AN EXTENSION OF AN OLD PROBLEM OF DIOPHANTUS AND EULER

Andrej Dujella

Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia

Diophantus studied the following problem: Find three (rational) numbers such that the product of any two increased by the sum of those two gives a square. He obtained the solutions $\{4, 9, 28\}$ and $\{\frac{3}{10}, \frac{21}{5}, \frac{7}{10}\}$ (see [3]). Euler treated the same problem with four numbers (see [2]). He found the solution $\{\frac{65}{224}, \frac{9}{224}, \frac{9}{56}, \frac{5}{2}\}$. Indeed, we have

$$\frac{65}{224}\cdot\frac{9}{224}+\frac{65}{224}+\frac{9}{224}=(\frac{131}{224})^2, \qquad \frac{65}{224}\cdot\frac{9}{56}+\frac{65}{224}+\frac{9}{56}=(\frac{79}{112})^2,$$
$$\frac{65}{224}\cdot\frac{5}{2}+\frac{65}{224}+\frac{5}{2}=(\frac{15}{8})^2, \qquad \frac{9}{224}\cdot\frac{9}{56}+\frac{9}{224}+\frac{9}{56}=(\frac{51}{112})^2,$$
$$\frac{9}{224}\cdot\frac{5}{2}+\frac{9}{224}+\frac{5}{2}=(\frac{13}{8})^2, \qquad \frac{9}{56}\cdot\frac{5}{2}+\frac{9}{56}+\frac{5}{2}=(\frac{7}{4})^2.$$

In the present paper we will construct the set of **five** numbers with the above property.

Let $\{x_1, \ldots, x_m\}$ be a set of rational numbers such that $x_ix_j + x_i + x_j$ is a perfect square for all $1 \le i < j \le m$. Since

$$x_ix_j + x_i + x_j = (x_i + 1)(x_j + 1) - 1,$$

if we put $x_i + 1 = a_i$, $i = 1, \ldots, m$, we obtain the set $\{a_1, \ldots, a_m\}$ with the property that the product of its any two distinct elements diminished by 1 is a perfect square. Such a set is called *a (rational) Diophantine m-tuple with the property* $D(-1)$ (see [4, p. 75]). If a_i's are positive integers, such a set is also called *a P_{-1}-set of size m*. The conjecture is that there does not exist a P_{-1}-set of size 4. Let us mention that in [1, 6, 7], it was proved that some particular P_{-1}-sets of size 3 cannot be extended to a P_{-1}-set of size 4. In [5], some consequences of the above conjecture were considered.

We will derive a two-parametric formula for Diophantine quintuples, and as a consequence we will obtain a rational Diophantine quintuple with the property $D(-1)$.

We will consider quintuples of the form $\{A, B, C, D, x^2\}$ with the property $D(\alpha x^2)$, where A, B, C, D, x, α are integers. Furthermore, we will use the following simple result known already to Euler: If $BC+n = k^2$, then the set $\{B, C, B+C\pm 2k\}$ has the property $D(n)$.

Therefore, if we assume that

$$BC + \alpha x^2 = k^2, \quad A = B + C - 2k, \quad D = B + C + 2k,$$

then the set $\{A, B, C, D, x^2\}$ has the property $D(\alpha x^2)$ if and only if $AD+\alpha x^2$ is a perfect square. Hence, we reduced the original $\binom{5}{2} = 10$ conditions to only two conditions:

$$(b^2 - \alpha)(c^2 - \alpha) + \alpha x^2 = k^2, \tag{1}$$

$$(a^2 - \alpha)(d^2 - \alpha) + \alpha x^2 = y^2. \tag{2}$$

Our assumptions

$$(b^2 - \alpha) + (c^2 - \alpha) - 2k = a^2 - \alpha, \quad (b^2 - \alpha) + (c^2 - \alpha) + 2k = d^2 - \alpha$$

imply that $4k = (d + a)(d - a)$. Let $d + a = 2p$, $d - a = 2r$. It implies that $k = pr$ and

$$b^2 + c^2 - \alpha = \frac{1}{2}(a^2 + d^2) = p^2 + r^2. \tag{3}$$

Let us rewrite condition (2) in the form

$$(ad - \alpha)^2 - \alpha(d - a)^2 = y^2 - \alpha x^2.$$

Thus, we may take

$$y = ad - \alpha, \qquad x = d - a = 2r. \tag{4}$$

Substituting (3) and (4) into (1) we obtain

$$p^2r^2 - b^2c^2 = 4\alpha r^2 - \alpha(b^2 + c^2 - \alpha) = \alpha(3r^2 - p^2). \tag{5}$$

At this point we make the further assumption (motivated by (3) and (5)):

$$b + c = p + r. \tag{6}$$

Now (3) implies

$$pr - bc = \frac{\alpha}{2}, \tag{7}$$

and (5) implies

$$pr + bc = 2(3r^2 - p^2). \tag{8}$$

Adding (7) and (8) yields

$$\alpha = 4p^2 + 4pr - 12r^2. \tag{9}$$

From (6) and (7) we conclude that b and c are the solutions of the quadratic equation

$$z^2 - (p+r)z + (pr - \frac{\alpha}{2}) = 0.$$

The discriminant of this equation has to be a perfect square. Thus,

$$(p-r)^2 + 2\alpha = q^2. \tag{10}$$

Substituting (9) into (10) we have, finally,

$$(3p+r)^2 - 24r^2 = q^2. \tag{11}$$

Hence, we reduce our problem to the solving equation (11). However, the general solution of the equation $u^2 - 24v^2 = w^2$ with $(u, v, w) = 1$ is given by $u = e^2 + 6f^2$, $v = ef$, $w = |e^2 - 6f^2|$ or $u = 2e^2 + 3f^2$, $v = ef$, $w = |2e^2 - 3f^2|$ (see [8, p. 225]). Thus we proved

Theorem 1 *If* $e \equiv 0 \pmod 3$ *or* $e \equiv f \pmod 3$*, then the set*

$$\begin{aligned} &\{\tfrac{1}{3}(e^2 + 6ef - 18f^2)(2f^2 + 2ef - e^2),\ \tfrac{1}{3}e^2(e+5f)(3f-e), \\ &f^2(e-2f)(5e+6f),\ \tfrac{1}{3}(e^2 + 4ef - 6f^2)(6f^2 + 4ef - e^2),\ 4e^2f^2\} \end{aligned} \tag{12}$$

has the property $D(\frac{16}{9}e^2f^2(e^2 - ef - 3f^2)(e^2 + 2ef - 12f^2))$*, and the set*

$$\begin{aligned} &\{\tfrac{1}{3}(9f^2 + 6ef - 2e^2)(2e^2 + 2ef - f^2),\ \tfrac{1}{3}e^2(5f-2e)(2e+3f), \\ &f^2(e+f)(5e-3f),\ \tfrac{1}{3}(3f^2 + 4ef - 2e^2)(2e^2 + 4ef - 3f^2),\ 4e^2f^2\} \end{aligned} \tag{13}$$

has the property $D(\frac{16}{9}e^2f^2(e^2 - ef - 3f^2)(4e^2 + 2ef - 3f^2))$.

Substituting $e = 5$ and $f = 2$ in (12) we obtain the following two corollaries.

Corollary 1 *The set* $\{\frac{13}{40}, \frac{25}{8}, \frac{37}{10}, 10, \frac{533}{40}\}$ *is a rational Diophantine quintuple with the property* $D(-1)$.

Corollary 2 *The five numbers* $-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}$ *have the property that the product of any two of them increased by the sum of those two gives a perfect square.*

References

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