Fluid/solid coupled convection/diffusion in unidirectional flows.

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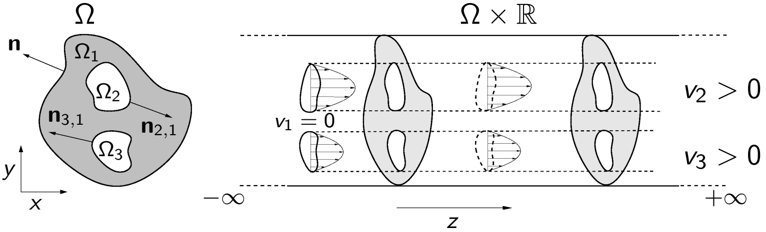
1- The problem : physical configuration



Heat transfer in an infinite cylinder with cross-section Ω.

- 3 sub-domains : Ω_1 (solid), $\Omega_{2,3}$ (fluid).
- Laminar steady flow : $\overrightarrow{\mathbf{v}} = v(x,y)\overrightarrow{\mathbf{e}}_z$.
- $v_i = v_{|\Omega_i}$, here : $v_1 = 0$ (solid), v_2, $v_3 \neq 0$ (fluid).
- Heterogeneous conductivities k : $k_i = k_{|\Omega_i}$, $k_i \neq k_j$.
- $\Gamma_{i,j}$ interface between Ω_i and Ω_j, $\mathbf{n}_{i,j}$ normal to $\Gamma_{i,j}$.

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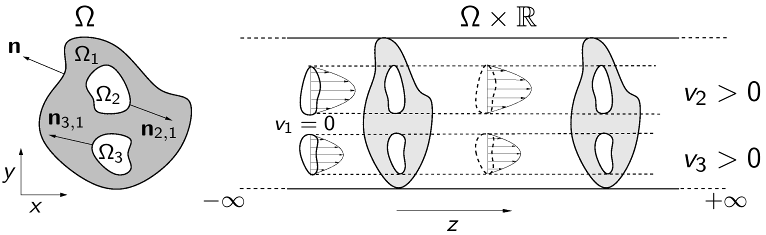

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- co-current flows,
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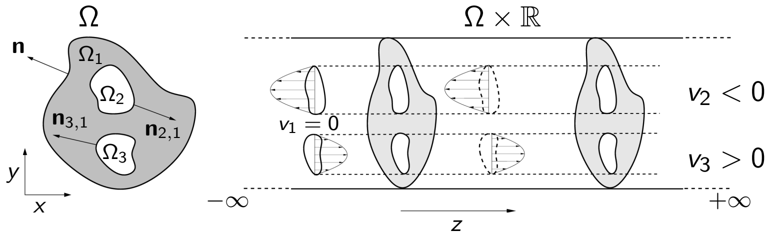

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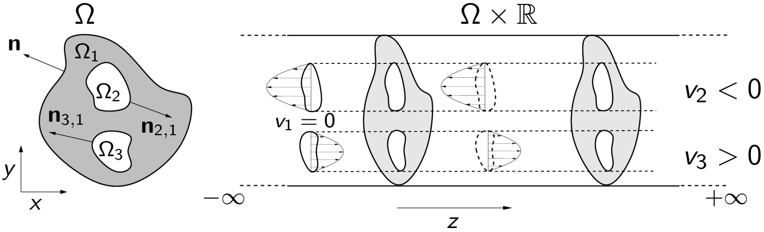

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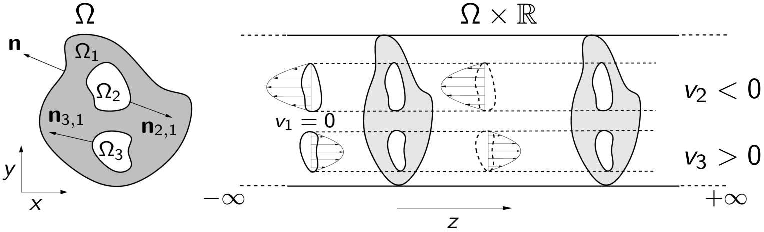
It has extension to :

- planar configurations (unbounded in x),
- periodic configurations.

2- The problem : mathematical formulation



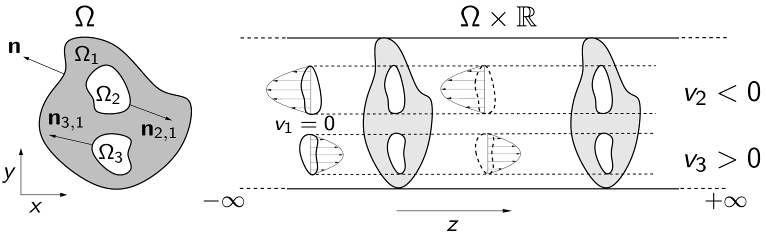
Energy equation on the temperature T : $v = v(x, y)$, $k = k(x, y)$,

$$\operatorname{div}(k\nabla T) \;+\; k\partial_z^2 T \;=\; v\partial_z T \;,$$

$+$ Continuity coupling conditions between the sub-domains :

$$T_i \;=\; T_j \;, \quad k_i\nabla T_i \cdot \mathbf{n}_{i,j} \;=\; k_j\nabla T_j \cdot \mathbf{n}_{i,j} \quad \text{on} \quad \Gamma_{i,j} \;,$$

2- The problem : mathematical formulation



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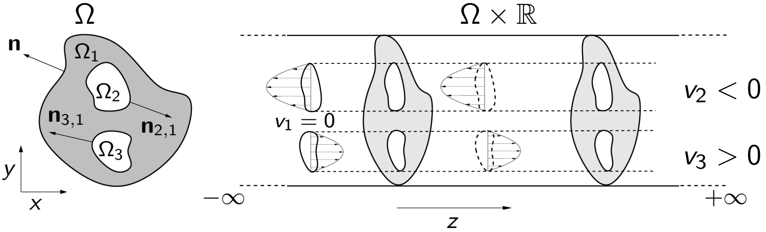
$$\mathrm{div}(k\nabla T) \;+\; k\partial_z^2 T \;=\; v\partial_z T \;,$$

$+$ Boundary conditions on $\partial\Omega$: Dirichlet with jump at $z = 0$,
$+$ Limit conditions at $\pm\infty$:

$$\text{on} \quad \partial\Omega \;:\; \begin{cases} T = 1 \;,\; z < 0 \\ T = 0 \;,\; z > 0 \end{cases} \quad \text{and} \quad \begin{cases} T \to 1 \;,\; z \to -\infty \\ T \to 0 \;,\; z \to +\infty \end{cases} \quad .$$

3- Objectives



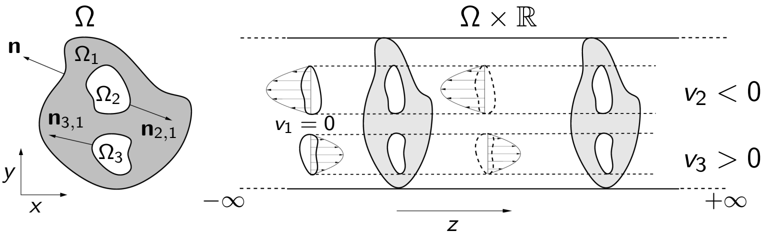
1. Macroscopic description of *an average temperature* $T^\star(z)$:

$$T^\star(z) \simeq C_1\ e^{\lambda_1\ z}\ +\ C_2\ e^{\lambda_2\ z}\ +\ \dots$$

2. Exchanges between sub-domains description :

$$\int_{\Gamma_{i,j}} k_i \nabla T_i \cdot \mathbf{n}_{i,j}\ ds\quad .$$

4- Pending questions



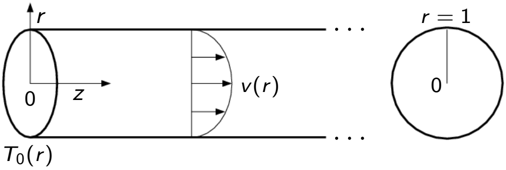
1. Does T read : $T(x,y,z) = \sum_{\lambda \in \Lambda} c_\lambda \; t_\lambda(x,y) \; e^{\lambda \; z}$?

2. Location of the "*spectrum*" Λ, get a computation method for the *eigenvalues/eigenfunctions* λ, $t_\lambda(x,y)$.

3. Computation of the constants c_λ : searching an *orthogonality property* for the t_λ.

Introductory exemple : the Graetz problem

Semi-infinite tube (radius 1)
1 fluid phase
Axi-symmetry
High Péclet : $Pe \gg 1$



Taylor approximation $\longrightarrow$ axial diffusion $\partial_z^2 T$ neglected

"*Directional*" problem $\longrightarrow$ entry condition $T_0(r)$ given

$$\frac{1}{r}\partial_r \left(r\ \partial_r T\right) = v(r)\ \partial_z T\ ,\quad T(r,0) = T_0(r)\ ,\quad T(1,z) = 0\ .$$

Separate variable $\longrightarrow$ $T = t(r)\ e^{\lambda z}$

Eigenvalue problem $\longrightarrow$ λ, $t(r)$ read :

$$\frac{1}{r}\partial_r \left(r\ \partial_r t\right) = \lambda\ Pe\ v(r)\ t\ ,\quad t(1) = 0\ .$$

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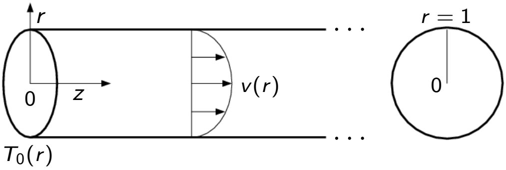


Self-adjoint, negative and compact problem :

$\Rightarrow$ Complete orthogonal system of eigenfunctions $(t_i(r))_i$,

with eigenvalues $0 > \lambda_1 \geq \lambda_2 \geq \cdots \rightarrow \; -\infty$.

$\Rightarrow$ Analytical solution :

$$T(r,z) \; = \; \sum_{i \in \mathbb{N}} c_i \;\, t_i(r) \;\, e^{\lambda_i z} \quad , \qquad c_i \; = \; \int_0^1 t_i \;\, T_0 \;\, r \;\, dr \quad .$$

Generalisation 1 : extended Graetz

Axial diffusion $\partial_z^2 T$ is no longer neglected.

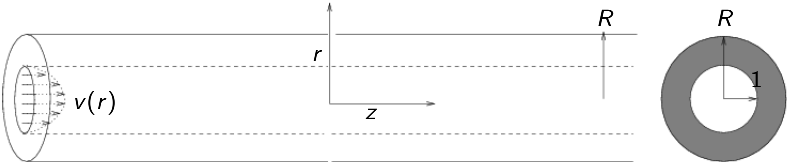
1. Separate variable $\longrightarrow$ $T = t(r)\ e^{\lambda z}$
   $\longrightarrow$ do not provide an eigenvalue problem

$$\frac{1}{r}\partial_r\left(r\ \partial_r t\right) = \left(\lambda\ v(r)\ -\ \lambda^2\right)\ t\ .$$

2. No symmetry property available :
   $\longrightarrow$ problem for the spectrum location : $\Lambda \in \mathbb{R}$?
   $\longrightarrow$ computational problem for the c_λ.

3. The problem is not *directional* any more :
   entry condition $T_0(r)$ not relevant.
   $\longrightarrow$ switch to limit conditions in $\pm\infty$.

Generalisation 2 : conjugated Graetz



Coupling with a solid wall where diffusion occurs.

$$1 < r < R \;:\; \frac{1}{r}\partial_r\left(r\partial_r T\right) + \partial_z^2 T = 0$$

$$r = 1 \;:\; T(1^+, z) = T(1^-, z)\ ,\quad \partial_r T(1^+, z) = k\ \partial_r T(1^+, z)\ .$$

⇒ **Same difficulties as before :**

1. no *real* eigenvalue problem,
2. no symmetry property ,
3. problem not *directional*.

Mixed reformulation : statement

One reformulate the initial problem

$$\mathrm{div}(k\nabla T) \; + \; k\partial_z^2 T \; = \; v\partial_z T \; ,$$

adding a vectorial unknown $\mathbf{X} = \mathbf{X}(x, y, z)$:

$$k \; \partial_z T = v \;\; T \; - \; \mathrm{div}(\mathbf{X})$$
$$\partial_z \mathbf{X} = k\nabla T$$

→ Introducing the operator A :

$$\partial_z \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. \; = \; A \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. \; , \quad A = \begin{pmatrix} v \; k^{-1} & -k^{-1}\mathrm{div} \\ k \; \nabla & \end{pmatrix}$$

Mixed reformulation : analysis

Theorem 1. The unbounded operator $A: \; D(A) \subset \mathcal{H} \mapsto \mathcal{H}$,

$$\mathcal{H} = L^2(\Omega) \times L^2(\Omega)^2 \quad , \quad D(A) = H^1_0(\Omega) \times H(div, \Omega) \quad :$$

1. is self adjoint,
2. is diagonal on an eigenfunctions orthogonal system,
3. $\lambda_0 = 0$ excepted, all eigenvalues have finite order.

Its spectrum Λ reads

$$\Lambda = \{\lambda_0\} \; \cup \; \Lambda^+ \; \cup \; \Lambda^- \quad :$$

- Λ^+=**downstream** modes : $0 > \lambda_1^+ \geq \lambda_2^+ \geq \cdots \to -\infty$
  $\to$ related to the $z > 0$ region.
- Λ^-=**upstream** modes : $0 < \lambda_1^+ \leq \lambda_2^+ \leq \cdots \to +\infty$
  $\to$ related to the $z < 0$ region.

Mixed reformulation : solution definition

Analytical solution : defined from

- Downstream eigenvalues / eigenfunctions : λ_n^+, $t_n^+(x,y)$
- Upstream eigenvalues / eigenfunctions : λ_n^-, $t_n^-(x,y)$
- The coefficients α_n

$$\alpha_n := \frac{1}{\lambda_n^2} \int_{\partial\Omega} k \nabla t_n \cdot \mathbf{n} \, ds \ ,$$

Corollary . The sought temperature field reads :

$$T(x,y,z) = \begin{cases} 1 + \sum_n \alpha_n^- \ t_n^-(x,y) \ e^{\lambda_n^- z} & z \leq 0 \\ -\sum_n \alpha_n^+ \ t_n^+(x,y) \ e^{\lambda_n^+ z} & z \geq 0 \end{cases}$$

Some numerical analysis

The following eigen-pronlem has to be solved :

$$\text{find } \lambda \in \mathbb{R}, \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. \in \; D(A) \; : \quad A \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. = \lambda \left| \begin{array}{l} T \\ \mathbf{X} \end{array} \right. . \qquad (1)$$

Theorem 2. Eigen-problem (1) is equivalent to the following variational problem :

find $\lambda \in \mathbb{R}$ and $(T, \mathbf{X}) \in L^2(\Omega) \times H(\mathrm{div}, \Omega)$,

such that $\forall (u, \mathbf{Y}) \in L^2(\Omega) \times H(\mathrm{div}, \Omega)$:

$$\begin{aligned} \int_\Omega T\, u\, v\, dx \; - \; \int_\Omega u\, \mathrm{div}(\mathbf{X})\, dx \; &= \; \lambda \int_\Omega T\, u\, k\, dx \\ - \int_\Omega T\, \mathrm{div}(\mathbf{Y})\, dx \; &= \; \lambda \int_\Omega \mathbf{X} \cdot \mathbf{Y}\, k^{-1} dx \, . \end{aligned}$$

Axi-symmetric convergence analysis

Discretisation using mixed finite element spaces :

$$e.\ g.\ T_h \in \mathbb{P}^0,\ \mathbf{X}_h \in RT_0.$$

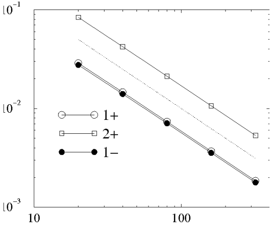
Evaluation of the method on the conjugated Graetz problem :

→ Reduction to a 1D numerical problem (axi-symmetry).

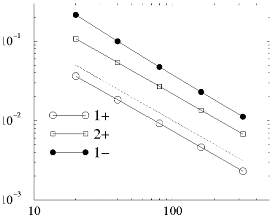
→ Comparison with analytical reference solutions.

Relative error on the first eigenfunctions : T_1^+, T_2^+ and T_1^-, with respect to the nodes number.

Dashed line : slope = -1



$Pe = 0.1$



$Pe = 10$

Convergence rate on the eigenvalues → same order 1.

Conclusion

- *Nice mathematical framework* : orthogonality properties, problem analysis on a compete orthogonal base.
- Natural mixed numerical formulation.
- From 3D to 2D problem reduction,
  Only smallest modulus eigenvalues to be computed (principal modes),
  Numerical validation on a test case.

Conclusion and perspectives

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- General 2D implementation,
- heat exchanger shape optimisation.