# THE PROBLEM OF DIOPHANTUS FOR INTEGERS OF $\mathbb{Q}(\sqrt{-3})$

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ABSTRACT. We solve the problem of Diophantus for integers of the quadratic field $\mathbb{Q}(\sqrt{-3})$ by finding a $D(z)$-quadruple in $\mathbb{Z}[(1+\sqrt{-3})/2]$ for each $z$ that can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{-3})$, up to finitely many possible exceptions.

## 1. INTRODUCTION AND PRELIMINARIES

Let $R$ be a commutative ring with unity 1 and $n \in R$. The set of nonzero and distinct elements $\{a_1, a_2, a_3, a_4\}$ in $R$ such that $a_i a_j + n$ is a perfect square in $R$ for $1 \leq i < j \leq 4$ is called a *Diophantine quadruple with the property $D(n)$* in $R$ or just a *$D(n)$-quadruple*. If $n = 1$ then a quadruple with a given property is called a *Diophantine quadruple*. The problem of constructing such sets was first studied by Diophantus of Alexandria who found the rational quadruple $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ with the property $D(1)$. Fermat found the first Diophantine quadruple in the ring of integers $\mathbb{Z}$ - the set $\{1, 3, 8, 120\}$.

The problem on the existence of $D(n)$-quadruples has been studied in different rings, but mainly in rings of integers of numbers fields. The following assertion is shown to be true in many cases: *There exists a $D(n)$-quadruple if and only if $n$ can be represented as a difference of two squares, up to finitely many exceptions.* In the ring $\mathbb{Z}$ one part of the assertion is proved independently by several authors (Brown, Gupta, Singh, Mohanty, Ramamsamy, see [6, 25, 27]), and another by Dujella in [7]. The set of possible exceptions $S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$ is still an open problem studied by many authors. The conjecture is that for $n \in S$ there does not exist a Diophantine quadruple with the property $D(n)$.

In the ring of integers $\mathbb{Z}$ well studied is the case of $n = -1$. There is a conjecture that $D(-1)$-quadruple does not exist in $\mathbb{Z}$. That is known as the $D(-1)$-*quadruple conjecture* and it was presented explicitly in [11] for the first time. While it is conjectured that $D(-1)$-quadruples do not exist in integers (see [11]), it is known that no $D(-1)$-quintuple exists and that if $\{a, b, c, d\}$ is a $D(-1)$-quadruple with $a < b < c < d$, then $a = 1$ (see [16]). It is proved that some infinite families of $D(-1)$-triples cannot be extended to a $D(-1)$-quadruple. The non-extendibility of $\{1, b, c\}$ was confirmed for $b = 2$ by Dujella in [10], for $b = 5$ partially by Abu Muriefah and Al Rashed in [2], and completely by Filipin in [18]. The statement was also proved for $b = 10$ by Filipin in [18], and for $b = 17, 26, 37, 50$ by Fujita in [24]. Dujella, Filipin and Fuchs in [13] proved that there are at most finitely many $D(-1)$-quadruples, by giving an upper bound of $10^{903}$ for the number of $D(-1)$-quadruples. This bound was improved several times: to $10^{356}$ by Filipin and Fujita ([19]), to $4 \cdot 10^{70}$ by Bonciocat, Cipu and Mignotte ([5]) and very recently to $5 \cdot 10^{60}$ by Elsholtz, Filipin and Fujita ([15]).

In the ring of Gaussian integers $\mathbb{Z}[i]$ the above assertion was proved in [9]. Namely, if $a + bi$ is not representable as a difference of the squares of two elements in $\mathbb{Z}[i]$, and in contrary if $a + bi$ is not of such form and $a + bi \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then $D(a + bi)$-quadruple exists. Franušić in [20, 21, 22] found that a similar statement is true for rings of integers of some real quadratic fields, i.e. it can be seen that there exist infinitely many $D(n)$-quadruples if and only if $n$ can be represented as a difference of two squares of integers. To be more precise, assuming the solvability of certain Pellian equation ($x^2 - dy^2 = 2$ or $x^2 - dy^2 = 4$ in odd numbers) we are able to obtain an effective characterization of integers that can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{d})$ and then apply some polynomial formulas for Diophantine quadruples in a combination with elements of a small norm. Also, in [23] the existence problem in the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{2})$ has been completely solved.

The case of complex quadratic fields is more demanding because the set of elements with a small norm is poor (while in the real case there exist infinitely many units). A group of authors ([1, 17, 28]) worked on the problem of the existence of $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-2}]$ and contributed that the problem is almost completely solved. As a prominent case, there appear the case $z = -1$, which could not be solved by the standard method via polynomial formulas. In [29] and [30] Soldo studied $D(-1)$-triples of the form $\{1, b, c\}$ and the existence of $D(-1)$-quadruples of the form $\{1, b, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-t}], t > 0$, for $b = 2, 5, 10, 17, 26, 37$ or $50$. He proved a more general result i.e. if positive integer $b$ is a prime, twice prime or twice prime squared such that $\{1, b, c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-t}], t > 0$, then $c$ has to be an integer. As a consequence of this result, he showed that for $t \notin \{1, 4, 9, 16, 25, 36, 49\}$ there does not exist a subset of $\mathbb{Z}[\sqrt{-t}]$ of the form $\{1, b, c, d\}$ with the property that

the product of any two of its distinct elements diminished by 1 is a square of an element in $\mathbb{Z}[\sqrt{-t}]$. For those exceptional cases of $t$ he showed that there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-t}]$.

In this paper, we verify assertion on the existence of $D(z)$-quadruples in complex quadratic field $\mathbb{Q}(\sqrt{-3})$, i.e. in the corresponding ring of integers $\mathbb{Z}[(1+\sqrt{-3})/2]$. In other words, we show the following theorems.

THEOREM 1.1. *There exists a $D(z)$-quadruple in the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-3})$ if and only if $z$ can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{-3})$, up to possible exceptions $z \in \{-1, 3, \frac{1}{2} - \frac{1}{2}\sqrt{-3}, \frac{1}{2} + \frac{1}{2}\sqrt{-3}\}$.*

THEOREM 1.2. *There exists a $D(z)$-quadruple in the ring $\mathbb{Z}[\sqrt{-3}]$ if and only if $z$ can be represented as a difference of two squares of elements in $\mathbb{Z}[\sqrt{-3}]$, up to possible exceptions $z \in \{-4, -1, 3, 2 - 2\sqrt{-3}, 2 + 2\sqrt{-3}\}$.*

Although we have mentioned that the case of complex quadratic fields is rather complicated, observe that the Pellian equation $x^2 - dy^2 = 4$ is solvable for $d = -3$ in $\mathbb{Z}$ (the only solution is $1 + \sqrt{-3}$). To begin with, we will list briefly all statements that we require for the proofs of Theorem 1.1 and Theorem 1.2.

LEMMA 1.3 ([8, Theorem 1]). *Let $R$ be a commutative ring with the unity 1 and $m, k \in R$. The set*

$$(1.1) \quad \{m, m(3k+1)^2 + 2k, m(3k+2)^2 + 2k + 2, 9m(2k+1)^2 + 8k + 4\}$$

*has the $D(2m(2k+1)+1)$-property.*

The set (1.1) is a $D(2m(2k+1)+1)$-quadruple if it contains no equal elements or elements equal to zero.

LEMMA 1.4. *If $u$ is an element of a commutative ring $R$ with the unity 1 and $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$-quadruple in $R$, then $\{w_1 u, w_2 u, w_3 u, w_4 u\}$ is a $D(wu^2)$-quadruple in $R$.*

LEMMA 1.5 ([14, Theorem 1]). *An integer $z \in \mathbb{Q}(\sqrt{-3})$ can be represented as a difference of two squares of elements in $\mathbb{Z}[\sqrt{-3}]$ if and only if is one of the following forms*

$$2m + 1 + 2n\sqrt{-3}, \ 4m + 4n\sqrt{-3}, \ 4m + 2 + (4n+2)\sqrt{-3},$$

$m, n \in \mathbb{Z}$.

LEMMA 1.6 ([14, Theorem 2]). *An integer $z \in \mathbb{Q}(\sqrt{-3})$ can be represented as a difference of two squares of elements in $\mathbb{Z}[(1+\sqrt{-3})/2]$ if and only if is one of the following forms*

$2m + 1 + 2n\sqrt{-3}, \ 2m + (2n+1)\sqrt{-3}, \ 4m + 4n\sqrt{-3}, \ 4m + 2 + (4n+2)\sqrt{-3},$

$$\frac{2m+1}{2} + \frac{2n+1}{2}\sqrt{-3},$$

$m, n \in \mathbb{Z}$.

LEMMA 1.7 ([22, Lemma 5]). *For each $M, N \in \mathbb{Z}$, there exist $k \in \mathbb{Z}[(1 + \sqrt{-3})/2]$ such that*

1. $2M + 1 + 2N\sqrt{-3} = 2m(k+1) + 1$, *where* $m = \frac{1}{2} + \frac{1}{2}\sqrt{-3}$,
2. $4M + 3 + (4N + 2)\sqrt{-3} = 2m(2k+1) + 1$, *where* $m = 1 + \sqrt{-3}$,
3. $2M + (2N + 1)\sqrt{-3} = m(2k+1) + 1$, *where* $m = 1 + \sqrt{-3}$,
4. $2M + 1 + (2N + 1)\sqrt{-3} = m(2k+1) + 1$, *where* $m = \frac{1}{2} + \frac{1}{2}\sqrt{-3}$,
5. $\frac{2M+1}{2} + \frac{2N+1}{2}\sqrt{-3} = \frac{m}{2}(2k+1) + 1$, *where* $m = 1 + \sqrt{-3}$.

By using Lemmas 1.3, 1.4 and 1.7, we effectively construct Diophantine quadruples for integers of the forms given in Lemmas 1.5 and 1.6. The following assertion gives the nonexistence of a $D(z)$-quadruple in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ if $z$ cannot be represented as a difference of two squares in $\mathbb{Z}[(1 + \sqrt{-3})/2]$, i.e. if and only if $z$ is of the form $4m + 2 + 4n\sqrt{-3}$, $4m + (4n + 2)\sqrt{-3}$, $2m + 1 + (2n + 1)\sqrt{-3}$.

LEMMA 1.8 ([22, Theorem 2]). *If $z$ has one of the forms*

$$4m + 2 + 4n\sqrt{-3}, \ 4m + (4n + 2)\sqrt{-3}, \ 2m + 1 + (2n + 1)\sqrt{-3},$$

*where $m, n \in \mathbb{Z}$, then a $D(z)$-quadruple in $\mathbb{Z}[(1 + \sqrt{-3})/2]$ does not exist.*

The nonexistence of a $D(z)$-quadruple in $\mathbb{Z}[\sqrt{-3}]$ if $z$ cannot be represented as a difference of two squares of in $\mathbb{Z}[\sqrt{-3}]$ follows partially from Lemma 1.8 (if $z = 4m + 2 + 4n\sqrt{-3}$ or $z = 4m + (4n + 2)\sqrt{-3}$) and from the following assertion.

LEMMA 1.9. *Let $d \in \mathbb{Z}$ is not a perfect square. Then there is no $D(m + (2n + 1)\sqrt{d})$-quadruple in the ring $\mathbb{Z}[\sqrt{d}]$.*

PROOF. The proof of Proposition 1 in [1] given for $d = 2$ can be immediately rewritten for an arbitrary $d$.                                          □

## 2. $D(z)$ QUADRUPLES IN $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$

Let us denote the set

$$D_4 = \{mu, (m(3k+1)^2 + 2k)u, (m(3k+2)^2 + 2k + 2)u, (9m(2k+1)^2 + 8k + 4)u\}.$$

According to Lemmas 1.3 and 1.4, $D_4$ is $D((2m(2k + 1) + 1)u^2)$-quadruple if it contains no equal elements or elements equal to zero. This polynomial formula combining with specific values of $m$ and $u$ solves our problem, up to finitely many cases. Our results are listed in the tables of the following subsections.

## 2.1. $D(2m+1+2n\sqrt{-3})$-quadruples.

In this subsection, for integers $A$ and $B$, we will separate the cases of $z = 4A + 3 + (4B + 2)\sqrt{-3}$ and $z = 4A + 1 + 4B\sqrt{-3}$ to corresponding subcases.

| $z$ | $k$ | $m$ | $u$ | $D_4$ in | exceptions of $z$ |
|---|---|---|---|---|---|
| $4A+3+4B\sqrt{-3}$ | $A+B\sqrt{-3}$ | $1$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $-1, 3$ |
| $4A+1+(4B+2)\sqrt{-3}$ | $\frac{-1+2A}{2} + \frac{1+2B}{2}\sqrt{-3}$ | $1$ | $1$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-$ |
| $8A+3+(8B+2)\sqrt{-3}$ | $\frac{A+3B}{2} + \frac{-A+B}{2}\sqrt{-3}$ | $1+\sqrt{-3}$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $3+2\sqrt{-3}$ |
| $8A+7+(8B+6)\sqrt{-3}$ | $\frac{A+3B+2}{2} + \frac{-A+B}{2}\sqrt{-3}$ | $1+\sqrt{-3}$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $-1-2\sqrt{-3}$ |
| $8A+7+(8B+2)\sqrt{-3}$ | $\frac{A-3B-1}{2} + \frac{A+B+1}{2}\sqrt{-3}$ | $1-\sqrt{-3}$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $-1+2\sqrt{-3}$ |
| $8A+3+(8B+6)\sqrt{-3}$: | $\frac{A-3B-3}{2} + \frac{A+B+1}{2}\sqrt{-3}$ | $1-\sqrt{-3}$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $3-2\sqrt{-3}$ |
| $8A+5+8B\sqrt{-3}$ | $A+B\sqrt{-3}$ | $2$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $5, -3$ |
| $8A+1+(8B+4)\sqrt{-3}$ | $\frac{2A-1}{2} + \frac{2B+1}{2}\sqrt{-3}$ | $2$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $-$ |
| $8A+1+8B\sqrt{-3}$ | $A-1+B\sqrt{-3}$ | $4$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $1, 9, -7$ |
| $8A+5+(8B+4)\sqrt{-3}$ | $\frac{4A-2B-3}{4} + \frac{2B+1}{4}\sqrt{-3}$ | $4$ | $1$ | $\mathbb{Z}[\sqrt{-3}]$ | $-$ |

TABLE 1

It is easy to check that for those exceptions of $z$ in Table 1, the polynomial formula $D_4$ gives the set with two equal elements, or some element is equal to zero. Therefore, in those exceptions of $z$ (and all further exceptions), we used the method for the first time described in [8] (but only for quadruples in $\mathbb{Z}$), to construct $D(z)$-quadruples with all distinct elements, of the form $\{u, v, u+v+2r, u+4v+4r\}$, for some $u, v, r \in \mathbb{Z}[\sqrt{-3}]$, or $u, v, r \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, respectively. Except in cases of $z = -1, 3$, we found the following $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-3}]$:

- $\{3+\sqrt{-3}, 1-\sqrt{-3}, -2, -5-3\sqrt{-3}\}$ is the $D(3+2\sqrt{-3})$-quadruple,
- $\{3-\sqrt{-3}, 1+\sqrt{-3}, -2, -5+3\sqrt{-3}\}$ is the $D(3-2\sqrt{-3})$-quadruple,
- $\{1+3\sqrt{-3}, -1+\sqrt{-3}, 2, 1-\sqrt{-3}\}$ is the $D(-1-2\sqrt{-3})$-quadruple,
- $\{1-3\sqrt{-3}, -1-\sqrt{-3}, 2, 1+\sqrt{-3}\}$ is the $D(-1+2\sqrt{-3})$-quadruple,
- $\{8, 1+\sqrt{-3}, 1-\sqrt{-3}, -4\}$ is the $D(5)$-quadruple,
- $\{\sqrt{-3}, 3\sqrt{-3}, 8\sqrt{-3}, 120\sqrt{-3}\}$ is the $D(-3)$-quadruple,
- $\{1, 3, 8, 120\}$ is the $D(1)$-quadruple,
- $\{6, -2-2\sqrt{-3}, -2+2\sqrt{-3}, -14\}$ is the $D(9)$-quadruple,
- $\{2+2\sqrt{-3}, 1+\sqrt{-3}, 1-\sqrt{-3}, 2-2\sqrt{-3}\}$ is the $D(-7)$-quadruple.

### 2.2. $D(2m + (2n + 1)\sqrt{-3})$-quadruples.

| $z$ | $k$ | $m$ | $u$ | $D_4$ in | exceptions of $z$ |
|---|---|---|---|---|---|
| $4A + (4B + 3)\sqrt{-3}$ | $\frac{A+3B+1}{2} + \frac{-A+B+1}{2}\sqrt{-3}$ | $\frac{1+\sqrt{-3}}{2}$ | $1$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-\sqrt{-3}$ |
| $4A + 2 + (4B + 1)\sqrt{-3}$ | $\frac{A+3B}{2} + \frac{-A+B}{2}\sqrt{-3}$ | $\frac{1+\sqrt{-3}}{2}$ | $1$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $2 + \sqrt{-3}$ |
| $4A + (4B + 1)\sqrt{-3}$ | $\frac{A-3B}{2} - 1 + \frac{A+B}{2}\sqrt{-3}$ | $\frac{1-\sqrt{-3}}{2}$ | $1$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $\sqrt{-3}$ |
| $4A + 2 + (4B + 3)\sqrt{-3}$ | $\frac{A-3B-3}{2} + \frac{A+B+1}{2}\sqrt{-3}$ | $\frac{1-\sqrt{-3}}{2}$ | $1$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $2 - \sqrt{-3}$ |

TABLE 2

For the exceptions noted in Table 2, we found the following $D(z)$-quadruples in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$:

- $\{\frac{1}{2} - \frac{3}{2}\sqrt{-3}, -1, \frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{3}{2} + \frac{1}{2}\sqrt{-3}\}$ is the $D(-\sqrt{-3})$-quadruple,
- $\{\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{7}{2} + \frac{1}{2}\sqrt{-3}, -2, -\frac{23}{2} + \frac{1}{2}\sqrt{-3}\}$ is the $D(2+\sqrt{-3})$-quadruple,
- $\{\frac{1}{2} + \frac{3}{2}\sqrt{-3}, -1, \frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{3}{2} - \frac{1}{2}\sqrt{-3}\}$ is the $D(\sqrt{-3})$-quadruple,
- $\{\frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{7}{2} - \frac{1}{2}\sqrt{-3}, -2, -\frac{23}{2} - \frac{1}{2}\sqrt{-3}\}$ is the $D(2-\sqrt{-3})$-quadruple.

### 2.3. $D(4m + 4n\sqrt{-3})$-quadruples.

| $z$ | $k$ | $m$ | $u$ | $D_4$ in | exceptions of $z$ |
|---|---|---|---|---|---|
| $8A + 8B\sqrt{-3}$ | $\frac{-A+3B-2}{2} - \frac{A+B}{2}\sqrt{-3}$ | $\frac{1}{2}$ | $1 + \sqrt{-3}$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $0$ |
| $8A + 4 + (8B + 4)\sqrt{-3}$ | $\frac{-A+3B-1}{2} - \frac{A+B+1}{2}\sqrt{-3}$ | $\frac{1}{2}$ | $1 + \sqrt{-3}$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-4 + 4\sqrt{-3}$ |

TABLE 3

The set $\{1, 2 - 2\sqrt{-3}, 5, 13 - 4\sqrt{-3}\}$ is the $D(-4 + 4\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$ and it is easy to see that there exits infinitely many $D(0)$-quadruples.

We obtain a $D(8A + 4 + 8B\sqrt{-3})$-quadruple by multiplying elements of a $D(2m + 1 + 2n\sqrt{-3})$-quadruple by $u = 2$ except for $z = -4, 12$, but

$$\{1, \frac{7}{2} + \frac{1}{2}\sqrt{-3}, \frac{7}{2} - \frac{1}{2}\sqrt{-3}, 13\}$$

is the $D(-4)$-quadruple, and

$$\{-2, 7 + \sqrt{-3}, 7 - \sqrt{-3}, 30\}$$

is the $D(12)$-quadruple. Also, a $D(8A + (8B + 4)\sqrt{-3})$-quadruple is obtained by multiplying elements of a $D(2m + (2n + 1)\sqrt{-3})$-quadruple by $u = 2$. Obviously, the resulting sets are subsets of $\mathbb{Z}[\sqrt{-3}]$ (except for $z = -4$).

### 2.4. $D(4m + 2 + (4n + 2)\sqrt{-3})$-quadruples.

| $z$ | $k$ | $m$ | $u$ | $D_4$ in | exceptions of $z$ |
|---|---|---|---|---|---|
| $8A+2+(8B+2)\sqrt{-3}$ | $-A-1-B\sqrt{-3}$ | $\frac{1+\sqrt{-3}}{4}$ | $1+\sqrt{-3}$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-6+2\sqrt{-3},$ $2+2\sqrt{-3}$ |
| $8A+6+(8B+6)\sqrt{-3}$ | $-\frac{2A+3}{2}-\frac{2B+1}{2}\sqrt{-3}$ | $\frac{1+\sqrt{-3}}{4}$ | $1+\sqrt{-3}$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-$ |
| $8A+2+(8B+6)\sqrt{-3}$ | $-(A+1)-(B+1)\sqrt{-3}$ | $\frac{1-\sqrt{-3}}{4}$ | $1-\sqrt{-3}$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-6-2\sqrt{-3},$ $2-2\sqrt{-3}$ |
| $8A+6+(8B+2)\sqrt{-3}$ | $-\frac{2A+3}{2}-\frac{2B+1}{2}\sqrt{-3}$ | $\frac{1-\sqrt{-3}}{4}$ | $1-\sqrt{-3}$ | $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $-$ |

TABLE 4

While the polynomial formula $D_4$ gives sets with two equal elements, for those exceptions of $z$ of Table 4, we found the following $D(z)$-quadruples in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$:

- $\{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -9-2\sqrt{-3}, -\frac{25}{2}-\frac{1}{2}\sqrt{-3}, -\frac{85}{2}-\frac{11}{2}\sqrt{-3}\}$ is the $D(-6+2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{5}{2}+\frac{3}{2}\sqrt{-3}, -1+2\sqrt{-3}, -\frac{13}{2}+\frac{13}{2}\sqrt{-3}\}$ is the $D(2+2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -9+2\sqrt{-3}, -\frac{25}{2}+\frac{1}{2}\sqrt{-3}, -\frac{85}{2}+\frac{11}{2}\sqrt{-3}\}$ is the $D(-6-2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -\frac{5}{2}-\frac{3}{2}\sqrt{-3}, -1-2\sqrt{-3}, -\frac{13}{2}-\frac{13}{2}\sqrt{-3}\}$ is the $D(2-2\sqrt{-3})$-quadruple.

### 2.5. $D(\frac{2m+1}{2} + \frac{2n+1}{2}\sqrt{-3})$-quadruples.

We derive $D(\frac{2m+1}{2} + \frac{2n+1}{2}\sqrt{-3})$-quadruples from $D(2m + 1 + 2n\sqrt{-3})$ and $D(2m+(2n+1)\sqrt{-3})$ -quadruples by multiplying them by $\frac{1+\sqrt{-3}}{2}$ and $\frac{1-\sqrt{-3}}{2}$.

- Multiplying the elements of a $D(2m + 1 + 2n\sqrt{-3})$-quadruple by $u = \frac{1+\sqrt{-3}}{2}$ we obtain a $D((2m + 1 + 2n\sqrt{-3})u^2)$-quadruple except for $z = \frac{1}{2} - \frac{1}{2}\sqrt{-3}, -\frac{3}{2} + \frac{3}{2}\sqrt{-3}$. The number $(2m + 1 + 2n\sqrt{-3})u^2$ is of the form $\frac{2A+1}{2} + \frac{2B+1}{2}\sqrt{-3}$ and for given $A, B \in \mathbb{Z}$ the equation

$$(2.2) \qquad (2m + 1 + 2n\sqrt{-3})u^2 = \frac{2A + 1}{2} + \frac{2B + 1}{2}\sqrt{-3}$$

  has an integer solution $(m, n \in \mathbb{Z})$ if and only if $-A+3B \equiv 1 \pmod 4$ and $A+B \equiv 3 \pmod 4$, i.e. $(A, B) \mod 4 \in \{(0,3),(1,2),(2,1),(3,0)\}$. Concerning exceptions, the set

$$\{\frac{1}{2} + \frac{1}{2}\sqrt{-3}, -\frac{5}{2} - \frac{3}{2}\sqrt{-3}, -1 - 2\sqrt{-3}, -\frac{15}{2} - \frac{15}{2}\sqrt{-3}\}$$

represents the $D(-\frac{3}{2}+\frac{3}{2}\sqrt{-3})$-quadruple, while we could not find the $D(\frac{1}{2}-\frac{1}{2}\sqrt{-3})$-quadruple.

- Multiplying the elements of a $D(2m+1+2n\sqrt{-3})$-quadruple by $u = \frac{1-\sqrt{-3}}{2}$ we obtain a $D((2m+1+2n\sqrt{-3})u^2)$-quadruple except for $z = \frac{1}{2}+\frac{1}{2}\sqrt{-3}$, $-\frac{3}{2}-\frac{3}{2}\sqrt{-3}$. For given $A, B \in \mathbb{Z}$ the equation (2.2) has an integer solution if and only if $A + 3B \equiv 0 \pmod 4$ and $A - B \equiv 0 \pmod 4$, i.e. $(A, B) \mod 4 \in \{(0,0),(1,1),(2,2),(3,3)\}$. The set

$$\{\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -\frac{5}{2}+\frac{3}{2}\sqrt{-3}, -1+2\sqrt{-3}, -\frac{15}{2}+\frac{15}{2}\sqrt{-3}\}$$

  is the $D(-\frac{3}{2}-\frac{3}{2}\sqrt{-3})$-quadruple and we have not detected a $D(\frac{1}{2}+\frac{1}{2}\sqrt{-3})$-quadruple.

- Multiplying the elements of a $D(2m+(2n+1)\sqrt{-3})$-quadruple by $u = \frac{1+\sqrt{-3}}{2}$ we obtain a $D(\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3})$-quadruple. For given $A, B \in \mathbb{Z}$ the equation

$$(2.3) \qquad (2m+(2n+1)\sqrt{-3})u^2 = \frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3}$$

  has an integer solution if and only if $-A+3B \equiv 3 \pmod 4$ and $A+B \equiv 1 \pmod 4$, i.e. $(A, B) \mod 4 \in \{(0,1),(1,0),(3,2),(2,3)\}$.

- Multiplying the elements of a $D(2m+(2n+1)\sqrt{-3})$-quadruple by $u = \frac{1-\sqrt{-3}}{2}$ we obtain a $D(\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3})$-quadruple. For given $A, B \in \mathbb{Z}$ the equation (2.3) has an integer solution if and only if $A + 3B \equiv 2 \pmod 4$ and $-A + B \equiv 2 \pmod 4$, i.e. $(A, B) \mod 4 \in \{(0,2),(2,0),(1,3),(3,1)\}$.

## 3. $D(z)$ QUADRUPLES IN $\mathbb{Z}[\sqrt{-3}]$

In the previous section we see that some $D(z)$-quadruples that have been constructed already lie in $\mathbb{Z}[\sqrt{-3}]$ but some of them do not although $z$ can be represented as a difference of squares in $\mathbb{Z}[\sqrt{-3}]$. Here we show that this can be improved.

### 3.1. $D(2m+1+2n\sqrt{-3})$-quadruples.

| $z$ | $k$ | $m$ | $u$ | $D_4$ in | exceptions of $z$ |
|---|---|---|---|---|---|
| $4A+1+(4B+2)\sqrt{-3}$ | $\frac{2A-2B+1}{2}+\frac{A+1}{2}\sqrt{-3}$ | $\sqrt{-3}/3$ | $\sqrt{-3}$ | $\mathbb{Z}[\sqrt{-3}]$ | $-3-2\sqrt{-3},$ $-3+2\sqrt{-3}$ |

TABLE 5

The set $\{-\sqrt{-3}, -2+\sqrt{-3}, -2, -8+3\sqrt{-3}\}$ is a $D(-3-2\sqrt{-3})$, while the set $\{\sqrt{-3}, -2-\sqrt{-3}, -2, -8-3\sqrt{-3}\}$ is a $D(-3+2\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$.

### 3.2. $D(4m + 2 + (4n + 2)\sqrt{-3})$-quadruples.

Since there exist a $D(\frac{2m+1}{2} + \frac{2n+1}{2}\sqrt{-3})$-quadruple in $\mathbb{Z}[(1 + \sqrt{-3})/2]$, by multiplying by 2 the elements of this quadruple we obtain a $D(4m + 2 + (4n + 2)\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$, up to $z = 2 - 2\sqrt{-3}, 2 + 2\sqrt{-3}$.

### 3.3. $D(4m + 4n\sqrt{-3})$-quadruples.

We have shown in § 2.3. that $D(8m + (8n + 4)\sqrt{-3})$ and $D(8m + 4 + 8n\sqrt{-3})$-quadruples in $\mathbb{Z}[\sqrt{-3}]$ are obtained by multiplying by 2 the elements of $D(2m + (2n+1)\sqrt{-3})$ and $D(2m + 1 + 2n\sqrt{-3})$-quadruples in $\mathbb{Z}[(1 + \sqrt{-3})/2]$ up to the the $D(-4)$-quadruple whose elements are not in $\mathbb{Z}[\sqrt{-3}]$.

The set

$$\{1, 9k^2 - 8k, 9k^2 - 2k + 1, 36k^2 - 20k + 1\}$$

is $D(8k)$-quadruple ([7]) if $k \neq 0, 1$, so there exists a $D(8m + 8n\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$.

| $z$ | $k$ | $m$ | $u$ | $D_4$ in | exceptions of $z$ |
|---|---|---|---|---|---|
| $8A + 4 + (8B + 4)\sqrt{-3}$ | $\frac{3A - 2B + 4}{2} + \frac{A+2}{2}\sqrt{-3}$ | $\frac{\sqrt{-3}}{6}$ | $2\sqrt{-3}$ | $\mathbb{Z}[\sqrt{-3}]$ | $-12 - 4\sqrt{-3}$ $-12 + 4\sqrt{-3}$ |

TABLE 6

It is easy to check that for those exceptions of $z$ in Table 6, the polynomial formula $D_4$ gives the set with two equal elements. Therefore for certain $z$, we found the following $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-3}]$:

- $\{2 + \sqrt{-3}, 2 - 2\sqrt{-3}, 2 - 3\sqrt{-3}, 6 - 11\sqrt{-3}\}$ is the $D(-12 + 4\sqrt{-3})$-quadruple,
- $\{2 - \sqrt{-3}, 2 + 2\sqrt{-3}, 2 + 3\sqrt{-3}, 6 + 11\sqrt{-3}\}$ is the $D(-12 - 4\sqrt{-3})$-quadruple,
- $\{2 + \sqrt{-3}, -2 + 2\sqrt{-3}, -2 + \sqrt{-3}, -10 + 5\sqrt{-3}\}$ is the $D(8)$-quadruple.

REMARK 3.1. Concerning the list of possible exceptions given in Theorem 1.1 and Theorem 1.2, we can easily observe that $3 = -1 \cdot (\sqrt{-3})^2$, $-4 = -1 \cdot 2^2$, $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = -1 \cdot (\frac{1}{2} \mp \frac{1}{2}\sqrt{-3})^2$ and $2 \pm 2\sqrt{-3} = -1 \cdot (1 \mp \sqrt{-3})^2$. So, we are not surprised that the key point lies in an investigation on the existence of $D(-1)$-quadruples in rings $\mathbb{Z}[(1 + \sqrt{-3})/2]$ and $\mathbb{Z}[\sqrt{-3}]$. In an analogy to $D(-1)$-quadruple conjecture in the ring of integers and the problem of existence of $D(-1)$-quadruples in $\mathbb{Z}[\sqrt{-t}], t > 0$ studied in [29] and [30], we might expect that for such $z$ there does not exists a $D(z)$-quadruple.

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