ON THE HIGH RANK $\pi/3$ AND $2\pi/3$-CONGRUENT NUMBER ELLIPTIC CURVES

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Abstract. Consider the elliptic curves given by

$$E_{n,\theta}: \quad y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x$$

where $0 < \theta < \pi$, $\cos(\theta) = s/r$ is rational with $0 \leq |s| < r$ and $\gcd(r, s) = 1$. These elliptic curves are related to the θ-congruent number problem as a generalization of the congruent number problem. For fixed θ this family corresponds to the quadratic twist by n of the curve $E_\theta: y^2 = x^3 + 2sx^2 - (r^2 - s^2)x$. We study two special cases $\theta = \pi/3$ and $\theta = 2\pi/3$. We have found a subfamily of $n = n(w)$ having rank at least 3 over $\mathbb{Q}(w)$ and a subfamily with rank 4 parametrized by points of an elliptic curve with positive rank. We also found examples of n such that $E_{n,\theta}$ has rank up to 7 over $\mathbb{Q}$ in both cases.

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1. Introduction

The construction of high rank elliptic curves is an important problem concerning elliptic curves. Dujella [5] collected a list of high rank elliptic curves with prescribed torsion groups. The largest known rank, found by Elkies [8] in 2006, is 28. In this work we search for high ranks in the family of elliptic curves related with $\pi/3$ and $2\pi/3$ congruent problem.

Let us briefly describe the problem. Consider $0 < \theta < \pi$ such that $\cos(\theta) = s/r$ with r and s in $\mathbb{Q}$, $0 \leq |s| < r$ and $\gcd(r, s) = 1$. A positive integer n is called a θ-congruent number if there exists a triangle with rational sides and area equal to $n\alpha_\theta$, where $\alpha_\theta = \sqrt{r^2 - s^2}$. It is clear that if a positive integer n is θ-congruent, then so is nt^2, for any integer t, so we concentrate on square-free positive integers.

The problem of determining θ-congruent numbers is related to the problem of finding non-2-torsion points on the family of elliptic curves which are called θ-congruent number elliptic curves,

$$E_{n,\theta}: y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x,$$

where r and s are as above, see [28]. Observe that this curve is the quadratic twist by n of the curve $E_{1,\theta}$.

This family of elliptic curves was introduced by Koblitz in [13, Section I.2, Exercise 3], and systematically studied by Fujiwara [9, 10]. Let $E_{n,\theta}(\mathbb{Q})$ be the group of rational points on $E_{n,\theta}$ and denote by $r_\theta(n)$ its (algebraic) rank.

An ordinary congruent number is nothing but a $\pi/2$-congruent number and hence a congruent number elliptic curve is just a $\pi/2$-congruent number elliptic curve. Rogers

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[19, 20] and Dujella, Janfada and Salami [6], exhibited recently a list of congruent number elliptic curves with $r_{\pi/2}(n)$ up to 7.

We restrict our search for high rank θ-congruent number elliptic curves to the cases $\theta = \pi/3$ and $2\pi/3$.

In this paper we present a family of values of $n = n(w)$ such that the curves $E_{n(w),2\pi/3}$ have rank at least 3 over $\mathbb{Q}(w)$. An equivalent result is valid for the $\pi/3$ case. We also exhibit examples of curves with rank up to 7 in both cases, $\pi/3$ and $2\pi/3$.

Yoshida [24] proved that $r_{\pi/3}(6) = 1$, $r_{\pi/3}(39) = 2$ and also $r_{2\pi/3}(5) = 1$, $r_{2\pi/3}(14) = 2$. These are the smallest positive integers corresponding to the given Mordell-Weil ranks. In this paper, we find the smallest positive integers n for which $r_{\pi/3}(n) = 3, 4, 5$ in one case and $r_{2\pi/3}(n) = 3, 4$ on the other (the result for $r_{\pi/3}(n) = 4$ is conditional, assuming the BSD and GRH).

In our computations we use the Pari/Gp software [18], William Stein's SAGE software [26] and Cremona's mwrank program [4] and the program package Magma [2].

2. Preliminary results

In this section we recall some results about θ-congruent number elliptic curves, in particular, a criterion for a square-free positive integer to be a θ-congruent number. This, jointly with the subfamilies mentioned before, are the starting point for our search of good candidates for high rank curves.

We use the Mestre-Nagao sum, the Mestre's conditional upper bound for the rank of elliptic curves over $\mathbb{Q}$ and the root number as sieving tools in order to reduce the size of the lists and selecting only the best candidates for high rank. We briefly describe these items below.

It is known that for the usual congruent numbers there exist a close relation with elliptic curves, and in fact the following classical result holds: n is a congruent number if and only if $r_{\pi/2}(n) > 0$, see e.g. [13, Section I.9, Proposition 18]. A similar theorem was proved by Fujiwara, see [9], for θ-congruent numbers.

Theorem 1. *Let n be arbitrary square-free positive integer and consider the elliptic curve $E_{n,\theta}$ as above. Then*

- **i):** *n is a θ-congruent number if and only if there exists a non-2-torsion point in $E_{n,\theta}(\mathbb{Q})$;*
- **ii):** *for $n \neq 1, 2, 3, 6$, n is a θ-congruent number if and only if $r_\theta(n) > 0$.*

Kan [12] proved the following result which gives a family of θ-congruent numbers for every $0 < \theta < \pi$.

Lemma 2. *A square-free positive integer n is a θ-congruent number if and only if n is the square-free part of*

$$pq(p+q)(2rq + p(r-s)), \tag{1}$$

for some positive integers p, q with $\gcd(p, q) = 1$.

Yoshida [24, 25] proved important results concerning θ-congruent numbers. In particular, in [24] he gave the root numbers for the cases $\pi/3$ and $2\pi/3$ (see Table 1).

Now we recall the Mestre-Nagao sum for an elliptic curve E over $\mathbb{Q}$. Reduce E modulo a prime p and suppose that N_p is the number of points on E with coordinates on F_p.

TABLE 1. Root-numbers

		$2\pi/3$	$\pi/3$
$n \equiv 1,2,3,6,7,11,13,14,18$	(mod 24)	$+1$	-1
$n \equiv 5,9,10,15,17,19,21,22,23$	(mod 24)	-1	$+1$

For any positive integer t, let $\mathbf{P}_t$ be the set of all primes less than t and $a_p = p+1-N_p$. The Mestre-Nagao sum is defined by

$$S(t,E) = \sum_{p\in\mathbf{P}_t} \left(1 - \frac{p-1}{N_p}\right)\log p = \sum_{p\in\mathbf{P}_t} \frac{-a_p+2}{N_p}\log p.$$

It is experimentally known [14, 17] that high rank curves have large values $S(t,E)$. We cite [3] for a heuristic argument which links the Mestre-Nagao sum to the Birch and Swinnerton-Dyer conjecture [1].

Now we describe the Mestre's conditional (assuming the Birch and Swinnerton-Dyer conjecture and GRH) upper bound (see [15, 7]) for the rank of an elliptic curve over $\mathbb{Q}$. Let E be an elliptic curve with conductor N. For an integer $m \geq 1$, let

$$b(p^m) = \begin{cases} 0 & \text{if } p|N, \\ \alpha_p^m + \alpha_p'^m & \text{if } p \not| N, \end{cases}$$

where α_p and α_p' are the roots of $x^2 - a_p x + p$. Let

$$F(x) = \begin{cases} (1-x)\cos(\pi x) + \sin(\pi x)/\pi & \text{if } x \in [0,1], \\ 0 & \text{if } x > 1. \end{cases}$$

Take a positive real number λ and write

$$M(\lambda) = 2\big(\log(2\pi) + \int_0^\infty \big(F(x/\lambda)/(e^x-1) - e^{-x}/x\big)\,dx\big).$$

The Mestre's conditional upper bound for the rank of E is defined as

$$M(\lambda,E) = \frac{\pi^2}{8\lambda}\Big(\log(N) - 2\sum_{p^m \leq e^\lambda} b(p^m)F(m\log(p)/\lambda)\frac{\log(p)}{p^m} - M(\lambda)\Big).$$

3. A family with generic rank at least 3

3.1. **Twists.** Observe that, once θ is fixed, the curve

$$E_{n,\theta}: \quad y^2 = x^3 + 2snx^2 - (r^2-s^2)n^2x$$

is the quadratic twist with parameter n of the curve $E_{1,\theta}: \quad y^2 = x^3 + 2sx^2 - (r^2-s^2)x$. General results about twists can be applied for any θ and we can find families of rank at least 2 over $\mathbb{Q}(r,s)$ by direct applications of results given in Mestre [16] or Rubin and Silverberg [21], [22] (see also [11, 27]).

In our particular cases, $\theta = \pi/3$ corresponds to $s=1$ and $r=2$ and $\theta = 2\pi/3$ to $s=-1$ and $r=2$, we are lead to study the quadratic twists of the curves

$$E_{\pi/3}: \quad y^2 = x^3 + 2x^2 - 3x$$
$$E_{2\pi/3}: \quad y^2 = x^3 - 2x^2 - 3x$$

Each curve is the twist of the other by -1 so their twists can be studied jointly.

3.2. **A family of twists for $\theta = 2\pi/3$ with rank ≥ 3.**

3.2.1. *Rank* 1. We start with the twists of the curve

$$E_{2\pi/3}: \quad y^2 = x^3 - 2x^2 - 3x$$

with parameter $(u+a)(u+b)(u+c)$, so we have the family of twists $y^2 = x^3 + Ax^2 + Bx$ where

$$\begin{aligned} A =& -2(u+a)(u+b)(u+c), \\ B =& -3(u+a)^2(u+b)^2(u+c)^2. \end{aligned}$$

Now we impose $-(b+u)(c+u)^2$ as the x-coordinate of a new point. This is the same as choosing

$$c = \frac{-3a - 4u + abw^2 + auw^2}{1 + bw^2 + uw^2}$$

With this choice we get a family of twists with rank at least 1 over $\mathbb{Q}(b, u, w)$ which, after clearing denominators, can be written as $y^2 = x^3 + A_1x^2 + B_1x$ with

$$\begin{aligned} A_1 =& -2(b+u)(-3+bw^2+uw^2)(1+bw^2+uw^2), \\ B_1 =& -3(b+u)^2(-3+bw^2+uw^2)^2(1+bw^2+uw^2)^2. \end{aligned}$$

The x-coordinate of the infinite order point is $x_1 = -(b+u)(-3+bw^2+uw^2)^2$.

3.2.2. *Rank* 2. We proceed by forcing $3(b+u)(1+bw^2+uw^2)$ as the x-coordinate of a new point in the previous rank 1 family of twists. For this purpose it is enough to choose

$$b = -\frac{-4 + u^2 + uw^2 + u^3w^2}{(1+u^2)w^2}.$$

Now the new family of twist can be written as $y^2 = x^3 + A_2x^2 + B_2x$ with

$$\begin{aligned} A_2 =& -10(-2+u)(2+u)(-1+2u)(1+2u)(1+u^2), \\ B_2 =& -75(-2+u)^2(2+u)^2(-1+2u)^2(1+2u)^2(1+u^2)^2. \end{aligned}$$

The x-coordinates of the two infinite order points are

$$\begin{aligned} x_1 =& (-2+u)(2+u)(-1+2u)^2(1+2u)^2(1+u^2), \\ x_2 =& -15(-2+u)(2+u)(1+u^2)^2. \end{aligned}$$

These two points are independent, so the new family has rank at least 2 over $\mathbb{Q}(u)$.

3.2.3. *Rank* 3. Finally we choose

$$u = -\frac{70 - 10w + w^2}{3(5+w^2)}$$

in order to get $5(-2+u)^2(-1+2u)^2(1+u^2)$ as x-coordinate of a new point in the rank 2 family. In this way we get $y^2 = x^3 + A_3x^2 + B_3x$ with

$$\begin{aligned} A_3 =& -2(-5+w)(-2+w)(4+w)(25+w)(31-4w+w^2) \\ & (100-10w+7w^2)(1025-280w+66w^2-4w^3+2w^4), \\ B_3 =& -3(-5+w)^2(-2+w)^2(4+w)^2(25+w)^2(31-4w+w^2)^2 \\ & (100-10w+7w^2)^2(1025-280w+66w^2-4w^3+2w^4)^2. \end{aligned}$$

The x-coordinates of the three independent points are given by

$$\begin{aligned}x_1 =&\frac{-1}{9(5+w^2)^2}(-5+w)^2(-2+w)(4+w)(25+w)^2(31-4w+w^2)^2\\ &(100-10w+7w^2)(1025-280w+66w^2-4w^3+2w^4),\\ x_2 =&3(-2+w)(4+w)(100-10w+7w^2)(1025-280w+66w^2-4w^3+2w^4)^2,\\ x_3 =&(31-4w+w^2)^2(100-10w+7w^2)^2(1025-280w+66w^2-4w^3+2w^4).\end{aligned}$$

For $w=10$, after reducing coefficients, we get the rank 3 curve given by $y^2=x^3-442x^2-146523x$. The specialized points are

$$P_1=\{-2873/81, 1562912/729\},\quad P_2=\{867, 13872\},\quad P_3=\{2873/4, 48841/8\}.$$

A calculation with mwrank [4] shows that these three points are independent. An argument of specialization [23] proves that this family has rank at least 3 over $\mathbb{Q}(w)$.

Observe that the parameter for the rank 3 family of twists can be made both positive and negative for infinitely many values of w, so we get a family of rank 3 twist for both $2\pi/3$ and $\pi/3$ congruent number problem.

3.3. **A subfamily with rank ≥ 4.** We can find a subfamily with rank ≥ 4 in the family $y^2=x^3+A_3x^2+B_3x$ by forcing

$$-\frac{1}{4}(-5+w)^2(-2+w)^2(4+w)^2(25+w)^2(31-4w+w^2)(100-10w+7w^2)$$

to be the x-coordinate of a point on the curve. We get the condition

$$25w^4-26w^3+699w^2-3770w+13300=z^2. \tag{2}$$

It can be transformed to the elliptic curve

$$Y^2=X^3+X^2-17220X-352800$$

with positive rank (rank is equal 2 with generators $[255, 3450]$, $[-22, 126]$, corresponding to the points $(w,z)=[315/74, 695275/5476]$, $[8,-342]$ on the quartic (2)). Hence, we get infinitely many rational parameters w for which the curve $y^2=x^3+A_3x^2+B_3x$ has the rank ≥ 4.

4. Strategies and results

4.1. **General setting.** Now we attempt to find high rank elliptic curves $E_{n,\theta}$ in two cases $\theta=\pi/3$ and $2\pi/3$. We will use the expression (1) and the families given in the previous section as sources for good candidates for high rank curves. We shall use the following notations: $r_\theta(n)$ for the rank and $s_\theta(n)$ for the 2-Selmer rank (see e.g. [6]), which is an upper bound for the rank; that is $r_\theta(n)\leq s_\theta(n)$.

We proceed in three steps, depending on the range and the form of the square-free positive integers n.

Step (I) In this step we take all the square-free positive integers $n\leq 5\times 10^6$. By a direct computation with mwrank, we find the 2-Selmer rank of $E_{n,\theta}$ for all square-free n in that range and in each case $\theta=\pi/3$ and $2\pi/3$. Our computations show that there are no integers n with $s_\theta(n)\geq 6$. Table 2 presents the distribution of the number of these square-free integers according to the values of $s_\theta(n)$. Finally, we compute directly rank $r_\theta(n)$ with mwrank to find the smallest n's with $r_{\pi/3}(n)=3,4,5$ as well as the smallest n's with $r_{2\pi/3}(n)=3,4$.

TABLE 2. Distribution of $s_\theta(n)$

$s_\theta(n)$	0	1	2	3	4	5	≥ 6	Total
$\theta = \pi/3$	783043	1401045	734290	116158	5045	52	0	3039633
$\theta = 2\pi/3$	760511	1374165	751192	144641	9038	86	0	3039633

Step (II) We consider all square-free θ-congruent numbers $n > 5\times 10^6$ of the form (1) in Lemma 2 with $1 < p, q \leq 10^4$, $\gcd(p, q) = 1$, and having at least 4 odd prime factors. We get a list with more than 7×10^6 elements for each of the cases $\theta = \pi/3$ and $\theta = 2\pi/3$. Using the Mestre-Nagao sum, we reduce by Pari/Gp program the length of this list. In fact, we choose the n with $S(10^3, E_{n,\theta}) > 15, S(10^4, E_{n,\theta}) > 20, S(10^5, E_{n,\theta}) > 40$, for which $s_{\pi/3}(n) \geq 6$, and $s_{2\pi/3}(n) \geq 5$. After computing the values of $r_\theta(n)$ for these candidates by mwrank, we finally select the n with $r_{\pi/3}(n) = 6, 7$ and the n with $r_{2\pi/3}(n) = 5, 6$. In the cases in which mwrank do not give exact value $r_\theta(n)$ we compute the Mestre's conditional upper bound $M(\lambda, E_{n,\theta})$ for $r_\theta(n)$ with $15 \leq \lambda < 24$.

Step (III) In this part we use the families in section 3 in order to search for good candidates for high rank. Since curves in the families with rank 3 and 4 have large coefficients, we find the family with rank 2 the most suitable for our purpose. The search for rank 6 curves is conducted upon the rank 2 family with $u = p/q$ for $1 < p < q < 4000$ with sieving conditions $S(523, E_{n,\theta}) > 18$, $S(1979, E_{n,\theta}) > 28$ and the Selmer rank ≥ 6.

The search for rank 7 is made in the same family of twists with $u = p/q$ for $1 < p < q < 13000$ with the following conditions, root number equal to -1, $S(523, E_{n,\theta}) > 20$, $S(1979, E_{n,\theta}) > 30$ and the Selmer rank ≥ 7. The ranks are calculated with mwrank. For the case $2\pi/3$ for $p = 4127$ and $q = 10004$, i.e. for $n = 12748697412909916241$ the corresponding curve has rank 7. In this case, the direct application of mwrank gives only $6 \leq$ rank ≤ 7, but applying mwrank to an isogenous curve give the seventh independent point.

In the next subsections, we collect the results. We find the smallest integers n such that $r_{\pi/3}(n) = 3, 4, 5$ and $r_{2\pi/3}(n) = 3, 4$, and we exhibit examples of curves with rank up to 7 in both cases.

4.2. **The case $\theta = \pi/3$.** Rank 3: The integers 407 and 646 are the two smallest ones among 116158 integers n less than 5×10^6 with $s_{\pi/3}(n) = 3$. We have $r_{\pi/3}(646) = 3$, while for $n = 407$ Magma gives that the analytic rank is 1, so by Kolyvagin's theorem $r_{\pi/3}(407) = 1$. Therefore, the value $n = 646$ is the minimum value producing a curve with rank 3.

Rank 4: The smallest n that we have found with rank 4 is $n = 172081$. There are 63 integers n less than 172081 with $s_{\pi/3}(n) = 4$. For 29 cases mwrank gives $0 \leq r_{\pi/3}(n) \leq 4$, and for all these cases the 4-descent implemented in Magma gives that the rank is ≤ 2. In the remaining 34 cases, mwrank gives $2 \leq r_{\pi/3}(n) \leq 4$. In the most of these cases the 4-descent shows that rank is equal to 2. However, in three cases: $n = 31622$, 143222, 150866, we are not able to show that rank < 4 unconditionally. In these cases, we use Mestre's conditional upper bound (with $\lambda = 11$), which gives $r_{\pi/3}(n) \leq 2$, so $r_{\pi/3}(n) = 2$ (conditionally). Thus, the value $n = 172081$ is, conditionally (assuming BSD and GRH), the minimum value giving a curve with rank 4.

Rank 5: The direct computation shows that $n = 221746$ is the smallest among 52 integers n in the observed range with $s_{\pi/3}(n) = 5$, and since $r_{\pi/3}(221746) = 5$, $n = 221746$ is the smallest positive integer giving rank 5.

Rank 6: The smallest n that we have found with rank 6 is $n = 11229594411$. We do not know if it is the smallest one with this property. The values of n given in Table 3 also give curves with rank 6.

TABLE 3. Case $\pi/3$. Other n with rank 6

40004232681,	158763281079,	167514827545,	198606002595,
251819173095,	271314827665,	3302971161265,	3492293850595,
5144668978371,	6634009064865,	17073273800095,	40582123000419,
45563330326345,	7658263493840940211.		

Rank 7: The only n that we have found giving rank 7 is $n = 365803464586$. We do not know if it is the smallest one.

TABLE 4. Ranks in the cases $\theta = \pi/3$

$r_\theta(n)$	n	Generators of $E_{n,\theta} : y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x$
3	646	[-722,34656], [6137,521645], [-1216,40432].
4	172081	[-505141,-61627202], [-58621,-78669382], [-440076,-143244738], [224175,92987790].
5	221746	[345450,207822720], [-15792,49357896], [994896,1130036040], [-13254,-45063600], [-386575,-255989965].
6	11229594411	[904103532759/25,-992069570757491352/125], [1541731888897/16,2090318638263775025/64], [265444083202036/2025,463638744073698265813 4/91125], [719501508201/64,40873417425022581/512], [13006760076899764/269361,169318158533140400026749 8/139798359], [50286669020153449/278784,11896090671289659453790795/147197952].
7	365803464586	[433764757524,212456676940982628], [1291274050073,-1689545579159165609], [-59335333874904423/3644281,-57054165989043197679051 4695/6956932429], [11954902524369/4,-4527746699608451 6865/8], [2138828658027602/5329,568903954835494296233 12/389017], [78676918101443355 4/80089,72198240738053669208885 2160/22665187], [-5622360281643737653 42/540237049, 361716521043536662555944519736 0/12556729729907].

4.3. **The case $\theta = 2\pi/3$.** Rank 3: The smallest n with $s_{2\pi/3}(n) = 3$ is $n = 221$. Since $r_{2\pi/3}(221) = 3$, we conclude that $n = 211$ is the smallest n for which the rank is 3.

Rank 4: The smallest n that we have found with rank 4 is $n = 12710$. There are two smaller positive integers with Selmer rank equal to 4 ($n = 4718$ and $n = 6398$) but having analytic rank 0, so by Kolyvagin's theorem the algebraic rank is also 0. Thus the minimality of $n = 12710$ follows.

Rank 5: The smallest n that we have found with rank 5 is $n = 16470069$. We do not know if it is the smallest one with this property.

Rank 6: We have found several positive integers n with $r_{2\pi/3}(n) = 6$ where $n = 456249066$ is the smallest one. Other values are given in Table 5.

TABLE 5. Case $2\pi/3$. Other n with rank 6

764046470,	902472906,	5062245006,	9667090290,
11801899970,	19969987310,	20240772006,	23819599518,
24080567966,	30834423438,	39360775454,	58181539130,
64256704710,	98708770590,	106366008126,	148280772990,
181684390314,	292826163630,	309000045354,	333515184002,
554883184814,	653918457570,	685374515826,	713465075246,
860842004286,	1185986591790,	1248260820170,	2004510092970,
2743972777910,	10745486363210,	55967962170246,	90952836208430,
104732378607110,	177348563238770,	219163751391326,	1459584795789354,
29410732919116094,	40315634933149394,	30375400815771401390.	

TABLE 6. Ranks in the cases $\theta = 2\pi/3$

$r_\theta(n)$	n	Generators of $E_{n,\theta} : y^2 = x^3 + 2snx^2 - (r^2 - s^2)n^2x$
3	221	[-204,1734], [-169,2704], [4131,-249696].
4	12710	[-310,384400], [-9920,-1153200], [48050,5381600], [76880,16337000].
5	16470069	[-3115959/4,-198146948769/8], [-16255958103/1024,-813789518594283/32768], [118172745075/1849,-21701053829180880/79507], [174895662711/3481,-10850526914590440/205379], [18013358979/361,-275820552686448/6859].
6	4562490669	[1372171206,2930957696016], [24303608784,3714988879700280], [1677715326,-33259028622624], [3635049873,-183588193835865], [27273656667348/18769,39342846732689875284/2571353], [36967427406/25,2217080599939296/125].

Rank 7: The integer $n = 12748697412909916241$ with $r_{2\pi/3}(n) = 7$ has been found within the family of rank 2 of section 3. It correspond to $u = \frac{4127}{10004}$ in such family. The

data for this curve are too large to fit in the table, so we give them here. The rank and independent points were found by applying mwrank [4] to one of its 2-isogenous curves. The curve is

$$y^2 = x^3 - 2549739482581983248 2x^2 - 4875878571778079741951246524489067102 43x$$

and x-coordinates of 7 independent points are:

$$-3478204633589378700,$$

$$-11685945449719133341,$$

$$-65741795518552997301837420589901614595094 81/57559883687779698597002 5,$$

$$25824931965925746931591311990861035046105913 21/64687220044469657223311844,$$

$$93780507427270339924086066690295941912574093 0561/18861375626453019864153493504,$$

$$11046597353655160109746607088518021300436780651968665388467892941687327986508750549446041 3809/11748335750378251588082756719839642493430053195853237296682112610898176,$$

$$60467717012835269079195671205236355556884315494450504834931884615838950389619168774330021927698790345943423897889681/24233017022312587640055367659184083618942065345759517265987410838049572492835809955736272757210 0.$$

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