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PARABOLIC INDUCTION FROM TWO SEGMENTS, LINKED UNDER CONTRAGREDIENT, WITH A ONE HALF CUSPIDAL REDUCIBILITY, A SPECIAL CASE

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Abstract. In this paper, we determine the composition series of the induced representation $\delta([\nu^{-a}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma$ where $a, b, c \in \mathbb{Z} + \frac{1}{2}$ such that $\frac{1}{2} \leq a < b < c$, ρ is an irreducible cuspidal unitary representation of a general linear group and σ is an irreducible cuspidal representation of a classical group such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces.

1. Introduction

The problem of determining the composition series of induced representations is important for the representation theory. Namely, classes of representations of certain interest, like irreducible, unitary, tempered representations or discrete series, are often placed inside parabolically induced representations, raising a question of their position. Despite the interest, complete description of the composition factors of induced representations is known only in some special cases, such as [2], [7], [9], [14] and [24], and for some low-rank groups.

This paper is a continuation of the effort from [3] and [4] to study classes of parabolically induced representations similar to ones appearing in the Mœglin-Tadić classification of discrete series ([11], [13]). To explain this we introduce some notation. Fix a local non-archimedean field F of characteristic different than two. Let ρ be an irreducible cuspidal unitary representation of some $GL(m, F)$, and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{\geq 0}$. By Zelevinsky classification, the set $\Delta = [\nu^{x}\rho, \nu^{y}\rho] = \{\nu^{x}\rho, ..., \nu^{y}\rho\}$ is called a segment. We have a

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unique irreducible subrepresentation

$$\delta(\Delta)=\delta([\nu^x\rho,\nu^y\rho])\hookrightarrow\nu^y\rho\times\cdots\times\nu^x\rho,$$

of the parabolically induced representation. If $\Delta\subseteq\Delta'$, then $\delta(\Delta)\times\delta(\Delta')\cong\delta(\Delta')\times\delta(\Delta)$ is irreducible. Set $e(\Delta)=(x+y)/2$. Given a sequence of segments $\Delta_1,...,\Delta_k$, such that $e(\Delta_1)\geq\cdots\geq e(\Delta_k)>0$ and an irreducible tempered representation τ, of a symplectic or (full) orthogonal group, we have a unique irreducible quotient, called the Langlands quotient,

$$\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\tau\twoheadrightarrow L(\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\tau),$$

of the parabolically induced representation. Let ρ be as above, and σ an irreducible cuspidal representation of a symplectic or (full) orthogonal group such that $\nu^{\frac{1}{2}}\rho\rtimes\sigma$ reduces. Let $a,b,c\in\mathbb{Z}+\frac{1}{2}$ such that $\frac{1}{2}\leq a<b<c$. In [3], we determined composition series of induced representation

$$\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma.$$

The search for composition factors relied on decomposing kernels of intertwining operators using results of [14]. This approach held in [4], where we considered an arbitrary number of segments Δ_i. There, one of conditions was on segments involved: induced representations $\delta(\Delta_i)\times\delta(\Delta_j)$ and $\delta(\Delta_i)\times\widetilde{\delta(\Delta_j)}$, $i\neq j$, are to be irreducible, where $\tilde{}$ stands for the contragredient. As a result, all discrete series there appeared as subrepresentations. Neither this property nor the condition holds in the present paper, where we consider composition series of induced representation

$$\delta([\nu^{-a}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma,$$

even though all irreducible subquotients, including discrete series, are preserved. Moreover, decomposing kernels of intertwining operators requires more basic tools developed in [14], complicating our search.

To describe our results, we introduce some discrete series, appearing as only irreducible subrepresentations in the following induced representations (for more details see Section 3):

$$\sigma_a\hookrightarrow\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma,\text{ and similarly for }\sigma_b\text{ and }\sigma_c.\text{ Further}$$
$$\sigma_{b,c}^+\hookrightarrow\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_c,\quad\sigma_{b,c}^++\sigma_{b,c}^-\hookrightarrow\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma,$$

and similarly for $\sigma_{a,c}^\pm$. Finally $\sigma_{b,c,a}^\pm\hookrightarrow\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_{b,c}^\pm$. Now we have

THEOREM 1.1. *Let* $\psi = \delta([\nu^{-a}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma$ *and define representations*

$$\begin{aligned}
W_1 =&\sigma^+_{b,c,a} + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^-_{b,c}),\\
W_2 =&L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^+_{b,c}) + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c)+\\
&L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^-_{a,c}) + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a),\\
W_3 =&L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c}) + L(\delta([\nu^{-a}\rho, \nu^c\rho]) \rtimes \sigma_b)+\\
&\sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma),\\
W_4 =&L(\psi).
\end{aligned}$$

Then there exists a sequence $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \psi$*, such that*

$$V_i/V_{i-1} \cong W_i, \quad i = 1, \ldots, 4.$$

Further, W_1 *is chosen to be the largest possible, then* W_2*, and so on.*

Now we describe the content of the paper.
After Preliminaries, we fix the notation in Section 3 and collect some reducibility results. Intertwining operators and an approach to decompose the induced representation are considered in Section 4. In Section 5, we determine the occuring discrete series. The remaining non-tempered candidates, not provided in Section 4, are listed in Section 6. Their occurrence is confirmed in Sections 7-9. Composition factors are described in Section 10. To determine composition series, we decompose kernels of intertwining operators in Sections 11-13, and provide the main result in Section 14.

2. PRELIMINARIES

Let F be a local non-archimedean field of characteristic different than two. As in [13], fix a tower of symplectic or orthogonal non-degenerate F vector spaces V_n, $n \geq 0$ where n is the Witt index. We denote by G_n the group of isometries of V_n. It has split rank n. Also, we fix the set of standard parabolic subgroups in the usual way. Standard parabolic proper subgroups of G_n are in bijection with the set of ordered partitions of positive integers $m \leq n$:

$$\begin{aligned}
&\{s = (n_1, \ldots, n_k) \mid n_1 + \cdots + n_k = m, k > 0\} \longleftrightarrow P_s,\\
&P_s = M_s N_s, \quad \text{Levi factorization with } M_s \text{ Levi factor},\\
&M_s \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n-m}.
\end{aligned}$$

By Alg G_n we denote smooth representations of G_n, Irr G_n irreducible representations, and subscript *f.l.* means finite length, *u* unitary, and *cusp* cuspidal. Also denote Alg $G = \cup_{n\geq 0}$Alg G_n, and so on. We use a similar notation for $GL(n,F)$. For $\delta_i \in \text{Alg}\, GL(n_i,F)$, $i=1,...,k$ and $\tau \in \text{Alg}\, G_{n-m}$, let $\pi = \delta_1 \otimes \cdots \otimes \delta_k \otimes \tau \in \text{Alg}\, M_s$ and

$$\delta_1 \times \cdots \times \delta_k \rtimes \tau = \text{Ind}_{M_s}^{G_n}(\pi)$$

be the representation induced from π using normalized parabolic induction. If $\sigma \in \text{Alg}\, G_n$ we denote by $\mathrm{r}_s(\sigma) = \mathrm{r}_{M_s}(\sigma) = \text{Jacq}_{M_s}^{G_n}(\sigma)$ the normalized Jacquet module of σ. We have the Frobenius reciprocity

$$\text{Hom}_{G_n}(\sigma, \text{Ind}_{M_s}^{G_n}(\pi)) \cong \text{Hom}_{M_s}(\text{Jacq}_{M_s}^{G_n}(\sigma), \pi).$$

Let $\rho \in \text{Irr}_{u,cusp} GL$ and $x,y \in \mathbb{R}$, such that $y-x+1 \in \mathbb{Z}_{\geq 0}$. The set

$$\Delta = [\nu^x \rho, \nu^y \rho] = \{\nu^x \rho, ..., \nu^y \rho\}$$

is called a segment. We have a unique irreducible subrepresentation

$$\delta([\nu^x \rho, \nu^y \rho]) \hookrightarrow \nu^y \rho \times \cdots \times \nu^x \rho,$$

of the induced representation, and it is essentially square integrable. We also denote $e([\nu^x\rho, \nu^y\rho]) = e(\delta([\nu^x\rho, \nu^y\rho]) = \frac{x+y}{2}$. For $y-x+1 \in \mathbb{Z}_{<0}$ define $[\nu^x\rho, \nu^y\rho] = \emptyset$ and $\delta(\emptyset)$ is the irreducible representation of the trivial group. Let $\widetilde{\Delta} = [\nu^{-y}\widetilde{\rho}, \nu^{-x}\widetilde{\rho}]$ where $\widetilde{\rho}$ denotes the contragredient of ρ. We have $\delta(\Delta)\widetilde{\ } = \delta(\widetilde{\Delta})$. By [24] if $\delta \in \text{Irr}\, GL$ is essentially square integrable, there exists a segment Δ such that $\delta = \delta(\Delta)$. Let $\delta_i = \delta(\Delta_i), e_i = e(\delta_i), i = 1,2$. We have

$$\delta_1 \times \delta_2 \text{ reduces} \Leftrightarrow \Delta_1 \cup \Delta_2 \text{ is a segment and } \Delta_1 \not\subseteq \Delta_2, \Delta_2 \not\subseteq \Delta_1.$$

In that case, if $e_1 \geq e_2$, the induced representation has a unique irreducible quotient, called Langlands quotient, and a unique irreducible subrepresentation. They swap positions in $\delta_2 \times \delta_1$ and make composition factors. We have an exact sequence.

$$\delta(\Delta_1 \cup \Delta_2) \times \delta(\Delta_1 \cap \Delta_2) \hookrightarrow \delta_1 \times \delta_2 \twoheadrightarrow L(\delta_1 \times \delta_2) = L(\delta_1, \delta_2).$$

Given a sequence $\delta_i = \delta(\Delta_i)$, $i = 1, \ldots, k$ such that $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ and $\tau \in \text{Irr}\, G$, tempered, the Langlands quotient is a unique irreducible quotient:

$$\delta_1 \times \cdots \times \delta_k \rtimes \tau \twoheadrightarrow L(\delta_1 \times \cdots \times \delta_k \rtimes \tau),$$

and it appears with multiplicity one in the induced representation. It is also a unique irreducible subrepresentation of

$$\widetilde{\delta_1} \times \cdots \times \widetilde{\delta_k} \rtimes \tau \cong (\delta_1 \times \cdots \times \delta_k \rtimes \tau)\widetilde{\ }.$$

Permuting δ_i-s and possibly taking contragredients does not change composition factors. Every irreducible representation of G_n can be written as a Langlands quotient.

If σ is a discrete series representation of G_n then by the Mœglin-Tadić, now unconditional, classification ([11],[13]), it is described by an admissible triple

$$(\mathrm{Jord}, \sigma_{cusp}, \epsilon).$$

Here Jord is a set of pairs (a, ρ) where $\widetilde{\rho} \cong \rho \in \mathrm{Irr}_{u,cusp}GL$ and $a \in \mathbb{Z}_{>0}$, of parity depending on ρ, such that $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$ is irreducible, but there exists an integer $a' > a$, of the parity same as a such that the induced representation reduces when we replace a with a'. We write

$$\mathrm{Jord}_\rho = \{a : (a, \rho) \in \mathrm{Jord}\}$$

and for $a \in \mathrm{Jord}_\rho$ let a_- be the largest element of Jord_ρ strictly less than a, if such exists. Next, there exists a unique, up to an isomorphism, $\sigma_{cusp} \in \mathrm{Irr}_{cusp}G$, such that there exists $\pi \in \mathrm{Irr}GL$ and $\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}$. It is called the partial cuspidal support of σ. Finally, ϵ is a function from a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ into $\{\pm 1\}$. Assume $(a, \rho) \in \mathrm{Jord}$ and a is even. Then $\epsilon(a, \rho)$ is defined, and if $a = \min(\mathrm{Jord}_\rho)$

$$\epsilon(a, \rho) = 1 \Leftrightarrow \exists \pi' \in \mathrm{Irr}G, \quad \sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi',$$

while if a_- exists

$$\epsilon(a, \rho)\epsilon(a_-, \rho)^{-1} = 1 \Leftrightarrow \exists \pi'' \in \mathrm{Irr}G, \quad \sigma \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi''.$$

Now we recall the Tadić formula for computing Jacquet modules of induced representations. Let $R(G_n)$ be the Grothendieck group of the category of smooth representations of G_n of finite length. It is the free Abelian group generated by classes of irreducible representations of G_n. If σ is a smooth finite length representation of G_n denote by s.s.(σ) the semisimplification of σ, that is the sum of classes of composition series of σ. Put $R(G) = \oplus_{n \geq 0} R(G_n)$. Let $R_0^+(G)$ be a $\mathbb{Z}_{\geq 0}$ subspan of classes of irreducible representations. For $\pi_1, \pi_2 \in R(G)$ we define $\pi_1 \leq \pi_2$ if $\pi_2 - \pi_1 \in R_0^+(G)$. Similarly define $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$. We have the map $\mu^* : R(G) \to R(GL) \otimes R(G)$ defined by

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} \mathrm{s.s.}(r_{(k)}(\sigma)), \ \sigma \in R(G_n).$$

The following result derives from Theorems 5.4 and 6.5 of [20], see also Section 1. in [13]. They are based on Geometrical Lemma (2.11 of [1]).

Theorem 2.1. *Let $\sigma \in Alg_{f.l.}G$, and $[\nu^x\rho, \nu^y\rho] \neq \emptyset$ a segment. Then*

$$\mu^*(\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma) = \sum_{\delta' \otimes \sigma' \leq \mu^*(\sigma)} \sum_{i=0}^{y-x+1} \sum_{j=0}^{i} \delta([\nu^{i-y}\widetilde{\rho}, \nu^{-x}\widetilde{\rho}]) \times \delta([\nu^{y+1-j}\rho, \nu^y\rho]) \times \delta' \otimes \delta([\nu^{y+1-i}\rho, \nu^{y-j}\rho]) \rtimes \sigma' \tag{2.1}$$

where $\delta' \otimes \sigma'$ denotes an irreducible subquotient in the appropriate Jacquet module.

Now we provide results of [10], about Jacquet modules of some irreducible representations. Consider induced representation

$$\pi = \delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma$$

where $\sigma \in \mathrm{Irr}_{cusp}G$, $\widetilde{\rho} \cong \rho \in \mathrm{Irr}_{u,cusp}GL$, such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces, and $a, c \in \mathbb{Z} + \frac{1}{2}$, such that $c - a \geq 0$. We define terms $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma)$, $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{-}; \sigma)$ and $L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma)$. Each of them is either an irreducible representation or zero.

If $\frac{1}{2} < -a$, then π is irreducible and $L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma) = \pi$, while other terms are zero.

If $-a \leq \frac{1}{2}$ then π reduces. We denote by $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma)$ a unique irreducible subquotient that has in its minimal standard Jacquet module at least one irreducible subquotient whose all exponents are non-negative.

If $-a = \frac{1}{2}$, then π is of length two, $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma)$ is a discrete series subrepresentation and $L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma)$ is the Langlands quotient of π. The remaining term is zero.

If $-a \leq -\frac{1}{2}$, and $a = c$, then π is a direct sum of two tempered representations, $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma)$ and $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{-}; \sigma)$. The remaining term is zero.

If $-a \leq -\frac{1}{2}$, and $a \neq c$, then π is of length three. It has two discrete series representations, $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma)$ and $\delta([\nu^{-a}\rho, \nu^{c}\rho]_{-}; \sigma)$, and $L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma)$ is the Langlands quotient.

We have in $R(G)$

$$\begin{aligned} \delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma = & \delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma) + \delta([\nu^{-a}\rho, \nu^{c}\rho]_{-}; \sigma) + \\ & L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma). \end{aligned} \tag{2.2}$$

where the right hand side equals to

$$\begin{aligned} \frac{1}{2} < -a: &\quad L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma), \\ \frac{1}{2} = -a: &\quad \delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma) + L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma), \\ c = a, -a \leq -\frac{1}{2}: &\quad \delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma) + \delta([\nu^{-a}\rho, \nu^{c}\rho]_{-}; \sigma), \\ c \neq a, -a \leq -\frac{1}{2}: &\quad \delta([\nu^{-a}\rho, \nu^{c}\rho]_{+}; \sigma) + \delta([\nu^{-a}\rho, \nu^{c}\rho]_{-}; \sigma) + L(\delta([\nu^{-a}\rho, \nu^{c}\rho]); \sigma). \end{aligned}$$

Now we have
(2.3)

$$\mu^*(\delta([\nu^{-a}\rho,\nu^c\rho]_\pm;\sigma)) = \sum_{i=-a-1}^{\pm\frac{1}{2}-1} \delta([\nu^{-i}\rho,\nu^a\rho]) \times \delta([\nu^{i+1}\rho,\nu^c\rho]) \otimes \sigma +$$

$$\sum_{i=-a-1}^{a}\sum_{j=i+1}^{c} \delta([\nu^{-i}\rho,\nu^a\rho]) \times \delta([\nu^{j+1}\rho,\nu^c\rho]) \otimes \delta([\nu^{i+1}\rho,\nu^j\rho]_\pm;\sigma) +$$

$$\sum_{\substack{-a-1\le i\le a\\ i+j<-1}}\ \sum_{i+1\le j\le a} \delta([\nu^{-i}\rho,\nu^a\rho]) \times \delta([\nu^{j+1}\rho,\nu^c\rho]) \otimes L(\delta([\nu^{i+1}\rho,\nu^j\rho]);\sigma).$$

If we write $\delta([\nu^{\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho]_+;\sigma)$ for σ, we have

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^c\rho]_+;\sigma)) = \sum_{j=-\frac{1}{2}}^{c} \delta([\nu^{j+1}\rho,\nu^c\rho]) \otimes \delta([\nu^{\frac{1}{2}}\rho,\nu^j\rho]_+;\sigma). \tag{2.4}$$

And for $a < \frac{1}{2}$ or $\frac{1}{2} \le a < c$ we have
(2.5)

$$\mu^*(L(\delta([\nu^{-a}\rho,\nu^c\rho]);\sigma)) = \sum_{i=\frac{1}{2}}^{c} L(\delta([\nu^{-i}\rho,\nu^a\rho]),\delta([\nu^{i+1}\rho,\nu^c\rho])) \otimes \sigma +$$

$$\sum_{\substack{-a-1\le i\le c\\ 0\le i+j}}\ \sum_{i+1\le j\le c} L(\delta([\nu^{-i}\rho,\nu^a\rho]),\delta([\nu^{j+1}\rho,\nu^c\rho])) \otimes L(\delta([\nu^{i+1}\rho,\nu^j\rho]);\sigma).$$

3. Notation and basic reducibilities

In this section, we fix the notation and prepare some reducibility results. Let ρ be an irreducible unitary cuspidal representation of $GL(m_\rho,F)$ and σ an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces. By Proposition 2.4 of [17] ρ is self-dual. We consider

$$\frac{1}{2} \le a,b,c \in \mathbb{Z}+\frac{1}{2},$$

that need not be fixed, but when appearing together in a formula, we have, depending on which appears, $a<b<c$. We denote the representation we want to decompose

$$\psi = \delta([\nu^{-a}\rho,\nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho]) \rtimes \sigma.$$

Further, we shorten some notations from (2.2):

$$\sigma_a = \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]_+;\sigma),\quad \sigma_{b,c}^- = \delta([\nu^{-b}\rho,\nu^c\rho]_-;\sigma),\quad \sigma_{b,c}^+ = \delta([\nu^{-b}\rho,\nu^c\rho]_+;\sigma).$$

The following result is Theorem 2.3 from [14].

THEOREM 3.1. *With discrete series being subrepresentations, we have in $R(G)$*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \sigma_a + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma),$$
$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma = \sigma^+_{b,c} + \sigma^-_{b,c} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma).$$

Here

$$Jord(\sigma_a) = \{(2a+1, \rho)\} \cup Jord(\sigma),$$
$$Jord(\sigma^+_{b,c}) = Jord(\sigma^-_{b,c}) = \{(2b+1, \rho), (2c+1, \rho)\} \cup Jord(\sigma).$$

Further, $\epsilon_{\sigma_a}, \epsilon_{\sigma^+_{b,c}}$, *and* $\epsilon_{\sigma^-_{b,c}}$ *extend* ϵ_σ, *such that* $\epsilon_{\sigma_a}(2a+1, \rho) = 1$, *and* $\epsilon_{\sigma^+_{b,c}}(2b+1, \rho) = \epsilon_{\sigma^+_{b,c}}(2c+1, \rho) = 1, \epsilon_{\sigma^-_{b,c}}(2b+1, \rho) = \epsilon_{\sigma^-_{b,c}}(2c+1, \rho) = -1$.

The next proposition follows from Theorem 2.1 of [14].

PROPOSITION 3.2. *We use* $\sigma^+_{b,c,a} = \sigma^+_{a,b,c}$, $\sigma^-_{b,c,a}$ *and* $\sigma^-_{a,b,c}$ *to denote non-isomorphic discrete series, such that in $R(G)$ we have*

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a = \sigma^+_{b,c,a} + \sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a) \quad \text{and}$$
$$\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c = \sigma^+_{b,c,a} + \sigma^-_{a,b,c} + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c).$$

These discrete series appear as subrepresentations in induced representations. Also

$$Jord(\sigma^+_{b,c,a}) = Jord(\sigma^-_{b,c,a}) = Jord(\sigma^-_{a,b,c}) = \{(2a+1, \rho), (2b+1, \rho), (2c+1, \rho)\} \cup Jord(\sigma)$$

and $\epsilon_{\sigma^+_{b,c,a}}$, $\epsilon_{\sigma^-_{b,c,a}}$ *and* $\epsilon_{\sigma^-_{a,b,c}}$ *extend* ϵ_σ *such that*

$$\begin{aligned}
&\epsilon_{\sigma^+_{b,c,a}}(2a+1, \rho) = 1, && \epsilon_{\sigma^+_{b,c,a}}(2b+1, \rho) = 1, && \epsilon_{\sigma^+_{b,c,a}}(2c+1, \rho) = 1,\\
&\epsilon_{\sigma^-_{b,c,a}}(2a+1, \rho) = 1, && \epsilon_{\sigma^-_{b,c,a}}(2b+1, \rho) = -1, && \epsilon_{\sigma^-_{b,c,a}}(2c+1, \rho) = -1,\\
&\epsilon_{\sigma^-_{a,b,c}}(2a+1, \rho) = -1, && \epsilon_{\sigma^-_{a,b,c}}(2b+1, \rho) = -1, && \epsilon_{\sigma^-_{a,b,c}}(2c+1, \rho) = 1.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\mu^*(\sigma^+_{b,c,a}) \geq \delta([\nu^{-a}\rho, \nu^b\rho]) \otimes \sigma_c + \delta([\nu^{-b}\rho, \nu^c\rho]) \otimes \sigma_a,\\
&\mu^*(\sigma^-_{b,c,a}) \geq \delta([\nu^{-b}\rho, \nu^c\rho]) \otimes \sigma_a,\\
&\mu^*(\sigma^-_{a,b,c}) \geq \delta([\nu^{-a}\rho, \nu^b\rho]) \otimes \sigma_c.
\end{aligned} \tag{3.1}$$

We finished introducing notation and state some more reducibility results.

Here is a consequence of Theorem 6.3 of [19], see also section 3 there.

PROPOSITION 3.3. *We have in $R(G)$, with multiplicity one:*

$$\nu^a\rho \times \cdots \times \nu^{\frac{1}{2}} \rtimes \sigma \geq \sigma_a.$$

The next lemma follows from Theorem 5.1 of [14], ii).

LEMMA 3.4. *We have in* $R(G)$,

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma_b = \sigma^+_{a,b} + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma_b), \text{ and}$$
$$\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho]) \rtimes \sigma_a = \sigma^+_{a,b} + L(\delta([\nu^{-a}\rho,\nu^b\rho]) \rtimes \sigma) + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma_b) + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho]) \rtimes \sigma_a).$$

By Proposition 2.4 of [3] we have

LEMMA 3.5. *We have in* $R(G)$

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma^{\pm}_{b,c} = \sigma^{\pm}_{b,c,a} + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma^{\pm}_{b,c}), \text{ so} \tag{3.2}$$
$$\mu^*(\sigma^{\pm}_{b,c,a}) \geq \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \otimes \sigma^{\pm}_{b,c}. \tag{3.3}$$

The next is Proposition 3.2 of [8].

THEOREM 3.6. *With discrete series being a subrepresentation, we have in* $R(G)$

$$\delta([\nu^{-a}\rho,\nu^c\rho]) \rtimes \sigma_b = \sigma^+_{b,c,a} + L(\delta([\nu^{-a}\rho,\nu^c\rho]) \rtimes \sigma_b) + L(\delta([\nu^{-a}\rho,\nu^b\rho]) \rtimes \sigma_c) + L(\delta([\nu^{-b}\rho,\nu^c\rho]) \rtimes \sigma_a).$$

Finally we have the main result of [3].

THEOREM 3.7. *With discrete series being a subrepresentation, we have in* $R(G)$

$$\delta([\nu^{-b}\rho,\nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma = L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma^+_{b,c}) + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma^-_{b,c}) + \sigma^+_{b,c,a} + \sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho,\nu^c\rho]) \rtimes \sigma_a) + L(\delta([\nu^{-b}\rho,\nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho]) \rtimes \sigma).$$

4. DECOMPOSING MIXED CASE

As we are interested in the composition series of induced representations, we shall need a result that follows from proofs of Theorems 2-1 and 2-6 from [5].

THEOREM 4.1. *There exists a contravariant exact functor:*

$$Alg\ G_n \xrightarrow{\wedge} Alg\ G_n,$$

such that

$$\hat{\pi} \cong \pi, \quad \pi \in Irr\ G_n,$$

and if $\delta_i \in Irr\ GL(n_i,F)$, $i = 1,...,k$, $m = n_1 + \cdots + n_k$ *and* $\tau \in Irr\ G_{n-m}$ *we have*

$$(\delta_1 \times \cdots \times \delta_k \rtimes \tau)^\wedge \cong \widetilde{\delta_1} \times \cdots \times \widetilde{\delta_k} \rtimes \tau.$$

Proof. We follow the same lines as in proofs of Theorems 2-1 and 2-6 from [5]. If G_n is orthogonal, then $\wedge$ is just contragredient. If G_n is symplectic, by a result of Waldspurger ([12] Chapter 4, II.1), for any element $\eta \in GSp(2n)$ of similitude -1, and $\pi \in IrrG_n$ we have $\widetilde{\pi} \cong \pi^\eta$, where $\pi^\eta(g) = \pi(\eta g \eta^{-1})$. Now choose an element of the form $\eta = (id, \eta') \in GL(n,F) \times GSp(0,F) = GL(n,F) \times F^\times$, identified with the Levi subgroup of the appropriate maximal parabolic subgroup of $GSp(2n,F)$, where η' is an element with similitude equal to -1, as η is. For $\wedge = \eta \circ \sim$, we have

$$(\delta_1 \times \cdots \times \delta_k \rtimes \tau)^{\sim\eta} \cong (\widetilde{\delta_1} \times \cdots \times \widetilde{\delta_k} \rtimes \widetilde{\tau})^\eta \cong \widetilde{\delta_1} \times \cdots \times \widetilde{\delta_k} \rtimes \tau.$$

□

Consider some standard intertwining operators

$$\begin{tikzcd}
\delta([\nu^{-a}\rho,\nu^c\rho]) \times \delta([\nu^{\frac12}\rho,\nu^b\rho]) \rtimes \sigma \arrow[r, "\cong"] \arrow[d, "\cong"', "f_0"] & \delta([\nu^{\frac12}\rho,\nu^b\rho]) \times \delta([\nu^{-a}\rho,\nu^c\rho]) \rtimes \sigma \arrow[d, "\cong"', "g_0"] \\
\delta([\nu^{\frac12}\rho,\nu^b\rho]) \times \delta([\nu^{-a}\rho,\nu^c\rho]) \rtimes \sigma \arrow[d, "f_1"] & \delta([\nu^{-a}\rho,\nu^c\rho]) \times \delta([\nu^{\frac12}\rho,\nu^b\rho]) \rtimes \sigma \arrow[d, "g_1"] \\
\delta([\nu^{\frac12}\rho,\nu^b\rho]) \times \delta([\nu^{-c}\rho,\nu^a\rho]) \rtimes \sigma \arrow[d, "f_2"] & \delta([\nu^{-a}\rho,\nu^c\rho]) \times \delta([\nu^{-b}\rho,\nu^{-\frac12}\rho]) \rtimes \sigma \arrow[d, "g_2"] \\
\delta([\nu^{-c}\rho,\nu^a\rho]) \times \delta([\nu^{\frac12}\rho,\nu^b\rho]) \rtimes \sigma \arrow[d, "f_3"] & \delta([\nu^{-b}\rho,\nu^{-\frac12}\rho]) \times \delta([\nu^{-a}\rho,\nu^c\rho]) \rtimes \sigma \arrow[d, "g_3"] \\
\delta([\nu^{-c}\rho,\nu^a\rho]) \times \delta([\nu^{-b}\rho,\nu^{-\frac12}\rho]) \rtimes \sigma \arrow[r, "\cong"] & \delta([\nu^{-b}\rho,\nu^{-\frac12}\rho]) \times \delta([\nu^{-c}\rho,\nu^a\rho]) \rtimes \sigma
\end{tikzcd}$$

We denoted $\psi = \delta([\nu^{-a}\rho,\nu^c\rho]) \times \delta([\nu^{\frac12}\rho,\nu^b\rho]) \rtimes \sigma$. Also for all $i \geq 1$ denote $K_i = \mathrm{Ker}\ f_i$, and $H_i = \mathrm{Ker}\ g_i$. By Theorems 3.1 and 4.1, we have

$$\begin{aligned}
K_1 &\cong H_3{}^\wedge \cong \delta([\nu^{\frac12}\rho,\nu^b\rho]) \rtimes \sigma^+_{a,c} + \delta([\nu^{\frac12}\rho,\nu^b\rho]) \rtimes \sigma^-_{a,c},\\
K_2 &\cong H_2^\wedge \cong \delta([\nu^{-c}\rho,\nu^b\rho]) \times \delta([\nu^{\frac12}\rho,\nu^a\rho]) \rtimes \sigma,\\
K_3 &\cong H_1^\wedge \cong \delta([\nu^{-c}\rho,\nu^a\rho]) \rtimes \sigma_b,
\end{aligned}$$

and no kernel contains $L(\psi)$. Thus we have:

$$Im(f_3 \circ \cdots \circ f_0) \cong L(\psi) \cong Im(g_3 \circ \cdots \circ g_0),$$

and the diagram is commutative up to a constant. We have in $R(G)$:

$$\forall i \quad K_i \leq \psi \leq K_1 + K_2 + K_3 + L(\psi). \tag{4.1}$$

The composition factors of K_2 and K_3 are determined by Theorems 3.6 and 3.7. So in search of remaining subquotients, we need to decompose K_1. After that, we determine the multiplicities of all subquotients of ψ. Note that by Lemma 8.1 of [13], all tempered subquotients of ψ are discrete series.

5. Discrete series subquotients

Here we determine discrete series subquotients in three induced representations

$$\begin{aligned}\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma \geq \\ \delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}+\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{-}_{a,c},\end{aligned} \tag{5.1}$$

where the inequality follows from Theorem 3.1. We start with candidates.

Lemma 5.1. *Only possible discrete series subquotients appearing in*

$$\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma$$

are $\sigma^{+}_{b,c,a}$, $\sigma^{-}_{b,c,a}$ *and* $\sigma^{-}_{a,b,c}$.

Proof. Consider cuspidal support of the induced representation and Mœglin Tadić classification of discrete series. Since $\nu^{\pm\frac{1}{2}}\rho$ appears 3 times in the cuspidal support, possible discrete series are subrepresentations of a representation of the form

$$\delta([\nu^{-y}\rho,\nu^{z}\rho])\rtimes\sigma_x,$$

for some $\frac{1}{2}\leq x,y,z\in\mathbb{Z}+\frac{1}{2}$, where either $0<x<y<z$ or $0<y<z<x$, and σ_x is a unique irreducible subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho,\nu^{x}\rho])\rtimes\sigma$. We look at the first case. Here z is the largest such that $\nu^{\pm z}\rho$ appears once in the cuspidal support, so we must have $z=c$. Further, y is the largest such that $\nu^{\pm y}\rho$ appears two times in the cuspidal support, so it must be b. Finally, x is the largest such that $\nu^{\pm x}\rho$ appears three times in the cuspidal support, so it must be a. Same reasoning goes for the second case, where we have $x=c$, $z=b$, and $y=a$. By Theorem 2.1 of [14], we look for subrepresentations

$$\sigma^{+}_{b,c,a}\oplus\sigma^{-}_{a,b,c}\hookrightarrow\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c \quad\text{and}\quad \sigma^{+}_{b,c,a}\oplus\sigma^{-}_{b,c,a}\hookrightarrow\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a,$$

□

To determine which of these discrete series do appear in (5.1), and what are their multiplicities, we need a couple of lemmas.

Lemma 5.2. *We have in $R(G)$, with maximum multiplicity*

$$\begin{aligned}\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{c}\rho])\rtimes\sigma \geq \\ 1\cdot\sigma^{+}_{b,c}+1\cdot\sigma^{-}_{b,c}+1\cdot L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma).\end{aligned} \tag{5.2}$$

Proof. Check multiplicity two of $\delta([\nu^{-b}\rho,\nu^{c}\rho])\otimes\sigma$ and one of $\delta([\nu^{-c}\rho,\nu^{b}\rho])\otimes\sigma$ in $\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{c}\rho])\rtimes\sigma)$ and use Theorem 3.1. □

Lemma 5.3. *We have in $R(G)$, with maximum multiplicities:*

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\times\delta([\nu^{-a}\rho,\nu^c\rho])\rtimes\sigma)\geq 1\cdot\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\otimes\sigma^+_{b,c}+1\cdot\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\otimes\sigma^-_{b,c}.$$

Proof. The claim can be deduced from (2.1) and Lemma 5.2. ∎

Lemma 5.4. *We have in $R(G)$, with maximum multiplicities:*

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma^+_{a,c})\geq 1\cdot\delta([\nu^{-a}\rho,\nu^b\rho])\otimes\sigma_c +1\cdot\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\otimes\sigma^+_{b,c}+0\cdot\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\otimes\sigma^-_{b,c}.$$

Proof. By (2.1) consider $0\leq s\leq r\leq b+\frac{1}{2}$, $\delta'\otimes\sigma'\leq\mu^*(\sigma^+_{a,c})$, and

$$\delta([\nu^{r-b}\rho,\nu^{-\frac{1}{2}}\rho])\times\delta([\nu^{b-s+1}\rho,\nu^b\rho])\times\delta'\otimes\delta([\nu^{b+1-r}\rho,\nu^{b-s}\rho])\rtimes\sigma'. \tag{5.3}$$

First we look for $\delta([\nu^{-a}\rho,\nu^b\rho])\otimes\sigma_c$. Observe that $\nu^c\rho$ is not in a cuspidal support of δ', so in (2.3) we have $j=c$ and

$$\delta'\otimes\sigma'\leq\sum_{i=-a-1}^{a}\delta([\nu^{-i}\rho,\nu^a\rho])\otimes\delta([\nu^{i+1}\rho,\nu^c\rho]_+;\sigma).$$

Searching for $\nu^{-a}\rho$ in cuspidal support in (5.3), left of $\otimes$, we have options

- $r-b=-a$, so $b+1-s>\frac{1}{2}$, and we have $i=-\frac{1}{2}$, $s=b-a$ and $\sigma'=\delta([\nu^{\frac{1}{2}}\rho,\nu^c\rho]_+;\sigma)=\sigma_c$.
- $b+1-s=-a$, so $s>b+\frac{1}{2}$ and this is not possible.
- $-i=-a$, this is not possible since $\delta([\nu^{a+1}\rho,\nu^c\rho]_+;\sigma)$ is zero.

Looking for $\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\otimes\sigma^-_{b,c}$, we have $r=b+\frac{1}{2}$ and $s=0$. Thus we search in

$$\delta'\otimes\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma'.$$

Now in (2.3) we have $j=c$ and $i=-\frac{1}{2}$, so $\sigma'=\delta([\nu^{\frac{1}{2}}\rho,\nu^c\rho]_+;\sigma)=\sigma_c$. But, $\sigma^-_{b,c}\not\leq\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_c$, and $\sigma^+_{b,c}$ appears there once, by Lemma 3.4. ∎

Lemma 5.5. *We have in $R(G)$, with maximum multiplicities:*

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma^-_{a,c})\geq 0\cdot\delta([\nu^{-a}\rho,\nu^b\rho])\otimes\sigma_c+0\cdot\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\otimes\sigma^-_{b,c}.$$

Proof. The proof goes as in Lemma 5.4, with the difference that one now obtains $\sigma'=\delta([\nu^{\frac{1}{2}},\rho,\nu^c]_-;\sigma)$, but the term on the right-hand side is zero, by (2.2). ∎

Now we determine all discrete series that appear on the left-hand side of (5.1).

PROPOSITION 5.6. *Writting all discrete series, with multiplicities, we have in $R(G)$:*

$$\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma\geq 1\cdot\sigma^{+}_{b,c,a}+1\cdot\sigma^{-}_{b,c,a},$$
$$\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}\geq 1\cdot\sigma^{+}_{b,c,a},$$
$$\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{-}_{a,c}\geq 0.$$

PROOF. By (5.1) and Lemma 5.1, only possible discrete series subquotients in all of these representations are $\sigma^{+}_{b,c,a}$, $\sigma^{-}_{b,c,a}$ and $\sigma^{-}_{a,b,c}$. By Theorem 3.7 and (4.1), $\sigma^{+}_{b,c,a}$ and $\sigma^{-}_{b,c,a}$ appear in $\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma$. Lemmas 3.5 and 5.3 show that they appear with multiplicity one. Now (3.1) and Lemma 5.4 show that we have only one discrete series in $\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}$, which is $\sigma^{+}_{b,c,a}$, and none in the $\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{-}_{a,c}$, by Lemma 5.5. Since we have no $\sigma^{-}_{a,b,c}$ in neither $\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}$ nor $\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{-}_{a,c}$, Theorems 3.6 and 3.7, and (4.1), show that we have no $\sigma^{-}_{a,b,c}$ in $\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma$. □

6. NON-TEMPERED CANDIDATES

As noted in Section 4, we search for possible remaining non-tempered subquotients in

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}+\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{-}_{a,c}.$$

We have

PROPOSITION 6.1. *If π is a non-tempered subquotient of $\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}$, different from its Langlands quotient, then π is either $L(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma^{+}_{b,c})$ or $L(\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_{c})$.*

PROOF. We use Lemma 2.2 of [14] (in terms of that lemma $\pi\leq\delta([\nu^{-l_1}\rho,\nu^{l_2}\rho])\rtimes\sigma$, $-l_1=\frac{1}{2}$, $l_2=b$ and $\sigma=\sigma^{+}_{a,c}$). So we look for possible embeddings

$$\pi\hookrightarrow\delta([\nu^{-\alpha_1}\rho,\nu^{\beta_1}\rho])\rtimes\pi', \tag{6.1}$$

where $-\alpha_1+\beta_1<0$ and π' is irreducible. By the lemma, there exists an irreducible representation σ_1 such that

$$\begin{cases}\mu^{*}(\sigma^{+}_{a,c})\geq\delta([\nu^{\frac{1}{2}}\rho,\nu^{\beta_1}\rho])\otimes\sigma_1\\ \pi'\leq\delta([\nu^{\alpha_1+1}\rho,\nu^{b}\rho])\rtimes\sigma_1\end{cases} \tag{6.2}$$

and we must have

$$\begin{cases}-\frac{1}{2}\leq\beta_1\\ b\geq\alpha_1>\beta_1,-\frac{1}{2}\\ \alpha_1\geq\frac{1}{2}.\end{cases} \tag{6.3}$$

We have two possible cases:

a) $\beta_1 = -\frac{1}{2}$. Now $\sigma_1 = \sigma^+_{a,c}$.

- Assume that π' is tempered. We may take $2\alpha_1+1 \in \mathrm{Jord}_\rho(\sigma^+_{a,c})$. Now $\alpha_1 = c$ is not possible, since that would imply $b \geq c$. Thus $\alpha_1 = a$, and
$$\pi' \leq \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c}.$$
Looking at the cuspidal support on the right hand side, Lemma 8.1 of [13] implies that π' is a discrete series. Thus $\pi' = \sigma^+_{b,c}$ or $\pi' = \sigma^-_{b,c}$. By Section 8. of [22], we have $\pi' = \sigma^+_{b,c}$. So (6.1) is written as
$$\pi \hookrightarrow \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma^+_{b,c}.$$
Thus, we have $\pi \cong L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^+_{b,c})$.
- If π' is not tempered, by Lemma 2.2 of [14], there exist $\beta_2+1 \in \mathrm{Jord}_\rho(\sigma^+_{a,c})$ and $(\beta_2+1)_- = (\beta_2)_- + 1 \in \mathrm{Jord}_\rho(\sigma^+_{a,c}) = \{2a+1, 2c+1\}$, such that (in terms of the lemma $\alpha_1 \leq (\beta_2)_- < \beta_2 < \alpha_2 \leq l_2$) we have
$$\alpha_1 \leq a < c < \alpha_2 \leq b.$$
So $c < b$, but this is a contradiction.

b) $\beta_1 > \frac{1}{2}$. Then, by the lemma, $2\beta_1+1 \in \mathrm{Jord}_\rho(\sigma^+_{a,c}) = \{2a+1, 2c+1\}$. Since
$$c > b \geq \alpha_1 > \beta_1, -\frac{1}{2}$$
we have $\beta_1 = a$ and $\alpha_1 > a$. Now (6.2) gives
$$\mu^*(\sigma^+_{a,c}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma_1.$$
To determine σ_1, we look at $\mu^*(\sigma^+_{a,c})$. In (2.3) it is necessary to pick $j = c$ and $i = -\frac{1}{2}$, and we have
$$\sigma_1 = \delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]_+; \sigma) = \sigma_c.$$
So far, we have
$$\begin{cases} \pi \hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^a\rho]) \rtimes \pi', \\ \pi' \leq \delta([\nu^{\alpha_1+1}\rho, \nu^b\rho]) \rtimes \sigma_c, \\ a < \alpha_1 \leq b. \end{cases} \tag{6.4}$$
If $\alpha_1 = b$ then $\pi' \cong \sigma_c$ and $\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma_c$, so
$$\pi \cong L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c),$$
as expected. Thus, we assume
$$a < \alpha_1 < b.$$

Since $\mathrm{Jord}_\rho(\sigma_c) \cap [2\alpha_1+1, 2b+1] = \{2c+1\} \cap [2\alpha_1+1, 2b+1] = \emptyset$, Proposition 3.1 ii) of [14] implies that $\delta([\nu^{\alpha_1+1}\rho, \nu^b\rho]) \rtimes \sigma_c$ is irreducible. So

$$\pi' \cong \delta([\nu^{\alpha_1+1}\rho, \nu^b\rho]) \rtimes \sigma_c \cong \delta([\nu^{-b}\rho, \nu^{-\alpha_1-1}\rho]) \rtimes \sigma_c,$$

and finally

$$\pi \hookrightarrow \delta([\nu^{-\alpha_1}\rho, \nu^a\rho]) \times \delta([\nu^{-b}\rho, \nu^{-\alpha_1-1}\rho]) \rtimes \sigma_c.$$

Now, by Lemma 5.5 of [6], either $\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^a\rho]) \rtimes \sigma_c$ or there exists an irreducible representation $\pi'' \leq \delta([\nu^{-\alpha_1}\rho, \nu^a\rho]) \rtimes \sigma_c$ such that

$$\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^{-\alpha_1-1}\rho]) \rtimes \pi''.$$

The second case is not possible since, similar to (6.3), we would obtain $-\frac{1}{2} \leq -\alpha_1 - 1 (\leq -\frac{1}{2} - 1)$, a contradiction.

□

Using the same methods as above, we obtain

Proposition 6.2. *If π is a non-tempered subquotient of $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^-$, different from its Langlands quotient, then π is $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^-)$.*

7. Multiplicity of $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^\pm)$

Here we write explicitly $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^\pm)$ as a non-tempered subquotient of $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^\pm$, different from its unique Langlands quotient, as claimed by Lemma 6.2 of [15]. We start with a couple of lemmas.

Lemma 7.1. *Discrete series $\sigma_{b,c}^+$ and $\sigma_{b,c}^-$ appear with multiplicity one in equations*

$$1 \cdot \sigma_{b,c}^+ \leq \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^+,$$
$$1 \cdot \sigma_{b,c}^- \leq \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^-,$$
$$1 \cdot \sigma_{b,c}^+ + 1 \cdot \sigma_{b,c}^- \leq \delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^c\rho]) \rtimes \sigma.$$

Proof. This follows from Section 8. of [22] and (2.1). □

The next two lemmas follow.

Lemma 7.2. *Both $\delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \otimes \sigma_{b,c}^+$ and $\delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \otimes \sigma_{b,c}^-$ appear with multiplicity one in $\mu^*(\delta([\nu^{-a}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma)$.*

Lemma 7.3. *For $\epsilon = \pm$ irreducible representation $\delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \otimes \sigma_{b,c}^\epsilon$ appears with multiplicity one in $\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^\epsilon)$.*

Using Lemmas 7.2 and 7.3 we have

Proposition 7.4. *With all multiplicities being one, we have in* $R(G)$

$$\begin{aligned}\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma^+_{a,c} &\geq 1\cdot L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma^+_{b,c}),\\ \delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma^-_{a,c} &\geq 1\cdot L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma^-_{b,c}),\\ \delta([\nu^{-a}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma &\geq 1\cdot L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma^+_{b,c})+\\ &\quad 1\cdot L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma^-_{b,c}).\end{aligned}$$

8. Multiplicity of $L(\delta([\nu^{-a}\rho,\nu^b\rho])\rtimes\sigma_c)$

By Theorem 3.6 and (4.1), this subquotient does appear in $\delta([\nu^{-a}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma$, but also as a candidate in $\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma^+_{a,c}$, by Proposition 6.1, so we need to check its multiplicity. We want to obtain a subquotient in some of its Jacquet module, used to identify it.

First we state a result of Proposition 3.9 of [23].

Lemma 8.1. *In appropriate Grothendieck group we have*

$$\begin{aligned}\nu^{\frac{1}{2}}\rho\times\nu^{\frac{1}{2}}\rho\rtimes\sigma = \delta([\nu^{-\frac{1}{2}}\rho,\nu^{\frac{1}{2}}\rho]_+;\sigma)+\delta([\nu^{-\frac{1}{2}}\rho,\nu^{\frac{1}{2}}\rho]_-;\sigma)+\\ L(\nu^{\frac{1}{2}}\rtimes\sigma_{\frac{1}{2}})+L(\nu^{\frac{1}{2}}\rho\times\nu^{\frac{1}{2}}\rho\rtimes\sigma).\end{aligned}$$

Lemma 8.2. *We have in* $R(G)$*, with multiplicity one*

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma \geq 1\cdot L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_b).$$

Proof. By Lemma 8.1 we may assume $b\geq\frac{3}{2}$. Obviously, $L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_b)$ does appear as a subquotient, so we need to prove multiplicity one. Let us denote

$$\pi=\delta([\nu^{\frac{3}{2}}\rho,\nu^b\rho])\times\delta([\nu^{\frac{3}{2}}\rho,\nu^b\rho])\otimes L(\nu^{\frac{1}{2}}\rho\rtimes\sigma_{\frac{1}{2}}).$$

It is enough to see that π appears in both

i) $\mu^*(L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_b))$ and ii) $\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma)$,

and that its multiplicity is one in ii).
We start with i). By Theorem 5.1 of [14] ii), and Lemma 5.4 there, we have in $R(G)$

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_b=\sigma_{temp}+L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_b),$$

where σ_{temp} is an irreducible tempered subquotient of $\delta([\nu^{-b}\rho,\nu^b\rho])\rtimes\sigma$. By (2.3), $\pi\nleq\mu^*(\sigma_{temp})$. On the other hand, by (2.4) $\mu^*(\sigma_b)\geq\delta([\nu^{\frac{3}{2}}\rho,\nu^b\rho])\otimes\sigma_{\frac{1}{2}}$, so from (2.1) we have

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_b)\geq\delta([\nu^{\frac{3}{2}}\rho,\nu^b\rho])\times\delta([\nu^{\frac{3}{2}}\rho,\nu^b\rho])\otimes\nu^{\frac{1}{2}}\rho\rtimes\sigma_{\frac{1}{2}}\geq\pi.$$

To we prove ii), look for $0 \leq i \leq j \leq b + \frac{1}{2}$ and $0 \leq s \leq r \leq b + \frac{1}{2}$ such that

$$\begin{aligned}\delta([\nu^{\frac{3}{2}}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{3}{2}}\rho, \nu^b\rho]) \leq & \delta([\nu^{i-b}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{b+1-j}\rho, \nu^b\rho]) \\ & \times \delta([\nu^{r-b}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{b+1-s}\rho, \nu^b\rho]),\end{aligned}$$

and

$$L(\nu^{\frac{1}{2}}\rho \rtimes \sigma_{\frac{1}{2}}) \leq \delta([\nu^{b+1-i}\rho, \nu^{b-j}\rho]) \times \delta([\nu^{b+1-r}\rho, \nu^{b-s}\rho]) \rtimes \sigma.$$

The first equation implies $i = r = b + \frac{1}{2}$ and $j = s = b - \frac{1}{2}$. The second is

$$L(\nu^{\frac{1}{2}}\rho \rtimes \sigma_{\frac{1}{2}}) \leq \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma.$$

Multiplicity one in the second follows from Lemma 8.1. □

LEMMA 8.3. *We have in $R(G)$, with multiplicity one*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma \geq 1 \cdot L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_c).$$

PROOF. Use Lemma 3.4, (2.1) and (2.3) to show that $\delta([\nu^{b+1}\rho, \nu^c\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_b)$ appears in $\mu^*(L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_c))$, and Lemma 8.2 that it appears once in $\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma)$. □

Now we provide a subquotient that can be used to identify $L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c)$.

LEMMA 8.4. *We have in $R(G)$*

$$\mu^*(L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c)) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_c).$$

PROOF. By Proposition 3.2, we have

$$\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c = \sigma^+_{b,c,a} + \sigma^-_{a,b,c} + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c).$$

To prove the claim, one can use (2.1) and a square integrability criterium, see the end of Section 2 of [21]. □

LEMMA 8.5. *We have in $R(G)$, with maximum multiplicity*

$$\mu^*(\delta([\nu^{-a}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma) \geq 1 \cdot \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_c).$$

PROOF. Use (2.1) and Lemma 8.3. □

LEMMA 8.6. *We have in $R(G)$, with maximum multiplicity*

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c}) \geq 1 \cdot \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_c).$$

PROOF. By (3.1) $\mu^*(\sigma^+_{a,c}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma_c$. The inequality follows by (2.1) and multiplicity one by (5.1) and Lemma 8.5. □

Finally we have

PROPOSITION 8.7. *Both induced representations*

$$\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma \text{ and } \delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c},$$

contain $L(\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c)$ *as a subquotient, with multiplicity one.*

PROOF. Theorem 3.6 and (4.1) give existence in the first representation, and Lemmas 8.4 and 8.5 multiplicity one. For the second, use (5.1), and Lemma 8.6. □

9. MULTIPLICITY OF $L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)$

We observe that by Theorems 3.6 and 3.7, $L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)$ appears two times in (4.1). So we determine its multiplicity. First we state a result that we use later.

LEMMA 9.1. *We have in* $R(G)$*, with multiplicity one*

$$\mu^*(L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a))\geq 1\cdot\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\otimes L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma).$$

PROOF. Denote the subquotient by π. By (2.4) $\mu^*(\sigma_a)\geq\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\otimes\sigma$, so

$$\mu^*(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)\geq\pi.$$

Comparing Proposition 3.2 and Lemma 3.5, it is enough to show

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma^{\pm}_{b,c})\not\geq\pi.$$

So by (2.1) consider $0\leq j\leq i\leq a+\frac{1}{2}$, $\delta'\otimes\sigma'\leq\mu^*(\sigma^{\pm}_{b,c})$ and

$$\delta([\nu^{i-a}\rho,\nu^{-\frac{1}{2}}\rho])\times\delta([\nu^{a-j+1}\rho,\nu^{a}\rho])\times\delta'\otimes\delta([\nu^{a+1-i}\rho,\nu^{a-j}\rho])\rtimes\sigma'.$$

As δ' should not contain neither $\nu^{b}\rho$ nor $\nu^{c}\rho$ in its cuspidal support, by (2.3), we have $\delta'\otimes\sigma'=1\otimes\sigma^{\pm}_{b,c}$, and $i=j=a+\frac{1}{2}$, giving $\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\otimes\sigma^{\pm}_{b,c}\not\geq\pi$. □

PROPOSITION 9.2. *We have in* $R(G)$*, with multiplicity one*

$$\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma\geq 1\cdot L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a).$$

PROOF. It is easy to check that $\delta([\nu^{-c}\rho,\nu^{b}\rho])\otimes\sigma_a$ appears with multiplicity one in $\mu^*(\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma)$. □

10. COMPOSITION FACTORS

Here we determine composition factors of the kernel K_1 from Section 4.

PROPOSITION 10.1. *We have in* $R(G)$

$$\begin{aligned}\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c}=&L(\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^{+}_{a,c})+\sigma^{+}_{b,c,a}+\\&L(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma^{+}_{b,c})+L(\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c).\end{aligned}$$

Proof. Discrete series subquotients are determined in Proposition 5.6. Remaining irreducible subquotients are described by Propositions 6.1, 7.4 and 8.7. ∎

Proposition 10.2. *We have in* $R(G)$

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{a,c}^- = L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{a,c}^-) + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_{b,c}^-).$$

Proof. Possible discrete series subquotients are discussed in Proposition 5.6. Remaining irreducible subquotients are described by Propositions 6.2 and 7.4. ∎

Theorem 10.3. *We have in* $R(G)$

$$\begin{aligned}\delta([\nu^{-a}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma = {} & L(\delta([\nu^{-a}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma)+\\ & L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{a,c}^+) + L(\delta([\nu^{-a}\rho,\nu^c\rho])\rtimes\sigma_b)+\\ & \sigma_{b,c,a}^- + L(\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma)+\\ & L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_{b,c}^+) + L(\delta([\nu^{-a}\rho,\nu^b\rho])\rtimes\sigma_c)+\\ & L(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{a,c}^-) + L(\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_a)+\\ & \sigma_{a,b,c}^+ + L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_{b,c}^-).\end{aligned}$$

Proof. By (4.1), Theorems 3.6 and 3.7, and Propositions 10.1 and 10.2, we listed all irreducible subquotients, up to multiplicities. Proposition 5.6 shows multiplicity one for discrete series. Propositions 7.4, 8.7 and 9.2, show multiplicity one for $L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_{b,c}^+)$, $L(\delta([\nu^{-a}\rho,\nu^b\rho])\rtimes\sigma_c)$, $L(\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_a)$, and $L(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_{b,c}^-)$. Remaining irreducible subquotients appear with a multiplicity one in $K_1+K_2+K_3+L(\psi)$, on the right hand side of (4.1), by Theorems 3.6 and 3.7, and Propositions 10.1 and 10.2.

∎

11. Composition series of $\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{a,c}^+$ and $\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{a,c}^-$.

Here we determine the composition series of the kernel K_1, from Section 4. For the first representation, we show that a discrete series is a subrepesentation, and use intertwining operators to position other subquotients.

The first lemma follows directly from (2.3) and (2.4).

Lemma 11.1. *We have in* $R(G)$*, with maximum multiplicities:*

$$\mu^*(\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_c)\geq 2\cdot\delta([\nu^{-a}\rho,\nu^b\rho])\otimes\sigma_c.$$

The next proposition gives positions of both $\sigma_{b,c,a}^+$ and $L(\delta([\nu^{-a}\rho,\nu^b\rho])\rtimes\sigma_c)$.

Proposition 11.2. *We have embeddings*

$$\sigma^{+}_{b,c,a} \hookrightarrow \delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \sigma_c / \sigma^{-}_{a,b,c} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma^{+}_{a,c}.$$

Proof. By Theorem 2-6. of [5] and Lemma 3.4 we have an epimorphism

$$\delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_c \twoheadrightarrow \sigma^{+}_{a,c}.$$

Now we have a composition of an embedding and an epimorphism

$$\begin{aligned}\delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \sigma_c &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \times \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_c \\ &\twoheadrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma^{+}_{a,c}.\end{aligned} \tag{11.1}$$

By Propositions 3.2, and 5.6 and Lemma 11.1, all representations in (11.1), have $\sigma^{+}_{b,c,a}$ as a subquotient, with multiplicity one, and the last doesn't contain $\sigma^{-}_{a,b,c}$. □

Now we want to see a position of $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c})$. First we state two lemmas that can be proved using (2.1), (2.3), Proposition 3.2, and Lemma 3.5.

Lemma 11.3. *We have in $R(G)$, with multiplicity one:*

$$\mu^*(L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c})) \geq 1 \cdot \delta([\nu^{-b}\rho, \nu^{c}\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma). \tag{11.2}$$

Lemma 11.4. *We have in $R(G)$, with maximum multiplicities:*

$$\begin{aligned}\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_c) &\geq 1 \cdot \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \otimes \sigma^{+}_{b,c} \\ &+1 \cdot \delta([\nu^{-b}\rho, \nu^{c}\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma).\end{aligned}$$

The next proposition gives position of $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c})$

Proposition 11.5. *There exists an embedding*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma^{+}_{a,c}.$$

Proof. Denote $\pi = \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma_c$. By Lemma 3.4, we have

$\delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma^{+}_{a,c} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_c \cong \pi,$ and by (3.2)

$\sigma^{+}_{b,c,a} + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c}) \overset{R(G)}{=} \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c} \hookrightarrow \pi.$

It is enough to show that both $\sigma^{+}_{b,c,a}$ and $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c})$ appear in π once. This follows by Lemmas 3.5, 11.3 and 11.4. □

Finally

Proposition 11.6. *Induced representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^+$ has a unique irreducible subrepresentation and unique quotient. We have an exact sequence*

$$L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^+) + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c) \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^+/\sigma_{a,b,c}^+ \twoheadrightarrow L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^+).$$

Proof. Composition factors are determined by Proposition 10.1. Position of irreducible subquotients are determined by Propositions 3.2, 11.2, 3.5 and 11.5. ∎

By Theorem 4.1, we have

Corollary 11.7. *Induced representation $\delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{a,c}^+$ has a unique irreducible subrepresentation and a unique irreducible quotient. We have an exact sequence*

$$L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^+) + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c) \hookrightarrow \delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{a,c}^+/L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^+) \twoheadrightarrow \sigma_{a,b,c}^+.$$

Now we state the composition series of $\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^-$ as a direct consequence of Proposition 10.2.

Proposition 11.8. *We have a non split exact sequence*

$$L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^-) \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^- \twoheadrightarrow L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^-).$$

By Theorem 4.1, we have

Corollary 11.9. *We have a non split exact sequence*

$$L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{a,c}^-) \hookrightarrow \delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{a,c}^- \twoheadrightarrow L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^-).$$

12. Composition series of $\delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$

Here we determine the composition series of the kernel K_2 from Section 4.

Proposition 12.1. *Induced representation $\delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ has exactly one irreducible subrepresentation, and two irreducible quotients. We have an exact sequence*

$$\sigma_{b,c,a}^+ + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) + \sigma_{b,c,a}^- \hookrightarrow \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma/L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a) \twoheadrightarrow L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^+) + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_{b,c}^-).$$

Proof. Denote the induced representation by π. By Theorem 3.7, π is a multiplicity one representation with irreducible subquotients listed in the claim. Consider embeddings

$$\begin{aligned}\delta([\nu^{-c}\rho,\nu^{b}\rho])\rtimes\sigma_a &\hookrightarrow\pi,\\ \delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma) &\hookrightarrow\pi,\\ \delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma) &\hookrightarrow\hat{\pi},\\ \delta([\nu^{-a}\rho,\nu^{-\frac12}\rho])\rtimes\sigma^{\pm}_{b,c} &\hookrightarrow\hat{\pi}.\end{aligned}$$

We shall describe representations on the left and the claim will follow.

• By Proposition 3.2, $\delta([\nu^{-c}\rho,\nu^{b}\rho])\rtimes\sigma_a$ has a unique irreducible subrepresentation:
$L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)$, and two irreducible quotients: $\sigma^{\pm}_{b,c,a}$.

• By Corrolary 4.1 of [3], $\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma)$ is a quotient of $\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma$ containing a unique irreducible subrepresentation: $L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)$ and a quotient:
$L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma)$.

• By Corrolary 4.1 of [3], $\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma)$ is a quotient of $\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma$, containing a unique irreducible quotient:

$$L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma)$$

and two irreducible subrepresentations: $L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma^{\pm}_{b,c})$. Thus π has a quotient, containing two irreducible quotients: $L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma^{\pm}_{b,c})$ and a unique irreducible subrepresentation: $L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma)$.

• Similarly, by Lemma 3.5, for every $\epsilon\in\{+,-\}$ π has a quotient, containing a unique irreducible subrepresentation $\sigma^{\epsilon}_{b,c,a}$ and a unique irreducible quotient $L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma^{\epsilon}_{b,c})$. □

By Theorem 4.1, we have

Corollary 12.2. *Induced representation*
$\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{-a}\rho,\nu^{-\frac12}\rho])\rtimes\sigma$ *has exactly two irreducible subrepresentations and one irreducible quotient. We have an exact sequence*

$$\begin{aligned}&\sigma^{+}_{b,c,a}+L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\times\sigma)+\sigma^{-}_{b,c,a}\hookrightarrow\\ &\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{-a}\rho,\nu^{-\frac12}\rho])\rtimes\sigma\\ &\qquad\qquad/(L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma^{+}_{b,c})+L(\delta([\nu^{\frac12}\rho,\nu^{a}\rho])\rtimes\sigma^{-}_{b,c}))\\ &\twoheadrightarrow L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a).\end{aligned}$$

13. Composition series of $\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b$

Here we determine composition series of the kernel $K_3\cong\delta([\nu^{-c}\rho,\nu^{a}\rho])\rtimes\sigma_b$ from Section 4. First we determine subrepresentations.

Proposition 13.1. *We have a unique subrepresentation*

$$\sigma^{+}_{b,c,a} \hookrightarrow \delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma_b.$$

Proof. Compare Theorem 3.6 and Proposition 11.2 with

$$\delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma_b \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \times \delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma.$$

∎

Now we determine position of $L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a)$.

Lemma 13.2. *We have in $R(G)$, with maximum multiplicities*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma_a \geq 1 \cdot \sigma^{+}_{b,c,a} + 1 \cdot \sigma^{-}_{b,c,a} + 1 \cdot L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a),$$

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma^{+}_{a,b} \geq 1 \cdot \sigma^{+}_{b,c,a} + 0 \cdot \sigma^{-}_{b,c,a} + 1 \cdot L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a).$$

Proof. By Proposition 3.2, $\sigma^{\pm}_{b,c,a}$ appears in the first formula. For the multiplicity one use (2.1), (3.3) and (5.2) to search for $\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \otimes \sigma^{\pm}_{b,c}$. For the third summand use Lemmas 3.4 and 9.1. Similarly show the second formula. ∎

Lemma 13.3. *We have an embedding*

$$\delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma_b \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma^{+}_{a,b}.$$

Proof. By Lemma 3.4 we have composition of an embedding and an epimorphism

$$\begin{aligned}\delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma_b &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \times \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_b \\ &\twoheadrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma^{+}_{a,b}.\end{aligned}$$

Lemmas 3.4 and 3.5, Theorem 3.6, and Lemma 13.2, imply that all representations here have $\sigma^{+}_{b,c,a}$ as a subquotient, with multiplicity one. By Proposition 13.1, $\sigma^{+}_{b,c,a}$ is a unique irreducible subrepresentation of $\delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma_b$. The claim follows. ∎

Lemma 13.4. *We have an embedding*

$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a / \sigma^{-}_{b,c,a} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma^{+}_{a,b}.$$

Proof. Consider a composition of an embedding and an epimorphism

$$\begin{aligned}\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_a \\ &\twoheadrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma^{+}_{a,b}.\end{aligned}$$

By Proposition 3.2 $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a = \sigma^{+}_{b,c,a} + \sigma^{-}_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a)$, with discrete series being subrepresentations. Apply Lemma 13.2. ∎

The next proposition gives a position of $L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a)$.

Proposition 13.5. *We have an embedding*

$$\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a/\sigma^-_{b,c,a}\hookrightarrow\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b.$$

Proof. By Proposition 3.2 and Theorem 3.6 we have in $R(G)$

$$\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a/\sigma^-_{b,c,a}\leq\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b.$$

By Lemmas 13.3 and 13.4, these representations embed into $\delta([\nu^{\frac{1}{2}}\rho,\nu^{c}\rho])\rtimes\sigma^+_{a,b}$, which has irreducible subquotiens of the first representations with multiplicity one, by Lemma 13.2. □

Finally we determine position of $L(\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c)$.

Proposition 13.6. *We have an embedding*

$$\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c/\sigma^-_{a,b,c}\hookrightarrow\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b.$$

Proof. By Proposition 11.2 we have

$$\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c/\sigma^-_{a,b,c}\hookrightarrow\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma^+_{a,c},\ \text{so}$$

$$\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c/\sigma^-_{a,b,c}\hookrightarrow\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma,\ \text{and}$$
$$\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b\hookrightarrow\delta([\nu^{-a}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{b}\rho])\rtimes\sigma.$$

The claim follows, since by Theorem 10.3 the representation on the right is multiplicity one, and by Proposition 3.2 and Theorem 3.6 we have in $R(G)$

$$\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c/\sigma^-_{b,a,c}\leq\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b.$$

□

Now we write composition series for $\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b$.

Proposition 13.7. *Induced representation $\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b$ has a unique irreducible subrepresentation. We have an exact sequence*

$$L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)+L(\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c)\hookrightarrow$$
$$\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b/\sigma^+_{a,b,c}\twoheadrightarrow L(\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b).$$

Proof. Composition factors are determined by Theorem 3.6. Positions of irreducible subquotients are determined by Propositions 3.2, 13.1, 13.5 and 13.6. □

By Theorem 4.1, we have

Corollary 13.8. *Induced representation $\delta([\nu^{-c}\rho,\nu^{a}\rho])\rtimes\sigma_b$ has a unique irreducible quotient. We have an exact sequence*

$$L(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_a)+L(\delta([\nu^{-a}\rho,\nu^{b}\rho])\rtimes\sigma_c)\hookrightarrow$$
$$\delta([\nu^{-c}\rho,\nu^{a}\rho])\rtimes\sigma_b/L(\delta([\nu^{-a}\rho,\nu^{c}\rho])\rtimes\sigma_b)\twoheadrightarrow\sigma^+_{a,b,c}.$$

14. The main result

Here we give the main result, a composition series of the representation ψ.

THEOREM 14.1. *Let* $\psi = \delta([\nu^{-a}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma$ *and define representations*

$$\begin{aligned}W_1 =&\sigma^{+}_{b,c,a} + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{-}_{b,c}),\\ W_2 =&L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma^{+}_{b,c}) + L(\delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \sigma_c)+\\ &L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma^{-}_{a,c}) + L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_a),\\ W_3 =&L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma^{+}_{a,c}) + L(\delta([\nu^{-a}\rho, \nu^{c}\rho]) \rtimes \sigma_b)+\\ &\sigma^{-}_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma),\\ W_4 =&L(\psi).\end{aligned}$$

Then there exists a sequence $\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \psi$*, such that*

$$V_i/V_{i-1} \cong W_i, \quad i = 1, \ldots, 4.$$

Further, W_1 *is chosen to be the largest possible, then* W_2*, and so on.*

PROOF. We use the notation K_i, H_i, and f_i from Section 4. Composition series of K_1, K_2 and K_3 are determined by Propositions 11.6, 11.8, 12.1, and Corollary 13.8. Composition series of H_1, H_2 and H_3 are determined by Proposition 13.7 and Corollaries 12.2, 11.9 and 11.7. For all $i \geq 1$ denote

$$k_i = K_i \cap Im(f_{i-1} \circ \cdots \circ f_0), \quad h_i = H_i \cap Im(g_{i-1} \circ \cdots \circ g_0).$$

Set $K_0 = k_0 = \{0\}$. Let π be an irreducible subquotient of ψ, and $1 \leq i \leq 3$. Obviously

$$\pi \leq Im(f_{i-1} \circ \cdots \circ f_0) \iff \forall j < i \quad \pi \nleq K_j.$$

This implies that if $\pi \leq K_j$, for some $(0 <)j < i$, taking j minimal, we have $\pi \in Im(f_{j-1} \circ \cdots \circ f_0)$. Thus $\pi \leq k_j$ for some $j < i$. Now we have

$$\begin{aligned}\exists j < i \quad \pi \leq K_j &\iff \exists j < i \quad \pi \leq k_j \text{ and so}\\ \pi \leq Im(f_{i-1} \circ \cdots \circ f_0) &\iff \forall j < i \quad \pi \nleq k_j.\end{aligned}$$

We conclude that k_i consists of irreducible subquotients of K_i that do not appear in any of $k_{i-1}, \ldots, k_0$. It is clear that we can write this in $R(G)$ as

$$k_i = \lfloor K_i - k_{i-1} - \cdots - k_0 \rfloor_{R_0^+(G)}.$$

where for $\pi \in R(G), \pi = \sum_i m_i\pi_i$, $m_i \in \mathbb{Z}$, $\pi_i \ncong \pi_j \in IrrG$ for $i \neq j$, we define

$$\lfloor \pi \rfloor_{R_0^+(G)} = \sum_{\{i | m_i \geq 0\}} m_i\pi_i.$$

Now we have in $R(G)$

$$\begin{aligned} k_1 &= K_1, \quad k_2 = \lfloor K_2 - k_1 \rfloor_{R_0^+(G)}, \quad k_3 = \lfloor K_3 - k_2 - k_1 \rfloor_{R_0^+(G)}, \\ h_1 &= H_1, \quad h_2 = \lfloor H_2 - h_1 \rfloor_{R_0^+(G)}, \quad h_3 = \lfloor H_3 - h_2 - h_1 \rfloor_{R_0^+(G)}. \end{aligned}$$

We have $k_1 \cong K_1$ and $h_1 \cong H_1$. Further, calculating composition factors of k_2 and h_2 and comparing with composition series of K_2 and H_2, we see that k_2 and h_2 have exactly two irreducible quotients. Similarly, we determine k_3 and h_3. So we have exact sequences, and no irreducible subquotient can go on lower position:

(14.1)
$$L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^+_{b,c}) + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c) + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^-_{a,c}) \hookrightarrow$$
$$k_1/(\sigma^+_{a,b,c} + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^-_{b,c})) \twoheadrightarrow L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c}),$$
(14.2)
$$k_2/L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a) \cong \sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma),$$
(14.3)
$$k_3 \cong L(\delta([\nu^{-a}\rho, \nu^c\rho]) \rtimes \sigma_b),$$
(14.4)
$$L(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_a) + L(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma_c) \hookrightarrow$$
$$h_1/\sigma^+_{a,b,c} \twoheadrightarrow L(\delta([\nu^{-a}\rho, \nu^c\rho]) \rtimes \sigma_b),$$
(14.5)
$$h_2/(L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^+_{b,c}) + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma^-_{b,c})) \cong$$
$$\sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma),$$
(14.6)
$$h_3 \cong L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c}) + L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^-_{a,c}).$$

We define representations V_i, $i = 1, \ldots, 4$ as follows. By (14.1) $W_1 \hookrightarrow \psi$. Let V_1 be its image. Further, (14.1) and (14.4) show $W_2 \hookrightarrow \psi/V_1$. Let V_2 be the preimage of W_2 in ψ. Denote representations

$$\begin{aligned} \zeta &= L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c}), \\ \nu &= L(\delta([\nu^{-a}\rho, \nu^c\rho]) \rtimes \sigma_b), \\ M &= \sigma^-_{b,c,a} + L(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma), \\ \tau &= \psi/V_2. \end{aligned}$$

By (14.1) and (14.4) we have embeddings

$$\zeta \hookrightarrow \tau \hookleftarrow \nu. \tag{14.7}$$

By (14.1) and (14.2) we have an embedding and an epimorphism

$$M \hookrightarrow \tau/\zeta \overset{P_\zeta}{\twoheadleftarrow} \tau, \quad \text{thus} \quad P_\zeta^{-1}(M)/\zeta \cong M.$$

By (14.4) and (14.5) we have an embedding and an epimorphism

$$M \hookrightarrow \tau/\nu \overset{P_\nu}{\twoheadleftarrow} \tau, \quad \text{thus} \quad P_\nu^{-1}(M)/\nu \cong M.$$

We have in $R(G)$

$$P_\zeta^{-1}(M) = M + \zeta \text{ and } P_\nu^{-1}(M) = M + \nu. \tag{14.8}$$

We claim that in $R(G)$

$$P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) = M.$$

Inclusions $P_\zeta^{-1}(M) \hookrightarrow \tau$ and $P_\nu^{-1}(M) \hookrightarrow \tau$ induce an embedding

$$P_\zeta^{-1}(M)/(P_\zeta^{-1}(M) \cap P_\nu^{-1}(M)) \hookrightarrow \tau/P_\nu^{-1}(M).$$

This shows that in $R(G)$

$$P_\zeta^{-1}(M) - P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) \leq \tau - P_\nu^{-1}(M), \text{ and by (14.8)}$$
$$M + \zeta - P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) \leq \tau - M - \nu.$$

So, we consider in $R(G)$

$$M + \zeta + \nu \leq P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) + (\tau - M) \tag{14.9}$$

By Proposition 10.2, ψ is a multiplicity one, and so is τ. Also, in $R(G)$

$$\tau = \psi - V_2 = \psi - W_1 - W_2 \geq M.$$

Now, any irreducible subquotient of M does not appear in $\tau - M$, so by (14.9), it must appear in $P_\zeta^{-1}(M) \cap P_\nu^{-1}(M)$. We conclude that in $R(G)$

$$P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) \geq M.$$

Thus (14.8) shows that in $R(G)$

$$P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) = M.$$

Now, since τ/ζ is a multiplicity one, and P_ζ is injective on $P_\zeta^{-1}(M) \cap P_\nu^{-1}(M)$, both compositions

$$M \overset{i}{\hookrightarrow} \tau/\zeta \twoheadrightarrow (\tau/\zeta)/P_\zeta(P_\zeta^{-1}(M) \cap P_\nu^{-1}(M)) \text{ and}$$
$$P_\zeta^{-1}(M) \cap P_\nu^{-1}(M) \overset{P_\zeta}{\hookrightarrow} \tau/\zeta \twoheadrightarrow (\tau/\zeta)/i(M)$$

are zero. Thus $i(M) = P_\zeta(P_\zeta^{-1}(M) \cap P_\nu^{-1}(M))$.
We conclude $M \cong P_\zeta^{-1}(M) \cap P_\nu^{-1}(M)$, and have an embedding

$$M \hookrightarrow \tau. \tag{14.10}$$

Combining (14.7) and (14.10) we have an embedding

$$W_3 \hookrightarrow \psi/V_2.$$

Let V_3 be the preimage of W_3 in ψ. We see that in $R(G)$:

$$\psi \overset{R(G)}{=} W_1 + W_2 + W_3 + L(\psi).$$

We proved the filtration formula. Now we show the last claim, about maximality. Decompositions of k_1 and h_1, (14.1) and (14.4), show that no irreducible subquotient of W_2, can be a subrepresentation of ψ. They also show that $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c})$ and $L(\delta([\nu^{-a}\rho, \nu^c\rho]) \rtimes \sigma_b)$ can not be embedded into ψ/V_1. To see the same for factors of M, first assume that $\sigma^-_{b,c,a} \hookrightarrow \psi/V_1$. Since $k_1/V_1 \hookrightarrow \psi/V_1$, and k_1 doesn't contain $\sigma^-_{b,c,a}$, we obtain $\sigma^-_{b,c,a} \hookrightarrow \psi/k_1$. On the other hand $k_2 \hookrightarrow \psi/k_1$, and k_2 contains $\sigma^-_{b,c,a}$, but not as a subrepresentation. Since ψ is a multiplicity one, we got a contradiction. Similarly for the other factor of M. ◻

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