Homogenization of the Lévy operators with asymmetric Lévy measures

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Periodic homogenisation problem: (P)

$$\frac{\partial u_\varepsilon}{\partial t} + F(x, u_\varepsilon, \nabla u_\varepsilon, \nabla^2 u_\varepsilon) - c(\frac{x}{\varepsilon}) \int_{\mathbf{R}^\mathbf{N}} [u_\varepsilon(x+z)$$

$$-u_\varepsilon(x) - \mathbf{1}_{|z|\leq 1} \langle \nabla u_\varepsilon(x), z\rangle] q(z) dz = 0 \quad x \in \Omega,$$

$$u_\varepsilon(x,t) = \phi(x) \quad x \in \Omega^c, \quad t > 0,$$

$$u_\varepsilon(x,0) = u_0 \quad x \in \Omega.$$

- Is there a unique limit :

$$\exists lim_{\varepsilon \downarrow 0} u_\varepsilon(x) = \overline{u}(x) \quad ?$$

- Characterize $\overline{u}$ by an effective PIDE:

$$\frac{\partial \overline{u}}{\partial t} + \overline{I}(x, \overline{u}, \nabla \overline{u}, \nabla^2 \overline{u}, I[\overline{u}]) = 0 \qquad t > 0.$$

Type I (Pure Jump Process).

- Linear problem:

$$\textbf{Ex.} \quad F = 0.$$

- First-order nonlinear problem:

$$\textbf{Ex.} \quad F = a(x)|p|.$$

Type II (Jump-Diffusion process).

F : uniformly elliptic, i.e. $\exists \theta > 0$

$$F(x,u,p,Q+Q') \leq F(x,u,p,Q) - \theta Tr Q' \quad \forall Q' \geq O,$$

Ex. $F(x,u,\nabla u,\nabla^2 u) = -Tr(\nabla^2 u) = -\Delta u.$

Method.

In the case of PDE (elliptic, parabolic), the effective PDE is obtained by

- Formal asymptotic expansion

- Cell problem (ergodic problem of PDE)

- Averaging principle in the underlying stochastic process

- Rigorous justification : for nonlinear PDEs Perturbed test function method by using viscosity solutions

References. A. Bensoussain, J.-L. Lions, and G. Papanicolaou, P.-L. Lions, G. Papanicolaou, and S.R.S. Varadhan, L.C. Evans, etc.

M. A. Homogenizations of PIDE with Lévy operators, submitted.
M.A. Some remarks on the homogenizations of Lévy operators with asymmetric densities, in preparation.

Pure jump case.

$$-\int_{\mathbf{R^N}}[u(x+z)-u(x)-\mathbf{1}_{|z|\le 1}\langle\nabla u(x),z\rangle]q(z)dz$$

$$\text{s.t.} \quad \int_{|\mathbf{z}|\le\mathbf{1}}|z|^2q(z)dz+\int_{|\mathbf{z}|>\mathbf{1}}1q(z)dz<\infty.$$

Homogenisations

$\Updownarrow$

Regularization effect (averaging principle)

$\Updownarrow$

Singular Lévy density

Examples.

- Symmetric α-stable process ($-\Delta^{\frac{\alpha}{2}}$, $0<\alpha<2$)

  $$q(z)dz=\frac{1}{|z|^{N+\alpha}}(dz).$$

  Remark. As $\alpha\to 2$, the operator tends to $-\Delta$.

- α-Stable process ($N = 1$, $0 < \alpha < 2$)

$$q(z)dz = c_1 \frac{1}{|z|^{1+\alpha}}(dz) \quad z < 0,$$

$$= c_2 \frac{1}{|z|^{1+\alpha}}(dz) \quad z > 0$$

where c_1, $c_2 \geq 0$, and at least one $c_i \neq 0$.

- CGMY model ($N = 1$, $C > 0$, $G \geq 0$, $M \geq 0$, $0 < Y < 2$)

$$q(z)dz = C(I_{z<0}e^{-G|z|} + I_{z>0}e^{-M|z|}) \frac{1}{|z|^{1+Y}}(dz).$$

- Asymmetric singularity ($N = 1$, $0 < \alpha_1 < \alpha_2 < 2$)

$$q(z)dz = c_1 \frac{1}{|z|^{1+\alpha_1}}(dz) \quad z < 0,$$

$$= c_2 \frac{1}{|z|^{1+\alpha_2}}(dz) \quad z > 0$$

- Nonlinear operator ($N = 2$, $0 < \alpha_1, \alpha_2 < 2$)

$$\max\{-\int_{\mathbf{R}}[u(x_1+z_1,x_2)-u(x_1,x_2)$$

$$-\mathbf{1}_{|z_1|\le 1}\langle\nabla_{x_1}u(x),z_1\rangle]\frac{1}{|z_1|^{1+\alpha_1}}dz_1,$$

$$-\int_{\mathbf{R}}[u(x_1,x_2+z_2)-u(x_1,x_2)$$

$$-\mathbf{1}_{|z_2|\le 1}\langle\nabla_{x_2}u(x),z_2\rangle]\frac{1}{|z_2|^{1+\alpha_2}}dz_2\}$$

Jump diffusion case.

Homogenisations

$\Updownarrow$

Regularization effect (averaging principle)

$\Updownarrow$

Effect of the diffusion of "$-\Delta$"

Examples. (Bdd Lévy measures can be added.)

- Discrete Lévy measure

$$q(dz) = c\Sigma_{j=1}^{d} p_j \delta_{a_j}(dz),$$

  $p_j \geq 0$, $\Sigma_{j=1}^{d} p_j = 1$, $c > 0$: frequency of the jump, a_i: jump lengths.

- Gaussian distribution

$$q(z)dz = c\frac{1}{\sqrt{2\pi v}}\exp(-\frac{|z-m|^2}{2v})dz$$

$c > 0$: frequency of the jump; jump distribution: the normal distribution.

- Variance gamma process (c, c_1, $c_2 > 0$)

$$q(z)dz = c(I_{z<0}e^{-c_1|z|} + I_{z>0}e^{-c_2|z|})\frac{1}{|z|}dz.$$

Formal asymptotic expansion.
Type I (Pure Jump Process).
1. Symmetric α-stable process

$(1 < \alpha < 2)$

$$\frac{\partial u_\varepsilon}{\partial t} + |\nabla u_\varepsilon| - c(\frac{x}{\varepsilon}) \int_{\mathbf{R^N}} [u_\varepsilon(x+z) - u_\varepsilon(x)$$

$$-\mathbf{1}_{|z|\leq 1} \langle \nabla u_\varepsilon(x), z\rangle] \frac{1}{|z|^{N+\alpha}} dz - g(\frac{x}{\varepsilon}) = 0$$

$$\Downarrow$$

$$u_\varepsilon(x,t) = \overline{u}(x,t) + \varepsilon^\alpha v(\frac{x}{\varepsilon}, t) + o(\varepsilon^\alpha)$$

$$\nabla u_\varepsilon(x,t) = \nabla_x \overline{u}(x,t) + \varepsilon^{\alpha-1} \nabla_y v(\frac{x}{\varepsilon}, t).$$

$$\Downarrow$$

$$\frac{\partial \overline{u}}{\partial t} + |\nabla \overline{u}| - c(\frac{x}{\varepsilon}) \int_{\mathbf{R^N}} [\overline{u}(x+z) - \overline{u}(x)$$

$$-\mathbf{1}_{|z|\leq 1} \langle \nabla_x \overline{u}(x), z\rangle] \frac{1}{|z|^{N+\alpha}} dz$$

$$-\varepsilon^{\alpha} c(\frac{x}{\varepsilon}) \int_{\mathbf{R^N}} [v(\frac{x+z}{\varepsilon}) - v(\frac{x}{\varepsilon})$$

$$-\mathbf{1}_{|z|\leq 1} \left\langle \nabla_y v(\frac{x}{\varepsilon}), \frac{z}{\varepsilon} \right\rangle] \frac{1}{|z|^{N+\alpha}} dz - g(\frac{x}{\varepsilon}) = 0$$

$$\Downarrow$$

$$\frac{\partial \overline{u}}{\partial t} + |\nabla \overline{u}| - c(y) \int_{\mathbf{R^N}} [\overline{u}(x+z) - \overline{u}(x)$$

$$-\mathbf{1}_{|z|\leq 1} \langle \nabla_x \overline{u}(x), z \rangle] \frac{1}{|z|^{N+\alpha}} dz - c(y) \int_{\mathbf{R^N}} [v(y+z')$$

$$-v(y) - \mathbf{1}_{|z'|\leq \frac{1}{\varepsilon}} \left\langle \nabla_y v(y), z' \right\rangle] \frac{1}{|z'|^{N+\alpha}} dz - g(y) = 0$$

$$\Downarrow$$

Ergodic problem (Averaging principle)

$$-\exists \overline{I}(x, \overline{u}, \nabla \overline{u}, I) - c(y) I - c(y) \int_{\mathbf{R^N}} [v(y+z')$$

$$-v(y) - \mathbf{1}_{|z'|\leq \frac{1}{\varepsilon}} \left\langle \nabla_y v(y), z' \right\rangle] \frac{1}{|z'|^{N+\alpha}} dz - g(y) = 0,$$

M.A. Proc."Stoc. Processes and Applic. to Math. Finance", World Scientifics, (2007)

Effective integro-differential operators

- Uniform sub-ellipticity:

$$\overline{I}(x,r,p,I+I')\underline{\leq}\overline{I}(x,r,p,I) - \exists\theta I' \quad \forall I' > 0$$

- $\overline{I}(x,r,p,I+I') \in C(\Omega \times \mathbf{R} \times \mathbf{R}^{\mathbf{N}} \times \mathbf{R})$.

$$\Downarrow$$

Effective integro-differential equations

$\overline{u}$ is the unique solution of

$$\frac{\partial \overline{u}}{\partial t} + \overline{I}(x, \overline{u}, \nabla \overline{u}, I) = 0 \quad t > 0.$$

Remark. The formal argument is justified by the purturbed test fc. method.

Remark. Asymmetric α-Stable process

$$q(z)dz = \mathbf{1}_{z<0}\frac{c_1}{|z|^{1+\alpha}}dz + \mathbf{1}_{z>0}\frac{c_2}{|z|^{1+\alpha}}dz$$

can be treated similarly.

2. Asymmetric singularity

$(N = 1,\ 0 < \alpha_1 < \alpha_2 < 2)$

$$q(z)dz = \mathbf{1_{z<0}}\frac{c_1}{|z|^{1+\alpha_1}}dz + \mathbf{1_{z<0}}\frac{c_2}{|z|^{1+\alpha_2}}dz,$$

i.e.

$$\frac{\partial u_\varepsilon}{\partial t} + |\nabla u_\varepsilon| - c(\frac{x}{\varepsilon})\int_{-\infty}^{0}[u_\varepsilon(x+z) - u_\varepsilon(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u_\varepsilon(x), z\rangle]\frac{c_1}{|z|^{1+\alpha_1}}dz$$

$$-c(\frac{x}{\varepsilon})\int_{0}^{\infty}[u_\varepsilon(x+z) - u_\varepsilon(x)$$

$$-\mathbf{1}_{|z|\leq 1}\langle \nabla u_\varepsilon(x), z\rangle]\frac{c_2}{|z|^{1+\alpha_2}}dz - g(\frac{x}{\varepsilon}) = 0$$

$$\Downarrow$$

Stronger singularity dominates:

$$u_\varepsilon(x,t) = \overline{u}(x,t) + \varepsilon^{\alpha_2} v(\frac{x}{\varepsilon}, t) + o(\varepsilon^{\alpha_2})$$

3. Nonlinear operator ($N = 2$, $0 < \alpha_1, \alpha_2 < 2$)

$$\frac{\partial u_\varepsilon}{\partial t} + c(\frac{x}{\varepsilon}) \max\{- \int_{\mathbf{R}} [u_\varepsilon(x_1 + z_1, x_2) - u_\varepsilon(x_1, x_2)$$

$$-\mathbf{1}_{|z_1|\leq 1} \langle \nabla_{x_1} u_\varepsilon(x), z_1\rangle] \frac{1}{|z_1|^{1+\alpha_1}} dz_1,$$

$$- \int_{\mathbf{R}} [u_\varepsilon(x_1, x_2 + z_2) - u_\varepsilon(x_1, x_2)$$

$$-\mathbf{1}_{|z_2|\leq 1} \langle \nabla_{x_2} u_\varepsilon(x), z_2\rangle] \frac{1}{|z_2|^{1+\alpha_2}} dz_2\} - g(\frac{x}{\varepsilon}) = 0.$$

$$\Downarrow$$

Developpements in each directions:

$$u_\varepsilon(x_1, x_2, t) = \overline{u}(x, t)$$

$$+\varepsilon^{\alpha_1} v(\frac{x_1}{\varepsilon}, x_2, t) + \varepsilon^{\alpha_2} w(x_1, \frac{x_2}{\varepsilon}, t) + o(\varepsilon^\alpha)$$

Theorem. Let us consider the problem (P), which is either Type I or Type II. Let u_ε be the solution of (P). Then, there is a unique fonction $\lim_{\varepsilon\downarrow 0} u_\varepsilon = \overline{u}$ exists, which is the unique solution of

$$\frac{\partial \overline{u}}{\partial t} + \overline{I}(x, \overline{u}, \nabla \overline{u}, \nabla^2 \overline{u}, I[\overline{u}]) = 0 \quad t > 0,$$

with the same initial and boundary conditions.

Remark. The result is applied to a stochastic volatility model with jumps, in maths finances. (cf. Fouque, Papanicolaou, Sircar.)