Domain Decomposition for Multiscale PDEs

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in collaboration with

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Scaling Up & Modelling for Transport and Flow in Porous Media
Dubrovnik, Wednesday, October 15th 2008

Motivation: Groundwater Flow

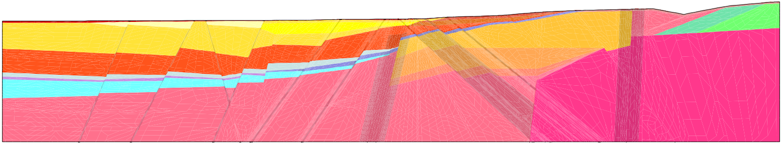
Safety assessment for radioactive waste disposal at Sellafield ©NIREX UK Ltd.

$$\textbf{Darcy's Law:} \qquad q + \mathcal{A}(x)\nabla p = f$$

$$\textbf{Incompressibility:} \qquad \nabla \cdot q = 0$$

$+$ **Boundary Conditions**

(More generally: Multiphase Flow in Porous Media, e.g. Oil Reservoir Modelling or CO_2 Sequestration)



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- Elliptic PDE in 2D or 3D bounded domain Ω

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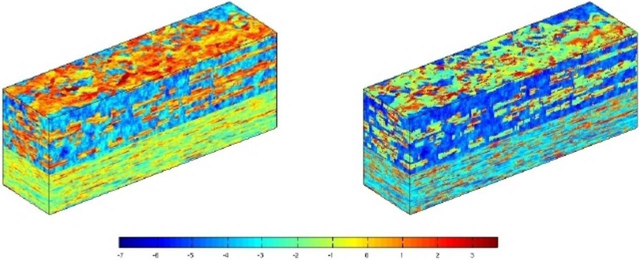
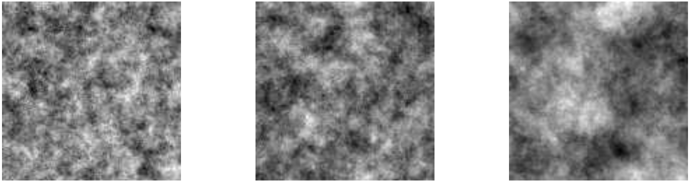
- **Aim:** Find efficient & robust preconditioner for A
(i.e. independent of variations in h **and** in $\alpha(x)$)

Heterogeneous multiscale deterministic media



Society of Petroleum Engineers (SPE) Benchmark SPE10

Multiscale stochastic media ($\lambda = 5h, 10h, 20h$)



Difficulties

- Requires very fine mesh resolution: $h \ll \text{diam}(\Omega)$
- A very large and very ill-conditioned, i.e.

$$\kappa(A) \lesssim \max_{\tau,\tau' \in \mathcal{T}^h} \left(\frac{\alpha_\tau}{\alpha_{\tau'}} \right) h^{-2}$$

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Meaning of $\lesssim$

Goals

- Efficient, scalable & parallelisable method,
  - robust w.r.t. problem size n and mesh resolution h
  - robust w.r.t. coefficients $\alpha(x)$ **!**
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  "handle" for choice of components

Possible Methods & Existing Theory

- Standard **Domain Decomposition** and **Multigrid** robust if coarse grid(s) **resolve(s)** coefficients
  [Chan, Mathew, Acta Numerica, 94], **[J. Xu, Zhu, Preprint, 07]**
- Otherwise: **coefficient-dependent** coarse spaces
  [Alcouffe, Brandt, Dendy et al, SISC, 81], **[Sarkis, Num Math, 97]**

- **Practically** most successful: **Algebraic Multigrid**
  No theory explaining coefficient robustness for standard AMG**!**

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Th^m. $\kappa(M^{-1}A) \lesssim \max_j \delta^2 \, \| \alpha |\nabla \Psi_j|^2 \|_{L_\infty(\Omega)} \, (1 + H/\delta)$

$\longrightarrow$ **low energy coarse spaces!**

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- **FETI (Finite Element Tearing & Interconnecting)**
  - **[Pechstein, Sch., Num Math, 08]** $\longleftarrow$ TODAY!

Finite Element Tearing & Interconnecting

(non-overlapping dual substructuring techniques)

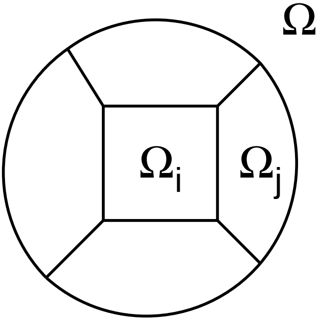
FETI methods – Idea

Domain decomposition

$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$

$\Gamma_i := \partial\Omega_i$

$H_i := \operatorname{diam} \Omega_i$



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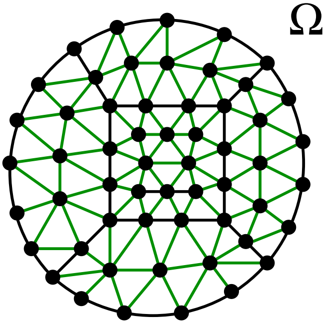
Conforming FE mesh on Ω
(p.w. linear FEs)

Mesh size on subdomain Ω_i: h_i



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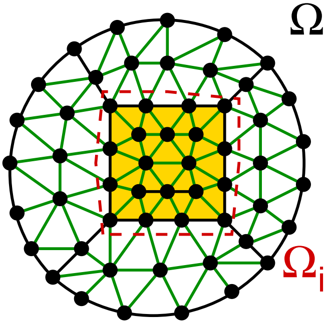
Mesh size on subdomain Ω_i: h_i

Subdomain stiffness matrix A_i
(including boundary, i.e. Neumann)



FETI methods – Idea

Tearing: Introduce local soln u_i, i.e. >1 dofs per interface node

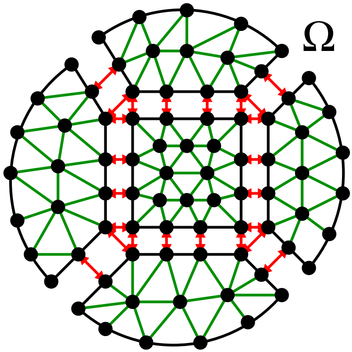
Interconnecting: Enforce continuity by pointwise constraints:

$u_i(x^h) - u_j(x^h) = 0, \; x^h \in \Gamma_i \cap \Gamma_j$

or compactly written,

$$B\,u \;:=\; \sum_i B_i u_i \;=\; 0$$

where $u := [u_1^\top u_2^\top \ldots u_N^\top]^\top$



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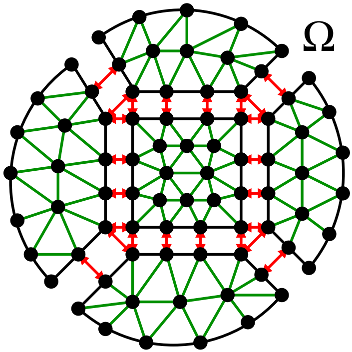

Interconnecting: Enforce continuity by pointwise constraints:

Introduce **Lagrange multipliers** to obtain the new global system:

$$\begin{bmatrix} A & B^\top \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

with $A := \text{diag}\,(A_i)$ & $f := [f_1^\top \dots f_N^\top]^\top$

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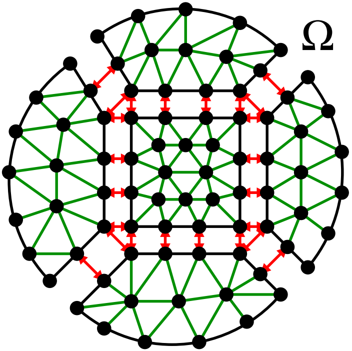


Interconnecting: Enforce continuity by pointwise constraints:

Eliminate u & solve dual problem

$$" B\,A^{-1}B^\top \lambda = B\,A^{-1} f "$$

with **preconditioner** $" \sum_i B_i \begin{bmatrix} 0 & 0 \\ 0 & S_i \end{bmatrix} B_i^\top "$ (Fully parallel!)

where $S_i := A_{i,\Gamma\Gamma} - A_{i,\Gamma I} A_{i,II}^{-1} A_{i,I\Gamma}$ (Schur complement).

Elimination of u (substructuring):

$$\begin{bmatrix} A_1 & 0 & & B_1^\top \\ 0 & \ddots & & \vdots \\ & & A_n & B_n^\top \\ B_1 & \dots & B_n & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \\ 0 \end{bmatrix}$$

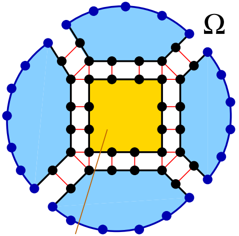
If $\partial\Omega_i \cap \Gamma_D \neq \emptyset$ **then** A_i is SPD:

$$u_i = A_i^{-1}[f_i - B_i^\top \lambda]$$

else (floating subdomains!)

$$u_i = A_i^\dagger [f_i - B_i^\top \lambda] + \text{kernel correction}$$

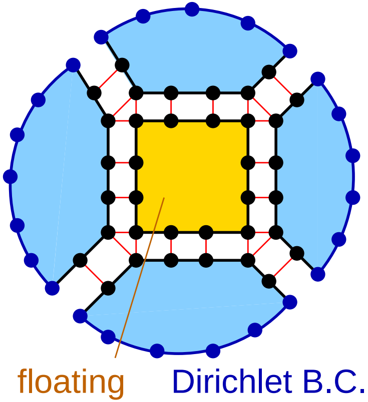
with compatibility condition on λ



FETI methods – Variants

"One-level" Methods

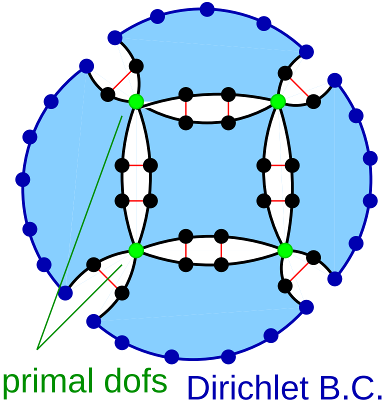
[Farhat & Roux, '91]



use projection to deal with kernel

Dual-primal Methods

[Farhat, Lesoinne, LeTallec et al, '01]



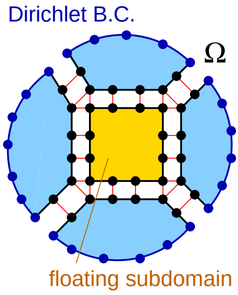
use primal dofs to avoid kernel

"One-level" FETI [Farhat & Roux, '91]

Projected Dual Problem:

$$P^\top \Big(\overbrace{\textstyle\sum_i B_i A_i^\dagger B_i^\top}^{=:F} \Big) \lambda = \mathrm{RHS}$$

$P = P(\alpha)$ ... α-**dependent** kernel projection involving **coarse solve**



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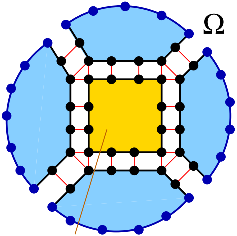
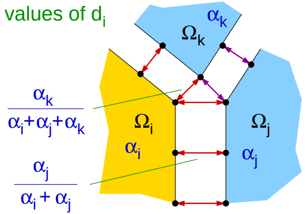


Preconditioner: [Klawonn, Widlund, '01]

$$P \Big(\overbrace{\sum_i D_i B_i \begin{bmatrix} 0 & 0 \\ 0 & S_i \end{bmatrix} B_i^\top D_i^\top}^{=:M^{-1}} \Big)$$

$D_i = D_i(\alpha)$ ... α-**weighted** diagonal scaling

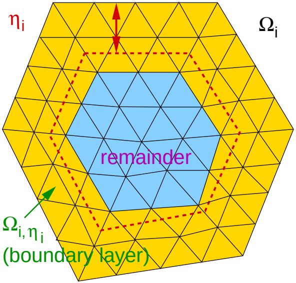


New Coefficient-Explicit FETI Theory

Boundary Layer: For $\eta_i > 0$ let

$$\underline{\alpha}_i^{\eta_i} \leq \alpha(x) \leq \overline{\alpha}_i^{\eta_i} \quad \text{for all } x \in \Omega_{i,\eta_i},$$

where $\Omega_{i,\eta_i} := \{x : \text{dist}(x, \Gamma_i) < \eta_i\}$ (boundary layer).



Arbitrary variation in remainder **!**

Theorem (Pechstein/Sch., '08)

Using $\overline{\alpha}_i^{\eta_i}$ *as weights in* D_i *and* P*:* *(in 2D and 3D!)*

$$\kappa(PM^{-1}P^\top F) \lesssim \max_j \left(\frac{H_j}{\eta_j}\right)^2 \max_i \frac{\overline{\alpha}_i^{\eta_i}}{\underline{\alpha}_i^{\eta_i}} \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2$$

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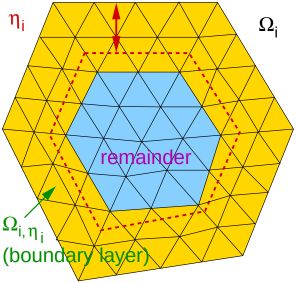
Additional Assumption:

$$\underline{\alpha}_i^{\eta_i} \lesssim \alpha(x) \quad \text{for } x \in \Omega \backslash \Omega_{i,\eta_i}$$



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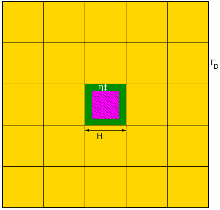
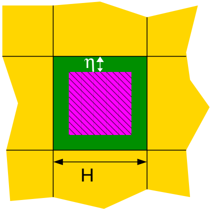
- Same theory for **FETI-DP** and other variants (e.g. Balancing Neumann-Neumann or BDDC)
- New **Poincaré-Friedrichs-type inequalities**

Numerical Results – One Island (PCG Its)





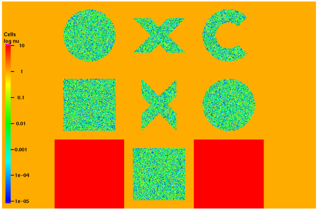
α_I = lognormal

Blue: $\alpha_I \geq 1 = \alpha_{BL}$

Red: $\alpha_I \leq 1 = \alpha_{BL}$

PCG Its	$\frac{H}{h} = 3$	6	12	24	48	96	192	384
$\frac{H}{\eta} = 3$	10 10	12 12	13 13	15 15	15 17	18 18	18 19	19 20
6	–	12 12	13 14	15 16	17 18	18 19	18 20	29 21
12	–	–	14 15	16 17	17 19	18 21	19 21	29 24
24	–	–	–	15 19	18 20	19 21	20 23	22 25
48	–	–	–	–	19 22	20 23	22 26	24 28
96	–	–	–	–	–	23 26	24 28	25 30
192	–	–	–	–	–	–	26 30	27 32
384	–	–	–	–	–	–	–	31 34
$\eta = 0$	10 11	13 14	15 17	17 19	19 23	21 26	24 32	26 39
$\alpha_I \equiv 1$	10	12	14	15	16	17	17	18

Numerical Results – Multiple Islands



$\boxed{\alpha_I = \text{lognormal}}$

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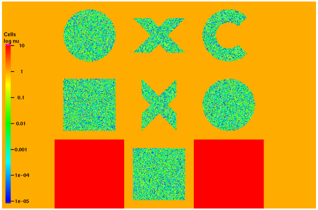
Red: $\alpha_I \leq 1 = \alpha_{BL}$

PCG Its	$\frac{H}{h} = 8$	16	32	64	128	256	512
$\frac{H}{\eta} = 8$	15 16	17 19	20 21	21 23	23 25	25 26	27 28
16	—	18 26	20 24	22 28	24 29	27 31	27 34
32	—	—	21 28	24 31	25 47	28 36	30 38
64	—	—	—	26 35	28 39	29 41	31 44
128	—	—	—	—	31 43	33 54	35 51
256	—	—	—	—	—	41 52	41 56
512	—	—	—	—	—	—	37 58

Condition Number Estimate (based on Ritz values)



α_I = lognormal

Green: $\alpha_I \geq 1 = \alpha_{BL}$

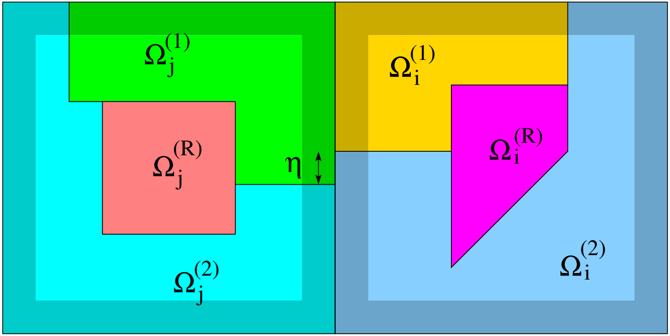
Orange: $\alpha_I \leq 1 = \alpha_{BL}$

	$\frac{H}{h} = 8$	16	32	64	128	256
$\frac{H}{\eta} = 8$	*3.7 4.2*	*4.6 5.6*	*5.6 7.1*	*6.8 8.9*	*8.2 10.7*	*9.7 12.6*
16	—	*5.6 28.1*	*6.5 11.7*	*7.6 14.3*	*8.8 17.2*	*10.2 20.2*
32	—	—	*9.3 18.1*	*10.1 22.1*	*11.1 85.2*	*12.2 32.5*
64	—	—	—	*16.3 33.2*	*17.1 41.3*	*18.0 49.4*
128	—	—	—	—	*28.8 58.6*	*30.6 81.5*
256	—	—	—	—	—	*55.5 93.4*

New Theory for Interface Variation

Per subdomain Ω_i, three materials are allowed:

- $\Omega_i^{(1)}$, $\Omega_i^{(2)}$ **connected** regions with mild variation
  (but possibly huge jumps between them!)
- $\Omega_i^{(R)}$ away from the interface, arbitrary variation

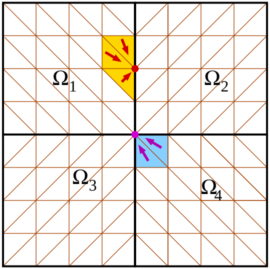


Define **nodal weights**:

$$\widehat{\alpha}_i(x) := \max_{T\in\mathcal{T}_i : x\in\overline{T}} \frac{1}{|T|}\int_T \alpha(x)\,dx$$

i.e. maximum on patch $\omega_x := \bigcup_{T:x\in\overline{T}} T$

"Superlumping" **[Rixen & Farhat, '98]**



Theorem (Pechstein/Sch., upcoming paper)

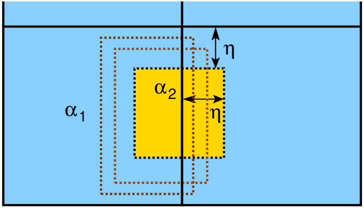
Using $\widehat{\alpha}_i(x)$ as weights in D_i and P and **all-floating** *FETI:*

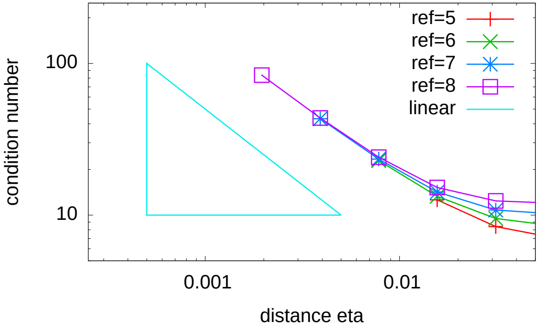
$$\kappa(PM^{-1}P^\top F) \lesssim \max_i \left(\frac{H_i}{\eta_i}\right)^{\beta} \left\{ \max_j \max_k \frac{\overline{\alpha}_j^{(k)}}{\underline{\alpha}_j^{(k)}} (1+\log(H_j/h_j))^2 \right\}$$

where $\boldsymbol{\beta}$ depends on exponent in **new** *weighted Poincaré inequality.*

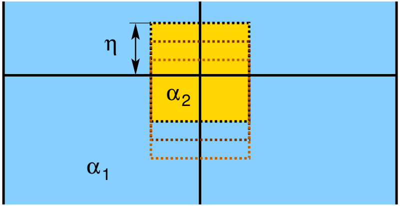
(For certain geometries we get $\boldsymbol{\beta}=2$, or if interior coefficient is larger, $\boldsymbol{\beta}=1$.)

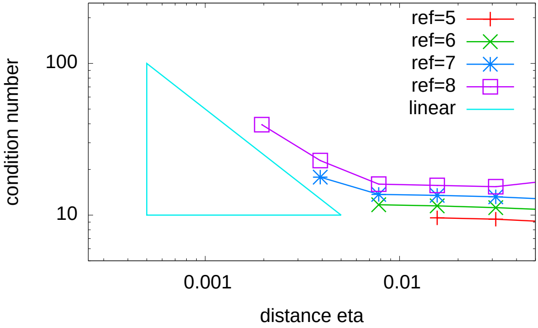
Numerical Results – Edge Island





Numerical Results – Cross Point Island





Conclusions

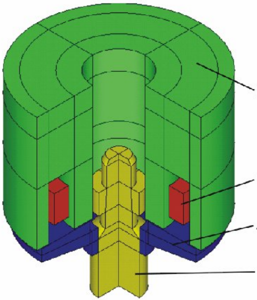
- Small modifications of standard DD methods render them **robust wrt. coefficient variation & mesh refinement**
- **Rigorous theory** even for non-resolved coefficients
- **Multilevel iterative solution** on fine grid asymptotically as costly/cheap as **numerical homogenisation/upscaling**
- Excellent **parallel efficiency** – results to come!

Nonlinear magnetostatics

$$-\nabla \cdot \left[\nu(|\nabla u|)\,\nabla u\right] = f \quad \text{in } \Omega$$

+ boundary conditions

+ interface conditions



Linearize via Newton

Nonlinear magnetostatics

$$-\nabla\cdot\left[\nu(|\nabla u|)\,\nabla u\right]=f\quad\text{in }\Omega$$

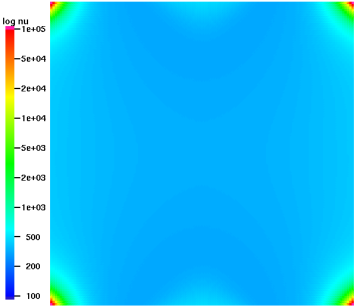
$+$ boundary conditions

$+$ interface conditions

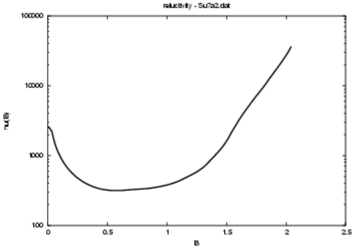
Linearize via Newton

Large variation of $\nu(\nabla u)$:

- from material to material: $\mathcal{O}(10^5)$ (discontinuous)
- within nonlinear material: $\mathcal{O}(10^3)$ (smooth)



reluctivity $|\nabla u|\mapsto\nu(|\nabla u|)$



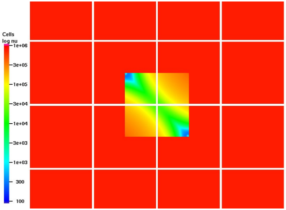
Strong variation along interface

One-level FETI with 16 subdomains:
($\varepsilon_{\text{lin}} = 10^{-8}$, $H = 1/4$, $h = 1/512$, $H/h = 128$)

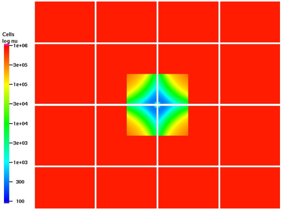
Problem	α	PCG Its	Cond #
homog.	1	13	8.26
pw. const.	1, 10^6	14	8.48
Case 1	$\nu(x)$	18.8	8.45
Case 2	$\nu(x)$	15.5	13.6

Variation of ν along interface in Case 2: ~ 2000!

> Contrary to common folklore:
> **Not necessarily best to allign sub-domains with material interfaces!**



Reluctivity $\nu(x)$ (**Case 1**)



Reluctivity $\nu(x)$ (**Case 2**)