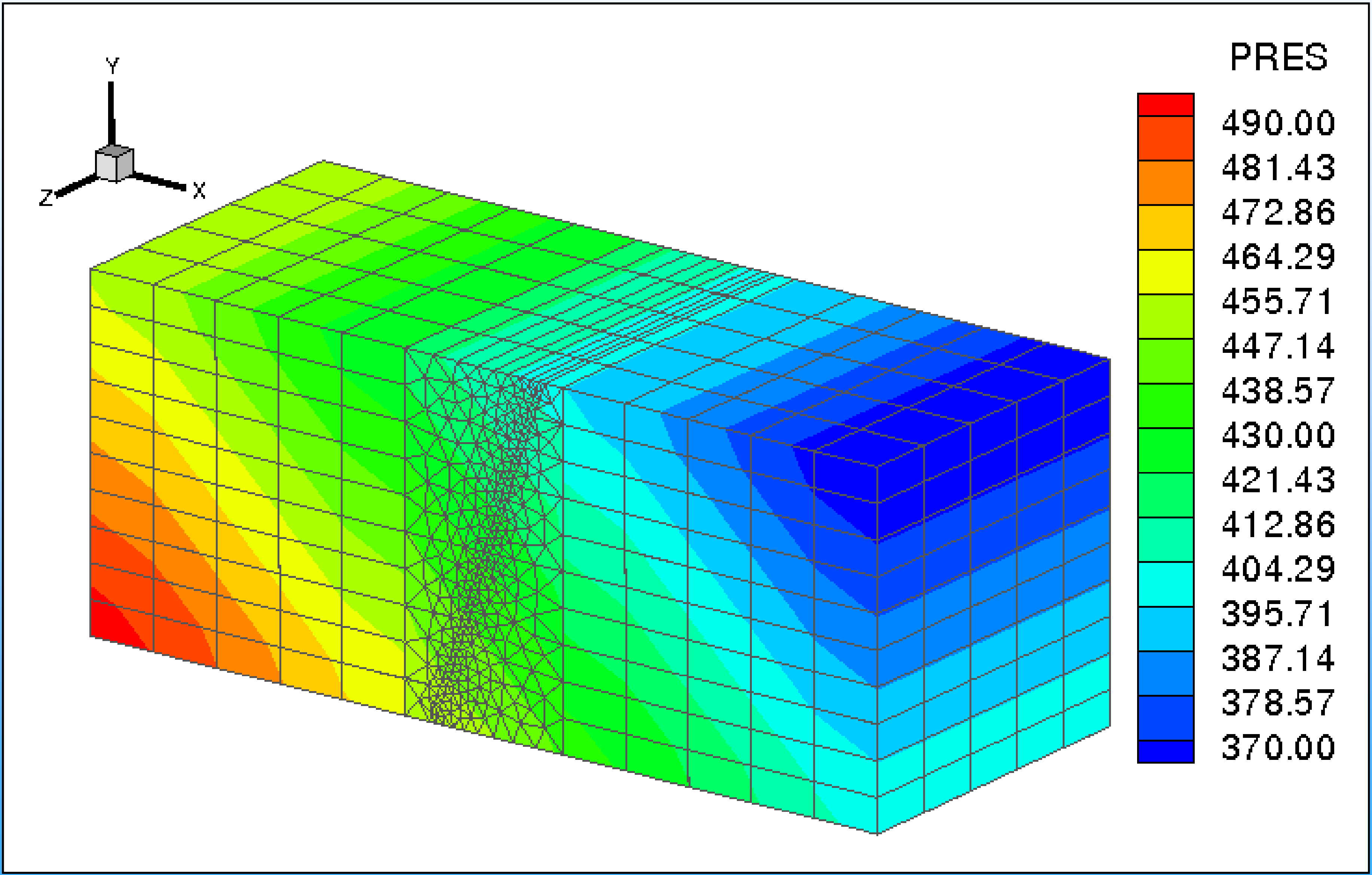
  500 psi at x=0

  400 psi at x=250,

  noflow o.w.
- r=2, k=0 (RT0), m=0

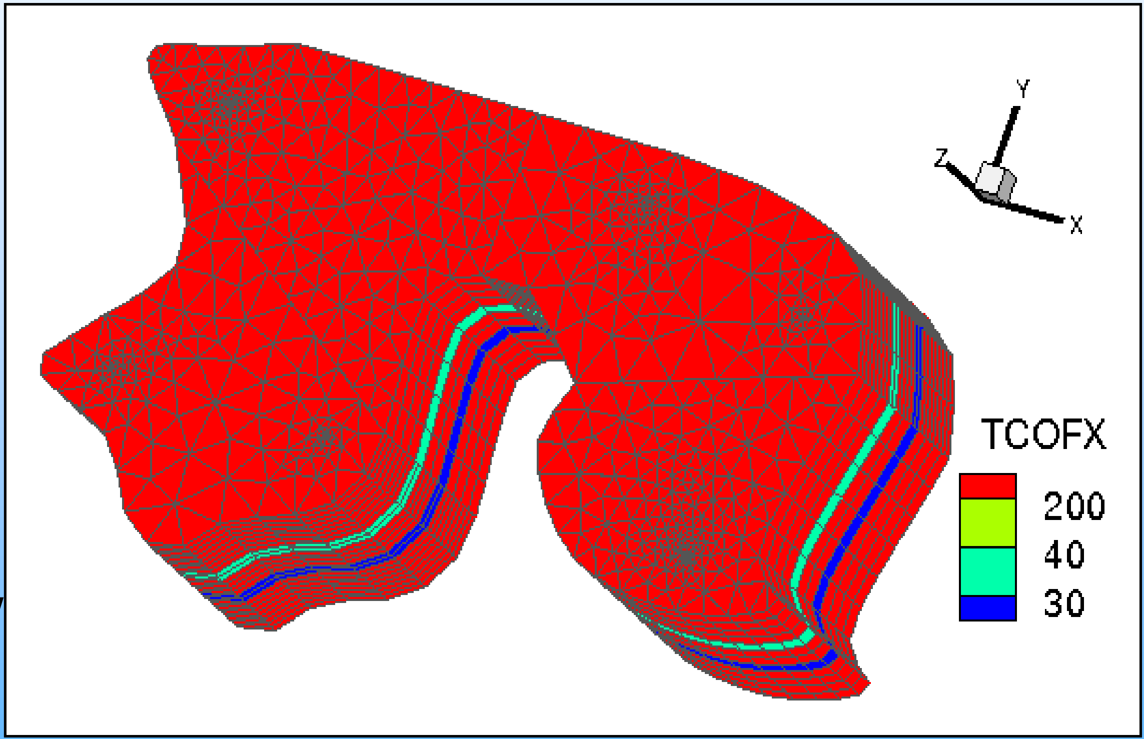


3 blocks with a fault : Solution

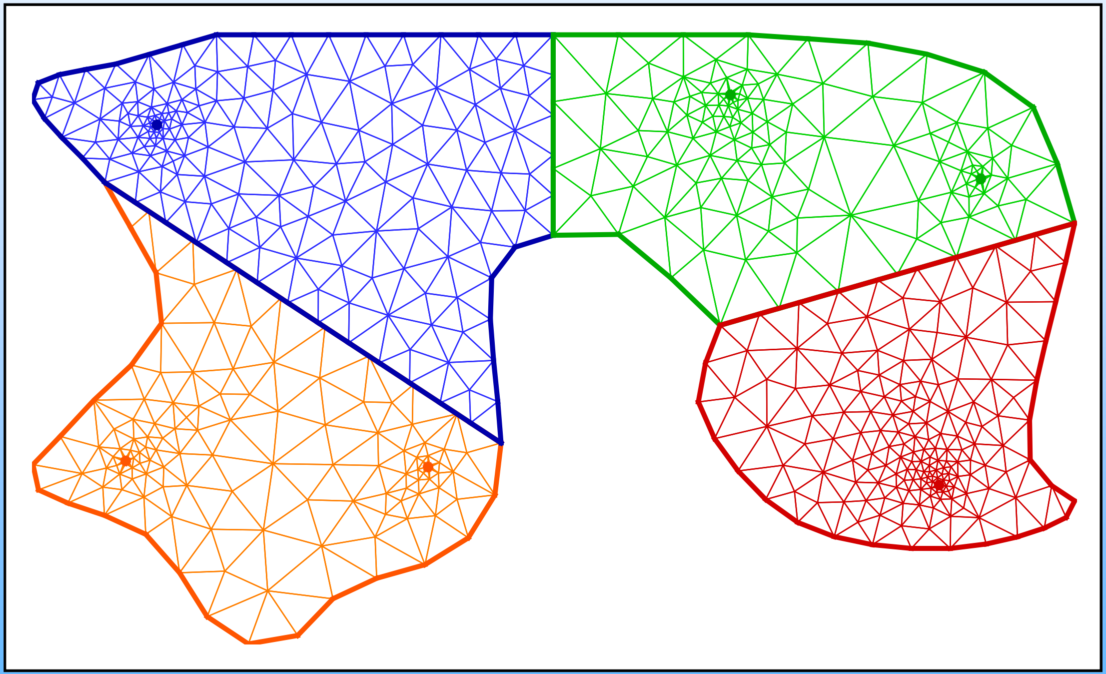


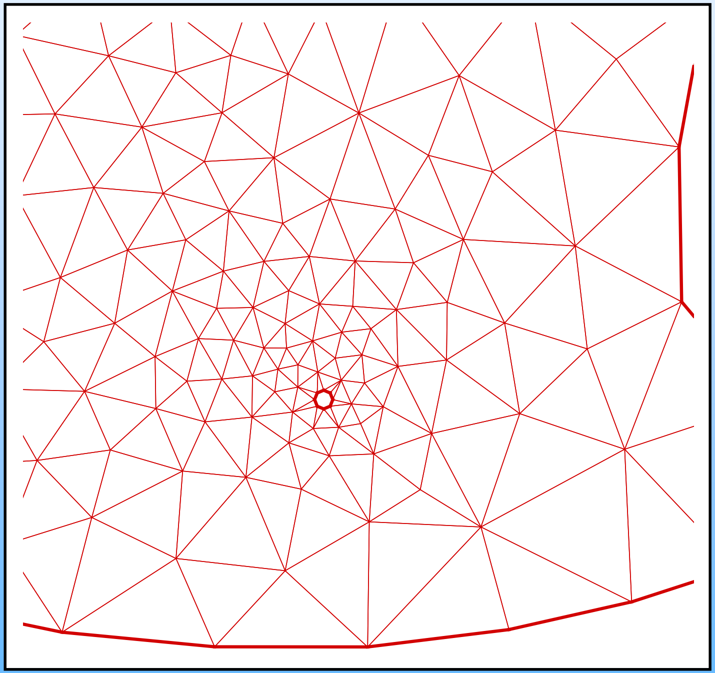
DG-DG, Oxbow problem

- 4 blocks with 6 wells
- 900 x 500 x 24 ft
- 1 production, 5 injection wells
- K_{xx} = K_{yy}={200,30,40}, K_{zz}={25,5,3}, ϕ={0.22,0.08,0.09}
- BC: P_{inj} = 700 psi, P_{prod} = 500 psi, noflow on the outer bdry
- nonmatching grids, r=2, m=1



Unstructured Mesh





Top view, 4 blocks

Magnified grid around well

Linear Elasticity Problem

Find $u \in H^1(\Omega)$ s.t.

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(u) &= f && \text{in } \Omega \\
u &= u_D && \text{on } \Gamma_D \\
\boldsymbol{\sigma}(u)\mathbf{n} &= t_N && \text{on } \Gamma_N
\end{aligned}
$$

$\boldsymbol{\sigma}(u) = \lambda \nabla \cdot u I + 2\mu \boldsymbol{\varepsilon}(u)$ - anisotropic Hooke's law

$\boldsymbol{\varepsilon}(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ - kinematic equation

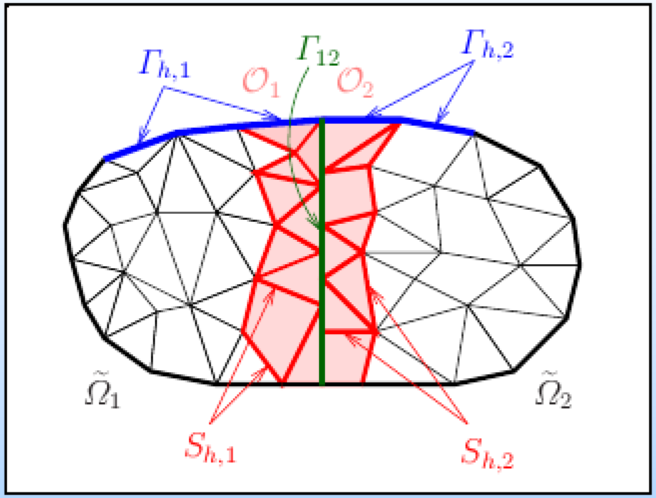
u : displacement

$\boldsymbol{\varepsilon}$: linearized strain tensor

$\boldsymbol{\sigma}$: Cauchy stress

$\lambda > 0, \mu > 0$: Lamé parameters

Domain Decomposition



$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma_{12}$

Find u with $u|_{\Omega_i} \in H^1(\Omega_i)$ s.t.

On each block Ω_i :

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}(u) &= f && \text{in } \Omega_i \\
u &= u_D && \text{on } \Gamma_{D_i} = \partial\Omega_i \cap \Gamma_D \\
\boldsymbol{\sigma}(u)\mathbf{n} &= t_N && \text{on } \Gamma_{N_i} = \partial\Omega_i \cap \Gamma_N
\end{aligned}
$$

On the interface Γ_{12} :

$[u] = 0, \quad [\boldsymbol{\sigma}(u)]\, n_1 = 0, \quad \text{on } \Gamma_{12} \quad :$ transmission conditions

Discrete Spaces

$\mathcal{T}_h(\Omega_i)$ – a conforming partition of $\Omega_i, i = 1, 2$

$\mathcal{E}_H$ – mortar finite element partition on Γ_{12}, independent of interior meshes

$$X_{h,i} = \{\boldsymbol{v}_{h,i} \in L^2(\Omega_i) \;:\; \boldsymbol{v}_{h,i}|_{\widetilde{\Omega}_i} \in H^1(\widetilde{\Omega}_i)\,,\, \forall E \in \mathcal{T}_{h,i}, \boldsymbol{v}_{h,i}|_E \in I\!P_k(E)\}$$

$$\Lambda_H = \{\boldsymbol{\lambda}_H \in L^2(\Gamma_{12}) : \forall \tau \in \mathcal{E}_H, \boldsymbol{\lambda}_H|_\tau \in I\!P_l(E)\}$$

$$\forall \boldsymbol{v}_h \in X_h\,,\; |\boldsymbol{v}_h|_{X_h} = \Big(\sum_{i=1}^{2} |\boldsymbol{v}_{h,i}|^2_{h,i}\Big)^{1/2}$$

$$|\boldsymbol{v}_{h,i}|^2_{h,i} = \lambda\left(\|\mathrm{div}\,\boldsymbol{v}_{h,i}\|^2_{L^2(\widetilde{\Omega}_i)} + \sum_{E \in \mathcal{O}_i} \|\mathrm{div}\,\boldsymbol{v}_{h,i}\|^2_{L^2(E)}\right)$$
$$+\,2\,\mu\left(\|\varepsilon(\boldsymbol{v}_{h,i})\|^2_{L^2(\widetilde{\Omega}_i)} + \sum_{E \in \mathcal{O}_i} \|\varepsilon(\boldsymbol{v}_{h,i})\|^2_{L^2(E)}\right)$$

Mortar Variational Form

$$\sum_{i=1}^{2}\Big(\int_{\Omega_i}\boldsymbol{\sigma}(\boldsymbol{u}_i):\boldsymbol{\epsilon}(\boldsymbol{v}_i)-\int_{\Gamma_{D_i}}\big(\boldsymbol{\sigma}(\boldsymbol{u}_i)\boldsymbol{n}_\Omega\cdot\boldsymbol{v}_i+s_D\boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_\Omega\cdot\boldsymbol{u}_i\big)$$

$$-\int_{\Gamma_{12}}\big(\boldsymbol{\sigma}(\boldsymbol{u}_i)\boldsymbol{n}_i\cdot\boldsymbol{v}_i-\boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_i\cdot\boldsymbol{u}_i\big)$$

$$+(\lambda+2\mu)\Big(\sum_{\gamma\in\Gamma_{h,i}}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\boldsymbol{u}_i\cdot\boldsymbol{v}_i+\sum_{\tau\in\mathcal{E}_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\boldsymbol{u}_i\cdot\boldsymbol{v}_i\Big)\Big)$$

$$=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}+\int_{\Gamma_N}\boldsymbol{t}_N\cdot\boldsymbol{v}$$

$$-\sum_{i=1}^{2}\Big(s_D\int_{\Gamma_{D_i}}\boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_\Omega\cdot\boldsymbol{u}_D-(\lambda+2\mu)\sum_{\gamma\in\Gamma_{h,i}}\frac{\sigma_\gamma}{h_\gamma}\int_\gamma\boldsymbol{u}_D\cdot\boldsymbol{v}_i\Big)$$

$$+\sum_{i=1}^{2}\Big(\int_{\Gamma_{12}}\boldsymbol{\sigma}(\boldsymbol{v}_i)\boldsymbol{n}_i\cdot\boldsymbol{\lambda}+(\lambda+2\mu)\sum_{\tau\in\mathcal{E}_H}\frac{\sigma_\tau}{H_\tau}\int_\tau\boldsymbol{\lambda}\cdot\boldsymbol{v}_i\Big),\quad\forall\boldsymbol{v}$$

$$\int_{\Gamma_{12}}[\boldsymbol{\sigma}(\boldsymbol{u})]_{12}\boldsymbol{n}_{12}\cdot\boldsymbol{\mu}-(\lambda+2\mu)\sum_{\tau\in\mathcal{E}_H}\frac{\sigma_\tau}{H_\tau}\sum_{i=1}^{2}\int_\tau(\boldsymbol{u}_i-\boldsymbol{\lambda})\cdot\boldsymbol{\mu}=0,\ \forall\boldsymbol{\mu}\in H^{\frac{1}{2}}(\Gamma_{12})$$

Error Estimates

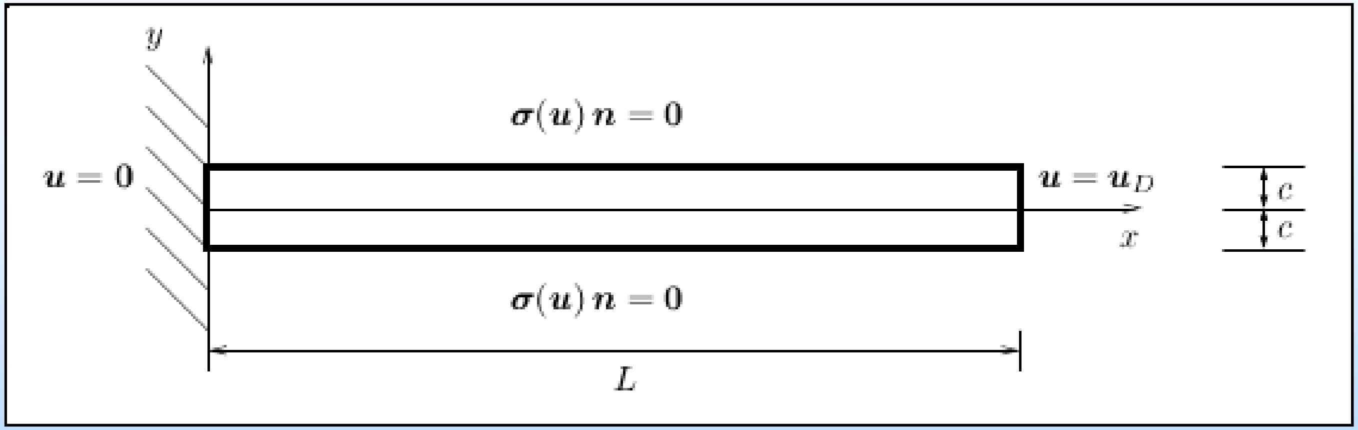
$$
\begin{aligned}
|\boldsymbol{u}_h - \boldsymbol{u}|_{X_h} &+ \Bigg((\lambda + 2\,\mu) \sum_{i=1}^{2} \Bigg(\sum_{\gamma \in \Gamma_{h,i}} \frac{\sigma_\gamma}{h_\gamma} \|\boldsymbol{u}_{h,i} - \boldsymbol{u}_D\|^2_{L^2(\gamma)} \\
&+ \sum_{\gamma \in S_{h,i}} \frac{\sigma_\gamma}{h_\gamma} \|[\boldsymbol{u}_{h,i}]_\gamma\|^2_{L^2(\gamma)} + \sum_{\tau \in \mathcal{E}_H} \frac{\sigma_\tau}{H_\tau} \|\boldsymbol{u}_{h,i} - \boldsymbol{\lambda}_H\|^2_{L^2(\tau)} \Bigg) \Bigg)^{1/2} \\
&\leq C \Bigg(h^{r-1} \Bigg(\left(\frac{h}{H}\right)^{1/2} + \left(\frac{H}{h}\right)^{1/2} \Bigg) + H^{\bar{r}-1/2} Z \Bigg),
\end{aligned}
$$

where $r = \min(k+1, s)$ and $\bar{r} = \min(\ell + 1, s - 1/2)$

$Z = 1,$ if DG strips are used

$Z = \left(\frac{H}{h}\right)^{1/2},$ if CG everywhere

Elastic Beam



$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

$$\boldsymbol{u} = \begin{pmatrix} -3\alpha x^2 y \\ \alpha x^3 + \frac{3\alpha\lambda}{\lambda+2\mu} x(y^2 - c^2) \end{pmatrix}$$

$$\boldsymbol{f} = \begin{pmatrix} 6\alpha\mu\frac{3\lambda+4\mu}{\lambda+2\mu} y \\ 0 \end{pmatrix}$$

$$E = 2.0\text{E+5 N m}^{-2}$$

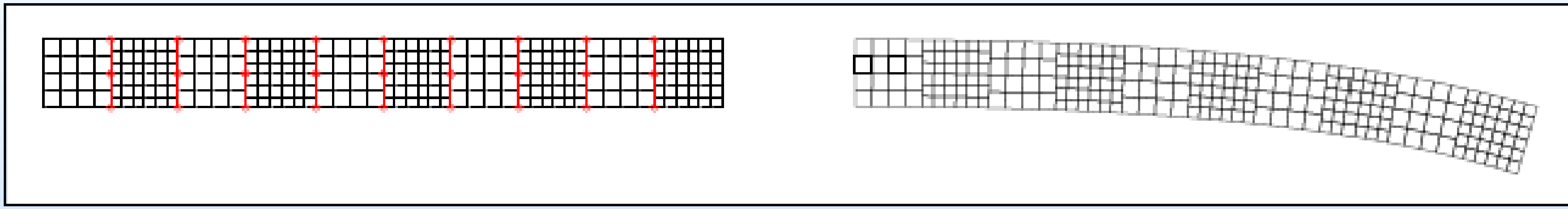
$$\nu = 0.3$$

$$L = 10 \text{ m}$$

$$c = 0.5\text{m}$$

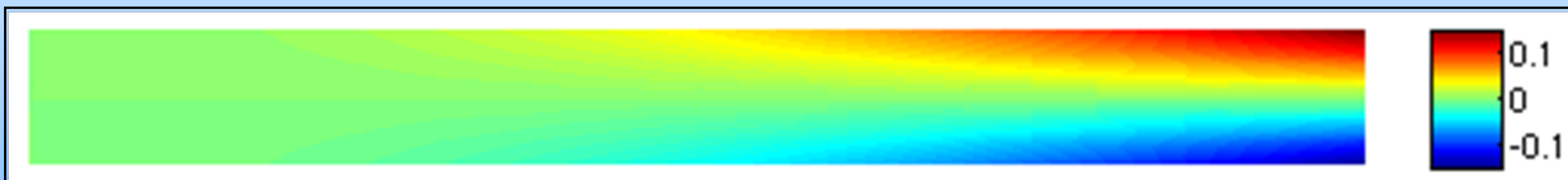
$$\alpha = -1.0\text{E-3 m}^{-2}$$

Meshes and Solution



Initial subdomain & mortar mesh

Deformed mesh



Displacement in x-direction

Displacement in y-direction

Convergence Rates

	Rectangular mesh				Triangular mesh			
	$l=1$		$l=2$		$l=1$		$l=2$	
$1/h$	Iter.	Error	Iter.	Error	Iter.	Error	Iter.	Error
4	45	3.98E+0	65	4.46E+0	44	1.10E+1	65	1.16E+1
8	60	1.53E+0	69	2.04E+0	59	3.87E+0	67	4.81E+0
16	78	6.52E-1	83	8.11E-1	79	1.40E+0	81	1.79E+0
32	106	3.05E-1	96	3.55E-1	107	6.46E-1	93	7.99E-1
64	146	1.48E-1	106	1.65E-1	148	3.24E-1	103	4.11E-1
128	205	7.33E-2	121	7.88E-2	207	1.57E-1	119	1.88E-1
OR		$\mathcal{O}(h^{1.14})$		$\mathcal{O}(h^{1.18})$		$\mathcal{O}(h^{1.21})$		$\mathcal{O}(h^{1.19})$
ER		$\mathcal{O}(h^{1.0})$		$\mathcal{O}(h^{0.7})$		$\mathcal{O}(h^{1.0})$		$\mathcal{O}(h^{0.7})$

Linear Subdomain Approx. (k=1)

	Rectangular mesh				Triangular mesh			
	$l=1$		$l=2$		$l=1$		$l=2$	
$1/h$	Iter.	Error	Iter.	Error	Iter.	Error	Iter.	Error
4	47	4.78E-1	73	1.44E-2	47	9.78E-1	72	1.05E-1
8	62	1.71E-1	101	3.10E-3	61	2.27E-1	96	1.14E-2
16	78	6.07E-2	136	6.83E-4	81	7.77E-2	132	2.40E-3
32	106	2.15E-2	189	1.60E-4	107	2.37E-2	185	5.35E-4
64	155	7.60E-3	264	3.85E-5	149	6.90E-3	260	1.26E-4
OR		$\mathcal{O}(h^{1.49})$		$\mathcal{O}(h^{2.14})$		$\mathcal{O}(h^{1.76})$		$\mathcal{O}(h^{2.38})$
ER		$\mathcal{O}(h^{1.5})$		$\mathcal{O}(h^{2.0})$		$\mathcal{O}(h^{1.5})$		$\mathcal{O}(h^{2.0})$

Quadratic Subdomain Approx. (k=2)

Conclusions

- Mortar methods defined and implemented for
  - Fully implicit multiscale method (MMFE) for multiphase flow coupled to a mixed/Godunov method for advection-diffusion-reaction problems on non-matching grids.
  - Elasticity
- Variably refined sub-domains results in significant savings in computational time (1 domain with fine grid takes twice the time as 3 domains) for multphase flow coupled to reactive transport
  - Multiblock domain solution agrees very well with single-domain fine-everywhere
- Mortar approach allows for legacy code reuse and
  - Cheaper way to handle irregular geometries
  - Ideally suited for handling geological faults, fractures, etc.

Current and Future Work

- Explore dynamic load balancing for treating reactions in parallel computations
- Apply error estimates for flow & transport to make suitable choice of sub-domain grids and mortar degrees of freedom
- Ading sharper *a posteriori* error estimators for adaptive mesh refinement (with M. Vohralík)
- Geomechanics model extensions to include permeability dependence on stress and coupling to nonisothermal EOS compositional flow model

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