Homogenization of Random Multilevel Junction

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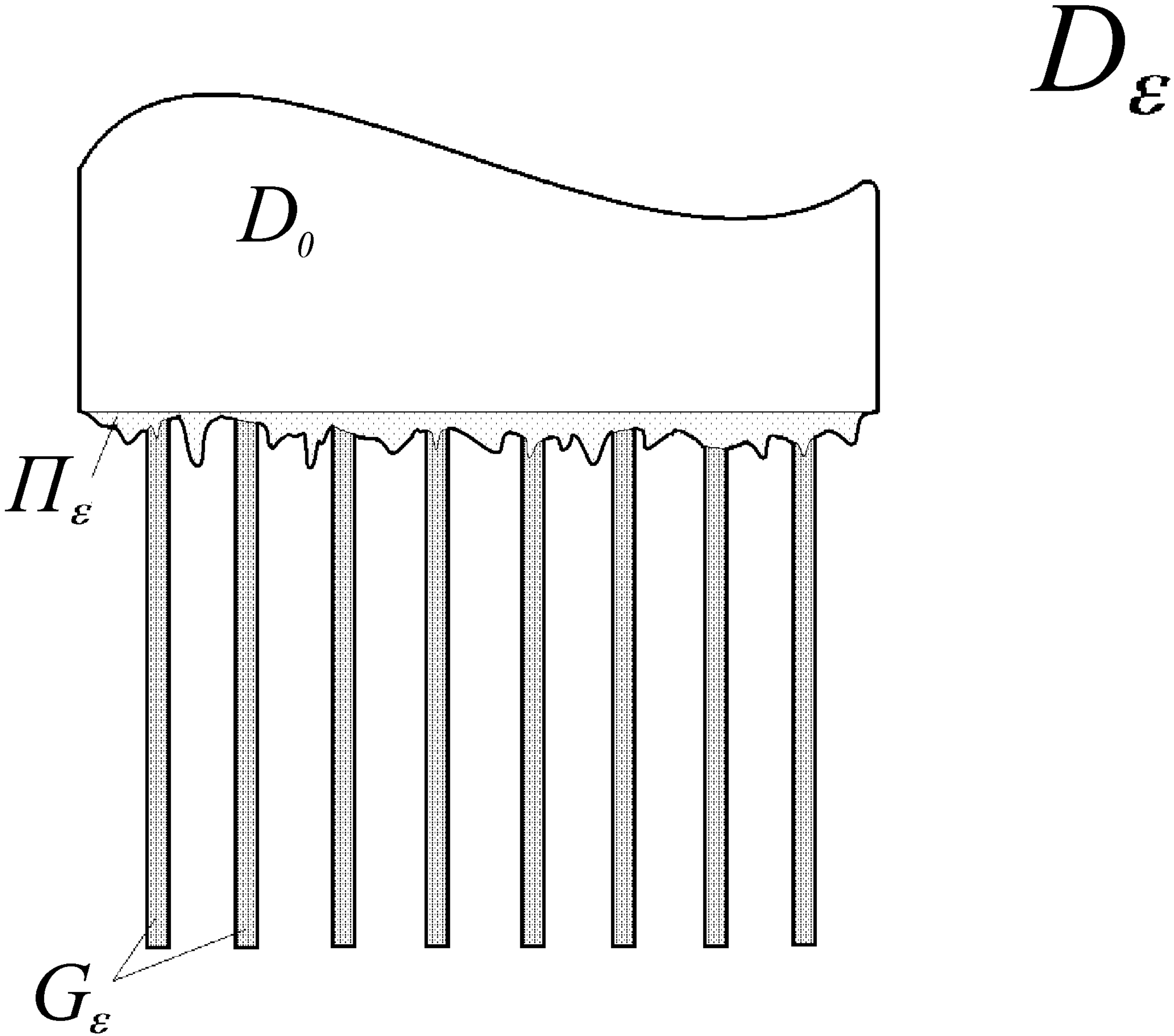
Figure 1: Cascade thick junction with random transmission zone



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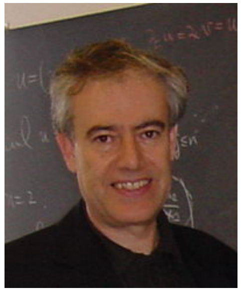 AMIRAT Y.

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Geometrical description

Let a, $h < 1$ be positive real numbers, $I_h(1/2) := \left(\frac{1-h}{2}\,,\,\frac{1+h}{2}\right)$ belong to $(0,1)$. The segment $I_0 := [0,a]$ consists of N subsegments $[\varepsilon j, \varepsilon(j+1)],\ j = 0, \ldots, N-1$. Here $\varepsilon = a/N$ is a small parameter.

Cascade thick junction D_ε *with random transmission zone* consists of a body

$$D_0 = \{x \in \mathbb{R}^2 :\ 0 < x_1 < a, \quad 0 < x_2 < \Phi(x_1)\,\},$$

$\Phi \in C^1([0,a]),\ \min\limits_{[0,a]} \Phi > 0$; thin rectangles

$$\widehat{G}_j(\varepsilon) = \left\{ x \in \mathbb{R}^2 :\ (x_1, 0) \in I_0, \left| x_1 - \varepsilon\,(j + \frac{1}{2}) \right| < \frac{\varepsilon h}{2},\ x_2 \in (-l, 0) \right\}, j = 0, \ldots, N-1$$

and thin layer with oscillating boundary

$$\Pi_\varepsilon = \left\{ x \in \mathbb{R}^2 \ :\ x_1 \in (0,a), \varepsilon\theta(x_1) F\left(\frac{x_1}{\varepsilon}, \omega\right) < x_2 \leq 0 \right\},$$

Geometrical description

where $\theta(x_1)$ is a smooth nonnegative function with $supp\,\theta(x_1) \subset I_0$ and $F(\xi_1, \omega)$ is a random statistically homogeneous ergodic nonpositive function with smooth realizations, ω is an element of a standard probability space $(\Omega, \mathcal{A}, \mu)$. Thus,

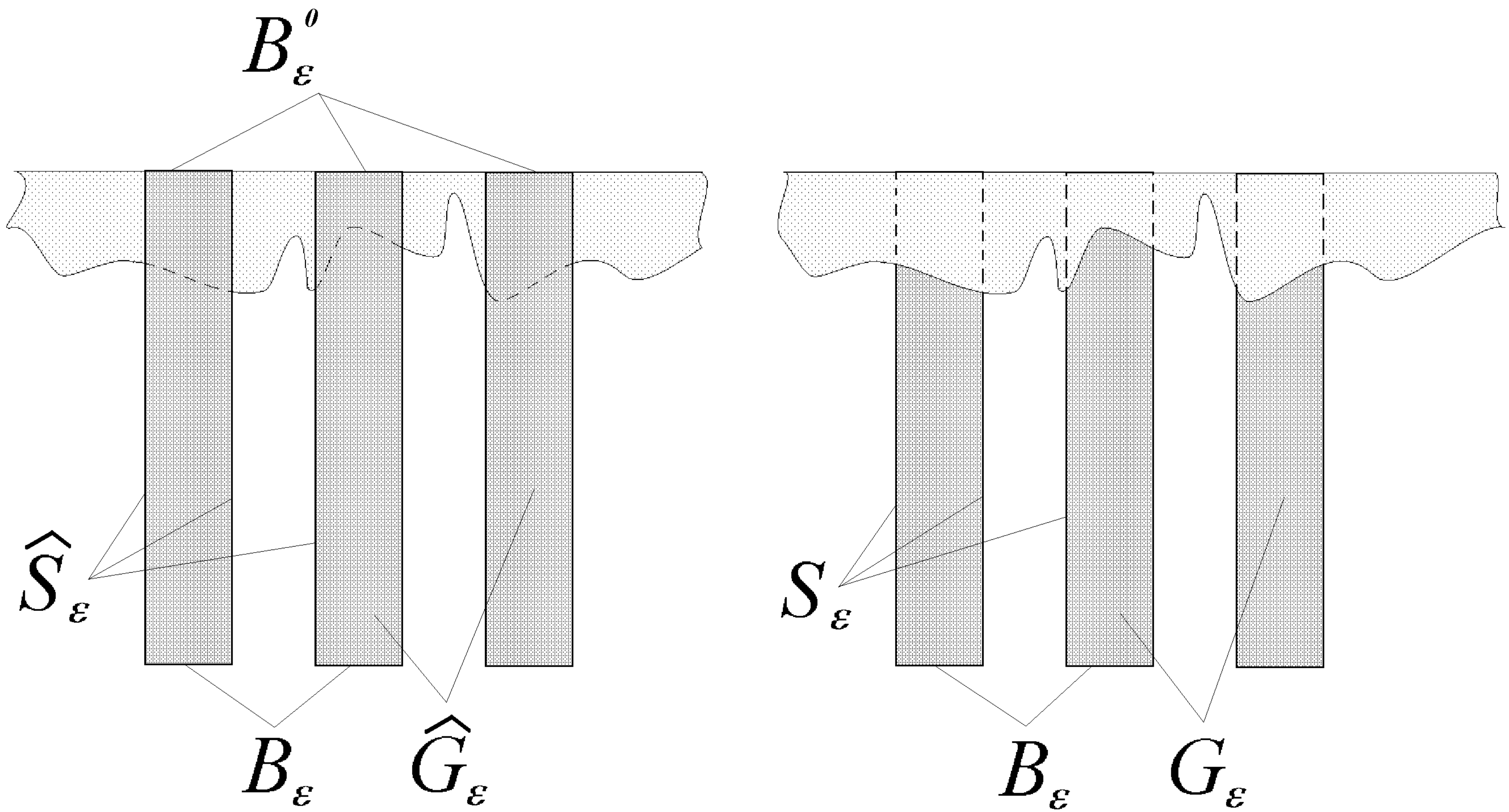
$$D_\varepsilon = D_0 \cup \Pi_\varepsilon \cup \widehat{G}_\varepsilon,$$

where

$$\widehat{G}_\varepsilon = \bigcup_{j=0}^{N-1} \widehat{G}_j(\varepsilon)$$

or $D_\varepsilon = D_0 \cup \Pi_\varepsilon \cup G_\varepsilon,$ where $G_\varepsilon = \widehat{G}_\varepsilon \backslash \Pi_\varepsilon$. We denote also
$B_\varepsilon^0 = \left\{ x \in \mathbb{R}^2 : \ \left| x_1 - \varepsilon(j + \frac{1}{2}) \right| < \frac{\varepsilon h}{2}, x_2 = 0 \right\}$,
$\Gamma_\varepsilon = \{ x \in \mathbb{R}^2 \ : \ x_1 \in (0,a) \backslash B_\varepsilon^0, \ \varepsilon \theta(x_1) F\left(\frac{x_1}{\varepsilon}, \omega\right) = x_2 \}$, $\widehat{\Upsilon}_\varepsilon := \partial \widehat{G}_\varepsilon \backslash \overline{B}_\varepsilon^0$ or
$\widehat{\Upsilon}_\varepsilon = \widehat{S}_\varepsilon \cup B_\varepsilon$, where $\widehat{S}_\varepsilon$ is the lateral surface of the set $\widehat{G}_\varepsilon$, B_ε is the lower surface of $\widehat{G}_\varepsilon$;
$\Upsilon_\varepsilon := \partial G_\varepsilon \backslash \partial \Pi_\varepsilon$, $\Upsilon_\varepsilon = S_\varepsilon \cup B_\varepsilon$, $\Gamma_1 = \{ x : \ x_2 = \Phi(x_1), \ x_1 \in [0,a] \}$,
$\gamma_\varepsilon = \partial D_\varepsilon \setminus \left(\Gamma_\varepsilon \cup \Upsilon_\varepsilon \cup \Gamma_1 \right)$.

Figure 2: Rectangles and random layer



Setting of the problem

In D_ε we consider the following problem:

$$\begin{cases} -\Delta_x\, u_\varepsilon(x) = f_\varepsilon(x), & x \in D_\varepsilon; \\ \partial_\nu u_\varepsilon(x) + \varepsilon^\tau\, \theta(x_1) p\left(\frac{x_1}{\varepsilon}, \omega\right) u_\varepsilon(x) = \theta\,(x_1)\, q\left(\frac{x_1}{\varepsilon}, \omega\right), & x \in \Gamma_\varepsilon; \\ \partial_\nu u_\varepsilon(x) + \varepsilon^\mu\, k\, u_\varepsilon(x) = \varepsilon^\beta\, g_\varepsilon(x), & x \in \Upsilon_\varepsilon; \\ u_\varepsilon(x) = 0, & x \in \Gamma_1; \\ \partial_\nu u_\varepsilon(x) = 0, & x \in \gamma_\varepsilon. \end{cases} \tag{1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the derivative with respect to the outer normal; the constant k is positive; the parameters $\beta \geq 1$, μ, τ are real; $p\,(\xi_1, \omega)$ and $q\,(\xi_1, \omega)$ are random statistically homogeneous ergodic positive functions. The functions p and q have smooth realizations. It should be noted that

$$[u_\varepsilon] = 0, \qquad [\partial_{x_2} u_\varepsilon] = 0 \qquad\qquad \text{on} \quad I_0 \cap \Pi_\varepsilon,$$

where $[\cdot]$ is the jump of a function.

Precise definition of randomness

Assume that $(\Omega, \mathcal{A}, \mu)$ is a probability space, i.e. the set Ω with σ-algebra $\mathcal{A}$ of its subsets and σ-additive nonnegative measure μ on $\mathcal{A}$ such that $\mu(\Omega) = 1$.

Definition 1. *A family of measurable maps*

$$T_{x_1} : \Omega \to \Omega, \quad x_1 \in \mathbb{R}$$

we call a dynamical system*, if the following properties hold true:*

- ***group property****:*

$$T_{x_1+y_1} = T_{x_1} T_{y_1} \quad \forall x_1, y_1 \in \mathbb{R}; \qquad T_0 = Id$$

(Id is the identical mapping);

- ***isometry property*** *(the mapping T_{x_1} preserves the measure μ on Ω):*

$$T_{x_1} A \in \mathcal{A}, \quad \mu(T_{x_1} A) = \mu(A) \quad \forall x_1 \in \mathbb{R},\ A \in \mathcal{A};$$

- ***measurability****: for any measurable functions $\Psi(\omega)$ on Ω the function $\Psi(T_{x_1}\omega)$ is measurable on $\Omega \times \mathbb{R}$ and continuous in x_1.*

Precise definition of randomness

Let $L^q(\Omega, \mu) \quad (q \geq 1)$ be the space of measurable functions integrable in the power q with respect to the measure μ. If $U_{x_1} : \Omega \to \Omega$ is a dynamical system, then in the space $L^2(\Omega, \mu)$ we define a parametric group of operators $\{U_{x_1}\}, x_1 \in \mathbb{R}$ (we keep the same notation) by the formula

$$(U_{x_1}\Psi)(\omega) := \Psi(U_{x_1}\omega), \quad \Psi \in L^2(\Omega, \mu).$$

From the condition 3) of the definition it follows that the group U_{x_1} is strongly continuous, i.e. for any $\Psi \in L^2(\Omega, \mu)$

$$\lim_{x_1 \to 0} \|U_{x_1}\Psi - \Psi\|_{L^2(\Omega,\mu)} = 0.$$

Definition 2. *Suppose that $\Psi(\omega)$ is a measurable function on Ω. The function $\Psi(T_{x_1}\omega)$ of $x_1 \in \mathbb{R}$ for fixed $\omega \in \Omega$ is called the* realization of the function Ψ.

Proposition 1. *Assume that $\Psi \in L^q(\Omega, \mu)$, then almost all realizations $\Psi(T_{x_1}\omega)$ belong to $L^q_{loc}(\mathbb{R})$. If the sequence $\Psi_k \in L^q(\Omega, \mu)$ converges in $L^q(\Omega, \mu)$ to the function Ψ, then there exists a subsequence k' such that almost all realizations $\Psi_{k'}(T_{x_1}\omega)$ converges in $L^q_{loc}(\mathbb{R})$ to the realization $\Psi(T_{x_1}\omega)$.*

Definition 3. *A measurable function $\Psi(\omega)$ on Ω is called* invariant*, if $\Psi(T_{x_1}\omega) = \Psi(\omega)$ for any $x_1 \in \mathbb{R}$ and almost all $\omega \in \Omega$.*

Definition 4. *The dynamical system T_{x_1} is called* ergodic*, if any invariant function almost everywhere coincides with a constant.*

Precise definition of randomness

We denote by $\mathcal{B}$ the natural Borel σ-algebra of subsets of the space $\mathbb{R}$. Suppose that $F(x_1) \in L^1_{loc}(\mathbb{R})$.

Definition 5. *We say that the function $F(x_1)$ has a spatial average, if the limit*

$$M(F) = \lim_{\varepsilon \to 0} \frac{1}{|B|} \int_B F\left(\frac{x_1}{\varepsilon}\right) dx_1 \tag{2}$$

does exist for any bounded Borel sets $B \in \mathcal{B}$ and does not depend on the choice of B, and $M(F)$ is called the spatial average value *of the function F.*

In equivalent form

$$M(F) = \lim_{t \to +\infty} \frac{1}{|B_t|} \int_{B_t} F(x_1)\, dx_1, \tag{3}$$

where $B_t = \{x \in \mathbb{R} \mid \frac{x_1}{t} \in B\}$.

Proposition 2. *Let the function $F(x_1)$ have a spatial meanvalue in $\mathbb{R}$, and suppose that the family $\{F\left(\frac{x_1}{\varepsilon}\right),\ 0 < \varepsilon \leq 1\}$ is bounded in $L^q(\mathcal{K})$ for some $q \geq 1$, where $\mathcal{K}$ is a compact in $\mathbb{R}$. Then*

$$F\left(\frac{x_1}{\varepsilon}\right) \rightharpoonup M(F) \quad \textit{weakly in} \quad L^q_{loc}(\mathbb{R}) \textit{ as } \varepsilon \to 0.$$

Precise definition of randomness

Proposition 3. (Birkhoff ergodic theorem) *Let T_{x_1} satisfy the Definition* 1 *and assume that* $\Psi \in L^q(\Omega, \mu), q \geq 1$. *Then for almost all $\omega \in \Omega$ the realization $\Psi(T_{x_1}\omega)$ has the spatial meanvalue* $M(\Psi(T_{x_1}\omega))$. *Moreover, the spatial meanvalue $M(\Psi(T_{x_1}\omega))$ is a conditional mathematical expectation of the function $\Psi(\omega)$ with respect to the σ-algebra of invariant subsets. Hence, $M(\Psi(T_{x_1}\omega))$ is an invariant function and*

$$\mathbb{E}(\Psi) \equiv \int_\Omega \Psi(\omega)\, d\mu = \int_\Omega M(\Psi(T_{x_1}\omega))\, d\mu. \tag{4}$$

In particular, if the dynamical system T_{x_1} is ergodic, then for almost all $\omega \in \Omega$ the following formula

$$\mathbb{E}(\Psi) = M(\Psi(T_{x_1}\omega))$$

holds true.

Definition 6. *A random function $\Psi(x_1, \omega)$ $(x_1 \in \mathbb{R}, \omega \in \Omega)$ is called* statistically homogeneous, *if the following representation*

$$\Psi(x_1, \omega) = \mathbf{\Psi}(T_{x_1}\omega)$$

holds for some function $\mathbf{\Psi}$, where T_{x_1} is a dynamical system in Ω.

For statistically homogeneous functions with smooth realizations we denote

$$\partial_\omega \mathbf{\Psi}(T_{x_1}\omega) := \partial_{x_1}\Psi(x_1, \omega).$$

Assumptions

We assume that the following conditions are fulfilled. Without loss of generality $f_\varepsilon \in L^2(D_1)$, where $\overline{D}_1 = \overline{D}_0 \cup \overline{D}_2$, $D_2 = (0,a) \times (-l,0)$, and

$$f_\varepsilon \to f_0 \quad \text{strongly in} \quad L^2(D_1) \quad \text{as} \quad \varepsilon \to 0. \tag{5}$$

The function $g_\varepsilon \in H^1(D_2)$ and

$$g_\varepsilon \rightharpoonup g_0 \quad \text{weakly in} \quad H^1(D_2) \quad \text{as} \quad \varepsilon \to 0. \tag{6}$$

Now let us formulate the conditions for functions p, q and F. We assume that p, q and F are statistically homogeneous, T_{x_1} is ergodic and a.s.

$$\mathbf{p}(\omega) \geq 0, \qquad \|\mathbf{p}\|_{L^\infty(\Omega,\mu)} < \infty, \qquad \|\mathbf{q}\|_{L^\infty(\Omega,\mu)} < \infty,$$

$$\|\mathbf{F}\|_{L^\infty(\Omega,\mu)} < \infty, \qquad \|\partial_\omega \mathbf{F}\|_{L^\infty(\Omega,\mu)} < \infty.$$

We define the continuation by zero for functions from $H^1(G_\varepsilon)$ in the following manner:

$$\widetilde{y_\varepsilon}(x) = \begin{cases} y_\varepsilon, & x \in G_\varepsilon, \\ 0, & x \in D_2 \setminus G_\varepsilon, \end{cases}$$

where $D_2 = (0,a) \times (-l,0)$.

Main results

Theorem 1 (The case $\tau \geq 0$ and $\mu \geq 1$). *The solution u_ε to the problem* (1) *for almost all ω (a.s.) satisfies*

$$\begin{array}{rclll} u_\varepsilon & \rightharpoonup & v_0^+ & \textit{in} & H^1(D_0,\Gamma_1), \\ \widetilde{\partial_{x_2} u_\varepsilon} & \rightharpoonup & h\,\partial_{x_2} v_0^- & \textit{in} & L^2(D_2), \end{array} \qquad \begin{array}{rclll} \widetilde{u_\varepsilon} & \rightharpoonup & h\,v_0^- & \textit{in} & L^2(D_2), \\ \widetilde{\partial_{x_1} u_\varepsilon} & \rightharpoonup & 0 & \textit{in} & L^2(D_2), \end{array} \tag{7}$$

as $\varepsilon \to 0$, where the function $v_0(x) = \begin{cases} v_0^+(x), & x \in D_0, \\ v_0^-(x), & x \in D_2, \end{cases}$ *is the unique solution to the problem*

$$\begin{cases} -\Delta_x\, v_0^+(x) = f_0(x), & x \in D_0 \\ v_0^+(x) = 0, & x \in \Gamma_1 \\ \partial_\nu v_0^+(x) = 0, & x \in \partial D_0 \setminus \big(\Gamma_1 \cup I_0\big), \\ -h\partial^2_{x_2 x_2} v_0^-(x) + 2\delta_{\mu,1} k v_0^-(x) = h f_0(x) + \delta_{\beta,1} g_0(x), & x \in D_2, \\ v_0^+(x_1,0) = v_0^-(x_1,0), & (x_1,0) \in I_0, \\ \Big(h\partial_{x_2} v_0^- - \partial_{x_2} v_0^+ + \delta_{\tau,0}(1-h)\theta(x_1)P(x_1)\, v_0^+\Big)(x_1,0) = (1-h)\theta(x_1)Q(x_1), & (x_1,0) \in I_0, \\ \partial_{x_2} v_0^-(x_1,-l) = 0, & (x_1,-l) \in I_l, \end{cases} \tag{8}$$

Main results

which is called homogenized problem *for the problem* (1). *Here*

$$I_l = \{x : \ x_2 = -l, \ x_1 \in (0,a)\};$$

$\delta_{\alpha,k}$ *is the Kroneker symbol;*

$$P(x_1) = \mathbb{E}\left(p(\xi_1,\omega)\sqrt{1+\left(\theta(x_1)\,\partial_{\xi_1}F(\xi_1,\omega)\right)^2}\right) = \mathbb{E}\left(\mathbf{p}(\omega)\sqrt{1+(\theta(x_1)\,\partial_\omega \mathbf{F}(\omega))^2}\right),$$

$$Q(x_1) = \mathbb{E}\left(q(\xi_1,\omega)\sqrt{1+\left(\theta(x_1)\,\partial_{\xi_1}F(\xi_1,\omega)\right)^2}\right) = \mathbb{E}\left(\mathbf{q}(\omega)\sqrt{1+(\theta(x_1)\,\partial_\omega \mathbf{F}(\omega))^2}\right).$$

Moreover the convergence of energy

$$E_\varepsilon(u_\varepsilon) := \int\limits_{D_\varepsilon} |\nabla_x u_\varepsilon|^2\,dx + \varepsilon^\tau \int\limits_{\Gamma_\varepsilon} \theta(x_1)\,p\left(\frac{x_1}{\varepsilon},\omega\right)\,u_\varepsilon^2\,d\sigma_x + \varepsilon^\mu k \int\limits_{\Upsilon_\varepsilon} u_\varepsilon^2\,d\sigma_x \longrightarrow$$
$$\longrightarrow \int\limits_{D_0} |\nabla v_0^+|^2\,dx + \int\limits_{D_2}\left(h|\partial_{x_2}v_0^-|^2 + 2\delta_{\mu,1}k|v_0^-|^2\right)dx + \tag{9}$$
$$+\delta_{\tau,0}(1-h)\int\limits_{I_0}\theta(x_1)P(x_1)|v_0^+(x_1,0)|^2\,dx_1 =: E_0(v_0)$$

holds true as $\varepsilon \longrightarrow 0$ *for almost all* ω.

Main results

Theorem 2 (The case $\tau < 0$ and $\mu \geq 1$). *For solutions u_ε to the problem* (1) *the limits*

$$\begin{array}{rclll} u_\varepsilon & \rightharpoonup & v_0^+ & \textit{in} & H^1(D_0, \Gamma_1), \\ \widetilde{\partial_{x_2} u_\varepsilon} & \rightharpoonup & h\,\partial_{x_2} v_0^- & \textit{in} & L^2(D_2), \end{array} \qquad \begin{array}{rclll} \widetilde{u_\varepsilon} & \rightharpoonup & h\,v_0^- & \textit{in} & L^2(D_2), \\ \widetilde{\partial_{x_1} u_\varepsilon} & \rightharpoonup & 0 & \textit{in} & L^2(D_2), \end{array} \tag{10}$$

as $\varepsilon \longrightarrow 0$ are valid for almost all ω, where the functions v_0^+ and v_0^- are respectively the solutions to the following problems:

$$\begin{cases} -\Delta_x\, v_0^+(x) = f_0(x), & x \in D_0 \\ v_0^+(x) = 0, & x \in \Gamma_1 \cup I_0 \\ \partial_\nu v_0^+(x) = 0, & x \in \partial D_0 \setminus \big(\Gamma_1 \cup I_0\big), \end{cases} \tag{11}$$

$$\begin{cases} -h\partial^2_{x_2x_2} v_0^-(x) + 2\,\delta_{\mu,1}\,k\,v_0^-(x) = h\,f_0(x) + \delta_{\beta,1}\,g_0(x), & x \in D_2, \\ v_0^-(x_1, 0) = 0, & (x_1, 0) \in I_0, \\ \partial_{x_2} v_0^-(x_1, -l) = 0, & (x_1, -l) \in I_l, \end{cases} \tag{12}$$

which together are called the homogenized problem *for the problem* (1).

Main results

Moreover the convergence of the energy integrals

$$
\begin{aligned}
E_\varepsilon(u_\varepsilon) \;\to\; \int\limits_{D_0} |\nabla v_0^+|^2\,dx \;+\; h \int\limits_{D_2} |\partial_{x_2} v_0^-|^2\,dx \,+ \\
+2\,\delta_{\mu,1}\,k \int\limits_{D_2} |v_0^-|^2\,dx =: E_0(v_0^+) + E_0(v_0^-).
\end{aligned}
\tag{13}
$$

holds true as $\varepsilon \longrightarrow 0$ for almost all ω.

Main results

Theorem 3 (The case $\mu < 1$). *For the solution u_ε to the problem* (1) *for almost all ω the limits*

$$\left.\begin{array}{llll} u_\varepsilon & \rightharpoonup & v_0^+ & \text{in } H^1(D_0, \Gamma_1), \\ \widetilde{u_\varepsilon} & \to & 0 & \text{in } L^2(D_2), \end{array}\right\} \text{ as } \varepsilon \to 0 \tag{14}$$

hold true, where the function v_0^+ is the solution to the problem

$$\begin{cases} -\Delta_x v_0^+(x) = f_0(x), & x \in D_0 \\ v_0^+(x) = 0, & x \in \Gamma_1 \cup I_0 \\ \partial_\nu v_0^+(x) = 0, & x \in \partial D_0 \setminus \big(\Gamma_1 \cup I_0\big). \end{cases} \tag{15}$$

Moreover, for almost all ω the following convergence

$$E_\varepsilon(u_\varepsilon) \to \int_{D_0} |\nabla v_0^+|^2 \, dx =: E_0(v_0^+) \tag{16}$$

is valid as $\varepsilon \to 0$.

Auxiliary Lemmas

Lemma 1. *Let $H(\xi_1, \omega)$ be a random statistically homogeneous function, such that $\|\mathbf{H}\|_{L^\infty(\Omega,\mu)} < \infty$ and*

$$\mathbb{E}(H(\xi_1, \omega)) \equiv 0. \tag{17}$$

Then a.s.

$$\int_{I_0} H(\frac{x_1}{\varepsilon}, \omega)\, u(x_1)\, v(x_1)\; dx_1 \longrightarrow 0 \tag{18}$$

as $\varepsilon \longrightarrow 0$ for any functions $u, v \in H^{\frac{1}{2}}(I_0)$.

Lemma 2. *For any $u, v \in H^1(D_\varepsilon)$ the following limit relations*

$$\left| \int_{\Gamma_\varepsilon} \theta(x_1)\, q\left(\frac{x_1}{\varepsilon}, \omega\right) v(x)\; d\sigma_x - (1-h) \int_{I_0} \theta(x_1)\, Q(x_1) v(x_1, 0)\; dx_1 \right| \to 0, \tag{19}$$

$$\left| \int_{\Gamma_\varepsilon} \theta(x_1) p\left(\frac{x_1}{\varepsilon}, \omega\right) v(x) u(x)\, d\sigma_x - (1-h) \int_{I_0} \theta(x_1) P(x_1) v(x_1, 0) u(x_1, 0)\, dx_1 \right| \to 0 \tag{20}$$

hold as $\varepsilon \longrightarrow 0$ for almost all ω.

Auxiliary Lemmas

Boundary value problems in dense junctions with different nonhomogeneous conditions on the boundary of thin subdomains have specific difficulties. To homogenize problems in such junctions we use special integral identities. In this case the identity has the following form:

$$\frac{\varepsilon h}{2} \int\limits_{\widehat{S}_\varepsilon} v \, dx_2 = \int\limits_{\widehat{G}_\varepsilon} v \, dx - \varepsilon \int\limits_{\widehat{G}_\varepsilon} Y_2\left(\frac{x_1}{\varepsilon}\right) \partial_{x_1} v \, dx, \qquad \forall \, v \in H^1(\widehat{G}_\varepsilon). \tag{21}$$

Here $Y_2(\xi) = -\xi + [\xi] + \frac{1}{2}$, where $[\xi]$ is the integer part of ξ; S_ε is the union of the lateral sides of the rectangles G_ε.
Keeping in mind that $\max\limits_{\mathbb{R}} |Y_2| \leq 1$, we get the inequality

$$\|v\|_{L^2(S_\varepsilon)} \leq C_2 \varepsilon^{-\frac{1}{2}} \|v\|_{H^1(G_\varepsilon)}. \tag{22}$$

Using the standard approach we obtain

$$\|v\|_{L^2(B_\varepsilon)} \leq C_3 \|v\|_{H^1(G_\varepsilon)}, \tag{23}$$

where $B_\varepsilon = \Upsilon_\varepsilon \backslash S_\varepsilon$.

Comments

For 3D model with variable cross section of the rods we change the function Y_2. Consider the following identity:

$$\varepsilon \int\limits_{S_\varepsilon} \frac{\varphi(x)\, d\sigma_x}{\sqrt{1+\varepsilon^2|\varrho'(x_3)|^2}} = \int\limits_{G_\varepsilon} \frac{l_\omega(x_3)}{|\omega(x_3)|}\,\varphi\, dx + \varepsilon \int\limits_{G_\varepsilon} \nabla_{\xi'} Y_2(\xi', x_3)\big|_{\xi'=\frac{x'}{\varepsilon}} \cdot \nabla_{x'}\varphi\, dx \qquad (24)$$

for any $\varphi \in H^1(G_\varepsilon)$. Here Y_2 is 1-periodic in ξ_1 and ξ_2 function which satisfies

$$\begin{cases} \Delta_{\xi'} Y_2(\xi', x_3) = \dfrac{l_\omega(x_3)}{|\omega(x_3)|} & \text{in } \omega(x_3), \\ \partial_{\nu'(\xi')} Y_2 = 1 & \text{on } \partial\omega(x_3), \\ \displaystyle\int\limits_{\omega(x_3)} Y_2(\xi', x_3)\, d\xi' = 0, & \end{cases} \qquad (25)$$

where $\xi' = (\xi_1, \xi_2),\ \nu'(\xi') = \big(\nu_1(\xi'), \nu_2(\xi')\big)$ is outer normal to D.