A two-level enriched finite element method for the Darcy equation

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Plan of the talk

- The Darcy equation and the enrichment strategy.
- The semi-discrete method and error estimates.
- The two-level method and its analysis.
- Numerical results.
- Concluding remarks.

The problem statement : Find $(\boldsymbol{u}, p)$ such that

$$\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot \boldsymbol{u} = g \quad \text{in } \Omega\,,$$
$$\boldsymbol{u}\cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,,$$

where $\int_\Omega g = 0\,.$

Weak problem : Find $(\boldsymbol{u}, p) \in H_0^{div}(\Omega) \times L_0^2(\Omega)$ such that

$$\boldsymbol{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{F}(\boldsymbol{v}, q) \qquad \forall\,(\boldsymbol{v}, q) \in H_0^{div}(\Omega) \times L_0^2(\Omega)\,,$$

where

$$\boldsymbol{A}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) := (\boldsymbol{u}, \boldsymbol{v})_\Omega - (p, \nabla\cdot \boldsymbol{v})_\Omega - (q, \nabla\cdot \boldsymbol{u})_\Omega\,,$$
$$\boldsymbol{F}(\boldsymbol{v}, q) := (\boldsymbol{f}, \boldsymbol{v})_\Omega - (g, q)_\Omega\,.$$

Derivation of the Method : Find $\boldsymbol{u}_H := \boldsymbol{u}_1 + \boldsymbol{u}_e \in \mathbb{P}_1(\Omega)^2 + H_0^{div}(\Omega)$ and $p_H := p_0 + p_e \in \mathbb{P}_0(\Omega) \oplus L_0^2(\mathcal{T}_H)$ such that

$$\boldsymbol{A}((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_0 + p_e), (\boldsymbol{v}_H, q_H)) = \boldsymbol{F}(\boldsymbol{v}_H, q_H),$$

for all $\boldsymbol{v}_H := \boldsymbol{v}_1 + \boldsymbol{v}_b \in \mathbb{P}_1(\Omega)^2 \oplus H_0^{div}(\mathcal{T}_H)$, $q_H = q_0 + q_e \in \mathbb{P}_0(\Omega) \oplus L_0^2(\mathcal{T}_H)$, where

$$H_0^{div}(\mathcal{T}_H) := \{\boldsymbol{w} \in L^2(\Omega)^2 \,:\, \boldsymbol{w}|_K \in H_0^{div}(K)\, \forall\, K \in \mathcal{T}_H\},$$
$$L_0^2(\mathcal{T}_H) := \{q \in L^2(\Omega) \,:\, q|_K \in L_0^2(K)\,, \forall\, K \in \mathcal{T}_H\}.$$

Equivalent system :

$$\boldsymbol{A}((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_0 + p_e), (\boldsymbol{v}_1, q_0)) = \boldsymbol{L}(\boldsymbol{v}_1, q_0) \qquad \forall\, (\boldsymbol{v}_1, q_0) \in \mathcal{V}_H \times Q_H\,,$$
$$(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_b)_K - (p_0 + p_e, \nabla \cdot \boldsymbol{v}_b)_K - (q_e, \nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e))_K$$
$$= (\boldsymbol{f}, \boldsymbol{v}_b)_K - (g, q_e)_K\,,$$

for all $(\boldsymbol{v}_b, q_e) \in H_0(div, K) \times L_0^2(K)$ and all $K \in \mathcal{T}_H$.

Strong problem for $(\boldsymbol{u}_e, p_e)$:

$$\boldsymbol{u}_e + \nabla p_e \;=\; -\boldsymbol{u}_1, \quad \nabla\cdot \boldsymbol{u}_e = C_K \quad \text{in } K,$$

$$\boldsymbol{u}_e \cdot \boldsymbol{n} = \alpha\, H_F\, [\![p_0]\!] \quad \text{on each } F \subseteq \partial K \cap \Omega\,.$$

In order to make this problem compatible, we set

$$C_K = \frac{1}{|K|} \sum_{i=1}^{3} \alpha\, H_{F_i} \int_{F_i} [\![p_0]\!]\,.$$

Derivation of the Method (continued)

- Splitting $\boldsymbol{u}_e = \boldsymbol{u}_e^M + \boldsymbol{u}_e^D$ and $p_e = p_e^M + p_e^D$

- $(\boldsymbol{u}_e^M, p_e^M)$ solves

$$\boldsymbol{u}_e^M + \nabla p_e^M = -\boldsymbol{u}_1, \quad \nabla\cdot \boldsymbol{u}_e^M = 0 \quad \text{in } K,$$
$$\boldsymbol{u}_e^M \cdot \boldsymbol{n} = 0 \quad \text{on } \partial K$$

- $(\boldsymbol{u}_e^D, p_e^D)$ solves

$$\boldsymbol{u}_e^D + \nabla p_e^D = \boldsymbol{0} \quad \text{in } K, \quad \nabla\cdot \boldsymbol{u}_e^D = C_K \quad \text{in } K,$$
$$\boldsymbol{u}_e^D \cdot \boldsymbol{n} = \alpha H_F \, [\![p_0]\!] \quad \text{on each } F \subseteq \partial K \cap \Omega \, .$$

Remarks:

- $\boldsymbol{u}_e^D$ is a Raviart-Thomas field. Indeed, there holds

$$\boldsymbol{u}_e^D = \sum_{F \subseteq \partial K \cap \Omega} \alpha H_F [\![p_0]\!] \boldsymbol{\varphi}_F ,$$

  where

$$\boldsymbol{\varphi}_F(\mathrm{x}) = \frac{|K|}{2H_F} (\mathrm{x} - \mathrm{x}_F) .$$

Returning to the first equation : For all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$:

$$(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_1)_\Omega - (p_0 + p_e, \nabla \cdot \boldsymbol{v}_1)_\Omega + (q_0, \nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e))_\Omega = \boldsymbol{F}(\boldsymbol{v}_1, q_0)\,.$$

Remark :

- $(p_e, \nabla\cdot \boldsymbol{v}_1)_K = 0$ for all $K \in \mathcal{T}_H$, and hence the enrichment of the pressure has no effect on the formulation.
- $(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_1)_\Omega = (\boldsymbol{u}_1 + \underbrace{\boldsymbol{u}_e^M(-\boldsymbol{u}_1)}_{=-\mathcal{M}_K(\boldsymbol{u}_1)}, \boldsymbol{v}_1)_\Omega + \sum_{K\in\mathcal{T}_H} (\boldsymbol{u}_e^D([\![p_0]\!]), \boldsymbol{v}_1)_K$;
- $(q_0, \nabla\cdot \boldsymbol{u}_e)_\Omega = \sum_{K\in\mathcal{T}_H} (\boldsymbol{u}_e^D \cdot \boldsymbol{n}, q_0)_{\partial K} = \sum_{F\in\mathcal{E}_H} (\alpha H_F [\![p_0]\!], [\![q_0]\!])_F$;

Find $(\boldsymbol{u}_1, p_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$ *such that*

$$\sum_{K\in\mathcal{T}_H} ((\mathcal{I}-\mathcal{M}_K)(\boldsymbol{u}_1), \boldsymbol{v}_1)_\Omega + \sum_{K\in\mathcal{T}_H} (\boldsymbol{u}_e^D([\![p_0]\!]), \boldsymbol{v}_1)_K - (p_0, \nabla\cdot\boldsymbol{v}_1)_\Omega$$

$$-(q_0, \nabla\cdot\boldsymbol{u}_1)_\Omega - \sum_{F\in\mathcal{E}_H} \alpha H_F\,([\![p_0]\!], [\![q_0]\!])_F = \boldsymbol{F}(\boldsymbol{v}_1, q_0)\,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$.

Lemma: The operator $\mathcal{M}_K$ satisfies

$$(\boldsymbol{v}-\mathcal{M}_K(\boldsymbol{v}), \mathcal{M}_K(\boldsymbol{w}))_K = 0 \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in L^2(K)^2\,.$$

Furthermore

$$\sum_{K\in\mathcal{T}_H} (\boldsymbol{u}_e^D([\![p_0]\!]), \boldsymbol{v}_1)_K \approx O(H^2)\,,$$

and then this term may be neglected.

Find $(\boldsymbol{u}_1, p_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$ *such that*

$$\boldsymbol{B}((\boldsymbol{u}_1, p_0), (\boldsymbol{v}_1, q_0)) = \boldsymbol{F}(\boldsymbol{v}_1, q_0)\,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$, where

$$\boldsymbol{B}((\boldsymbol{u}_1, p_0), (\boldsymbol{v}_1, q_0)) := \sum_{K \in \mathcal{T}_H} ((\mathcal{I} - \mathcal{M}_K)(\boldsymbol{u}_1), (\mathcal{I} - \mathcal{M}_K)(\boldsymbol{v}_1))_K$$

$$- (p_0, \nabla \cdot \boldsymbol{v}_1)_\Omega - (q_0, \nabla \cdot \boldsymbol{u}_1)_\Omega - \sum_{F \in \mathcal{E}_H} \alpha H_F\, ([\![p_0]\!], [\![q_0]\!])_F\,.$$

Remark : This method is symmetric.

Remark : $\boldsymbol{u}_H$ has discontinuous tangential component (unlike $\boldsymbol{u}_1$) and it satisfies the following local mass conservation property:

$$\int_K [\nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e^D) - g] = 0 \qquad \forall\, K \in \mathcal{T}_H\,.$$

The same argument may be applied to any jump-based stabilized method for the Darcy equation.

Numerical analysis of the semi-discrete problem :

Lemma: The bilinear forms $\boldsymbol{B}(.,.)$ satisfies

$$\boldsymbol{B}((\boldsymbol{v}_1, q_0), (\boldsymbol{v}_1, -q_0)) = \|(\mathcal{I} - \mathcal{M}_K)(\boldsymbol{v}_1)\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \tau_F \, \|[\![q_0]\!]\|_{0,F}^2 \,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$.

Lemma: There exists $C > 0$ such that

$$\|\boldsymbol{v}_1\|_{0,K} \,\le\, C\,(\|(\mathcal{I} - \mathcal{M}_K)(\boldsymbol{v}_1)\|_{0,K} + \|\nabla \cdot \boldsymbol{v}_1\|_{0,K}) \qquad \forall \boldsymbol{v}_1 \in \mathbb{P}_1(K)^2 \,.$$

The semi-discrete problem

Mesh-dependent norm :

$$\|(\boldsymbol{w}, t)\|_H^2 = \|\boldsymbol{w}\|_{div,\Omega}^2 + \alpha \|t\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \alpha H_F \|[\![t]\!]\|_{0,F}^2 .$$

Theorem: Let α small enough, then there exists $\beta > 0$, independent of H and α, such that

$$\sup_{(\boldsymbol{w}_1, t_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega) - \{\boldsymbol{0}\}} \frac{\boldsymbol{B}((\boldsymbol{v}_1, q_0), (\boldsymbol{w}_1, t_0))}{\|(\boldsymbol{w}_1, t_0)\|_H} \geq \beta \|(\boldsymbol{v}_1, q_0)\|_H ,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$.

The semi-discrete problem

Theorem: There exists $C > 0$ such that

$$\|(\boldsymbol{u} - \boldsymbol{u}_1, p - p_0)\|_H \leq CH\left(\|\boldsymbol{u}\|_{2,\Omega} + |p|_{1,\Omega}\right),$$

$$\|\boldsymbol{u} - (\boldsymbol{u}_1 + \boldsymbol{u}_e^D)\|_{div,\Omega} \leq C\,H\left(\|\boldsymbol{u}\|_{2,\Omega} + |p|_{1,\Omega}\right).$$

Remember: To implement the method, $\mathcal{M}_K(\boldsymbol{u}_1)$ must be computed, i.e., we must solve the local problem

$$\boldsymbol{u}_e^M + \nabla p_e^M = \boldsymbol{u}_1, \quad \nabla\cdot \boldsymbol{u}_e^M = 0 \quad \text{in } K,$$
$$\boldsymbol{u}_e^M \cdot \boldsymbol{n} = 0 \quad \text{on } \partial K$$

Starting remark :

$$\boldsymbol{v}_1 - \mathcal{M}_K(\boldsymbol{v}_1) = \nabla p_e(\boldsymbol{v}_1)\,.$$

Then our method may be rewritten in the following equivalent way

$$\sum_{K\in\mathcal{T}_H} (\nabla p_e(\boldsymbol{u}_1), \nabla p_e(\boldsymbol{v}_1))_K - (p_0, \nabla\cdot\boldsymbol{v}_1)_\Omega - (q_0, \nabla\cdot\boldsymbol{u}_1)_\Omega$$
$$- \sum_{F\in\mathcal{E}_H} \alpha H_F\,(\llbracket p_0\rrbracket, \llbracket q_0\rrbracket)_F = (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega - (g, q_0)\,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$. Here, $p_e(\boldsymbol{v}_1)$ solves

$$-\Delta p_e(\boldsymbol{v}_1) = -\nabla\cdot\boldsymbol{v}_1 \qquad \text{in } K\,,$$
$$\partial_{\boldsymbol{n}} p_e(\boldsymbol{v}_1) = \boldsymbol{v}_1\cdot\boldsymbol{n} \qquad \text{on } \partial K\,.$$

Discrete local problems : *Find* $p_h(\boldsymbol{v}_1) \in \boldsymbol{R}_h^K$ *such that*

$$\int_K \nabla p_h(\boldsymbol{v}_1) \cdot \nabla \xi_h = \int_K \boldsymbol{v}_1 \cdot \nabla \xi_h \qquad \forall \xi_h \in \boldsymbol{R}_h^K \,,$$

where $\boldsymbol{R}_h^K$ are Lagrangian finite elements of degree $l \geq 1$.

Two-level method : *Find* $(\boldsymbol{u}_{1,h}, p_{0,h}) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$ *such that:*

$$\boldsymbol{B}_h((\boldsymbol{u}_{1,h}, p_{0,h}), (\boldsymbol{v}_1, q_0)) = \mathrm{F}(\boldsymbol{v}_1, q_0) \qquad \forall\, (\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)\,,$$

where

$$\boldsymbol{B}_h((\boldsymbol{v}_1, q_0), (\boldsymbol{w}_1, t_0)) := \sum_{K \in \mathcal{T}_H} (\nabla p_h(\boldsymbol{v}_1), \nabla p_h(\boldsymbol{w}_1))_K - (q_0, \nabla \cdot \boldsymbol{w}_1)_\Omega$$
$$- (t_0, \nabla \cdot \boldsymbol{v}_1)_\Omega - \sum_{F \in \mathcal{E}_H} \tau_F \,([\![q_0]\!], [\![t_0]\!])_F \,.$$

Lemma: Let $\|\cdot\|_h$ be the mesh-dependent norm given by

$$\|(\boldsymbol{v}_1, q_0)\|_h^2 := \sum_{K \in \mathcal{T}_H} \|\nabla p_h(\boldsymbol{v}_1)\|_{0,K}^2 + \|\nabla \cdot \boldsymbol{v}_1\|_{0,\Omega}^2 +$$

$$\alpha \|q_0\|_{0,\Omega}^2 + \sum_{F \in \mathcal{E}_H} \tau_F \, \|[\![q_0]\!]\|_{0,F}^2 \,,$$

and let us suppose that there exists $C_0 > 0$ such that $h \leq C_0 H_K$. Then

$$\|(\boldsymbol{v}_1, q_0)\|_H \leq C \, \|(\boldsymbol{v}_1, q_0)\|_h \,.$$

Theorem: There exists $\beta_2 > 0$ independent of H, h and α such that

$$\sup_{(\boldsymbol{w}_1, t_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)} \frac{\boldsymbol{B}_h((\boldsymbol{v}_1, q_0), (\boldsymbol{w}_1, t_0))}{\|(\boldsymbol{w}_1, t_0)\|_H} \geq \beta_2 \, \|(\boldsymbol{v}_1, q_0)\|_H \,,$$

for all $(\boldsymbol{v}_1, q_0) \in \mathbb{P}_1(\Omega)^2 \times \mathbb{P}_0(\Omega)$.

Theorem: There exists $C > 0$ such that

$$\|(\boldsymbol{u} - \boldsymbol{u}_{1,h}, p - p_{0,h})\|_H \leq C \left(hH^t \, |g|_{t,\Omega} + (H + h) \, \|\boldsymbol{u}\|_{2,\Omega} + H \, |p|_{1,\Omega} \right) \, ,$$

for $t = 0, 1$.

Remark : The condition $h \leq C_0 H$ means that a fixed mesh may be used for all the elements and all the refinements, hence making the computation cheap. In fact, in all the numerical results, only one $\mathbb{P}_1$ element is used in each element.

Convergence analysis I : We consider $p(x,y) = \cos(2\pi x)\cos(2\pi y)$, $\boldsymbol{u} = -\nabla p$ ($\boldsymbol{f} = \boldsymbol{0}$, $g = 8\pi^2 \cos(2\pi x)\cos(2\pi y)$).

$$M_e := \max_{K \in \mathcal{T}_H} \frac{\left| \int_K \left(\nabla\cdot(\boldsymbol{u}_1 + \boldsymbol{u}_e^D) - g\right) d\,\mathrm{x} \right|}{|K|} \quad M_1 := \max_{K \in \mathcal{T}_H} \frac{\left| \int_K \left(\nabla\cdot \boldsymbol{u}_1 - g\right) d\,\mathrm{x} \right|}{|K|}.$$

h	0.5	0.125	6.25×10^{-2}	3.125×10^{-2}
M_e	6×10^{-14}	1.4×10^{-14}	2.1×10^{-14}	9.2×10^{-15}
M_1	9.2	3.4	1.1	0.28

Relative local mass conservation errors.

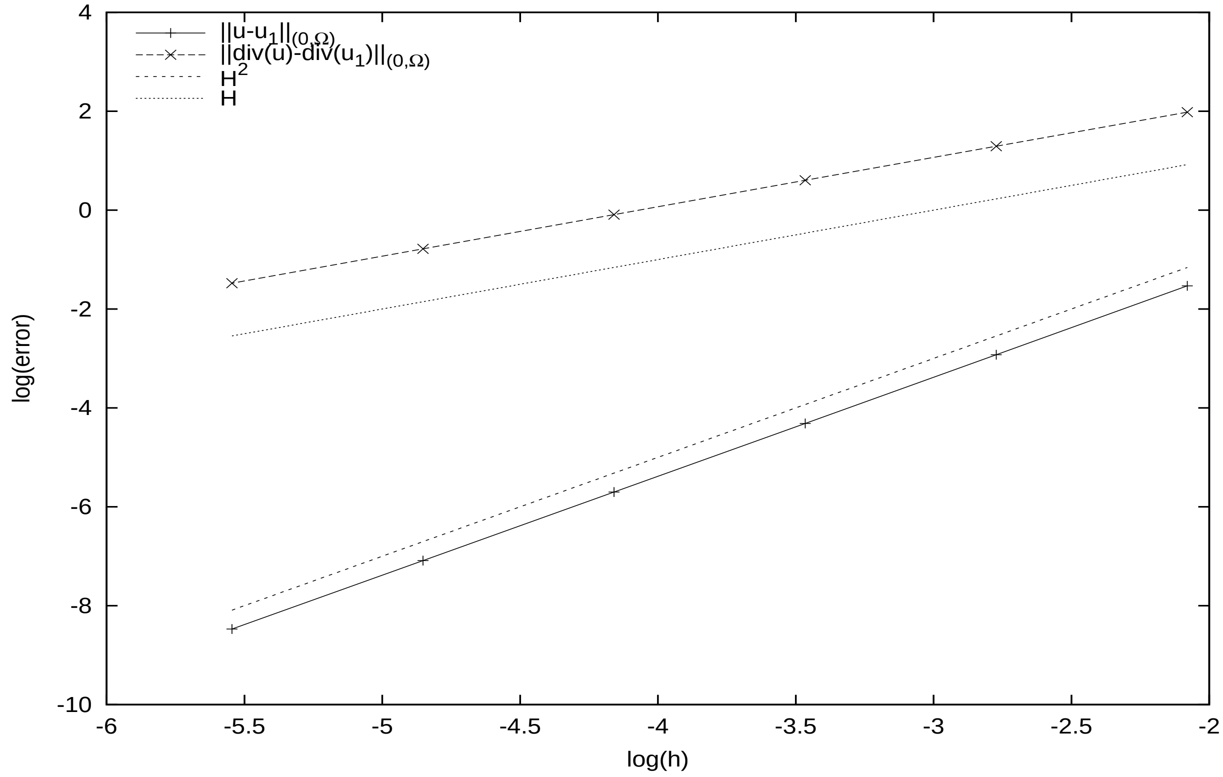


Figure 1: Convergence history of $\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_1)\|_{0,\Omega}$ and $\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,\Omega}$.

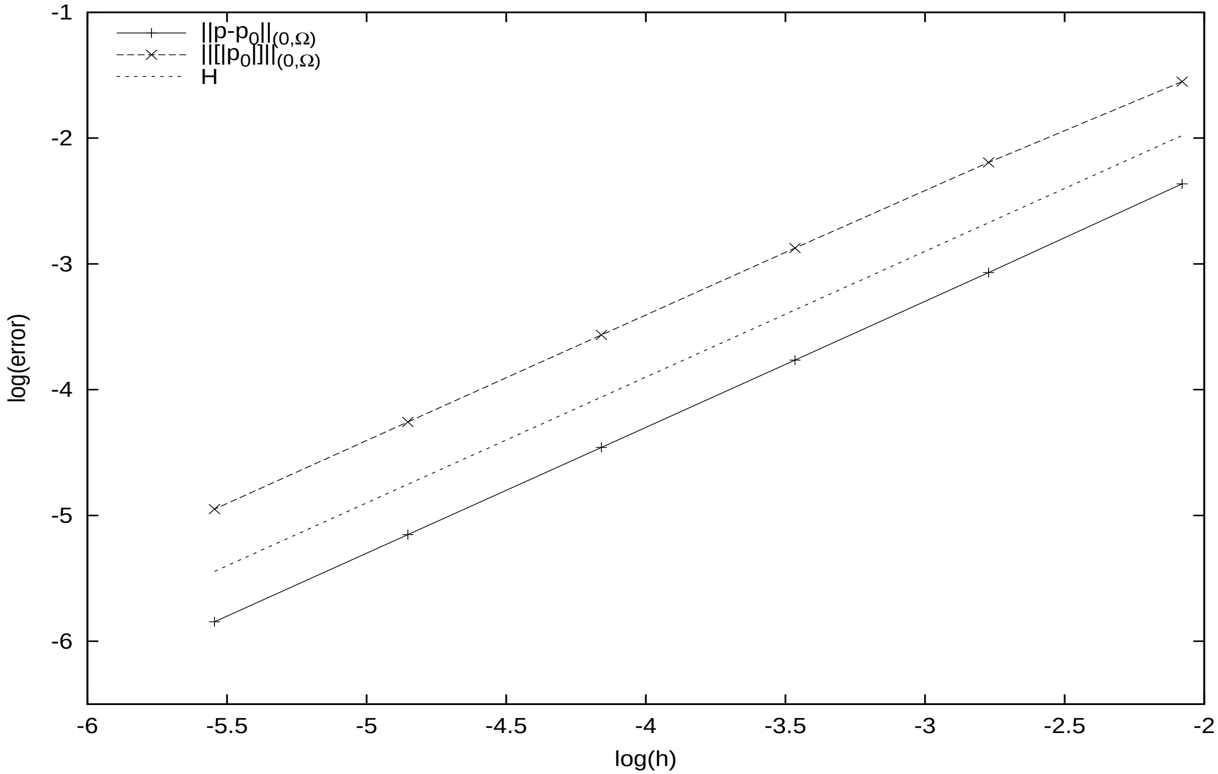
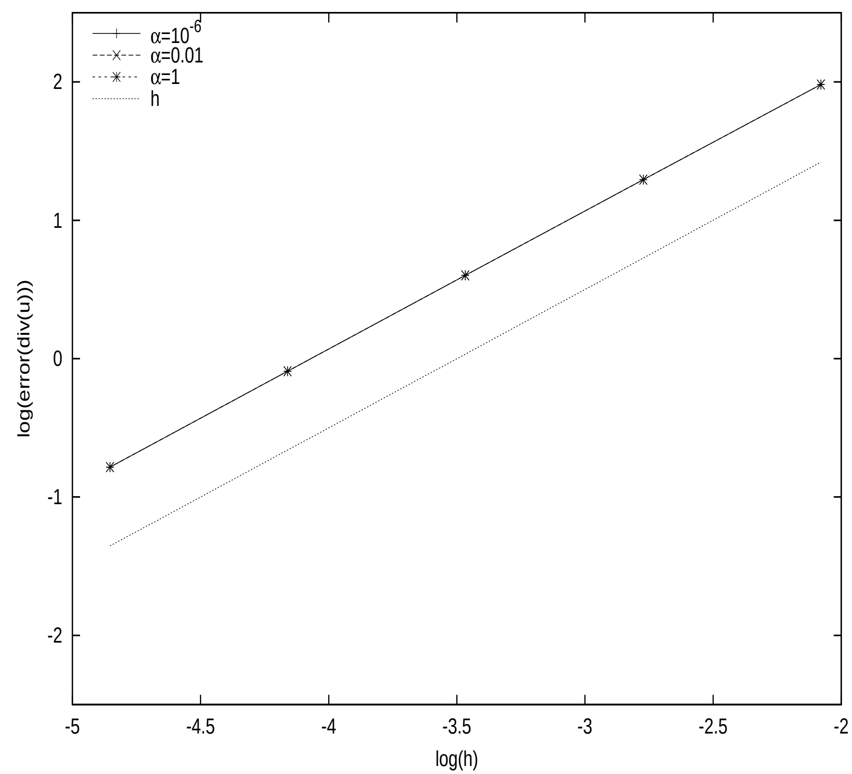
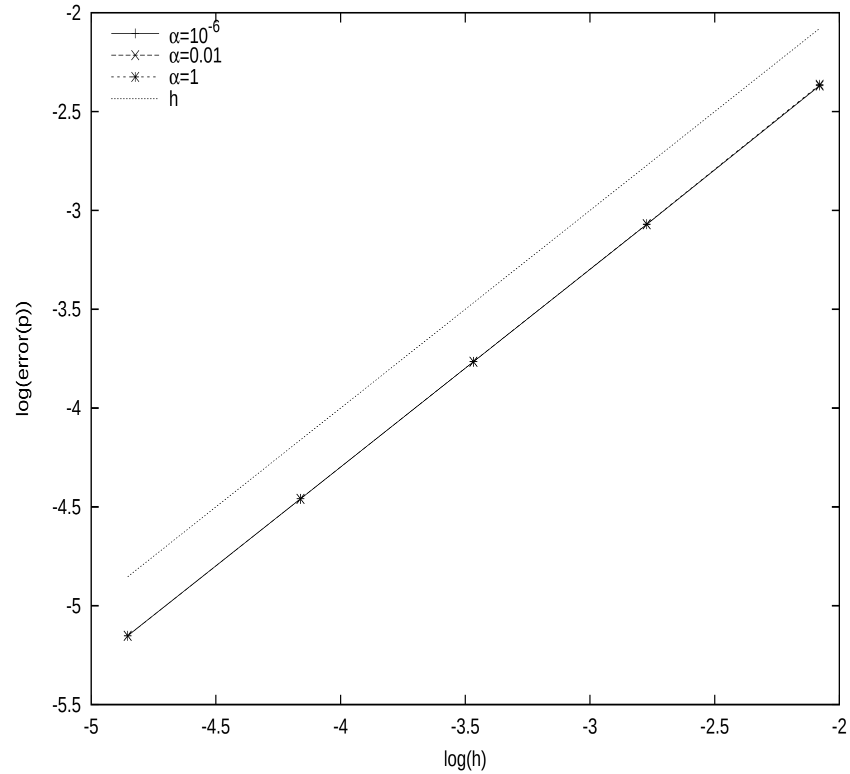


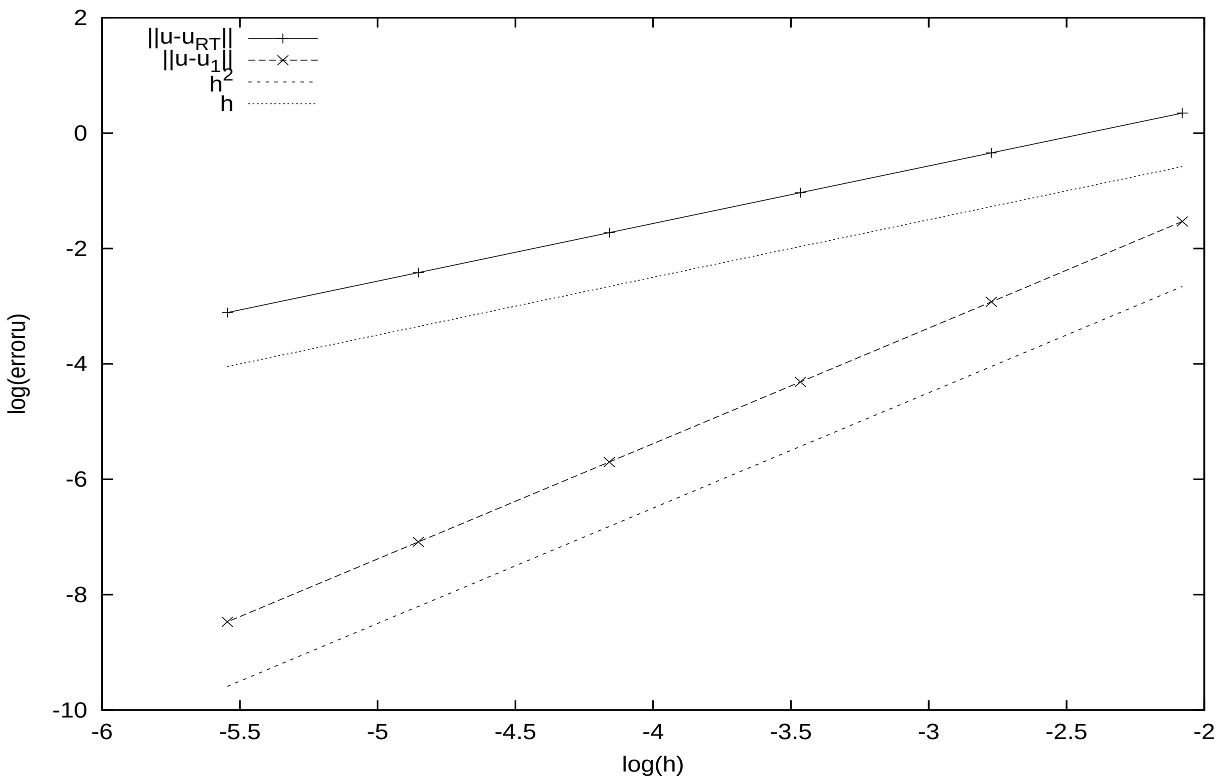
Figure 2: Convergence history of $\|p - p_0\|_{0,\Omega}$ and $|[\![p_0]\!]|_H$.

The sensitivity w.r. to α :





A caparison with the RT_0 method :



The checkerboard domain for the five-spot problem :

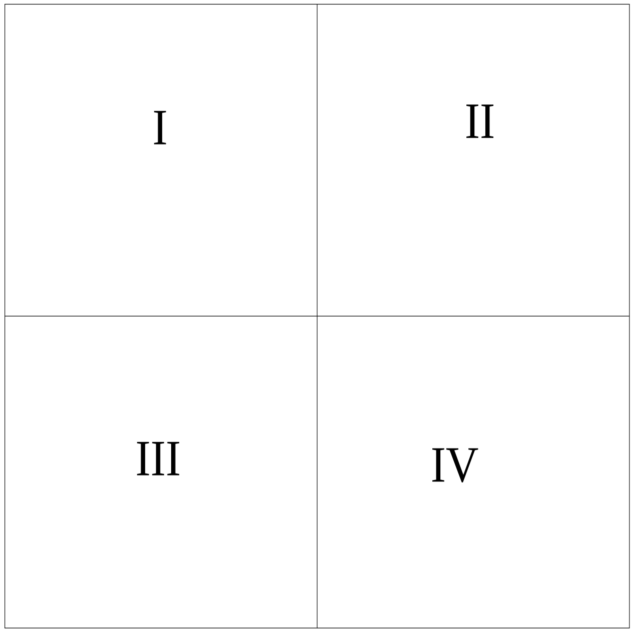


Figure 3: Checkerboard domain: $\sigma = 1$ in zones II-III and $\sigma = 10^{-9}$ in zones I-IV.

Coeficientes discontinuos

The boundary condition on $\boldsymbol{u}_e$ here reads:

$$\boldsymbol{u}_e \cdot \boldsymbol{n} = \frac{\alpha_F H_F}{\langle \sigma \rangle_F} [\![p_0]\!],$$

where

$$\langle \sigma \rangle_F = \frac{\sigma|_{K^+} + \sigma|_{K^-}}{2}.$$

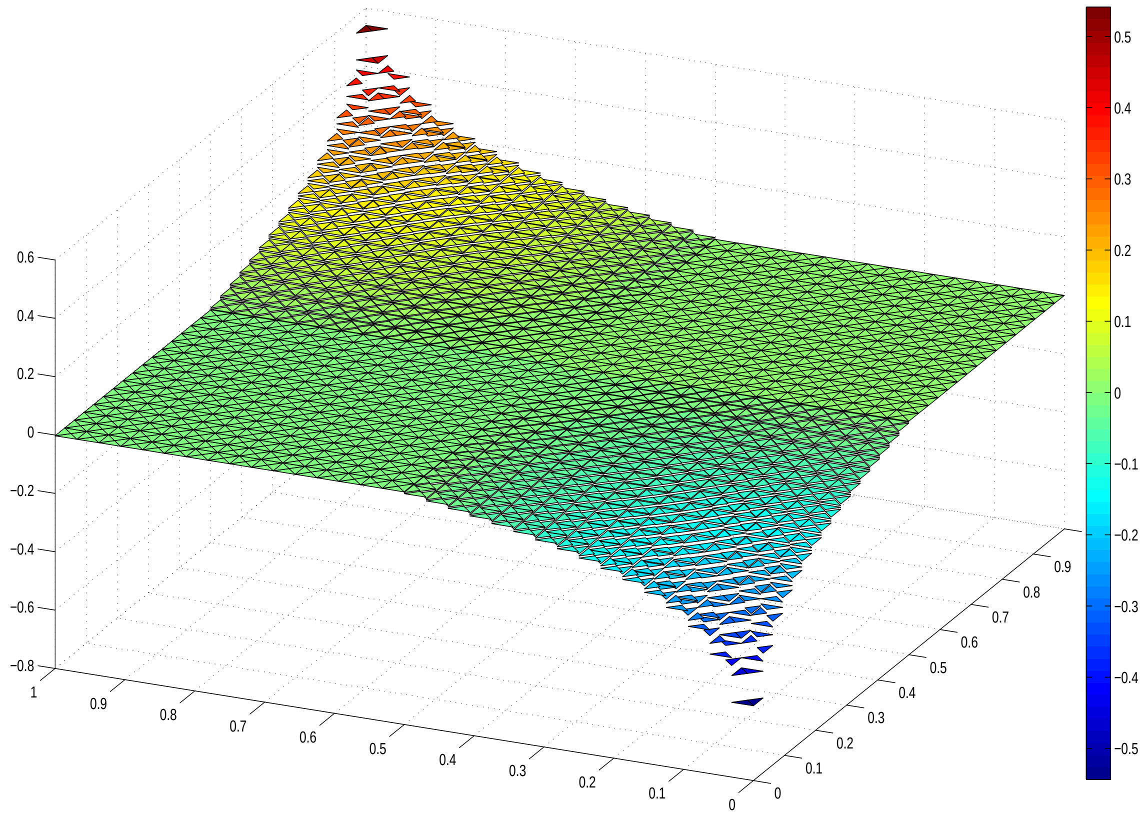


Figure 4: Pressure elevation for the checkerboard domain

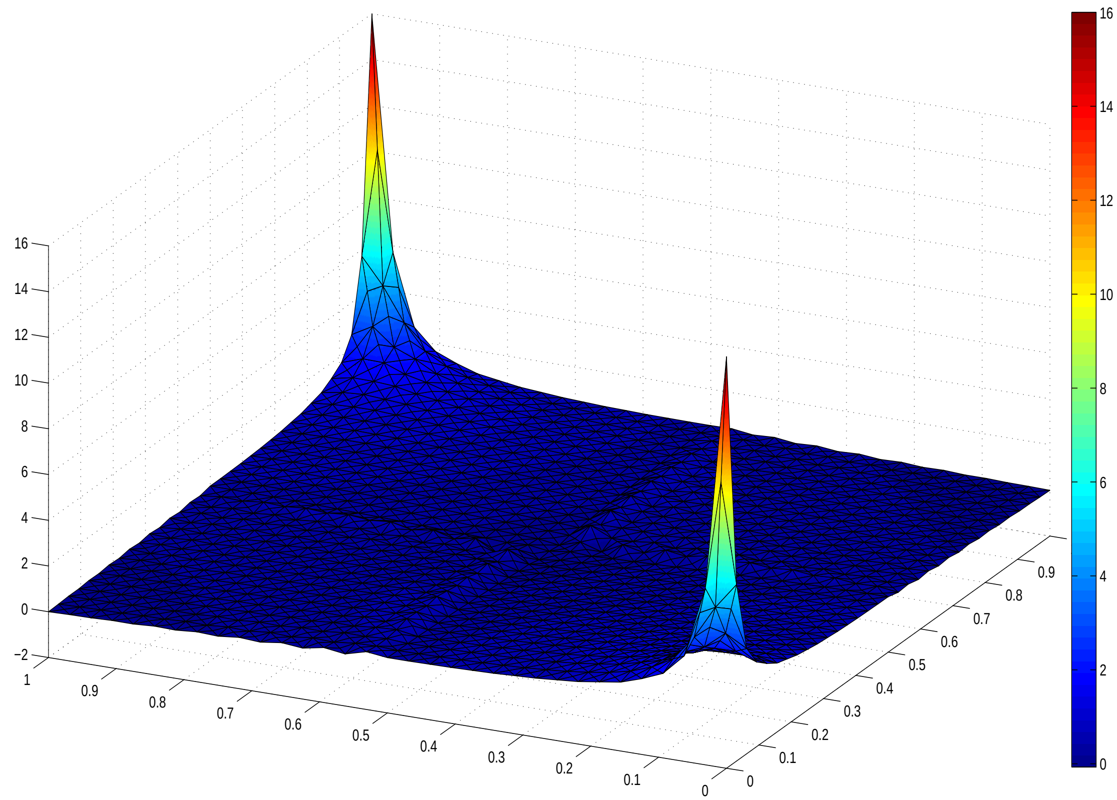


Figure 5: $|\boldsymbol{u}_1|$.

Concluding remarks

☞ The enrichment strategy has provided:

- ✔ Enrichment of the finite element space with local but not bubble functions.
- ✔ Theoretical justification for edge-based low-order stabilized method for the Darcy equation.
- ✔ A cheap computation of the local basis functions.

Concluding remarks

☞ Future extensions:

- ✔ Discontinuous and oscillating coefficients.
- ✔ Darcy-Stokes coupled problem.
- ✔ New enrichment functions treating convective flows.
- ✔ Time-dependent problems.