TOPOLOGICALLY FINITELY GENERATED HILBERT $C(X)$-MODULES

ILJA GOGIĆ

Abstract. For a Hilbert $C(X)$-module V, where X is a compact metrizable space, we show that the following conditions are equivalent: (i) V is topologically finitely generated, (ii) there exists $K \in \mathbb{N}$ such that every algebraically finitely generated submodule of V can be generated with $k \leq K$ generators, (iii) V is canonically isomorphic to the Hilbert $C(X)$-module $\Gamma(\mathcal{E})$ of all continuous sections of an (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ over X, whose fibres E_x have uniformly finite dimensions, and each restriction bundle of $\mathcal{E}$ over a set where $\dim E_x$ is constant is of finite type, (iv) there exists $N \in \mathbb{N}$ such that for every Banach $C(X)$-module W, each tensor in the $C(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C(X)} W$ is of (finite) rank at most N.

1. Introduction

It is well known that there exists a categorical equivalence between Hilbert $C(X)$-modules, where X is a compact Hausdorff space, and (F) Hilbert bundles over X (see [4, 21]). Since algebraically finitely generated Hilbert $C(X)$-modules are automatically projective (in a pure algebraic sense, see [23, Corollary 15.4.8]), by the celebrated Serre-Swan theorem [19, Theorem 1.6] they correspond to the classical locally trivial vector bundles over X. More generally, Goodearl has in [8] (see also [22]) generalized this result for paracompact spaces; if X is paracompact, then the category of algebraically finitely generated Hilbert $C_b(X)$-modules is equivalent to the category of locally trivial vector bundles over X which have the finite type property.

The purpose of this paper is to obtain a similar characterization for the larger class of topologically finitely generated Hilbert $C(X)$-modules, at least when X is compact and metrizable. The main difference between algebraically and topologically finitely generated Hilbert $C(X)$-modules is the fact that the dimensions of fibres of the underlying (F) Hilbert bundle may vary, even if X is connected. For example, let X be the unit interval $[0,1]$ and let $V := C_0((0,1])$ (i.e. the C^*-algebra of all continuous functions $f : [0,1] \to \mathbb{C}$ which vanish at zero). Then V becomes a Hilbert $C([0,1])$-module with respect to the standard action and inner product $\langle f, g\rangle = f^* g$. Note that V is topologically singly generated (for instance, the identity function $f(x) = x$ is such generator, by the Weierstrass approximation theorem). On the other hand, each fibre E_x of the underlying (F) Hilbert bundle $\mathcal{E}$ is isomorphic to $\mathbb{C}$, except E_0, which is zero. However, this phenomenon is in

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fact the only major difference between the classes of algebraically and topologically finitely generated Hilbert $C(X)$-modules. We state our first result:

Theorem 1.1. *Let X be a compact metrizable space and let V be a Hilbert $C(X)$-module with the associated canonical (F) Hilbert bundle $\mathcal{E}_V = (p, E, X)$ over X. Then the following conditions are equivalent:*

(i) *V is topologically finitely generated,*
(ii) *Each fibre E_x of $\mathcal{E}_V$ is finite dimensional, with*

$$\sup_{x \in X} \dim E_x < \infty,$$

and every restriction bundle of $\mathcal{E}_V$ over a set where $\dim E_x$ is constant is of finite type.

Using this result, we shall obtain some other characterizations of topologically finitely generated Hilbert $C(X)$-modules. More precisely, we shall consider the following conditions on a Hilbert $C(X)$-module V, where X is compact and metrizable:

(a) V is *weakly algebraically finitely generated*, which means that there exists $K \in \mathbb{N}$ such that every algebraically finitely generated submodule of V can be generated with $k \leq K$ generators,
(b) V satisfies the following factorization property: There exists $N \in \mathbb{N}$ such that for every column matrix $[a_i] \in \mathrm{M}_{\infty,1}(V)$ with $(a_i) \in \ell^1(V)$ there exits $n \leq N$, a matrix $[\varphi_{i,j}] \in \mathrm{M}_{\infty,n}(C(X))$ whose columns belong to $\ell^1(C(X))$, and a column matrix $[v_i] \in \mathrm{M}_{n,1}(V)$ such that

$$[a_i] = [\varphi_{i,j}] \cdot [v_i].$$

The latter means that

$$a_i = \sum_{j=1}^{n} \varphi_{i,j} v_j$$

for all $i \in \mathbb{N}$, as usual.
(c) V satisfies the following absorption property: There exists $M \in \mathbb{N}$ such that for every Banach $C(X)$-module W and every tensor t in the projective tensor product $V \overset{\pi}{\otimes} W$, its canonical image t_X in the $C(X)$-projective tensor product $V \overset{\pi}{\otimes}_{C(X)} W$ is a tensor of rank at most M. The latter means that t_X can be written in a form

$$t_X = \sum_{i=1}^{m} v_i \otimes_X w_i,$$

for some $v_i \in V$, $w_i \in W$ and $m \leq M$.

In Theorem 3.6, which is the main result of this paper, we shall prove that all these conditions are equivalent to the fact that V is topologically finitely generated.

At the end of this introductory, we would like to mention that it would be interesting to see if some of these results are still valid for the larger class of $C(X)$-locally convex modules arising as section modules of (F) Banach bundles (see [4, 7, 11, 12] for definition and properties of such modules).

2. Preliminaries

Let X be a locally compact Hausdorff space. In this paper by a *Hilbert $C_0(X)$-module* we mean a left $C_0(X)$-module V, equipped with a $C_0(X)$-valued inner product $\langle\cdot,\cdot\rangle$ which is $C_0(X)$-linear in the first and conjugate linear in the second variable, such that V is a Banach space with the norm $\|v\| := \|\langle v,v\rangle\|^{\frac{1}{2}}$.

We say that V is:

(i) *Algebraically finitely generated* if there exists a finite number of elements in V whose $C_0(X)$-linear span equals V,

(ii) *Topologically finitely generated* if there exists a finite number of elements in V whose $C_0(X)$-linear span is dense in V.

The basic theory of Hilbert C^*-modules (over the general C^*-algebras) can be found in [13, 15, 18, 23].

Following [4], by an *(F) Hilbert bundle* ((F) stands for Fell) we mean a triple $\mathcal{E} := (p, E, X)$ where E and X are topological spaces with a continuous open surjection $p : E \to X$, together with operations and norms making each *fibre* $E_x := p^{-1}(x)$ $(x \in X)$ into a complex Hilbert space, such that the following conditions are satisfied:

(H1) The maps $\mathbb{C} \times E \to E$, $E \times_X E \to E$ and $E \to \mathbb{R}$, given in each fibre by scalar multiplication, addition, and the norm, respectively, are continuous (where $E \times_X E$ denotes the Whitney sum),

(H2) The collection of all subsets of E of the form

$$\{e \in E \ : \ p(e) \in U, \ \|e\| < \varepsilon\},$$

where U is a neighborhood of x in X and $\varepsilon > 0$, form a basis of neighborhoods of the zero element 0_x of E_x in E.

As usual, we say that p is the *projection*, E is the *bundle space* and X is the *base space* of $\mathcal{E}$. Using a polarization identity together with the continuity of the norm and operations, it is an immediate consequence that the map $E \times_X E \to \mathbb{C}$ given by the inner product in each fibre is continuous.

If $\mathcal{E} = (p, E, X)$ is an (F) Hilbert bundle and $Y \subseteq X$ then we denote by

$$\mathcal{E}|_Y := (p|_{p^{-1}(Y)}, p^{-1}(Y), Y)$$

the *restriction* of $\mathcal{E}$ to Y.

For the (F) Hilbert bundles $\mathcal{E} = (p, E, X)$ and $\mathcal{E}' = (p', E', X')$ we say that $\Phi : \mathcal{E} \to \mathcal{E}'$ is a *Hilbert bundle map* if Φ is a pair $\Phi = (\phi, f)$ of maps, where $\phi : E \to E'$ and $f : X \to X'$ are continuous maps such that

(i) the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

is commutative,

(ii) for each $x \in X$, ϕ defines a linear map from E_x into $E'_{f(x)}$.

It is usually said that Φ *covers* f. If in addition ϕ defines an isometric isomorphism of each fibre E_x onto $E'_{f(x)}$, then we say that Φ is a *strong Hilbert bundle map*. If

$X' = X$, we write $\Phi : \mathcal{E} \cong \mathcal{E}'$ to say that Φ is an *isomorphism of Hilbert bundles*, that is, Φ is a strong Hilbert bundle map covering the identity map $\mathrm{id}_X : X \to X$.

If H is a Hilbert space then by the *product bundle over X with fibre H* we mean

$$\epsilon(X, H) := (p_1, X \times H, X),$$

where p_1 is a projection on the first coordinate. An (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ is said to be *trivial* if there exists a Hilbert space H such that $\mathcal{E} \cong \epsilon(X, H)$. We say that $\mathcal{E}$ is *locally trivial* if there exists a Hilbert space H and an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$ we have $\mathcal{E}|_U \cong \epsilon(U, H)$. If in addition X admits a finite open cover over which $\mathcal{E}$ is locally trivial, we say that $\mathcal{E}$ is of *finite type*.

If all fibres of an (F) Hilbert bundle $\mathcal{E}$ have the same finite dimension n, then we say that $\mathcal{E}$ is *n-homogeneous*. We shall frequently use the following fact:

Proposition 2.1. *If $\mathcal{E}$ is an n-homogeneous (F) Hilbert bundle, then $\mathcal{E}$ is locally trivial. In particular, if the base space X of $\mathcal{E}$ is compact, then $\mathcal{E}$ is of finite type.*

For the proof, see [3, Proposition 2.3]. If all fibres of $\mathcal{E}$ are finite dimensional with

$$n := \sup_{x \in X} \dim E_x < \infty,$$

then we say that $\mathcal{E}$ is *n-subhomogeneous*. In this case every restriction bundle of $\mathcal{E}$ over a set where $\dim E_x$ is constant is locally trivial, by Proposition 2.1. If in addition every such restriction bundle is of finite type, then we say that $\mathcal{E}$ is *n-subhomogeneous of finite type*.

By a *section* of an (F) Hilbert bundle $\mathcal{E} = (p, E, X)$ we mean a map $s : X \to E$ such that $p(s(x)) = x$ for all $x \in X$. By $\Gamma(\mathcal{E})$, $\Gamma_b(\mathcal{E})$ and $\Gamma_0(\mathcal{E})$ we respectively denote the set of all continuous sections, all bounded continuous sections, and all continuous sections of $\mathcal{E}$ which vanish at infinity. If X is compact, then obviously $\Gamma(\mathcal{E}) = \Gamma_b(\mathcal{E}) = \Gamma_0(\mathcal{E})$. Note that $\Gamma_b(\mathcal{E})$ (resp. $\Gamma_0(\mathcal{E})$) becomes a Hilbert $C_b(X)$-module (resp. Hilbert $C_0(X)$-module) with respect to the action

$$(\varphi s)(x) := \varphi(x) s(x)$$

and inner product

$$\langle s, u \rangle(x) := \langle s(x), u(x) \rangle_x,$$

where $\langle \cdot, \cdot \rangle_x$ denotes the inner product on fibre E_x.

We summarize the following important facts regarding continuous sections of (F) Hilbert bundles (see [5, Theorem C17], [5, Theorem II.14.8], [7, Theorem 4.2] and [3, Proposition 1.6]):

Theorem 2.2. *Let $\mathcal{E} = (p, E, X)$ be an (F) Hilbert bundle over a locally compact Hausdorff space X.*

(i) *For each $x \in X$ and $e_x \in E_x$ there exists $s \in \Gamma_0(\mathcal{E})$ such that $s(x) = e_x$.*

(ii) *If Y is a closed subset of X, then every section in $\Gamma(\mathcal{E}|_Y)$ extends to a section in $\Gamma(\mathcal{E})$. Moreover, the restriction map $\Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}|_Y)$, $s \mapsto s|_Y$ is a quotient map.*

(iii) *If X is compact then a $C(X)$-submodule $W \subseteq \Gamma(\mathcal{E})$ is dense in $\Gamma(\mathcal{E})$ if and only if for each $x \in X$,*

$$\{s(x) \ : \ s \in W\}$$

is dense in E_x.

(iv) *If* $s_1, \dots, s_m \in \Gamma(\mathcal{E})$, *then the set of points* $x \in X$ *for which the set*

$$\{s_1(x), \dots, s_m(x)\}$$

is linearly independent form an open subset of X.

The parts (ii) and (iii) of Theorem 2.2 are respectively known as Tietze extension theorem and Stone-Weierstrass theorem for (F) Hilbert bundles.

At the and of this section, let us briefly describe how for a given Hilbert $C(X)$-module V, where X is a compact Hausdorff space, one constructs a canonical Hilbert bundle $\mathcal{E}_V$. For $x \in X$ let I_x be the maximal ideal of $C(X)$ consisting of all functions which vanish at x, and put

$$J_x := I_x V = \{\varphi v \ : \ \varphi \in I_x, v \in V\}.$$

Then J_x is a closed submodule of V, by the Hewitt-Cohen factorization theorem [2, Theorem A.6.2]. Set $E_x := V/J_x$, let $\pi_x : V \to E_x$ be the quotient map, let

$$E := \bigsqcup_{x \in X} E_x$$

and let $p : E \to X$ be the canonical projection. Since for each $v \in V$ and $x \in X$ we have $\|\pi_x(v)\| = \sqrt{\langle v, v\rangle(x)}$ (by [18, Proposition 3.25]), the function $X \to \mathbb{R}$, $x \mapsto \|\pi_x(v)\|$ is continuous. Hence, by [5, Theorem II.13.18] there exists a unique topology on E (which is called *Fell's topology*) for which $\mathcal{E}_V := (p, E, X)$ becomes an (F) Hilbert bundle. We say that $\mathcal{E}_V$ is the *canonical (F) Hilbert bundle associated to* V. Moreover, by [4, Theorem 2.5] the *generalized Gelfand transform* $\Gamma_V : V \to \Gamma(\mathcal{E}_V)$ which sends $v \in V$ to $\hat{v} \in \Gamma(\mathcal{E}_V)$, where $\hat{v}(x) := v(x) := \pi_x(v)$, becomes an isometric $C(X)$-linear isomorphism between Hilbert $C(X)$-modules V and $\Gamma(\mathcal{E}_V)$.

3. Results

We shall first prove Theorem 1.1. To do this, we shall need the following two auxiliary results:

Lemma 3.1. *Let* $\mathcal{E}$ *be an* n*-homogeneous (F) Hilbert bundle over a locally compact paracompact Hausdorff space* X. *Then the following conditions are equivalent:*

(i) $\mathcal{E}$ *is of finite type,*

(ii) *There exists a finite number of sections* $s_1, \dots, s_m \in \Gamma_b(\mathcal{E})$ *such that*

$$\operatorname{span}_{\mathbb{C}}\{s_1(x), \dots, s_m(x)\} = E_x \tag{3.1}$$

for each $x \in X$.

If in addition X *is* σ*-compact, then* $s_1, \dots, s_m$ *can be chosen to lie in* $\Gamma_0(\mathcal{E})$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is known for vector bundles (for example, see [1, Remark 8.3.12] and [14, p. 733–734]). Since n-homogeneous (F) Hilbert bundles are precisely the classical n-dimensional complex vector bundles with Riemannian metric, the same arguments can be applied in this situation. Now suppose that X is σ-compact and that $s_1, \dots, s_m \in \Gamma_b(\mathcal{E})$ are sections which satisfy (3.1) for all $x \in X$. Choose a strictly positive function $\varphi \in C_0(X)$ (such φ clearly exists, since X is σ-compact). Then obviously $s_i' := \varphi s_i$ $(1 \le i \le m)$ lie in $\Gamma_0(\mathcal{E})$ and satisfy (3.1) for all $x \in X$. $\square$

Proposition 3.2. *Let* $\mathcal{E}$ *be an (F) Hilbert bundle over a compact metrizable space* X. *Then the following conditions are equivalent:*

(i) *$\mathcal{E}$ is subhomogeneous of finite type,*
(ii) *There exists a finite number of sections $s_1, \ldots, s_m \in \Gamma(\mathcal{E})$ which satisfy (3.1) for all $x \in X$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that the degree of subhomogenity of $\mathcal{E}$ is n. Let $X_0, X_1, \ldots, X_k$ be pairwise disjoint non-empty subsets of X covering X and let $0 \leq n_0 < n_1 < \cdots < n_k = n$ be integers such that for each i, $\mathcal{E}|_{X_i}$ is n_i-homogeneous. We shall prove the claim by (finite) induction on $k = k(\mathcal{E})$. Suppose that $k = 0$. Then $\mathcal{E}$ is n-homogeneous and claim follows directly from Lemma 3.1. Let $k > 0$ and suppose that the assertion is true for all n-subhomogeneous (F) Hilbert bundles $\mathcal{E}'$ of finite type for which $k(\mathcal{E}') = k - 1$. By Theorem 2.2 (iv), $U := X_k$ is open in X, so by Lemma 3.1 there exists a finite number of sections $\dot{u}_1, \ldots, \dot{u}_d \in \Gamma_0(\mathcal{E}|_U)$ such that

$$\operatorname{span}_{\mathbb{C}}\{\dot{u}_1(x), \ldots, \dot{u}_d(x)\} = E_x \tag{3.2}$$

for all $x \in U$. Let u_i $(1 \leq i \leq d)$ be sections of $\mathcal{E}$ defined by

$$u_i(x) := \begin{cases} \dot{u}_i(x) & : x \in U \\ 0_x & : x \in X \setminus U \end{cases}.$$

Since $\dot{u}_i \in \Gamma_0(\mathcal{E}|_U)$, note that Axiom (H2) in definition of an (F) Hilbert bundle guarantees that u_i are continuous, so that $u_i \in \Gamma(\mathcal{E})$ for all $1 \leq i \leq d$. On the other hand, for $Y := X \setminus U$ we have $k(\mathcal{E}|_Y) = k - 1$. Hence, our induction hypothesis implies that there exists a finite number of sections $\dot{v}_1, \ldots, \dot{v}_l \in \Gamma(\mathcal{E}|_Y)$ such that

$$\operatorname{span}_{\mathbb{C}}\{\dot{v}_1(x), \ldots, \dot{v}_l(x)\} = E_x \tag{3.3}$$

for all $x \in Y$. Since Y is closed, by Theorem 2.2 (ii) there exist sections $v_1, \ldots, v_l \in \Gamma(\mathcal{E})$ such that $v_i|_Y = \dot{v}_i$. Using (3.2) and (3.3) we see that

$$\operatorname{span}_{\mathbb{C}}\{u_1(x), \ldots, u_d(x), v_1(x), \ldots, v_l(x)\} = E_x$$

for all $x \in X$, so letting $m := d + l$ and

$$s_i := \begin{cases} u_i & : 1 \leq i \leq d \\ v_{i-d} & : d + 1 \leq i \leq m \end{cases},$$

we obtain desired sections.

(ii) $\Rightarrow$ (i). It is immediate that $\mathcal{E}$ is (say n-)subhomogeneous. Using the same notation from the first part of the proof, we need to prove that each $\mathcal{E}|_{X_i}$ $(1 \leq i \leq k)$ is of finite type. But this follows directly from Lemma 3.1, since $s_j|_{X_i} \in \Gamma_b(\mathcal{E}|_{X_i})$ for all $1 \leq j \leq m$. $\square$

Proof of Theorem 1.1. Let $\mathcal{E} := \mathcal{E}_V$ be the canonical (F) Hilbert bundle associated to V. Using the generalized Gelfand transform $\Gamma_V : V \to \Gamma(\mathcal{E})$, we shall identify V with $\Gamma(\mathcal{E})$.

(i) $\Rightarrow$ (ii). By assumption, there exists a finite number of sections $s_1, \ldots, s_m \in \Gamma(\mathcal{E})$ whose $C(X)$-linear span is dense in $\Gamma(\mathcal{E})$. Obviously

$$W_x := \operatorname{span}_{\mathbb{C}}\{s_1(x), \ldots, s_m(x)\}$$

is dense in E_x for each $x \in X$. Since every finite dimensional subspace of a Hilbert space is closed, we conclude that $W_x = E_x$ for each $x \in X$. The assertion now follows directly from Proposition 3.2.

(ii) $\Rightarrow$ (i). By Proposition 3.2, there exist $s_1, \ldots, s_m \in \Gamma(\mathcal{E})$ which satisfy (3.1) for all $x \in X$. It is an immediate consequence of Theorem 2.2 (iii) that a $C(X)$-module generated by these sections is dense in $\Gamma(\mathcal{E})$. $\square$

Now we shall obtain another characterizations of topologically finitely generated Hilbert $C(X)$-modules. First recall that if X is a locally compact Hausdroff space and if V and W are (left) Banach $C_0(X)$-modules, then the $C_0(X)$-*projective tensor product* $V \overset{\pi}{\otimes}_{C_0(X)} W$ of V and W is by definition the quotient of the (completed) projective tensor product $V \overset{\pi}{\otimes} W$ by the closure of the liner span of tensors of the form

$$\varphi v \otimes w - v \otimes \varphi w,$$

where $v \in V$, $w \in W$ and $\varphi \in C_0(X)$. For $t \in V \overset{\pi}{\otimes} W$, by t_X we denote the canonical image of t in $V \overset{\pi}{\otimes}_{C_0(X)} W$.

Also recall that a Banach $C_0(X)$-module V is said to be *non-degenerate* if the $C_0(X)$-linear span of V is dense in V. This is equivalent to say that if (φ_α) is an approximate unit for $C_0(X)$, then $\lim_\alpha \varphi_\alpha v = v$ for all $v \in V$. In fact, by the Hewitt-Cohen factorization theorem [2, Theorem A.6.2], V is non-degenerate if and only if each $v \in V$ can be factorized in a form $v = \varphi w$, for some $\varphi \in C_0(X)$ and $w \in V$.

Following [10, Definition 1.2], we introduce a notion of a $C_0(X)$-projective rank:

Definition 3.3. Let V be a non-degenerate Banach $C_0(X)$-module, where X is a locally compact Hausdorff space.

(i) If W is another Banach $C_0(X)$-module, then for $t \in V \overset{\pi}{\otimes} W$ we define a $C_0(X)$-*projective rank of* t, denoted by $\operatorname{rank}_X^\pi(t)$, as the smallest nonnegative integer k for which there exists a rank k tensor $u \in V \otimes W$ such that $t_X = u_X$ in $V \overset{\pi}{\otimes}_{C_0(X)} W$. If such k does not exist, we define $\operatorname{rank}_X^\pi(t) := \infty$.

(ii) If there exists $K \in \mathbb{N}$ such that for every Banach $C_0(X)$-module W, and every tensor $t \in V \overset{\pi}{\otimes} W$ we have $\operatorname{rank}_X^\pi(t) \le K$, then we say that V is of *finite* $C_0(X)$-*projective rank*. The smallest number K with this property is denoted by $\operatorname{rank}_X^\pi(V)$. If such K does not exist, we define $\operatorname{rank}_X^\pi(V) := \infty$.

We give a sufficient condition for V to be of finite $C_0(X)$-projective rank (note that this is just an equivalent form of the condition (b) from the introduction).

Proposition 3.4. *Let V be a non-degenerate Banach $C_0(X)$-module, where X is a locally compact Hausdorff space. Suppose that there exists $K \in \mathbb{N}$ such that for every sequence $(a_i) \in \ell^1(V)$ there exist $k \le K$, elements $v_1, \ldots, v_k \in V$ and sequences $(\varphi_{i,1})_i, \ldots, (\varphi_{i,k})_i \in \ell^1(C_0(X))$ such that*

$$a_i = \sum_{j=1}^{k} \varphi_{i,j} v_j \tag{3.4}$$

for all $i \in \mathbb{N}$. Then $\operatorname{rank}_X^\pi(V) \le K$.

Proof. Let W be another Banach $C_0(X)$-module and let $t \in V \overset{\pi}{\otimes} W$. By [20, Proposition 2.8] there exists a sequence $(a_i) \in \ell^1(V)$ and a sequence (b_i) of norm one elements in W such that $t = \sum_{i=1}^\infty a_i \otimes b_i$. By assumption, there exist $k \le K$, elements $v_1, \ldots, v_k \in V$ and sequences $(\varphi_{i,1})_i, \ldots, (\varphi_{i,k})_i \in \ell^1(C_0(X))$ such that

(3.4) holds. Note that for each $1 \le j \le k$ the series $\sum_{i=1}^{\infty} \varphi_{i,j} b_i$ is norm convergent in W, and denote its sum by w_j. Then in $V \overset{\pi}{\otimes}_{C_0(X)} W$ we have

$$\begin{aligned} t_X &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i \otimes_X b_i \right) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{k} \varphi_{i,j} v_j \right) \otimes_X b_i \right) \\ &= \lim_{n\to\infty} \left(\sum_{j=1}^{k} v_j \otimes_X \left(\sum_{i=1}^{n} \varphi_{i,j} b_i \right) \right) = \sum_{j=1}^{k} v_j \otimes_X \left(\lim_{n\to\infty} \sum_{i=1}^{n} \varphi_{i,j} b_i \right) \\ &= \sum_{j=1}^{k} v_j \otimes_X w_j. \end{aligned}$$

Thus, $\mathrm{rank}_X^{\pi}(V) \le K$. $\square$

Remark 3.5. Recall from the introduction, we say that a Banach $C_0(X)$-module V is weakly algebraically finitely generated if there exists $K \in \mathbb{N}$ such that every algebraically finitely generated submodule of V can be generated with $k \le K$ generators. Note that the condition of Proposition 3.4 in particulary implies that V is weakly algebraically finitely generated.

We are now ready to state our main result:

Theorem 3.6. *Let V be Hilbert $C(X)$-module, where X is a compact metrizable space. The following conditions are equivalent:*

(i) *V is topologically finitely generated,*
(ii) *V satisfies the condition of Proposition 3.4,*
(iii) *V is of finite $C(X)$-projective rank,*
(iv) *V is weakly algebraically finitely generated.*

Before proving this theorem we shall first need some facts.

Remark 3.7. Let X be a locally compact Hausdorff space and let V be a Banach $C_0(X)$-module. Note that $\ell^1(V)$ becomes a Banach $C_0(X)$-module with respect to the action

$$\varphi(v_i) := (\varphi v_i),$$

where $\varphi \in C_0(X)$ and $(v_i) \in \ell^1(V)$. If V is non-degenerate, note that $\ell^1(V)$ is non-degenerate as well. Indeed, let $(\varphi_\alpha)_{\alpha\in\mathbb{I}}$ be the approximate unit for $C_0(X)$. If $(v_i) \in \ell^1(V)$ and $\varepsilon > 0$, choose $k_0 \in \mathbb{N}$ and $\alpha_0 \in \mathbb{I}$ such that

$$\sum_{i=k_0+1}^{\infty} \|v_i\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|\varphi_\alpha v_k - v_k\| < \frac{\varepsilon}{2^{k+1}}$$

for all $1 \le k \le k_0$ and $\alpha \ge \alpha_0$. Then for $\alpha \ge \alpha_0$ we have

$$\begin{aligned} \|\varphi_\alpha(v_i) - (v_i)\|_{\ell^1(V)} &= \sum_{i=1}^{\infty} \|\varphi_\alpha v_i - v_i\| \le \sum_{i=1}^{k_0} \|\varphi_\alpha v_i - v_i\| + 2 \sum_{i=k_0+1}^{\infty} \|v_i\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, $\lim_{\alpha\in\mathbb{I}} \varphi_\alpha(v_i) = (v_i)$. In particular, by the Hewitt-Cohen factorization theorem [2, Theorem A.6.2], each $(v_i) \in \ell^1(V)$ can be factorized in a form $(v_i) = \varphi(v_i')$ for some $\varphi \in C_0(X)$ and $(v_i') \in \ell^1(V)$.

Proposition 3.8. *Let $\mathcal{E}$ be an n-homogeneous (F) Hilbert bundle of finite type over a locally compact paracompact Hausdorff space X. Then $V := \Gamma_0(\mathcal{E})$ satisfies the condition of Proposition 3.4.*

Proof. Choose a finite open cover $\{U_j\}_{1\le j\le m}$ of X such that each restriction bundle $\mathcal{E}|_{U_j}$ is trivial. Using a finite partition of unity argument, it is sufficient to prove the claim when $\mathcal{E}$ is already trivial. In this case, we shall identify $\mathcal{E}$ with the product bundle $\epsilon(X, \ell^2_n)$ (using a fixed isomorphism $\Phi : \mathcal{E} \cong \epsilon(X, \ell^2_n)$), so that $V = \Gamma_0(\mathcal{E}) = C_0(X, \ell^2_n) = \ell^2_n(C_0(X))$. Let $(a_i) \in \ell^1(V)$. Since V, as a Banach $C_0(X)$-module, is non-degenerate, by Remark 3.7 there exist $\varphi \in C_0(X)$ and $(a'_i) \in \ell^1(V)$ such that $(a_i) = \varphi(a'_i)$. Let $(\vec{e}_k)_{1\le k\le n}$ be the canonical basis for ℓ^n_2 and let $(e_k)_{1\le k\le n}$ be the constant functions in $C_b(X, \ell^2_n)$ defined by $e_k(x) := \vec{e}_k$ $(x \in X)$. Then

$$a'_i = \sum_{k=1}^{n} \varphi_{i,k} e_k,$$

for some functions $\varphi_{i,k} \in C_0(X)$ $(i \in \mathbb{N}, 1 \le k \le n)$. Obviously, $(\varphi_{i,1})_i, \ldots, (\varphi_{i,n})_i \in \ell^1(C_0(X))$. Hence,

$$a_i = \varphi a'_i = \sum_{k=1}^{n} \varphi_{i,k}(\varphi e_k)$$

for all $i \in \mathbb{N}$. Since $v_k := \varphi e_k$ lie in $C_0(X, \ell^2_n) = V$, the proof is finished. $\square$

Lemma 3.9. *Let H be a Hilbert space with the dual space H^*. For $\xi \in H$ let $\xi^* \in H^*$ be defined by $\xi^*(h) := \langle h, \xi\rangle$. If (ξ_i), (η_i) and (ζ_i) are sequences in $\ell^1(H)$ such that*

$$\sum_{i=1}^{\infty} \xi_i \otimes \xi_i^* = \sum_{i=1}^{\infty} \eta_i \otimes \zeta_i^* \tag{3.5}$$

(the equality of tensors in $H \overset{\pi}{\otimes} H^*$), then

$$\overline{\operatorname{span}}_{\mathbb{C}}\{\xi_i \ : \ i \in \mathbb{N}\} \subseteq \overline{\operatorname{span}}_{\mathbb{C}}\{\eta_i \ : \ i \in \mathbb{N}\},$$

where $\overline{\operatorname{span}}_{\mathbb{C}}$ denotes the closed $\mathbb{C}$-linear span.

Proof. Note that the equality (3.5) is equivalent to the equality

$$\begin{aligned}\sum_{i=1}^{\infty}\langle \xi_i, h\rangle\langle k, \xi_i\rangle &= \sum_{i=1}^{\infty} h^*(\xi_i)k^{**}(\xi_i^*) = \sum_{i=1}^{\infty} h^*(\eta_i)k^{**}(\zeta_i^*)\\ &= \sum_{i=1}^{\infty}\langle \eta_i, h\rangle\langle k, \zeta_i\rangle\end{aligned}$$

for all $h, k \in H$ (see [20]). In particular, for $h = k$ we obtain

$$\sum_{i=1}^{\infty} |\langle h, \xi_i\rangle|^2 = \sum_{i=1}^{\infty}\langle \eta_i, h\rangle\langle h, \zeta_i\rangle. \tag{3.6}$$

Let $h \in H$ be a vector which is orthogonal to the sequence (η_i). Then (3.6) implies that h is also orthogonal to the sequence (ξ_i). Since $h \in H$ was arbitrary, the claim follows. $\square$

Remark 3.10. If V and W are Banach spaces and if $q : V \to W$ is a quotient linear map, note that the induced map

$$\tilde{q} : \ell^1(V) \to \ell^1(W), \quad \tilde{q}((v_i)) := (q(v_i))$$

is also quotient. Indeed, this follows from the fact that $\ell^1(V)$ and $\ell^1(W)$ can be identified with $\ell^1 \overset{\pi}{\otimes} V$ and $\ell^1 \overset{\pi}{\otimes} W$, respectively (see [20, Example 2.6]), so that $\tilde{q} = \mathrm{id}_{\ell^1} \otimes q$. The claim now follows from [20, Proposition 2.5].

Proof of Theorem 3.6. Let $\mathcal{E} := \mathcal{E}_V$ be the canonical (F) Hilbert bundle associated to V. Like in the proof of Theorem 1.1, we shall identify V with $\Gamma(\mathcal{E})$, using the generalized Gelfand transform $\Gamma_V : V \to \Gamma(\mathcal{E})$.

(i) $\Rightarrow$ (ii). By Theorem 1.1, $\mathcal{E}$ is (say n-)subhomogeneous of finite type. Let $X_0, X_1, \ldots, X_k$ be pairwise disjoint non-empty subsets of X covering X and let $0 \leq n_0 < n_1 < \cdots < n_k = n$ be integers such that for each i, $\mathcal{E}|_{X_i}$ is n_i-homogeneous. Let $U := X_k$ and $Y := X \setminus U$. By Theorem 2.2 (iv) U is open (so that Y is closed, hence compact). We shall prove the claim by (finite) induction on $k = k(\mathcal{E})$. If $k = 0$, then the claim follows from Proposition 3.8. Let $k > 0$ and suppose that the assertion is true for all n-subhomogeneous (F) Hilbert bundles $\mathcal{E}'$ of finite type for which $k(\mathcal{E}') = k - 1$. Let $(a_i) \in \ell^1(\Gamma(\mathcal{E}))$ be given, and put $\dot{a}_i := a_i|_Y$. By induction hypothesis (applied to $\mathcal{E}|_Y$), there exist $M_1 \in \mathbb{N}$ (which depends only on $\mathcal{E}|_Y$), $m_1 \leq M_1$, sections $\dot{v}_1, \ldots, \dot{v}_{m_1} \in \Gamma(\mathcal{E}|_Y)$ and sequences $(\dot{\varphi}_{i,1})_i, \ldots, (\dot{\varphi}_{i,m_1})_i \in \ell^1(C(Y))$ such that

$$\dot{a}_i = \sum_{j=1}^{m_1} \dot{\varphi}_{i,j} \dot{v}_j. \tag{3.7}$$

Since the restriction map $C(X) \to C(Y)$, $\varphi \mapsto \varphi|_Y$ is quotient, by Remark 3.10 and Theorem 2.2 (ii), there exist elements $v_1, \ldots, v_{m_1} \in \Gamma(\mathcal{E})$ and sequences $(\varphi_{i,1})_i, \ldots,$ $(\varphi_{i,m_1})_i \in \ell^1(C(X))$ such that $v_j|_Y = \dot{v}_j$ and $\varphi_{i,j}|_Y = \varphi_{i,j}$ $(i \in \mathbb{N}, 1 \leq j \leq m_1)$. Define

$$b_i := a_i - \sum_{j=1}^{m_1} \varphi_{i,j} v_j. \tag{3.8}$$

Note that by (3.7), $b_i \in \Gamma_0(\mathcal{E}|_U)$, so that $(b_i) \in \ell^1(\Gamma_0(\mathcal{E}|_U))$. Since $\mathcal{E}|_U$ is n-homogeneous of finite type, by Proposition 3.8, there exist $M_2 \in \mathbb{N}$ (which depends only on $\mathcal{E}|_U$), $m_2 \leq M_2$, elements $v'_1, \ldots, v'_{m_2} \in \Gamma_0(\mathcal{E}|_U)$ and sequences $(\psi_{i,1})_i, \ldots, (\psi_{i,m_2})_i \in \ell^1(C_0(U))$ such that

$$b_i = \sum_{j=1}^{m_2} \psi_{i,j} v'_j. \tag{3.9}$$

As in the proof of Proposition 3.2 we may suppose (by extending with zeros) that $\psi_{i,j}$ and v'_j are globally defined, so that $(\psi_{i,j})_i \in \ell^1(C(X))$ and $v'_j \in \Gamma(\mathcal{E})$, with $\psi_{i,j}|_Y = 0$ and $v'_j|_Y = 0$. Finally, (3.8) and (3.9) imply that

$$a_i = \sum_{j=1}^{m_1} \varphi_{i,j} v_j + \sum_{j=1}^{m_2} \psi_{i,j} v'_j,$$

what completes the proof.

(ii) $\Rightarrow$ (iii). This follows from Proposition 3.4.

(iii) $\Rightarrow$ (i). We first prove that $\mathcal{E}$ is necessarily subhomogeneous. Indeed, let H be an infinite dimensional Hilbert space and let $W := C(X, H)$ (of course, we consider W as a Hilbert $C(X)$-module $\Gamma(\epsilon(X, H))$). By assumption, there exists $N \in \mathbb{N}$ such that $\operatorname{rank}_X^{\pi}(t) \leq N$ for every $t \in V \overset{\pi}{\otimes}_{C(X)} W$. Using [16, Proposition 1.5], we conclude that for each $x \in X$, every tensor in $E_x \overset{\pi}{\otimes} H$ is of (finite) rank at most N. But it is well known that this implies $\dim E_x \leq N$ for all $x \in X$.

Suppose that $\mathcal{E}$ is not of finite type and choose a sequence $(a_i) \in \ell^1(V)$ such that

$$\operatorname{span}_{\mathbb{C}}\{a_i(x) \ : \ i \in \mathbb{N}\} = E_x \tag{3.10}$$

for all $x \in X$ (such sequence clearly exists, since X is second-countable, and since each fibre E_x is finite dimensional, so that V is separable). Let $V^* = \operatorname{Hom}_{C(X)}(V, C(X))$ be a dual of V which consists of all bounded $C(X)$-linear maps $\theta : V \to C(X)$. We consider W as a Banach $C(X)$-module under the natural norm and action. For $a \in V$, let $a^* \in V^*$ be defined by $a^*(v) := \langle v, a\rangle$, and let V' be the image of V under the (anti $C(X)$-linear isometric) inclusion $V \hookrightarrow V^*$, $a \mapsto a^*$ (note that $V' \neq V^*$ in general, see [6, p. 88-89] and [15, Section 2.5]). By assumption, there exists $N \in \mathbb{N}$ such that $\operatorname{rank}_X^{\pi}(t) \leq N$ for every $t \in V \overset{\pi}{\otimes}_{C(X)} V'$. In particular, there are $n \leq N$, $s_1, \ldots, s_n \in V$ and $v_1, \ldots, v_n \in V$ such that

$$\sum_{i=1}^{\infty} a_i \otimes_X a_i^* = \sum_{i=1}^{n} s_i \otimes_X v_i^* \tag{3.11}$$

(the equality of tensors in $V \overset{\pi}{\otimes}_{C(X)} V'$). By Proposition 3.2, there exists a point $x_0 \in X$ such that

$$\operatorname{span}_{\mathbb{C}}\{s_1(x_0), \ldots, s_n(x_0)\} \subsetneqq E_{x_0}. \tag{3.12}$$

Note that the equality (3.11) implies the equality of tensors

$$\sum_{i=1}^{\infty} a_i(x_0) \otimes a_i(x_0)^* = \sum_{i=1}^{n} s_i(x_0) \otimes v_i(x_0)^*$$

in $E_{x_0} \overset{\pi}{\otimes} (E_{x_0})^*$, where for $a \in V$ and $e_{x_0} \in E_{x_0}$,

$$a(x_0)^*(e_{x_0}) := \langle e_{x_0}, a(x_0)\rangle_{x_0}.$$

Using (3.10) and Lemma 3.9, we conclude that

$$E_{x_0} = \operatorname{span}_{\mathbb{C}}\{a_i(x_0) \ : \ i \in \mathbb{N}\} \subseteq \operatorname{span}_{\mathbb{C}}\{s_1(x_0), \ldots, s_n(x_0)\},$$

a contradiction with (3.12). It follows that $\mathcal{E}$ is of finite type, so by Theorem 1.1, $V = \Gamma(\mathcal{E})$ is topologically finitely generated.

(ii) $\Rightarrow$ (iv). This follows from Remark 3.5.

(iv)$\Rightarrow$ (i). Let N be a natural number such that every algebraically finitely generated submodule of $\Gamma(\mathcal{E})$ can be generated with $n \leq N$ elements. We first prove that $\mathcal{E}$ must be m-subhomogeneous, where $m \leq N$. Suppose, to the contrary, that there exists $x_0 \in X$ such that $\dim E_{x_0} > N$. By Theorem 2.2 (i), we can find sections $s_1, \ldots, s_{N+1} \in \Gamma(\mathcal{E})$ such that $\dim W_{x_0} = N + 1$, where

$$W_{x_0} := \operatorname{span}_{\mathbb{C}}\{s_1(x_0), \ldots, s_{N+1}(x_0)\} \subseteq E_{x_0}.$$

By assumption, there exist $n \leq N$ and sections $v_1, \ldots, v_n \in \Gamma(\mathcal{E})$ such that

$$\operatorname{span}_{C(X)}\{s_1, \ldots, s_{N+1}\} = \operatorname{span}_{C(X)}\{v_1, \ldots, v_n\}.$$

In particular, this implies

$$W_{x_0} = \mathrm{span}_{\mathbb{C}}\{v_1(x_0), \ldots, v_n(x_0)\},$$

so that $\dim W_{x_0} \leq n$, a contradiction. It follows that $\mathcal{E}$ is subhomogeneous.

Now suppose that $\mathcal{E}$ is not of finite type. Let U and X_j be as in the proof (i) $\Rightarrow$ (ii). It follows that some of the homogeneous restriction bundles $\mathcal{E}|_{X_j}$ is not of finite type. Without loss of generality, we may assume that $\mathcal{E}|_U$ is not of finite type. Using [9, Lemma 2.10] and [9, Lemma 2.11], one can find a compact subset $K \subseteq U$ with the following property: If $s_1, \ldots, s_k$ are continuous sections of $\mathcal{E}|_K$ such that

$$\mathrm{span}_{\mathbb{C}}\{s_1(x), \ldots, s_k(x)\} = E_x \quad \text{for all } x \in K, \quad \text{then } k > N. \tag{3.13}$$

Since K is compact, $\Gamma(\mathcal{E}|_K)$ is an algebraically finitely generated (projective) $C(K)$-module (by the Serre-Swan theorem [19, Theorem 1.6]), and let $v_1, \ldots, v_d$ be its generators. On the other hand, by our assumption (together with Theorem 2.2 (ii)) there exist $n \leq N$ and sections $s_1, \ldots, s_n \in \Gamma(\mathcal{E}|_K)$ such that

$$\Gamma(\mathcal{E}|_K) = \mathrm{span}_{C(K)}\{v_1, \ldots, v_d\} = \mathrm{span}_{C(K)}\{s_1, \ldots, s_n\}.$$

In particular,

$$\mathrm{span}_{\mathbb{C}}\{s_1(x), \ldots, s_n(x)\} = E_x$$

for all $x \in K$. But (3.13) implies $n > N$, a contradiction. It follows that $\mathcal{E}$ is of finite type, so by Theorem 1.1, $V = \Gamma(\mathcal{E})$ is topologically finitely generated. $\square$

Problem 3.11. Generalize the results of Theorem 3.6 (if possible) for (unital) Banach $C(X)$-modules, in particular for (unital) $C(X)$-locally convex modules.

Problem 3.12. Suppose that $\mathcal{E}$ is an (F) Banach bundle over a compact Hausdorff space X. If $V := \Gamma(\mathcal{E})$ is an algebraically finitely generated $C(X)$-module, does it necessarily follow that V is projective? In particularly, if X is connected, does it necessarily follow that all fibres of $\mathcal{E}$ have the same finite dimension n?

Remark 3.13. Using [9, Theorem 2.4] together with results of [17], one can show that the answer to Problem 3.12 is positive for (continuous) C^*-bundles. Moreover, if the base space X is connected, then all fibres of this C^*-bundle must be pairwise $*$-isomorphic, in particularly isometrically isomorphic. On the other hand, next example shows that Problem 3.12 has a negative solution if one also considers (H) Banach bundles (see [4] for definition and properties of (H) Banach bundles).

Example 3.14. Let $V := C([0,1])$. We consider V as a Banach module over $C(S^1)$ (where S^1 is the unit circle), with respect to the action

$$(\varphi f)(x) := \varphi(e^{2\pi i x}) f(x),$$

where $\varphi \in C(S^1)$, $f \in C([0,1])$ and $x \in [0,1]$. Let $\mathcal{E}$ be the canonical (H) Banach bundle of V over S^1. Then the fibre of $\mathcal{E}$ at 1 is of dimension 2, rather than 1 as at other points in the unit circle S^1. On the other hand, [17, Example 3.7] shows that V is algebraically finitely generated.

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Department of Mathematics, University of Zagreb, Bijenička cesta 30, Zagreb 10000, Croatia

E-mail address: ilja@math.hr