GLASNIK MAT. - FIZ. I ASTR.
Tom 20. — No. 1—2 — 1965.

QUADRATIC AND SESQUILINEAR FUNCTIONALS

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1. Let $X = \{x, y, \ldots\}$ be a complex (quaternionic) vector space and B a function of two vectors which is linear in the first argument and antilinear in the second argument, i. e.

$$\left. \begin{aligned} B(\lambda_1 x_1 + \lambda_2 x_2, y) &= \lambda_1 B(x_1, y) + \lambda_2 B(x_2, y), \\ B(x, \mu_1 y_1 + \mu_2 y_2) &= \overline{\mu_1} B(x, y_1) + \overline{\mu_2} B(x, y_2), \end{aligned} \right\} \tag{1}$$

where $\overline{\lambda}$ denotes the conjugate of λ.

If we set $n(x) = B(x, x)$, then from (1) it follows that

$$n(x+y) + n(x-y) = 2n(x) + 2n(y) \tag{2}$$

holds for all $x, y \in X$ and also

$$n(\lambda x) = |\lambda|^2 n(x) \tag{3}$$

holds for every $x \in X$ and every complex (quaternionic) number λ. A functional $n(x)$ which satisfies (2) is termed a quadratic functional.

Prof. Israel Halperin in 1963 in Paris, in his lectures on Hilbert spaces raised the question which can be formulated as follows: *Does a quadratic functional n for which* (3) *holds possess the property that*

$$B(x, y) = m(x, y) + i\, m(x, iy) \tag{4}$$

in the case of a complex space and

$$B(x, y) = m(x, y) + i\, m(x, iy) + j\, m(x, jy) + k\, m(x, ky) \tag{5}$$

in the case of a quaternionic space is a sesquilinear functional with

$$m(x, y) = \frac{1}{4}(n(x+y) - n(x-y)). \tag{6}$$

A similar question was raised also for the case of a real vector space. In this case (3) is to be replaced by

$$n(tx) = t^2 n(x)$$

for all real numbers t and all vectors x. In [1] we have proved that *the answer to the Halperin's problem in the case of a real vector*

Ovaj rad je financirao Savezni fond za naučni rad i Republički fond za naučni rad SRH.

space is negative provided that the space is not one dimensional. It was also proved that *in an algebraic basic set* $\{e_k \mid 1 \leq k < \Omega\}$ *the quadratic functional n is given by*

$$n\left(\sum_{1 \leq k < \Omega} t_k e_k\right) = \sum_{1 \leq i, j < \Omega} b_{ij} t_i t_j + \sum_{1 \leq i < j < \Omega} \begin{vmatrix} a_{ij}(t_i) & a_{ij}(t_j) \\ t_i & t_j \end{vmatrix}, \quad (7)$$

where in the sum only a finite number of terms differs from zero; b_{ij} are constants and $a_{ij}(t)$ is a derivative on the set of all real numbers. By a derivative on the set of reals one understands a real-valued function f such that

$$f(t+s) = f(t) + f(s) \text{ and } f(ts) = tf(s) + sf(t)$$

holds for all real numbers t and s.

The object of this paper is to prove that *in the case of a complex or a quaternionic vector space* (2) *and* (3) *do imply that the functionals* (4) *and* (5) *are sesquilinear.* We derive these results by considering spaces as real vector spaces and then by using (7). It would be interesting to prove these results directly, i. e. without the use of a »negative« result in a real vector space. In addition to this in **2** we derive some results on functionals which satisfay (2) and which are defined on an arbitrary abelian group.

Since the case of a quaternionic vector space is a consequence of the situation in the complex case, the main result of this paper is given by the following theorem.

T h e o r e m 1. *Let X be a complex vector space and n a complex valued functional such that*

(i) $$n(x+y) + n(x-y) = 2n(x) + 2n(y)$$

and

(ii) $$n(\lambda x) = |\lambda|^2 n(x)$$

hold for all $x, y \in X$ *and all complex numbers* λ. *Under these conditions the functional*

$$B(x, y) = \frac{1}{4}[n(x+y) - n(x-y)] + \frac{i}{4}[n(x+iy) - n(x-iy)]$$

is linear in x and antilinear in y, i. e. $B(x, y)$ *is a sesquilinear functional on X and* $B(x, x) = n(x)$.

P r o o f. If n is a real-valued functional, then the functional m which is defined by (6) is real so that $n(ix) = n(x)$ implies

$$n(x+iy) - n(x-iy) = n[i(y-ix)] - n[i(-y-ix)] =$$
$$= [n(y-ix) - n(y+ix)],$$

which leads to $B(x, y) = \overline{B(y, x)}$. Similarily one finds $B(ix, y) = iB(x, y)$. Thus, if n is a real functional, then B is a Hermitian functional and it is sufficient to prove that B is linear in the first argument. On the other hand in the case of a complex (quaternionic) $n(x)$, (*i*) and (*ii*) imply that real and imaginary parts of $n(x)$ satisfy the same conditions. Hence, without loss of generality we can assume that n is a real functional. Since the functional $m(x, y)$ is additive in x, so is the functional B ([1], Lemma 1) and it is sufficient to prove that $B(tx, y) = tB(x, y)$, or equivalently that

$$m(tx, y) = tm(x, y) \tag{8}$$

holds for all real numbers t and all pairs $x, y \leqslant X$.

Suppose that x and y are given and that $y = \mu x$. Then

$$n(tx + y) - n(tx - y) = (|t + \mu|^2 - |t - \mu|^2)\, n(x) =$$
$$= t(|1 + \mu|^2 - |1 - \mu|^2)\, n(x) = t\, n(x + y) - t\, n(x - y)$$

implies (8), i. e. (8) holds if x and y are dependent.

If x and y are independent then they determine a two-dimensional subspace Y of X with $e_1 = x$ and $e_2 = y$ as a basic set. The restriction of n to Y, which we denote again by n, possesses properties (*i*) and (*ii*) on Y. Since e_1, e_2 is a basis set in Y, then e_1, ie_1, e_2, ie_2 is a basic set in Y considered as a real vector space. The functional n as a functional on the real vector space Y is quadratic, i. e. (2) holds and $n(tz) = t^2 n(z)$ for every real number t and $z \leqslant Y$. Applying (7) to the present situation we have

$$n(z) = n(t_1 e_1 + t_2 i e_1 + t_3 e_2 + t_4 i e_2) = \sum_{i,j=1}^{4} b_{ij} t_i t_j + \tag{9}$$
$$+ \sum_{1 \leq i < j \leq 4} \begin{vmatrix} a_{ij}(t_i) & a_{ij}(t_j) \\ t_i & t_j \end{vmatrix},$$

where b_{ij} are real constants and $a_{ij}(t)$ is a derivative on the set of all real numbers. Hence, $a_{ij}(r) = 0$ for every rational (and even for every algebraic) number r.

If we take z as a rational vector, i. e. all t_i are rational, then we get

$$n(z) = \sum_{i,j=1}^{4} b_{ij} t_i t_j . \tag{10}$$

Replacing in (9) z by

$$\lambda z = t_1' e_1 + t_2' i e_1 + t_3' e_2 + t_4' i e_2,$$

where λ is a complex rational number we get by using $n(\lambda z) = |\lambda|^2 n(z)$ the relation

$$|\lambda|^2 \sum_{i,j=1}^{4} b_{ij}\, t_i\, t_j = \sum_{i,j=1}^{4} b_{ij}\, t_i'\, t_j'. \tag{11}$$

Obviously, (11) holds for all complex numbers λ. This implies that the first sum in (9) satisfies conditions (*i*) and (*ii*) of Theorem 1. But then the functional

$$N(z) = \sum_{1 \le i < j \le 4} \begin{vmatrix} a_{ij}(t_i) & a_{ij}(t_j) \\ t_i & t_j \end{vmatrix} \tag{12}$$

also satisfies conditions (*i*) and (*ii*). We are going to prove that essentially (*ii*) implies $N(z) = 0$, for every $z \leqslant Y$.

For a vector $z = t\, e_1 + s\, i\, e_1$ from (12) we get

$$N(t\, e_1 + s\, i\, e_1) = \begin{vmatrix} a_{12}(t) & a_{12}(s) \\ t & s \end{vmatrix}.$$

On the other hand, $N(t\, e_1 + s\, i\, e_1) = N[(t + i\, s)\, e_1] = |t + i\, s|^2 N(e_1)$. We have, therefore, $(t^2 + s^2)\, N(e_1) = s\, a_{12}(t) - t\, a_{12}(s)$, which implies that a_{12} is a continuous function and therefore $a_{12} = 0$. Similarly $a_{34} = 0$. Thus,

$$\begin{aligned} N(t\, e_1 + t'\, i\, e_1 + s\, e_2 + s'\, i\, e_2) = & \begin{vmatrix} a(t) & a(s) \\ t & s \end{vmatrix} + \begin{vmatrix} b(t) & b(s') \\ t & s' \end{vmatrix} + \\ & + \begin{vmatrix} c(t') & c(s) \\ t' & s \end{vmatrix} + \begin{vmatrix} d(t') & d(s') \\ t' & s' \end{vmatrix}, \end{aligned} \tag{13}$$

where a, b, c and d are derivatives on the reals and t, t', s and s' are arbitrary real numbers. If we take $z = t\, e_1 + s'\, i\, e_2$ and $\lambda = \sigma + i\, \tau$ (σ, τ reals), then

$$(\sigma + i\tau)\, z = \sigma\, t\, e_1 + \tau\, t\, i\, e_1 + (-\tau\, s')\, e_2 + \sigma\, s'\, i\, e_2$$

together with (13) and (ii) implies

$$\begin{aligned} (\sigma^2 + \tau^2) \begin{vmatrix} b(t) & b(s') \\ t & s' \end{vmatrix} = & \begin{vmatrix} a(\sigma t) & a(-\tau s') \\ \sigma t & -\tau s' \end{vmatrix} + \begin{vmatrix} b(\sigma t) & a(\sigma s') \\ \sigma t & \sigma s' \end{vmatrix} + \\ & + \begin{vmatrix} c(\tau t) & c(-\tau s') \\ \tau t & -\tau s' \end{vmatrix} + \begin{vmatrix} d(\tau t) & d(\sigma s') \\ \tau t & \sigma s' \end{vmatrix}, \end{aligned} \tag{14}$$

for all real numbers σ, τ, t and s'. If we take σ and τ to be rational numbers and if we use $f(r\, t) = r\, f(t)$ for every derivative f and rational number r, then (14) turns out to be a polynomial in σ and

τ. But then the corresponding coefficients have to be equal. We have, therefore,

$$\begin{vmatrix} b(t) & b(s') \\ t & s' \end{vmatrix} = \begin{vmatrix} c(t) & -c(s') \\ t & -s' \end{vmatrix},$$

$$\begin{vmatrix} a(t) & -a(s') \\ t & -s' \end{vmatrix} + \begin{vmatrix} d(t) & d(s') \\ t & s' \end{vmatrix} = 0.$$

If we take $s' = 1$, we get

$$b(t) = -c(t) \quad \text{and} \quad a(t) = d(t). \tag{15}$$

Now, take σ in (14) to be rational. We get a polynomial in σ, which by the comparison of coefficients leads to

$$\begin{vmatrix} a(t) & -a(\tau s') \\ t & -\tau s' \end{vmatrix} + \begin{vmatrix} a(\tau t) & a(s') \\ \tau t & s' \end{vmatrix} = 0.$$

If we take $t = s' = 1$, we get $a(\tau) = 0$, for every $\tau \leqslant R$. Thus, $a = d = 0$. Now, we take $z = t\, e_1 + s\, e_2$ in (13) and we use $a = d = 0$, $b = -c$. We get $N(z) = 0$. This gives $N(\lambda z) = 0$, for $\lambda = \sigma + i\tau\,(\sigma, \tau \leqslant R)$. Using (13), for $\lambda z = \sigma t e_1 + \tau t i e_1 + \sigma s e_2 + \tau s i e_2$, we find

$$(\tau s\, b(\sigma t) - \sigma t\, b(\tau s)) + (-\sigma s\, b(\tau t) + \tau t\, b(\sigma s)) = 0,$$

which for $\sigma = s = t = 1$ implies $b(\tau) = 0$, for every $\tau \leqslant R$. Thus, $a = b = c = d = 0$ and, therefore, $N(z) = 0$, for every $z \leqslant Y$.

In such a way we have proved that in (9) all $a_{ij} = 0$. Hence,

$$n[(t_1 + i t_2) e_1 + (t_3 + i t_4) e_2] = \sum_{i,j=1}^{4} b_{ij}\, t_i\, t_j$$

for all real numbers t_i. If we set

$$\lambda_1 = t_1 + i t_2, \quad \lambda_2 = t_3 + i t_4,$$

then

$$t_1 = \frac{\lambda_1 + \bar{\lambda}_1}{2},\ t_2 = \frac{\lambda_1 - \bar{\lambda}_1}{2i},\ t_3 = \frac{\lambda_2 + \bar{\lambda}_2}{2} \text{ and } t_4 = \frac{\lambda_2 - \bar{\lambda}_2}{2i}$$

imply

$$n(\lambda_1 e_1 + \lambda_2 e_2) = \sum_{i,j=1}^{2} c_{ij}\, \lambda_i\, \bar{\lambda}_j, \tag{16}$$

where c_{11} and c_{22} are real numbers and $c_{12} = \bar{c}_{21}$. From (16) we find

$$m(\lambda_1 e_1, \lambda_2 e_2) = \frac{1}{2}(\bar{c}_{12}\, \lambda_1\, \bar{\lambda}_2 + \bar{c}_{12}\, \bar{\lambda}_1\, \lambda_2),$$

which for $\lambda_1 = t$ and $\lambda_2 = 1$ leads to

$$m(tx, y) = t\, m(x, y),$$

i. e. (8) also holds in the case of independent vectors x and y.

T h e o r e m 2. *Let X be a vector space over the field of quaternions and n a real functional on X such that*

$$\text{(a)} \qquad n(x+y) + n(x-y) = 2n(x) + 2n(y) \text{ and}$$
$$\text{(b)} \qquad n(\lambda x) = |\lambda|^2 n(x)$$

holds for all $x, y \leqslant X$ and all quaternions λ, where $|\lambda|^2 = \lambda \bar{\lambda}$ and $\bar{\lambda}$ is the conjugate quaternion of λ. Under these conditions the functional

$$B(x, y) = m(x, y) + i\, m(x, iy) + j\, m(x, jy) + k\, m(x, ky)$$

is a sesquilinear functional and $n(x) = B(x, x)$, where

$$m(x, y) = \frac{1}{4}(n(x+y) - n(x-y)).$$

P r o o f. If we consider X as a complex vector space, then the conditions of Theorem 1 are fulfiled so that $m(tx, y) = t\, m(x, y)$ holds for all real numbers t and all $x, y \leqslant X$. This implies the assertion of Theorem 2 by the same reasoning as in Theorem 1.

2. In this section X denotes an Abelian group, R the set of all reals and $n : X \to R$ a quadratic functional, i. e. a function which satisfies the equation

$$n(x+y) + n(x-y) = 2n(x) + 2n(y)$$

for all $x, y \leqslant X$. As in ([1], Lemma 1) one proves that a function

$$m(x, y) = \frac{1}{4}(n(x+y) - n(x-y))$$

is additive in each argument and that $m(x, y) = m(y, x)$ holds for all $x, y \leqslant X$.

A quadratic functional n is termed positive if

$$n(x) \geqq 0 \tag{17}$$

holds for all $x \leqslant X$. If $g : X \to R$ is any additive functional, i. e.

$$g(x+y) = g(x) + g(y) \quad (x, y \leqslant X), \tag{18}$$

then

$$n(x) = [g(x)]^2 \tag{19}$$

is a positive quadratic functional. Positive quadratic functionals on an Abelian group possess some properties of norms on unitary spaces. We have

Theorem 3. *Let $n : X \to R$ be a positive quadratic functional on an Abelian group X and $m(x, y)$ a biadditive functional defined by* (6).

I. *For any system of elements $x_1, \ldots, x_k \in X$ the matrix*

$$\Gamma(x_1, \ldots, x_k) = \begin{bmatrix} m(x_1, x_1) & m(x_1, x_2) & \ldots & m(x_1, x_k) \\ m(x_2, x_1) & m(x_2, x_2) & \ldots & m(x_2, x_k) \\ \cdot & \cdot & \cdot & \cdot \\ m(x_k, x_1) & m(x_k, x_2) & \ldots & m(x_k, x_k) \end{bmatrix} \tag{20}$$

is positive semidefinite.

II. *A mapping $x \to |x| = [n(x)]^{1/2}$ possesses the following properties*

$$|m(x, y)| \leq |x| \cdot |y| \tag{21}$$

and

$$|x + y| \leq |x| + |y|, \tag{22}$$

for all $x, y \in X$.

III. *The set $X_0 = \{x_0 \mid n(x_0) = 0, x_0 \in X\}$ is a subgroup of X and the functional $\hat{n} : X/X_0 \to R$ defined by*

$$\hat{n}(x + X_0) = n(x)$$

is a positive quadratic functional on X/X_0 with the property that $\hat{n}(x + X_0) = 0$ implies $x \in X_0$.

Proof. If p is any integer then $n(px) = p^2 n(x)$ holds for every $x \in X$. Hence, by using (2), we get

$$\frac{1}{2}[n(px + y) + n(px - y)] = n(px) + n(y) = p^2 n(x) + n(y)$$

and

$$\frac{1}{2}[n(px + y) - n(px - y)] = 2m(px, y) = 2p\, m(x, y).$$

If we add these two relations, we find

$$n(px + y) = p^2 n(x) + 2p\, m(x, y) + n(y), \tag{23}$$

for all $x, y \in X$ and any integer p.

If $x_1, \ldots, x_k$ are elements of X and $p_1, \ldots, p_k$ are integers, then by setting $p = p_1$, $x = x_1$ and

$$y = \sum_{i=2}^{k} p_i x_i$$

in (23) we get

$$n\left(\sum_{i=1}^{k} p_i x_i\right) = p_1^2 n(x_1) + 2 \sum_{i=2}^{k} p_1 p_i m(x_1, x_i) + n\left(\sum_{i=2}^{k} p_i x_i\right),$$

which leads to

$$n\left(\sum_{i=1}^{k} p_i x_i\right) = \sum_{i,j=1}^{k} p_i p_j m(x_i, x_j). \tag{24}$$

Now, suppose that $r_1, \ldots, r_k$ are rational numbers. Writing $r_i = p_i / p$ with integers p_i and a natural number p, we have

$$\sum_{i,j=1}^{k} r_i r_j m(x_i, x_j) = \frac{1}{p^2} \sum_{i,j=1}^{k} p_i p_j m(x_i, x_j).$$

Since $n(x) \geq 0$, for every $x \in X$, we find, by using (24),

$$\sum_{i,j=1}^{k} r_i r_j m(x_i, x_j) \geq 0. \tag{25}$$

By the continuity we derive from (25)

$$\sum_{i,j=1}^{k} t_i t_j m(x_i, x_j) \geq 0, \tag{26}$$

for all real numbers $t_1, \ldots, t_k$. Thus, the matrix (20) is positive semidefinite.

Since the matrix $\Gamma(x, y)$ is positive semidefinite, we have

$$\det \Gamma(x, y) = n(x)\, n(y) - m(x, y)\, m(y, x) \geq 0,$$

i. e. (21) holds for all $x, y \in X$. Now,

$$n(x + y) = n(x) + n(y) + 2\, m(x, y) \geq 0,$$

together with (21), implies (22).

In order to prove the third part of Theorem 3, we note that $x_0 \in X_0$, together with (21), implies $m(x_0, y) = 0$, for any $y \in X$. Thus, $x_0 \in X$ and $y \in X$ imply

$$n(x_0 + y) = n(x_0 - y). \tag{27}$$

Furthermore, $x_0, y_0 \in X_0$ and $n(z) \geq 0$, for all $z \in X$ imply

$$n(x_0 + y_0) + n(x_0 - y_0) = 0,$$

from which it follows that $n(x_0 + y_0) = n(x_0 - y_0) = 0$. Thus, X_0 is a subgroup of X.

If $x, x' \in X$ are such that $x_0 = x' - x \in X_0$, then

$$n(x') = n(x_0 + x) = n(x_0) + n(x) + 2m(x_0, x) = n(x)$$

implies that the functional $\hat{n}(x + X_0) = n(x)$ is well-defined. That it is positive definite and that $n(x + X_0) = 0 \Longleftrightarrow x \in X_0$ is obvious. This concludes the proof.

As we have remarked, a functional $x \to n(x) = [g(x)]^2$ is a positive quadratic functional for any additive functional $g: X \to R$. The following theorem gives necessary and sufficient conditions in order that a positive quadratic functional be of this form.

Theorem 4. *A positive quadratic functional $n: X \to R$ is of the form*

$$n(x) = [g(x)]^2,$$

where $g: X \to R$ is an additive functional, if and only if n satisfies the following subsidiary condition

$$[n(x+y) - n(x-y)]^2 = 16\, n(x)\, n(y), \tag{28}$$

i. e. if and only if det $\Gamma(x, y) = 0$, *for all* $x, y \in X$.

Proof. Let $n: X \to R$ be a positive quadratic functional and let it satisfy (28), i. e. let

$$[m(x, y)]^2 = n(x)\, n(y), \tag{29}$$

for all $x, y \in X$. If $n = 0$, then we can take $g = 0$ in order to satisfy (19). If $n \neq 0$, then a $y \in X$ can be found such that $n(y) > 0$. From this fact and (29) we conclude that

$$n(x) = \left[\frac{1}{\sqrt{n(y)}}\, m(x, y)\right]^2.$$

Thus, n is of the form (19) with

$$g(x) = \frac{1}{\sqrt{n(y)}}\, m(x, y),$$

which is an additive functional in x.

Since $x \to [g(x)]^2$ is a positive quadratic functional whenever $g: X \to R$ is additive, Theorem 4 is proved.

Corollary 1. *If $n: X \to R$ is a positive quadratic functional and $n(x) = g_1(x)\, g_2(x)$, where g_1 and g_2 are additive functionals, then g_1 and g_2 are proportional.*

Proof. Using $n(x) = g_1(x)\, g_2(x)$, we get

$$\det \Gamma(x, y) = -\frac{1}{4}[g_1(x)\, g_2(y) - g_1(y)\, g_2(x)]^2,$$

which, together with Theorem 3, leads to $g_1(x)\, g_2(y) = g_1(y)\, g_2(x)$, from which Corollary 1 follows.

In connection with the subsidiary condition (28) which appears in Theorem 4 we have the following

Theorem 5. *Suppose that $Q: R \to R$ is such a function that*

$$\det \begin{bmatrix} 4\,Q(x) & Q(x+y) - Q(x-y) \\ Q(x+y) - Q(x-y) & 4\,Q(y) \end{bmatrix} = 0, \text{ i. e.}$$

$$[Q(x+y) - Q(x-y)]^2 = 16\, Q(x)\, Q(y) \tag{30}$$

holds for all $x, y \leqslant R$. Then

$$Q(r\,x) = r^2\, Q(x) \tag{31}$$

holds for any $x \leqslant R$ and every rational number r.

If Q is a continuous function, then $Q(x) = x^2\, Q(1)$ holds for any $x \leqslant R$.

Proof. If in (30) we set $x = y = 0$, we get $[Q(0)]^2 = 0$, i. e. $Q(0) = 0$. Now, by setting $x = 0$ in (30), we find

$$[Q(y) - Q(-y)]^2 = 0,$$

i. e. Q is an even function. From (30) we see that Q is of constant sign on R.

Suppose that $Q(x) \geq 0$, for every $x \leqslant R$. From (30), for $x = y$, we get

$$[Q(2y)]^2 = 16\,[Q(y)]^2,$$

which together with $Q \geq 0$ leads to

$$Q(2y) = 4\, Q(y),$$

for any $y \leqslant R$.

Suppose that y is such that $Q(y) \neq 0$. If we set $x = 3y$ in (30) and if we use $Q(4y) = 4\, Q(2y) = 16\, Q(y)$, we get

$$[12\, Q(y)]^2 = 16\, Q(3y)\, Q(y),$$

from which

$$Q(3y) = 9\, Q(y)$$

follows.

Now supoose that

$$Q(k\,y) = k^2\, Q(y) \tag{32}$$

holds for all natural numbers $k \leq p$ $(p \geq 3)$. Let us prove that (32) is valid also for $k = p + 1$. If $p + 1$ is an even number, i. e. $p + 1 = 2q$ with a natural number q, then

$$Q[(p+1)\,y] = 4\, Q(q\,y) = 4\, q^2\, Q(y) = (p+1)^2\, Q(y)$$

is a consequence of $q \leq n$ and the inductive hypotheses (32).

If $p+1$ is an odd number, i. e. $p=2q$ with a natural number q, then $q+1 \leq p$, so that

$$Q[(q+1)y] = (q+1)^2 Q(y),$$

and

$$Q[(p+2)y] = Q[2(q+1)y] = 4Q[(q+1)y]$$

lead to

$$Q[(p+2)y] = (p+2)^2 Q(y). \tag{33}$$

If in (30) we set $x=(p+1)y$, we get by using (32) and (33)

$$[(p+2)^2 - p^2]^2 [Q(y)]^2 = 16 Q[(p+1)y] Q(y), \text{ i. e.}$$
$$Q[(p+1)y] = (p+1)^2 Q(y).$$

Thus, $Q(ky) = k^2 Q(y)$ holds for all integers k. Now, for any integer k, $k \neq 0$, we have

$$Q(y) = Q(k \frac{1}{k} y) = k^2 Q(\frac{1}{k} y),$$

from which

$$Q(\frac{1}{k} y) = \frac{1}{k^2} Q(y) \tag{34}$$

follows. Finally (34) and (32) imply

$$Q(ry) = r^2 Q(y), \tag{35}$$

for any rational r.

If $Q(y)=0$, for some $y \in X$, then $Q(ry)=0$, for any rational r, so that (35) holds in this case too. Indeed, otherwise one could find a rational number $r_0 \neq 0$ such that $Q(r_0 y) \neq 0$. This leads in the same way to $Q(r \cdot r_0 y) = r^2 Q(r_0 y)$, for any rational number r. Setting $r = 1/r_0$ we get

$$Q(y) = \frac{1}{r_0^2} Q(r_0 y) \neq 0$$

contrary to the assumption $Q(y)=0$. Thus, $Q(rx) = r^2 Q(x)$ holds for any $x \in X$ and for every rational number r.

We end this paper with a theorem about quadratic functionals on real partially ordered vector spaces. We have:

Theorem 6. *Suppose that X is a partially ordered vector space over reals. If $n: X \to R$ is a quadratic functional with the property that $x \leq y$ implies $n(x) \leq n(y)$, then*

$$m(x,y) = \frac{1}{4}[n(x+y) - n(x-y)]$$

is a bilinear functional on X.

Proof. Since $n(0)=0$, we conclude that $x \geq 0$ implies $n(x) \geq 0$. Now, set $n_x(t) = n(tx)$ for real number t and $x \geq 0$. Since

$t > s > 0$ $(t, s \in R)$ implies $tx \geq sx \geq 0$, we find that $0 < s < t$ *implies*

$$0 \leq n_x(s) \leq n_x(t),$$

i. e. the function $t \to n_x(t)$ is monotonic on $(0, \infty)$. Since the function $t \to n_x(t)$ satisfies the functional equation

$$n_x(t+s) + n_x(t-s) = 2n_x(t) + 2n_x(s)$$

and since it is monotonic, we find see (see [2]) that $n_x(t) = t^2 n_x(1)$, i. e. that

$$n(tx) = t^2 n(x), \tag{36}$$

for any real number t. Thus,

$$n(tx + y) = t^2 n(x) + 2m(tx, y) + n(y) \tag{37}$$

holds for all $x, y \geq 0$ and t (see [1], p. 26). On the other hand, $z \geq 0$ implies $n(z) \geq 0$ and $x, y \geq 0$, $t > 0$ implies $tx + y \geq 0$. Hence, by using (37), we get

$$2m(tx, y) \geq -t^2 n(x) - n(y),$$

which implies

$$\inf m(tx, y) > -\infty \quad (0 \leq t \leq 1)$$

so that the additive function $t \to m(tx, y)$ is bounded below on an interval. But then it is continuous and $m(tx, y) = t\, m(x, y)$. Thus,

$$m(tx, y) = t\, m(x, y) \tag{38}$$

holds for all $t \in R$ and $x, y \geq 0$.

Now any $y \in X$ can be written in the form

$$y = y_+ - y_- \quad (y_+, y_- \geq 0).$$

By the additive property of m we have, for $x \geq 0$,

$$m(tx, y) = m(tx, y_+ - y_-) = m(tx, y_+) - m(tx, y_-) =$$
$$= t\, m(x, y_+) - t\, m(x, y_-) = t\, m(x, y).$$

Hence, (38) holds for all $t \in R$, all $y \in X$ and $x \geq 0$.

Writing an arbitrary $x \in X$ in the form $x = x_+ - x_-$ we find in a similar way that (38) holds for all $x, y \in X$ and $t \in R$.

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KVADRATNI I SESKVILINEARNI FUNKCIONALI

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Sadržaj

U članku su dokazani ovi teoremi:

Teorem 1. *Neka je X kompleksan vektorski prostor, i n kompleksnoznačna funkcija takva da vrijedi*

(*i*) $n(x+y)+n(x-y)=2n(x)+2n(y)$ *i*

(*ii*) $n(\lambda x)=|\lambda|^2 n(x)$

za sve $x, y \in X$ *i sve kompleksne brojeve* λ. *Tada je funkcional*

$$B(x,y)=\frac{1}{4}[n(x+y)-n(x-y)]+\frac{i}{4}[n(x+iy)-n(x-iy)]$$

linearan u x i antilinearan u y (tj. $B(x,y)$ *je seskvilinearan funkcional na X) i* $B(x,x)=n(x)$.

Teorem 2. *Neka je X vektorski prostor nad tijelom kvaterniona i n realan funkcional definiran na X sa svojstvima* (*i*), (*ii*) *iz teorema* 1 *(pri tome je* $|\lambda|^2=\lambda\bar{\lambda}$ *i* $\bar{\lambda}$ *je konjugirani kvaternion kvaterniona* λ*).*

Tada je

$B(x,y)=m(x,y)+i\,m(x,iy)+j\,m(x,jy)+k\,m(x,ky)$ *seskvilinearan funkcional i* $n(x)=B(x,x)$. *Pri tome je*

$$m(x,y)=\frac{1}{4}[n(x+y)-n(x-y)].$$

Teorem 3. *Neka je R skup realnih brojeva, X Abelova grupa i neka* $n: X\to R$ *zadovoljava uvjete* (2) *i* (17) *za sve* $x, y \in X$. *Neka je nadalje* $m(x,y)$ *definirano sa* (6).

I. *Za bilo koji sistem elemenata* $x_1, \ldots, x_k \in X$ *matrica* (20) *je pozitivno semidefinitna.*

II. *Preslikavanje* $x\to|x|=[n(x)]^{1/2}$ *zadovoljava uvjete* (21) *i* (22).

III. *Skup* $X_0=\{x_0 \mid n(x_0)=0, x_0\in X\}$ *je podgrupa od X i funkcional* $\hat{n}: X/X_0\to R$ *definiran s* $\hat{n}(x+X_0)=n(x)$ *zadovoljava uvjete* (2) *i* (17), *i* $\hat{n}(x+X_0)=0$ *povlači* $x\in X_0$.

Teorem 4. *Ako su n, X i R isti kao u teoremu 3, onda je* $n(x)=[g(x)]^2$, *pri čemu* $g: X\to R$ *zadovoljava* (18) *za sve* $x, y\in X$, *onda i samo onda ako funkcional n zadovoljava uslov* (28) *za sve* $x, y \in X$.

T e o r e m 5. *Neka je R skup realnih brojeva i* $Q : R \to R$ *takva funkcija da vrijedi* (30) *za sve* $x, y \in R$. *Tada je* $Q(rx) = r^2 Q(x)$ *za svako* $x \in R$ *i svaki racionalni broj* r. *Ako je* Q *neprekidna funkcija onda je* $Q(x) = x^2 Q(1)$ *za svako* $x \in Q$.

T e o r e m 6. *Neka je R skup realnih brojeva i X parcijalno uređen vektorski prostor nad R. Ako funkcional* $n : X \to R$ *zadovoljava uslov* (2) *za sve* $x, y \in X$ *i ako* $x \leq y$ *povlači* $n(x) \leq n(y)$, *tada je* (6) *bilinearan funkcional na X.*

Teoremima 1 i 2 dan je pozitivan odgovor na jedan problem profesora I. H a l p e r i n a iz 1963. na koga je negativan odgovor u slučaju realnog prostora dan u [1].

(Primljeno 16. III 1964.; dopunjeno 22. I 1965.)