THE NON-EXTENSIBILITY OF $D(-2k+1)$-TRIPLES $\{1, k^2, k^2+2k-1\}$

BILGE PEKER, ANDREJ DUJELLA, AND SELIN (INAG) CENBERCI

Abstract. In this paper we prove that for an integer k such that $|k| \geq 2$, the $D(-2k+1)$-triple $\{1, k^2, k^2+2k-1\}$ cannot be extended to a $D(-2k+1)$-quadruple.

1. Introduction

Let n be an integer. A set of m distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ is said to have the property $D(n)$ if $a_i \cdot a_j + n$ is a perfect square for all $1 \leq i < j \leq m$. Such a set is called a Diophantine m-tuple with the property $D(n)$ (or $D(n)$-m-tuple, or P_n-set of size m). Fermat found the first example of a $D(1)$-quadruple, the set $\{1, 3, 8, 120\}$. Baker and Davenport [1] proved that Fermat's set cannot be extended to a $D(1)$-quintuple. Dujella and Pethő [7] proved that even the $D(1)$-pair $\{1, 3\}$ cannot be extended to a $D(1)$-quintuple. Dujella [4] proved that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples. Brown [2], Gupta and Singh [10] and Mohanty and Ramasamy [12] proved independently that if n is an integer such that $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple. On the other hand, Dujella [3] proved that if an integer n satisfies $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exist at least one $D(n)$-quadruple.

Many authors considered the problems of (non)extensibility of particular $D(n)$-triples, but also for parametric families of $D(n)$-triples (see references given in [6]). Let $\{a, b, c\}$ be a $D(n)$-triple. The problem of its extensibility to a $D(n)$-quadruple leads to a system of Diophantine equations. These equations are Pellian equations, unless one of the numbers (of polynomials in the parametric case) ab, ac and bc is a perfect square. One such case was considered by Fujita and Togbé [9]. Namely, they considered $D(-k^2)$-triple $\{1, k^2, k^2+1\}$ and they proved that if $\{1, k^2, k^2+1, d\}$ is a $D(-k^2)$-quadruple, then $d = 4k^2+1$, and in this case, $3k^2+1$ has to be a perfect square.

In the present paper, we will consider the $D(-2k+1)$-triple $\{1, k^2, k^2+2k-1\}$. This is indeed a $D(-2k+1)$-triple since $1 \cdot k^2 + (-2k+1) = (k-1)^2$,

---

2010 *Mathematics Subject Classification.* 11D45, 11D99.
Key words and phrases. Diophantine m-tuples, Pell equations.

$1 \cdot (k^2 + 2k - 1) + (-2k + 1) = k^2$ and $k^2 \cdot (k^2 + 2k - 1) + (-2k + 1) = (k^2 + k - 1)^2$. Note that for an integer k such that $k \neq 0, \pm 1$, the elements of $\{1, k^2, k^2 + 2k - 1\}$ are nonzero and distinct integers.

Our main result is the following theorem.

Theorem 1. *Let k be an integer such that $|k| \geq 2$. Then the $D(-2k+1)$-triple $\{1, k^2, k^2 + 2k - 1\}$ cannot be extended to a $D(-2k + 1)$-quadruple.*

Note that the statement of Theorem 1 is very simple for $k \equiv 2 \pmod 4$. Indeed, in this case we have the $D(n)$ triple $\{a, b, c\}$ with $n \equiv 5 \pmod 8$, $a \equiv 1 \pmod 8$, $b \equiv 4 \pmod 8$, $c \equiv 7 \pmod 8$. Assume that $\{a, b, c, d\}$ is a $D(n)$-quadruple. If d is odd, then $d \equiv a \pmod 4$ or $d \equiv c \pmod 4$, so one of the numbers $ad + n$, $cd + n$ is $\equiv 2 \pmod 4$, and thus it is not a perfect square. If d is even, then $bd + n \equiv 5 \pmod 8$ is not a perfect square.

In the proof of Theorem 1 we will use the fact that the product of first two elements of the triple $\{1, k^2, k^2 + 2k - 1\}$ is a perfect square. This will allow us to get simple upper bound for the solutions of the corresponding Diophantine equation. Other equation in the system will be a Pellian equation. We will determine the sequences to its solution by using tools of the Diophantine approximations (continued fractions). The comparison with previously obtained upper bounds for the solutions will leave only few initial elements in the sequences to be checked whether they satisfy the other equation of the system. Technical details (in particular, continued fraction expansion of the corresponding quadratic irrationals) differ slightly depending on whether the parameter k is positive or negative. We will handle positive case in Section 2 and negative case in Section 3.

2. Proof of Theorem 1 for positive k

Assume that there is a positive integer d such that the set $\{1, k^2, k^2 + 2k - 1, d\}$ is a $D(-2k + 1)$-quadruple. Then there exist nonnegative integers x, y, z satisfying

$$d - 2k + 1 = x^2, \tag{1}$$
$$dk^2 - 2k + 1 = y^2, \tag{2}$$
$$d(k^2 + 2k - 1) - 2k + 1 = z^2. \tag{3}$$

By eliminating d from the above equations, we obtain the system of equations

$$y^2 - k^2 x^2 = (k^2 - 1)(2k - 1), \tag{4}$$
$$k^2 z^2 - (k^2 + 2k - 1) y^2 = (2k - 1)^2. \tag{5}$$

From (4), we have $y^2 - k^2x^2 > 0$, which implies $0 \le kx \le y-1$. Thus, we have

$$y^2 \ge 2k^3 - k^2 - 2k + 1 > k^3 \tag{6}$$

and

$$y^2 \le (y-1)^2 + 2k^3 - k^2 - 2k + 1,$$

which implies

$$y \le (2k^3 - k^2 - 2k + 2)/2 < k^3. \tag{7}$$

Therefore,

$$k^{3/2} < y < k^3. \tag{8}$$

Next consider the equation (5). We have

$$0 < kz - \sqrt{k^2+2k-1}y = \frac{(2k-1)^2}{kz + \sqrt{k^2+2k-1}y} < \frac{4k^2}{2ky} = \frac{2k}{y}.$$

Thus, we get

$$\left| \frac{z}{y} - \frac{\sqrt{k^2+2k-1}}{k} \right| < \frac{2}{y^2},$$

so z/y is "almost" a convergent of continued fraction of $\frac{\sqrt{k^2+2k-1}}{k}$. If $\gcd(z,y) = g > 1$, $z = gz'$, $y = gy'$, then

$$\left| \frac{z'}{y'} - \frac{\sqrt{k^2+2k-1}}{k} \right| < \frac{1}{2y'^2},$$

and z'/y' is a convergent.

By applying the algorithm for continued fraction expansion of quadratic irrationals (see e.g. [13, Section 7.7]) to $\alpha = \frac{\sqrt{k^2+2k-1}}{k} = \frac{\sqrt{(k^2+2k-1)k^2}}{k^2}$, we get $s_0 = 0$, $t_0 = k^2$, $a_0 = 1$, $s_1 = k^2$, $t_1 = 2k-1$, $a_1 = k+2$, $s_2 = k^2+k-1$, $t_2 = 1$, $a_2 = 2k^2+2k-2$, $s_3 = k^2+k-1$, $t_3 = 2k-1$, $a_3 = k+1$, $s_4 = k^2$, $t_4 = k^2$, $a_4 = 2$, $(s_5, t_5) = (s_1, t_1)$. Thus, we have the following continued fraction expansion

$$\frac{\sqrt{k^2+2k-1}}{k} = [1, \overline{k+1, 2k^2+2k-2, k+1, 2}].$$

By [8, Lemma 1], we get

$$k^2z'^2 - (k^2+2k-1)y'^2 = 1, -2k+1 \text{ or } k^2.$$

Thus

$$(2k-1)^2 = g^2, g^2(-2k+1) \text{ or } g^2k^2,$$

so the only possibility is $g = 2k - 1$. Then we get the sequence of possible y' satisfying the equation

$$k^2 z'^2 - (k^2 + 2k - 1)y'^2 = 1.$$

Its initial values are $y_1' = k + 1$, $y_2'^5 + 20k^4 + 32k^3 + 16k^2 - k - 1, \ldots$. Since $y = gy'$, only the first value gives y satisfying the inequalities (8). Thus, the only candidate for a solution is $y = 2k^2+k-1$. By (4), it gives $x^2 = 4k^2+2k-2$, which is not possible (only solutions are $k = \pm 1$).

It remains to consider the case $g = 1$. Now candidates for solutions of (5) are not necessarily convergents, but also small linear combinations of successive convergents. More precisely, by Worley's theorem [14, 5], we have $z = rp_i \pm sp_{i-1}$, $y = rq_i \pm sq_{i-1}$, where $0 \le rs < 4$. By using [8, Lemma 1] again, we find that all solutions of (5) with $g = 1$ are given by $z = p_{4j+1} - 2p_{4j}$, $y = q_{4j+1} - 2q_{4j}$ and $z = 2p_{4j+2} + p_{4j+1}$, $y = 2q_{4j+2} + q_{4j+1}$. So, the sequence of solutions in y starts as

$$y_1 = k - 1,\ y_2 = 4k^3 + 8k^2 + k - 1,\ y_3 = 4k^5 + 12k^4 + 4k^3 - 8k^2 - k + 1,\ \ldots.$$

Thus, clearly there is no solutions satisfying the inequality (8).

3. Proof of Theorem 1 for negative k

The proof of Theorem 1 for negative k's follows the same lines as the proof for positive k's given in the previous section. So, we will give only the sketch of the proof here. We write $k = -K$, with positive integer $K \ge 2$. For $K = 2$, we have the $D(5)$-triple $\{1, 4, -1\}$ which clearly cannot be extended to a $D(5)$-quadruple because it contains elements with mixed signs. For $K = 3$, we have the $D(7)$-triple $\{1, 9, 2\}$ and its non-extensibility has been proved by Kaygisiz and Senay [11]. Thus, we may assume that $K \ge 4$.

Assume that there is a positive integer d such that the set $\{1, K^2, K^2 - 2K - 1, d\}$ is a $D(2K + 1)$-quadruple. Then there exist nonnegative integers x, y, z satisfying

$$d + 2K + 1 = x^2, \tag{9}$$

$$dK^2 + 2K + 1 = y^2, \tag{10}$$

$$d(K^2 - 2K - 1) + 2K + 1 = z^2. \tag{11}$$

Now, the system (4) and (5) becomes

$$y^2 - K^2x^2 = -(K^2 - 1)(2K + 1), \tag{12}$$

$$K^2z^2 - (K^2 - 2K - 1)y^2 = (2K + 1)^2. \tag{13}$$

In this case, from (12), we first get $y \le Kx - 1$, then $K^2x^2 \le (Kx - 1)^2 + 2K^3 + K^2 - 2K - 1$, which implies $x < 2K^2$, and finally

$$y < 2K^3. \tag{14}$$

Similarly as before, from the equation (5), we obtain

$$0 < Kz - \sqrt{K^2-2K-1}y = \frac{(2K+1)^2}{Kz+\sqrt{K^2-2K-1}y} < \frac{4K}{y},$$

and thus

$$\left|\frac{z}{y} - \frac{\sqrt{K^2-2K-1}}{K}\right| < \frac{4}{y^2}.$$

If $\gcd(z, y) = g > 1$, then from (10) and (11) we see that y and z cannot be both even, hence $g \geq 3$. Thus, by putting $z = gz'$, $y = gy'$, we obtain

$$\left|\frac{z'}{y'} - \frac{\sqrt{K^2-2K-1}}{K}\right| < \frac{1}{2y'^2},$$

and z'/y' is a convergent of the continued fraction expansion of $\alpha' = \frac{\sqrt{K^2-2K-1}}{K}$. We have the following continued fraction expansion

$$\frac{\sqrt{K^2-2K-1}}{K} = [0, 1, \overline{K-3, 1, 2K^2-2K-4, 1, K-3, 2}].$$

By [8, Lemma 1], we get

$K^2z'^2 - (K^2-2K-1)y'^2 = 1, 2K+1, K^2, -K^2+2K+1$ or $-2K^2+4K+4$.

Thus we have only two possibilities: the first is $g = 2K+1$, and the second can appear only if $2K+1$ is a perfect square, say $2K+1 = G^2$, in which case we may have $g = G$. The first possibility leads to the equation

$$K^2z'^2 - (K^2-2K-1)y'^2 = 1,$$

for which only solution satisfying the inequality (14) is $(y', z') = (K-1, K-2)$. This leads to $y = 2K^2-K-1$ and $x^2 = 4K^2-2K-2$, which has no solutions with $K \neq \pm 1$. For the second possibility, only solution of the corresponding Pellian equation

$$K^2z'^2 - (K^2-2K-1)y'^2 = 2K+1,$$

satisfying the inequality (14) is $(y', z') = (1, 1)$. Hence $y^2 = 2K+1$, and (10) implies that $d = 0$, which, by definition, is not considered as a proper extension to a quadruple.

It remains the case $g = 1$. Similarly as above, by applying Worley's theorem, we get that the only solutions of (13) with $g = 1$ in the range given by (14) is $(y, z) = (K+1, K)$. This gives $x^2 = 2(K+1)$. Therefore, this possibility can appear only if $2(K+1)$ is a perfect square. However, this possibility gives, by (9), that $d = 1$, which is again not considered as a proper extension to a quadruple, since the starting triple already contains an element 1.

Acknowledgement. This research is supported by TUBITAK (The Scientific and Technological Research Council of Turkey) and Necmettin Erbakan

University Scientific Research Project Coordinatorship (BAP). A.D. was supported by the Croatian Science Foundation under the project no. 6422.

References

[1] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[2] E. Brown, *Sets in which* $xy + k$ *is always a square*, Math. Comp. **45** (1985), 613–620.

[3] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65** (1993), 15–27.

[4] A. Dujella, *There are only finitely many Diophantine quintuples*, J. Reine Angew. Math. **566** (2004), 183–214.

[5] A. Dujella, *Continued fractions and RSA with small secret exponent*, Tatra Mt. Math. Publ. **29** (2004), 101–112.

[6] A. Dujella, *Diophantine m-tuples*, `http://www.math.hr/~duje/dtuples.html`

[7] A. Dujella and A. Pethő, *A generalization of theorem Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 291–306.

[8] A. Dujella and B. Jadrijević, *A family of quartic Thue inequalities*, Acta Arith. **111** (2004), 61–76.

[9] Y. Fujita and A. Togbé, *The extension of the* $D(-k^2)$*-pair* $\{k^2, k^2+1\}$, Period. Math. Hungar. **65** (2012), 75–81.

[10] H. Gupta and K. Singh, *On k-triad sequences*, Internat. J. Math. Math. Sci. **5** (1985), 799–804.

[11] K. Kaygisiz, H. Senay, *Constructions of some new nonextandable* P_k *sets*, Int. Math. Forum **2** (2007), 2869–2874.

[12] S. P. Mohanty and A. M. S. Ramasamy, *On* $P_{r,k}$ *sequences*, Fibonacci Quart. **23** (1985), 36–44.

[13] I. Niven, H. S. Zuckerman and H. L. Montgomery, An Introduction to the Theory of Numbers, Wiley, New York, 1991.

[14] R. T. Worley, *Estimating* $|\alpha - p/q|$, J. Austral. Math. Soc. Ser. A **31** (1981), 202–206.

Elementary Mathematics Education Programme, Ahmet Kelesoglu Education Faculty, Necmettin Erbakan University, Konya, Turkey
E-mail address: `bilge.peker@yahoo.com`

Department of Mathematics, University of Zagreb, Zagreb, Croatia
E-mail address: `duje@math.hr`

Secondary Mathematics Education Programme, Ahmet Kelesoglu Education Faculty, Necmettin Erbakan University, Konya, Turkey
E-mail address: `inag_s@hotmail.com`