Adaptation of a mortar method to model flow in large-scale fractured media

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Scaling up and Modeling for Transport and Flow in Porous Media

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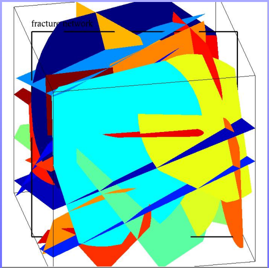
Motivation

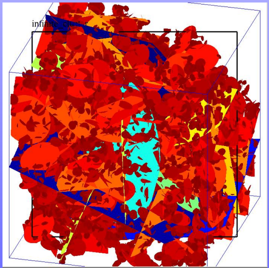
The simulation of the flow in Discrete Fracture Networks (DFNs).

Fracture network characteristics :

- Many fractures intersecting each other ($\approx 10^4$ fractures, $\approx 10^5$ intersections),
- Fractures with broad ranges of length, shape, orientation and position $\Rightarrow$ A stochastic discrete approach to model fractures
- A set of 2D domains (fractures) intersecting each other.



30 fractures/125 intersections



8064 fractures/12943 intersections

Some assumptions :

- The rock matrix is impervious : flow is only simulated in the fractures,
- Study of steady state flow,
- There is no longitudinal flux in the intersections of fractures.

Numerical method : a Mixed Hybrid Finite Element Method

- Makes it easy to deal with complex geometry (triangular elements) ;
- A linear system with only trace of pressure unknowns, the flux at the edges and the mean pressure are then easily derived locally on each triangle.

Two main difficulties :

1. Classical mesh generation can be insufficient due to the amount of intersections between fractures (FE with bad aspect ratio), e.g. success in only 222 networks for 1620 generated networks $\Rightarrow$ Local corrections are required, *J. Erhel et al., submitted 2008*
2. Matching grids at the intersection can be very costly (e.g. consider a small fracture with a fine mesh intersecting a large one).

Some assumptions :

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- Study of steady state flow,
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Our objective :

Allowing **independent mesh generation** within the fractures
$\Rightarrow$ Non matching grid at the intersections between fractures

The challenge :

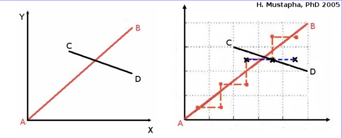
- Implementation of a Mortar method for each intersection between fractures to ensure continuity of the flux and trace of pressure at the intersections,
- The number of fractures and intersections can be large so that we have to deal with numerous cases of non matching grids,
- Handling the large variety of configurations leading to numerical difficulties.

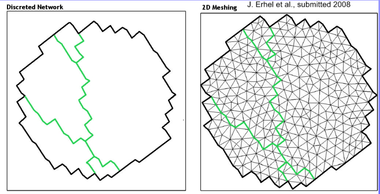
Mortar method : Bernardi, Maday et Patera, 1992 ; Arbogast, Cowsar, Wheeler et Yotov, 2000

Creating 3D Discrete Fracture Networks (DFNs) and meshing

A software included in the scientific platform Hydrolab (written in $C++$) (*Erhel et al., 2007*) :

- Allows the generation of random DFNs in a 3D domain,
- Includes a mesh generator for general DFN structures (using a procedure extracted from the software FreeFem++)
- Includes projection of the intersections on a regular grid (H. Mustapha, PhD, 2005) and local corrections to remove configurations that would lead to triangles with bad aspect ratio, (*J. Erhel, J.R. De Dreuzy and B. Poirriez, submitted 2008*)





Flow equations in each fracture Ω_f

$$
\begin{aligned}
\nabla \cdot \mathbf{u} = \mathbf{f}(\mathbf{x}) &\quad \text{for } \mathbf{x} \in \Omega_f \\
\mathbf{u} = -\mathcal{K}(\mathbf{x}) \nabla p(\mathbf{x}) &\quad \text{for } \mathbf{x} \in \Omega_f \\
p(\mathbf{x}) = p^D(\mathbf{x}) &\quad \text{on } \Gamma_D \\
\mathbf{u}(\mathbf{x}).\nu = q^N(\mathbf{x}) &\quad \text{on } \Gamma_N \\
\mathbf{u}(\mathbf{x}).\mu = 0 &\quad \text{on } \Gamma_f,
\end{aligned}
$$

ν (resp. μ) outer normal unit vector to the cube edge (resp.fracture side) ; $\mathcal{K}(\mathbf{x})$ is a given 2D permeability field ; $\mathbf{f}(\mathbf{x}) \in L^2(\Omega_f)$ represents the sources/sinks ;

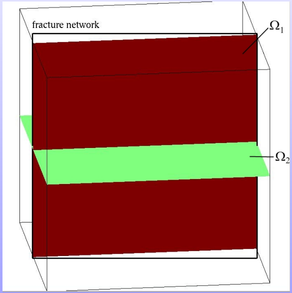
+ Continuity conditions at each intersection :

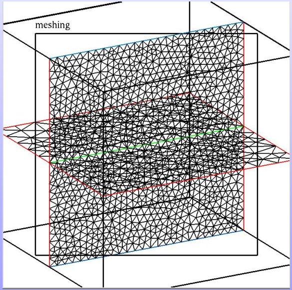
$$
p_{k,h} = p_k, \text{ on } \Sigma_k, \forall f \in F_k \quad \text{and} \quad \sum_{f \in F_k} \mathbf{u}_{k,f}.\mathbf{n}_{k,f} = 0 \text{ on } \Sigma_k,
$$

where $p_{k,h}$ is the trace of pressure and $\mathbf{n}_{k,f}$ is the normal unit vector on the boundary Σ_k of the fracture Ω_f,

Σ_k is the k-st intersection, F_k is the set of fractures with Σ_k as intersection.

On a simple example with two fractures





Geometry :

- A cubic domain $\Omega=[0, L]\times[0, L]\times[0, L]$,
- Two fractures Ω_1 and Ω_2, with $\Gamma = \Omega_1 \cap \Omega_2$.
- Ω_1 and Ω_2 independently meshed (mesh step in Ω_1 : 0.08 ; in Ω_2 : 0.2)
- Choice of a master intersection side (e.g. domain 1) and a slave intersection side (e.g. domain 2)

Remark : things will be more complicated with many fractures intersecting each other ...

Weak formulation of the problem

Géraldine Pichot, Jocelyne Erhel , Jean-Raynald De Dreuzy

- $P_d(K)$ space of polynoms of total degree d defined on K, $K \in \mathcal{T}_h$:

$$RT^0(K) = \{s \in (P_1(K))^2, s = (a + b\,x_1,\ c + b\,x_2), a, b, c \in \mathbb{R}\}$$

$$RT^0(\mathcal{T}_h) = \{\phi \in L^2(\Omega), \phi|_K \in RT^0(K), \forall K \in \mathcal{T}_h\}$$

- We also need a space $\mathcal{M}^0(\mathcal{T}_h)$ defined as :

$$\mathcal{M}^0(\mathcal{T}_h) = \{\varphi \in L^2(\Omega), \varphi|_K \in P_0(K), K \in \mathcal{T}_h\}$$

- $\mathcal{E}_{h,in}$ set of edges of the two meshes not belonging to Γ,
- $\mathcal{E}_{h,m}^G$ (resp. $\mathcal{E}_{h,s}^G$) : edges belonging to Γ on the master (resp. slave) side
- $\mathcal{E}_h = \mathcal{E}_{h,in} \cup \mathcal{E}_{h,m}^G \cup \mathcal{E}_{h,s}^G$.
- We define the multiplier spaces

$$\mathcal{N}^0(\mathcal{E}_h) = \{\lambda \in L^2(\mathcal{E}_h), \lambda|_E \in P_0(E), \forall E \in \mathcal{E}_h\}$$

$$\mathcal{N}_{g,D}^0(\mathcal{E}_h) = \{\lambda \in \mathcal{N}^0(\mathcal{E}_h), \lambda = g \text{ on } \Gamma^D\}$$

Weak MH Mortar formulation

Find $(\mathbf{u}_h, p_h, tp_h) \in RT^0(\mathcal{T}_h) \text{ x } \mathcal{M}^0(\mathcal{T}_h) \text{ x } \mathcal{N}^0_{p^D,D}(\mathcal{E}_h)$ such that :

$$
\begin{aligned}
&\int_\Omega \mathcal{K}^{-1}\mathbf{u}_h.\boldsymbol{\chi}_h d\mathbf{x} + \\
&\sum_{K\in\mathcal{T}_h}\int_{\partial K} tp_h \boldsymbol{\chi}_h.\nu_K dl &=& \quad \textstyle\sum_{K\in\mathcal{T}_h}\int_K p_h \nabla\cdot\boldsymbol{\chi}_h d\mathbf{x}, \forall \boldsymbol{\chi}_h \in RT^0(\mathcal{T}_h),\\
&\int_\Omega \nabla\cdot\mathbf{u}_h.\varphi_h d\mathbf{x} &=& \quad \textstyle\int_\Omega f\varphi_h d\mathbf{x}, \forall\varphi_h \in \mathcal{M}^0(\mathcal{T}_h),\\
&\sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathbf{u}_h.\nu_K\lambda_h dl &=& \quad \textstyle\int_{\partial\Omega} q^N\lambda_h dl, \forall\lambda_h \in \mathcal{N}^0_{0,D}(\mathcal{E}_{h,in})
\end{aligned}
$$

$$
\sum_{E\in\mathcal{E}^G_{h,m}}\int_E \mathbf{u}_h.\nu_E\eta_h = -\sum_{E'\in\mathcal{E}^G_{h,s}}\int_{E'} \mathbf{u}_h.\nu_{E'}\eta_h, \forall\eta_h \in M^m_h,
$$

$$
\sum_{E\in\mathcal{E}^G_{h,m}}\int_E tp_h\boldsymbol{\beta}_h.\nu_E dl = \sum_{E'\in\mathcal{E}^G_{h,s}}\int_{E'} tp_h\boldsymbol{\beta}_h.\nu_{E'} dl, \forall\boldsymbol{\beta}_h \in M^s_h.
$$

with $M^m_h = \mathcal{N}^0(\mathcal{E}^G_{h,m})$ and M^s_h the space spanned by the local RT basis functions on the slave element sides.

Weak MH Mortar formulation

Find $(\mathbf{u}_h, p_h, tp_h) \in RT^0(\mathcal{T}_h) \text{ x } \mathcal{M}^0(\mathcal{T}_h) \text{ x } \mathcal{N}^0_{p^D,D}(\mathcal{E}_h)$ such that :

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with $M^m_h = \mathcal{N}^0(\mathcal{E}^G_{h,m})$ and M^s_h the space spanned by the local RT basis functions on the slave element sides.

Those conditions can be written equivalently in matrix form :

- Continuity of the flux :

  $$Q^m = -C^T Q^s,$$

  with Q^m (resp. Q^s) flux unknowns along the master (resp. slave) side of Γ,
- Continuity of the trace of pressure :

  $$Tp^s = CTp^m,$$

with C a matrix representating the L^2-projection from one side to the other, of size $N_s \times N_m$, whose coefficients C_{ij}, $i \in 1, ..., N_s$, $j \in 1, ..., N_m$ are

$$C_{ij} = \left(\frac{|E_j^m \cap E_i^s|}{|E_i^s|} \right),$$

where $|E|$ denotes the length of the edge E, E_j^m (resp. E_i^s) denotes a master (resp. slave) edge.

Elimination of slave unknows

$$\begin{cases} D\,P - \begin{pmatrix} R_{in} & R_m + R_s C \end{pmatrix} \begin{pmatrix} Tp_{in} \\ Tp_m \end{pmatrix} = F, \\ \begin{pmatrix} M_{in} & M_m + M_s C \\ M_m^T + C^T M_s^T & B_m + C^T B_s C \end{pmatrix} \begin{pmatrix} Tp_{in} \\ Tp_m \end{pmatrix} \\ -\begin{pmatrix} R_{in}^T \\ R_m^T + C^T R_s^T \end{pmatrix} P - V = 0. \end{cases}$$

where Tp_{in} is the trace of pressure unknowns on edges in $\mathcal{E}_{h,in}$, Tp_m is the trace of pressure unknowns on edges in $\mathcal{E}_{h,m}$,
F vector of dimension $N_{\mathcal{T}}$ (source/sink and Dirichlet BC), V vector of dimension $N_{\mathcal{E}}$ (Dirichlet and Neumann BC), with $N_{\mathcal{E}}$ the cardinal of $\mathcal{E}_h$ and $N_{\mathcal{T}}$ the cardinal of $\mathcal{T}_h$.

This system can be rewritten under the form, with M symmetric

$$\begin{pmatrix} D & -R \\ -R^T & M \end{pmatrix} \begin{pmatrix} P \\ Tp \end{pmatrix} = \begin{pmatrix} F \\ V \end{pmatrix}.$$

The Schur complement matrix follows :

$$S = M - R^T D^{-1} R.$$

The Schur complement system becomes then

$$\begin{cases} S\,Tp = R^T D^{-1} F + V, \\ D\,P = R\,Tp + F; \end{cases}$$

with

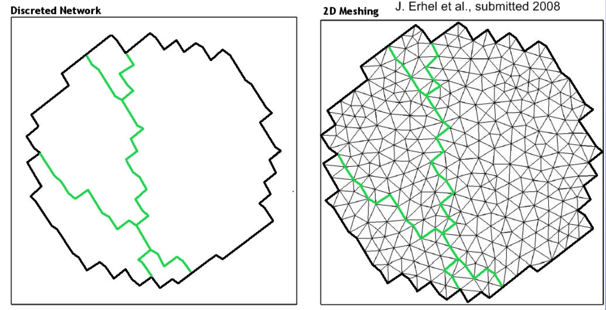
$$Tp = \begin{pmatrix} Tp_{in} \\ Tp_m \end{pmatrix}$$

$\Rightarrow$ A linear system in Tp.

Algorithm

1. Initialize geometry and physical parameters of the problem (a domain independant meshing process is now possible) ;
2. **Choice of slave and master sides for the intersections between fractures and compute the matrices C for each intersection ;**
3. Create the Schur complement matrix ;
4. Find Tp by solving the first system ;
5. Find P by solving the second system ;
6. **Find Tp^s thanks to Tp^m ;**
7. Loop on the triangle elements :
   - Compute the flux Q_K on each triangle thanks to local relations involving p_K and Tp_K.

Conflictual configurations



Discretization of the intersections within a fracture using the grid projection - J. Erhel et al, submitted 2008

Some edges may belong to several intersections.
What happens when an egde belongs simultaneously to a master and a slave intersection ?
$\Rightarrow$ We duplicate the edges in common (with care to keep a system like the one we gave previously - the one with the Schur complement)

Flow computation using the Mortar method

$\Rightarrow$ a code we wrote in Matlab :

- The Hydrolab mesh and intersections information are loaded in Matlab,
- The solver used in Matlab is the direct solver UMFPACK,
- Rules for the affectation of the master/slave properties :

Known	Property	Contains ME	Contains SE	Duplicated edges	Reused edges
No	Master	No	No	No	No
Master	Master	Yes	No	No	Master
Slave	Slave	Yes	No	M $\rightarrow$ S	No
Slave	Slave	No	Yes	No	Slave
No	Master	Yes	Yes	S $\rightarrow$ M	Master
No	Master	Yes	No	No	Master
No	Slave	No	Yes	No	Slave

Remark : Additionnal equality equations in the system between duplicated edges and their duplicata.

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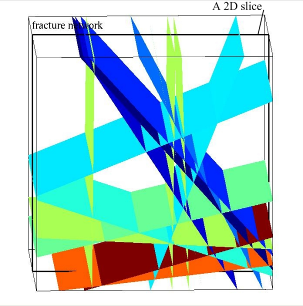
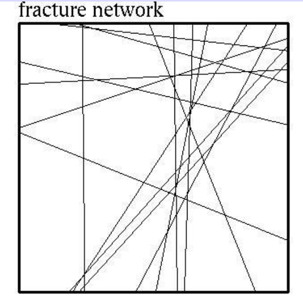
For some particular 3D geometries



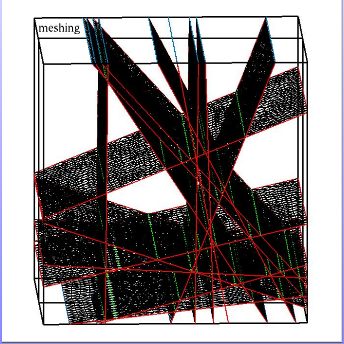


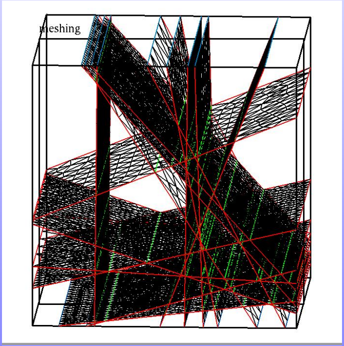
Example of fracture network with its 2D slice - 15 fractures

Imposed Boundary Conditions :

- On top of the cube : Dirichlet BC (imposed pressure =10) ;
- On the lateral sides : nul flux ;
- On bottom : Dirichlet BC (imposed pressure =0) ;

Meshing process

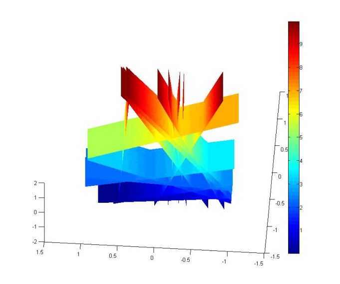


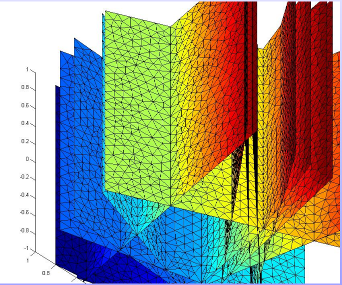


Matching grids
33164 egdes, mesh step=0.1

Non matching grids
23362 egdes, mesh steps from 0.3 to 0.08

Computed solution - Matching grid case

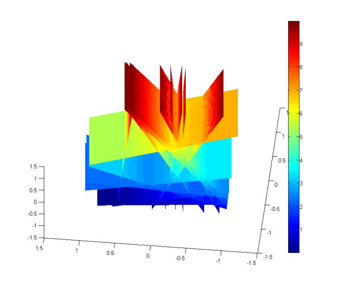


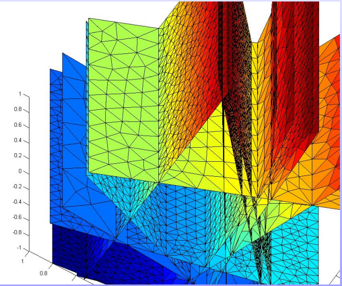


Computed mean pressure

- Relative error by comparison with the 2D solution : 4.25e-6,
- Input flux : $Q_{input} = 80.46\mathrm{m}^3.\mathrm{s}^{-1}$,
- Equivalent permeability : $K = \dfrac{Q_{input}}{L\,\delta h} = 4.0235\mathrm{m}^2.\mathrm{s}^{-1}$,
- Sum of flux on intersections : $3e - 13$
- Number of edges : 33164 ; Nb of intersections : 85

Computed solution - Non matching grid case





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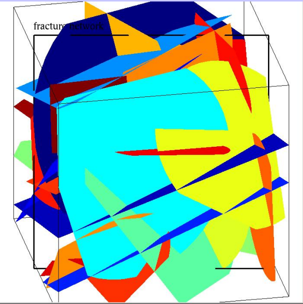
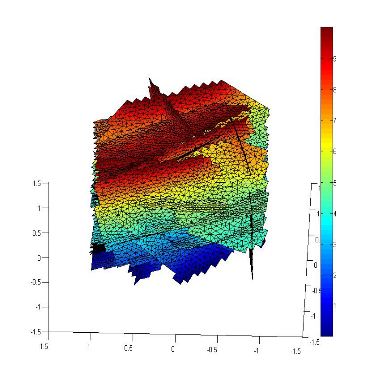
- Equivalent permeability : $K = \dfrac{Q_{input}}{L\,\delta h} = 4.0235 \mathrm{m}^2.\mathrm{s}^{-1}$,
- Sum of flux on intersections : $4e - 13$
- Number of edges : 23362

A more complex geometry - Matching grids

With 30 fractures of various lengths, mesh step : 0.08





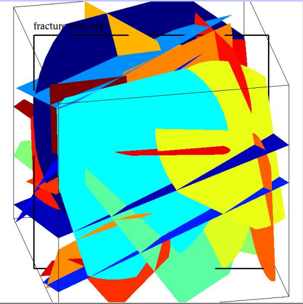
Initial geometry and computed mean pressure

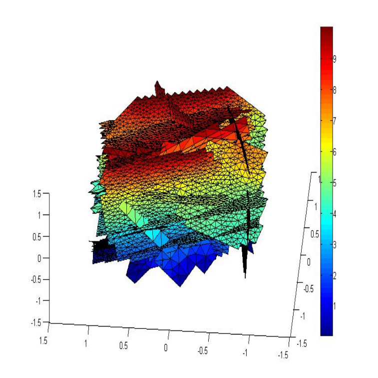
$Q_{input} = 49.53\mathrm{m}^3.\mathrm{s}^{-1}$; $K = 2.47\mathrm{m}^2.\mathrm{s}^{-1}$; Sum flux on intersections : 1e-13 ;
Nb of intersections : 114 ; Nb of edges : 37794 (1499 master - 1516 slave)

Nb of conflicts : 12 (slave)+ 31(master) ; Nb edges reused : 69 master + 77 slave

A more complex geometry - Non matching grids

With 30 fractures of various lengths, mesh step ranges from 0.07 to 0.2





Initial geometry and computed mean pressure

$Q_{input} = 49.14\mathrm{m}^3.\mathrm{s}^{-1}$; $K = 2.45\mathrm{m}^2.\mathrm{s}^{-1}$; Sum flux on intersections : 2e-13 ;
Nb of intersections : 114 ; Nb of edges : 31975 (1370 master - 1304 slave)

Nb of conflicts : 9 (slave)+ 36(master) ; Nb edges reused : 94 master + 76 slave

Conclusions and Perpectives

Conclusions

1. Validation of the method for some particular geometries,
2. Promising results for more general networks with many fractures in intersection.

Perpectives

1. Studying the properties of the Schur complement matrix with Mortar,
2. Reducing (if possible) the system to a system with only master unknowns, performing its parallel implementation and choosing the appropriate solver (B. Poirriez, PhD Inria),
3. Integrating the Mortar method into Hydrolab and performing simulations for larger networks,
4. Optimizing the mesh step within each fracture to keep a good precision on the results with a reduced number of unknowns.