LOCUS OF INTERSECTIONS OF EULER LINES

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Abstract. Let A, B, and C be vertexes of a scalene triangle in the plane. Answering a recent problem by Jordan Tabov, we show that the locus of all points P with the property that the Euler lines of the triangles BCP, CAP, and ABP are concurrent is a subset of the union of the circumcircle and the Neuberg cubic of the triangle ABC. Some new properties of this remarkable cubic have also been discovered.

1. Introduction

The origin of this paper is questions on geometry of triangles stated in Jordan Tabov's article "An extraordinary locus" in Mathematics and Informatics Quarterly (Vol. 4, No. 2, page 70). We shall present three different proofs of results that answer Tabov's problems. Our method is to use analytic geometry which is a usual approach in the identification of complicated curves.

Let us first recall basic definitions. Let ABC denote the triangle in the plane with vertexes A, B, C, angles A, B, C, and sides $a = |BC|$, $b = |CA|$, $c = |AB|$. When ABC is not an equilateral triangle, then the centroid G of ABC (intersection of medians) and the orthocenter H of ABC (intersection of altitudes) determine a unique line called the Euler line of ABC.

This line has many interesting properties and has been the source of many beautiful results and problems. One of them is Tabov's:

Superior Locus Problem: Find the locus of points P in the plane with the property that the Euler lines of the triangles BCP, CAP, and ABP are concurrent (see Figure 1).

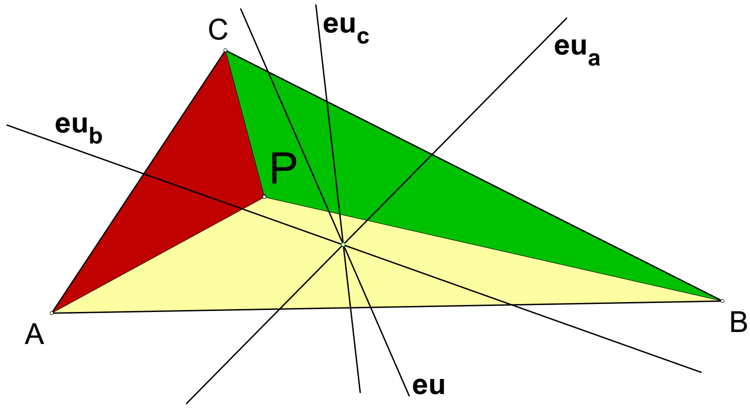

Figure 1: Euler lines of triangles BCP, CAP, and ABP are concurrent.

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As we shall see below, this locus in general is quite complicated. The locus is remarkable due to the fact that it includes some of the most interesting points related to the triangle ABC but as a curve of order five it is not tractable by elementary methods. It is essentially the union of the circumcircle and the Neuberg cubic that has been in the focus of recent renewal of interest into triangle geometry (see Figure 2 and [3], [4], [5], [19], [20], [23]).

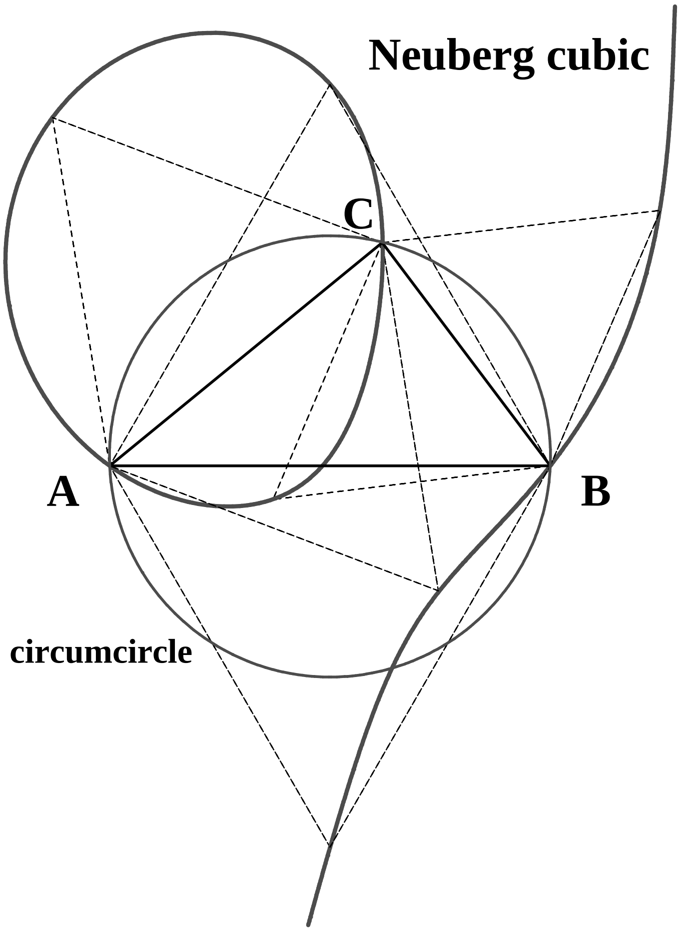


Figure 2: Union of the circumcircle and the Neuberg cubic minus the union of sidelines and the vertexes of equilateral triangles on sides answer Tabov's problem.

Following this introduction, in a preparatory section, we shall describe various kinds of coordinate systems that are usually used for exploration of properties of triangles and curves.

Our original solution of Tabov's problems was done on a computer entirely in Cartesian coordinates. While this approach requires only elementary knowledge of analytic geometry and could be easily followed by most readers, it is difficult for print and it does not fit well in the overall drive for simplicity of expressions in mathematics.

An outline of key steps for this direct method is followed by a detailed proof in areal coordinates. Several interesting side results related to Euler lines of the four triangles ABC, BCP, CAP, and ABP are also presented. Then we give the third complete proof using distance coordinates. This proof is entirely elementary and a bit longer since it involves the consideration of several cases that can arise. Those cases are interesting on their own and they give further insight into the problems.

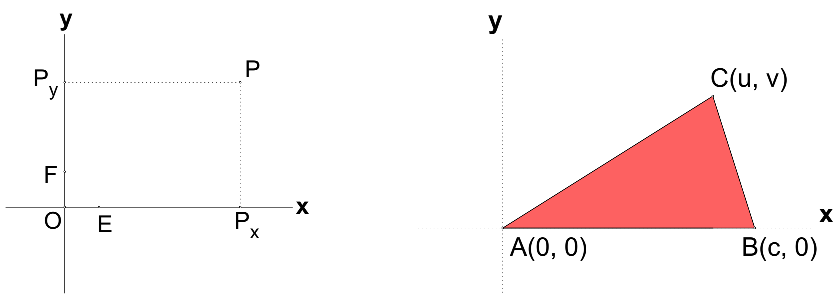
In the rest of the paper we shall give some known and some new information about the Neuberg cubic and it's properties. The style of our presentation here will be different because we shall omit proofs in order to make this a modest size paper. There are at least two other reasons for this. First, we want to encourage readers to try their own proofs perhaps using previous sections for suggestions. Second, we want to keep tradition of most papers from the golden era of triangle geometry that elegantly avoid too much details.

We hope that this is a contribution to optimistic tones in recent excellent articles [6], [8], and [11] on computer induced resurrection of interest into triangle geometry.

2. PRELIMINARIES

2.1. **Coordinate systems.** The position of a point P in the plane of the given reference triangle ABC can be described with several kinds of coordinates. In this paper we shall be working with Cartesian, areal, and distance coordinates while the normal coordinates will be used only occasionally. Each of these coordinates have their advantages and disadvantages. For example, in Cartesian coordinates the location of points is straightforward and the analytic geometry is well-known. However, for every selection of coordinate axes, most expressions become complicated and they do not reflect homogeneity and symmetry of triangles. In areal (or barycentric) and normal (or trilinear) coordinates the expressions can be significantly simpler since they are defined evenly with respect to the reference triangle. However, these coordinates are not so familiar, the expressions can become cumbersome especially for objects that are not centred, and the analytic geometry is tricky. Finally, distance coordinates are hidden in the literature and are useful only in some areas of triangle geometry.

It is fair to say that each coordinate system is best suited for certain purposes so that normally we must combine them. This has been common occurrence in recent papers. Some authors use in addition complex numbers which play no role in this article.



(a) Cartesian coordinate system.

(b) Placement of the triangle ABC.

Figure 3:

2.2. **Cartesian coordinates.** Let E and F be points on perpendicular lines x and y in the plane different from their intersection O. A point P in the plane is described by signed distances p and q of the origin O to the projections P_x and P_y of P into lines x and y. The number p is positive when points E and P_x are on the same side of the line y. Similarly, the number q is positive when points F and P_y are on the same side of the line x. We call p and q the (Cartesian) coordinates of P (with respect to the rectangular coordinate system (x, E, y, F)) and write $P(p, q)$ or (p, q) to indicate coordinates of the point P (see Figure 3 (a)).

The above definition has nothing to do with the triangle ABC so to make it useful for our purposes we shall assume that the origin is at A and that x is the line AB. In other words, we select the Cartesian coordinate system in the plane so that the coordinates of the vertexes A, B, and C are $(0, 0)$, $(c, 0)$, and (u, v), where c and v are positive real numbers and u is a real number (see Figure 3 (b)).

Let $X(f, g)$, $Y(h, k)$, and $Z(m, n)$ be points in the plane. Recall that the distance $|XY|$ and the signed area $|XYZ|$ are given by

$$|XY| = \sqrt{f^2 - 2\,fh + h^2 + g^2 - 2\,gk + k^2},$$

$$|XYZ| = \frac{1}{2}(fk - fn + nh - gh + gm - mk) = \frac{1}{2}\begin{vmatrix} f & g & 1 \\ h & k & 1 \\ m & n & 1 \end{vmatrix}.$$

Here we regard $|XYZ|$ positive provided going from X to Y and then to Z is done counterclockwise.

For sides a and b we get

$$a = \sqrt{c^2 - 2\,cu + u^2 + v^2}, \qquad b = \sqrt{u^2 + v^2}.$$

We can express u and v in terms of side lengths a, b, and c.

$$u = \frac{b^2 + c^2 - a^2}{2\,c}, \qquad v = \frac{\sqrt{2\,b^2c^2 + 2\,c^2a^2 + 2\,b^2a^2 - a^4 - b^4 - c^4}}{2\,c}.$$

2.3. **Areal coordinates.** Let $P(p, q)$ be a point whose position we wish to define with respect to the triangle ABC. The ratios of the signed areas

$$x = \frac{|BCP|}{|ABC|}, \qquad y = \frac{|CAP|}{|ABC|}, \qquad z = \frac{|ABP|}{|ABC|},$$

are called the *actual areal coordinates* of the point P.

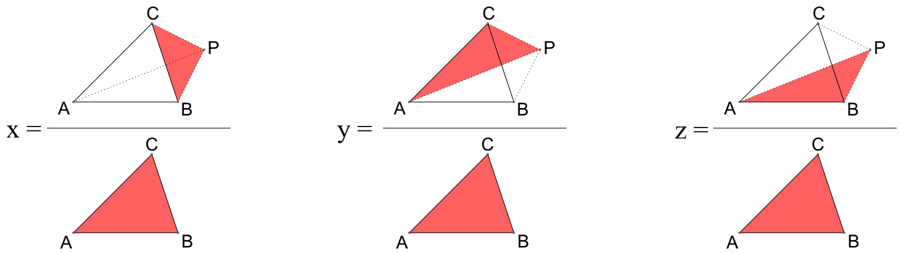


Figure 4: Actual areal coordinates are ratios of areas.

In finding the position of a point it is not necessary to know x, y, z but only their ratios. Indeed, since

$$\frac{x}{y} = \frac{c\,v - p\,v + q\,u - c\,q}{p\,v - q\,u}, \qquad \frac{y}{z} = \frac{p\,v - q\,u}{c\,q},$$

we can solve for p and q to get

$$p = \frac{y\,c + z\,u}{x + y + z}, \qquad q = \frac{z\,v}{x + y + z}.$$

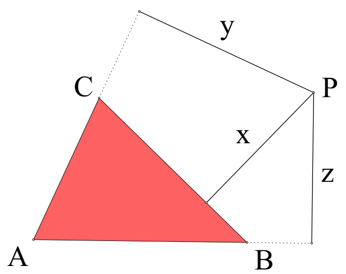
It is obvious that the above expressions for p and q remain unchanged when we replace x, y, and z with λx, λy, and λz, where λ is any real number different from zero. This is the reason why such triples $(\lambda x,\, \lambda y,\, \lambda z)$ are called *areal coordinates* or *barycentric coordinates* or just *areals* of the point P.

We shall write $Q[[f,\, g,\, h]]$ and $Q[x,\, y,\, z]$ to indicate that f, g, and h are actual areal coordinates of a point Q and that x, y, and z are areal coordinates of a point Q (i. e., numbers proportional to actual areal coordinates of Q). Recall [12, p.5] that

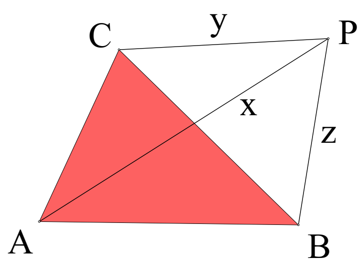
$$f+g+h=1, \quad f=\frac{x}{x+y+z}, \quad g=\frac{y}{x+y+z}, \quad h=\frac{z}{x+y+z}.$$

Sometimes it is necessary to use more precise notation $Q[x,\, y,\, z;\, ABC]$ in order to specify the reference triangle.

2.4. **Normal coordinates.** The *actual normal coordinates* or *actual trilinear coordinates* of a point P with respect to the triangle ABC are signed distances f, g, and h of P from the lines BC, CA, and AB (see Figure 5(a)). We shall regard P as lying on the positive side of BC if P lies on the same side of BC as A. Similarly, we shall regard P as lying on the positive side of CA if it lies on the same side of CA as B, and similarly with regard to the side AB.



(a) Actual normal coordinates are signed distances to sides.



(b) Distance coordinates are distances to vertexes.

Figure 5:

Ordered triples $(x,\, y,\, z)$ of real numbers proportional to $(f,\, g,\, h)$ (that is such that $x=\mu f$, $y=\mu g$, and $z=\mu h$, for some real number μ different from zero) are called *normal coordinates* or *trilinear coordinates* of P.

We shall write $Q\langle\langle f,\, g,\, h\rangle\rangle$ and $Q\langle x,\, y,\, z\rangle$ to indicate that f, g, and h are actual normal coordinates of a point Q and that x, y, and z are normal coordinates of a point Q, and $Q\langle x,\, y,\, z;\, ABC\rangle$ when we want to specify the reference triangle.

Both areal and normal coordinates are equally suitable for triangle geometry and it is only a matter of personal preference which one to choose.

If f, g, h are the actual normal coordinates of a point and x, y, z are the actual areal coordinates of the same point with respect to the same reference triangle with the area Δ, then we can transform from one system to the other by means of the formulas

$$x=\frac{a\,f}{2\,\Delta}, \qquad y=\frac{b\,g}{2\,\Delta}, \qquad z=\frac{c\,h}{2\,\Delta}.$$

2.5. **Distance coordinates.** The *tripolar* or *distance coordinates* of a point P with respect to the triangle ABC are distances from P to the vertexes A, B, and C (see Figure 5(b)).

We shall write $Q(x,\, y,\, z)$ to indicate that x, y, and z are distance coordinates of a point Q, and $Q(x,\, y,\, z;\, ABC)$ when we want to specify the reference triangle.

2.6. **Notation.** The most interesting properties and expressions related to triangles are miraculously simple provided we adopt proper notation and agreements that govern its use. Here we shall propose one such attempt at simplification in triangle geometry.

The expressions in terms of sides a, b, and c can be shortened using the following notation.

$$d_a = b - c, \quad d_b = c - a, \quad d_c = a - b, \quad z_a = b + c, \quad z_b = c + a, \quad z_c = a + b,$$
$$t = a + b + c, \quad t_a = b + c - a, \quad t_b = c + a - b, \quad t_c = a + b - c,$$
$$m = abc, \quad m_a = bc, \quad m_b = ca, \quad m_c = ab,$$
$$k = a^2 + b^2 + c^2, \quad k_a = b^2 + c^2 - a^2, \quad k_b = c^2 + a^2 - b^2, \quad k_c = a^2 + b^2 - c^2.$$

We shall use α, β, and γ for cotangents $cotA$, $cotB$, and $cotC$ of angles A, B, and C. The sum $cotA + cotB + cotC$ is shortened to ω. The cosine, sine, and tangent of angles A, B, and C are denoted α_κ, β_κ, γ_κ, α_σ, β_σ, γ_σ, α_τ, β_τ, and γ_τ.

The basic relationship connecting these values is

$$\beta\gamma + \gamma\alpha + \alpha\beta = 1.$$

Moreover, from sine and cosine rules and the fact that the circumradius R is equal to $\frac{m}{\delta}$, we get $\alpha = k_a/\delta$, $\beta = k_b/\delta$, and $\gamma = k_c/\delta$, where $\delta = \sqrt{t\, t_a\, t_b\, t_c}$ is four times the area Δ of the triangle ABC. Hence, we can express a^2, b^2, and c^2 as $\frac{\delta(\beta+\gamma)}{2}$, $\frac{\delta(\gamma+\alpha)}{2}$, and $\frac{\delta(\alpha+\beta)}{2}$, respectively.

Other frequently used relations are $a = 2\,R\,\alpha_\sigma$, $b = 2\,R\,\beta_\sigma$, $c = 2\,R\,\gamma_\sigma$,

$$2\,m_a\,\alpha_\kappa = k_a, \quad 2\,m_b\,\beta_\kappa = k_b, \quad 2\,m_c\,\gamma_\kappa = k_c, \quad \alpha\,\alpha_\tau = \beta\,\beta_\tau = \gamma\,\gamma_\tau = 1,$$
$$\alpha_\kappa + \beta_\kappa\,\gamma_\kappa = \beta_\sigma\,\gamma_\sigma, \quad \beta_\kappa + \gamma_\kappa\,\alpha_\kappa = \gamma_\sigma\,\alpha_\sigma, \quad \gamma_\kappa + \alpha_\kappa\,\beta_\kappa = \alpha_\sigma\,\beta_\sigma.$$

Since we shall deal mostly with permutations of finite sets on letters, we shall use short notation, for example $|x,\, z,\, y|$, for a permutation which takes x to itself, y to z, and z to y. This means that we consider sets of letters ordered by lexicographic order and it is always the first member in this order that is permuted. Therefore, it suffices to indicate only ordered set of images.

The expressions which appear in triangle geometry usually depend on sets that are of the form $\{a,\, b,\, c,\, \dots,\, x,\, y,\, z\}$ (that is, union of triples of letters). Let σ and τ stand for permutations $|\,b,\, c,\, a,\, \dots,\, y,\, z,\, x\,|$ and $|\,c,\, a,\, b,\, \dots,\, z,\, x,\, y\,|$.

Let $f = f(x,\, y,\, \dots)$ be an expression that depends on a set $S = \{x,\, y,\, \dots\}$ of variables and let $\varrho : S \to S$ be a permutation of S. Then f^ϱ is a short notation for $f(\varrho(x),\, \varrho(y),\, \dots)$. For permutations $\varrho,\, \dots,\, \xi$ of S we shall use $\mathbb{S}_{\varrho,\dots,\xi}\, f$ and $\mathbb{P}_{\varrho,\,\dots,\,\xi}\, f$ to shorten $f + f^\varrho + \dots + f^\xi$ and $f\, f^\varrho \dots f^\xi$. Finally, $\mathbb{S}f$ and $\mathbb{P}f$ replace $\mathbb{S}_{\sigma,\,\tau}\, f$ and $\mathbb{P}_{\sigma,\,\tau}\, f$.

Let P be a point in the plane of the triangle ABC. Let xP and yP be the Cartesian x-coordinate and y-coordinate of P and let $x[P]$, $y[P]$, and $z[P]$ be the first, the second, and the third areal coordinate of P. We shall use similar notation $x\langle P\rangle$, $y\langle P\rangle$, $z\langle P\rangle$ for normal and $x(P)$, $y(P)$, $z(P)$ for distance coordinates.

When $x[P] = f$, $y[P] = f^\sigma$, and $z[P] = f^\tau$, we shall write $P[f]$ and talk of f as an areal coordinate of P. Similar rules hold for normal and distance coordinates.

For example, if O is the circumcenter of ABC, then $O\langle\alpha_\kappa,\, \beta_\kappa,\, \gamma_\kappa,\, \rangle$, or, equivalently, $O\langle a\, k_a,\, b\, k_b,\, c\, k_c\rangle$. But, we shall write $O\langle\alpha_\kappa\rangle$ or $O\langle a\, k_a\rangle$ because the other two coordinates are built from the first using permutations σ and τ.

This rule will be further extended so that for example we shall usually define only an expression $name_a$ with the understanding that $name_b$ and $name_c$ are derived from $name_a$ with permutations σ and τ.

On the other hand, since points, lines, conics, ... associated to a triangle often appear in triples in which two members are built from a third not only by appropriate permutation but also with a shift of position, we shall give only one of them while the other two (*relatives*) are obtained from it by a procedure illustrated in the example below.

For example, $A_e[-a,\, b,\, c]$, $B_e[a,\, -b,\, c]$, and $C_e[a,\, b,\, -c]$ are excenters. We can get areals of B_e and C_e from those of A_e as follows:

$$x[B_e] = z[A_e]^\sigma, \qquad y[B_e] = x[A_e]^\sigma, \qquad z[B_e] = y[A_e]^\sigma,$$

$$x[C_e] = y[A_e]^\tau, \qquad y[C_e] = z[A_e]^\tau, \qquad z[C_e] = x[A_e]^\tau.$$

Here, B_e and C_e are relatives of A_e.

3. LOCUS RECOGNITION IN CARTESIAN COORDINATES

In this section we shall give an outline of the solution to Tabov's problem using Cartesian coordinates. All steps in this approach can be easily checked either by hand or by computer. However, most formulas that appear are complicated for print so that we left them out. The reason why this method is not entirely discarded lies in the fact that the author does not know how to draw curves in areal and normal coordinates. Hence, we shall use Cartesian coordinates for the dirty behind the scene work.

Let us select the Cartesian coordinate system in the plane so that the coordinates of points A, B, C, and P are $(0, 0)$, $(c, 0)$, (u, v), and (p, q), where c and v are positive real numbers and u, p, and q are real numbers. In order to find out the equations of the Euler lines eu, eu_a, eu_b, and eu_c of the triangles $t = ABC$, $t_a = BCP$, $t_b = ACP$, and $t_c = ABP$ we need to know how to draw a line through two points, determine the midpoint of a segment, and draw the perpendicular to a given line which passes through a given point. Once we do this, we find out that four Euler lines are represented by linear equations $a_{i1}\, x + a_{i2}\, y + a_{i3} = 0$ ($i = 1, 2, 3, 4$).

Here we have made implicitly the assumption that the points A, B, C, and P are in general position (that is, that P does not lie on lines BC, CA, and AB). It is obvious that the required locus consists only of points P with this property.

Our first task now is to find out when the Euler lines eu, eu_a, eu_b, and eu_c are undetermined (that is, when both the x-coefficient and the y-coefficient of their equations are zero). We have already observed that this will happen when triangles t, t_a, t_b, and t_c are equilateral. Hence, the solutions for the last three triangles will give us six points which are vertexes of equilateral triangles constructed on the sides of ABC. This can be done either towards outside (when we get points A_v, B_v, and C_v) or towards inside (when we get points A_u, B_u, and C_u). Those six points are definitely outside of the locus that we are looking for.

Once we have six important points that are not in the locus, we can immediately find the condition that coordinates of P must satisfy for Euler lines eu_a, eu_b, and eu_c to meet at a point. It is well known that the determinant

$$DET_{234} = a_{21}a_{32}a_{43} - a_{21}a_{33}a_{42} - a_{31}a_{22}a_{43} + a_{31}a_{23}a_{42} + a_{41}a_{22}a_{33} - a_{41}a_{23}a_{32}$$

of the matrix formed by the coefficients of their equations must be zero. By substituting the above values for the coefficients, we discover that DET_{234} can be represented as the product $c\, Q\, R$, where $Q = v\,(p^2 + q^2) - c\, v\, p + (c\, u - b^2)\, q$, and

$$R = (\, p^2 + q^2\,)\,(\, e\, p + f\, q\,) + g\, p^2 + h\, p\, q + k\, p + m\, q^2 + n\, q,$$

with

$$e = b^2 + 2\, u^2 - 3\, c\, u, \quad f = v\,(2\, u - c), \quad g = 3\, u\,(c^2 - b^2), \quad h = 2\, v\,(c^2 - b^2),$$

$$k = c\,(3\, u\, b^2 - c\, b^2 - 2\, c\, u^2), \quad m = u\,(c^2 - b^2), \quad \text{and} \quad n = c\, v\,(b^2 - 2\, c\, u).$$

We conclude that the required locus Z has the form $(K \bigcup N) \setminus V$, where V is the union of the lines BC, CA, and AB and the points A_u, A_v, B_u, B_v, C_u, and C_v, the set K is the set of all points P whose coordinates (p, q) satisfy the relation

$Q = 0$, while the set N consists of all points P whose coordinates are solutions of the equation $R = 0$.

The first equation $Q = 0$ obviously represents a circle and since coordinates of the points A, B, and C are among its solutions, we conclude that K is the circumcircle of ABC.

This was noticed in [21] and is an immediate consequence of the elementary geometry of a triangle. Indeed, for all points P on the circumcircle different from the vertexes A, B, C the four triangles ABC, BCP, CAP, and ABP have the same circumcircle and consequently, the four Euler lines intersect in the circumcentre. This is true when ABC does not have angles of either $\pi/3$ or $2\pi/3$ radians because then none of the four triangles is equilateral. In this exceptional cases, by considering the centre of an equilateral triangle as the degenerate Euler line, the statement of concurrence of the Euler lines (for points P on the circumcircle) remains true.

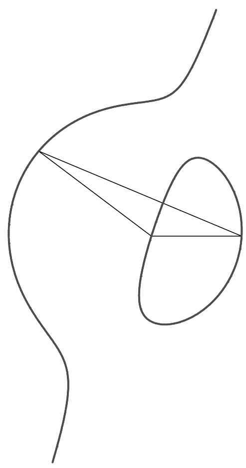


Figure 6: $\cot(\frac{A}{2}) = -3$.

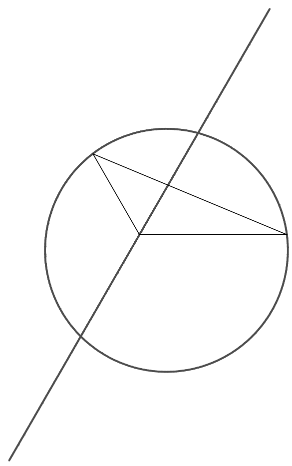


Figure 7: $\cot(\frac{A}{2}) = -\sqrt{3}$.

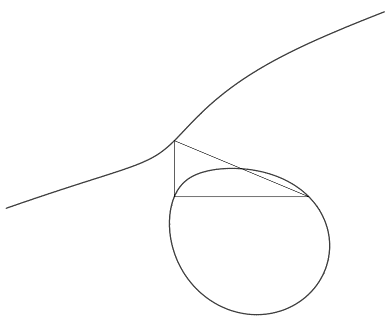


Figure 8: $\cot(\frac{A}{2}) = -1$.

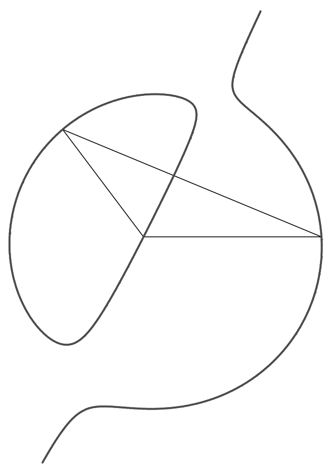


Figure 9: $\cot(\frac{A}{2}) = \frac{1}{2}$.

The second equation $R = 0$ will in general represent a curve N of order three in the plane (in the sense that every line either lies in N or intersects N in at most three points). As we shall see below, this curve is already extensively investigated and is known as the Neuberg cubic of the triangle ABC. However, as far as we know, nobody has observed the remarkable property of this cubic suggested by Tabov's problem that we verify in this paper.

The author is grateful to professor Vladimir Volenec who provided extensive help in references and suggested during our lecture in the Geometry Seminar of the University of Zagreb that the curve that answers J. Tabov's problem is in fact the Neuberg cubic. He also discovered that Tabov's Superior Locus Problem is given in [15] on page 200 as Exercise 20 in the section on the Neuberg cubic.

Our Figures 6 – 13 show the Neuberg cubic of a triangle ABC with $\cot(\frac{B}{2}) = 5$ and with stated values of $\cot(\frac{A}{2})$.

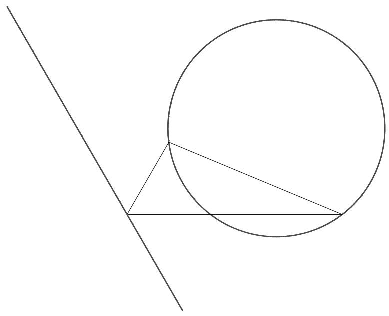


Figure 10: $\cot(\frac{A}{2}) = \sqrt{3}$.

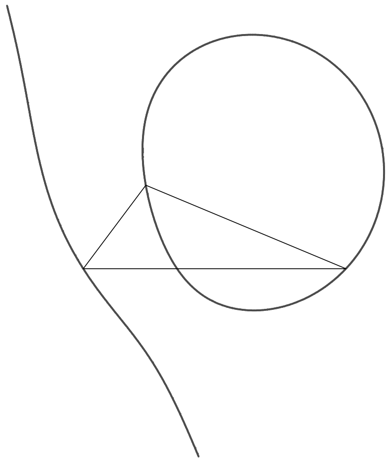


Figure 11: $\cot(\frac{A}{2}) = 2$.

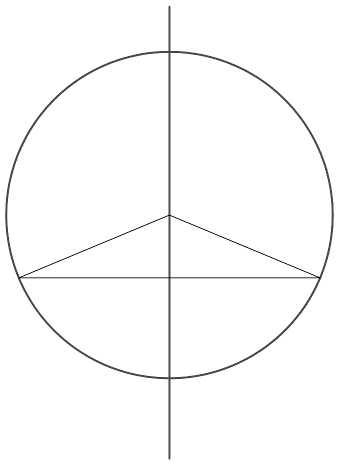


Figure 12: $\cot(\frac{A}{2}) = 5$.

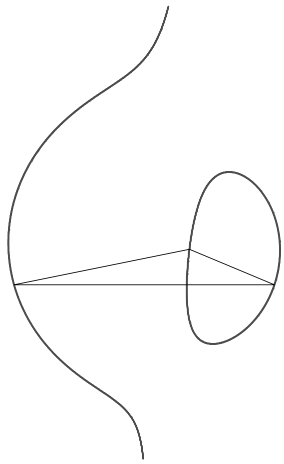


Figure 13: $\cot(\frac{A}{2}) = 10$.

It is interesting to observe that the analogous determinants DET_{123}, DET_{124}, and DET_{134} for other triplets among the Euler lines eu, eu_a, eu_b, and eu_c have up to a product with a constant the same form. This observation gives the proof of the **Appendix to the Superior Locus Problem** in [21] which asks to prove if any three among the lines eu, eu_a, eu_b, and eu_c are concurrent, then all four are concurrent. Of course, for this to be true the triangle ABC can not be equilateral. Hence, if ABC is not equilateral, and P is from the locus Z, then the Euler lines eu_a, eu_b, and eu_c intersect at the point on eu.

The Neuberg cubic has several equivalent geometric descriptions. He himself has discovered in [17] that it is a part (in addition to the circumcircle) of the locus of all points P with the property that the lines AO_a, BO_b, and CO_c are concurrent, where O_a, O_b, and O_c denote circumcenters of the triangles BCP, CAP, and ABP, respectively (see Figure 14).

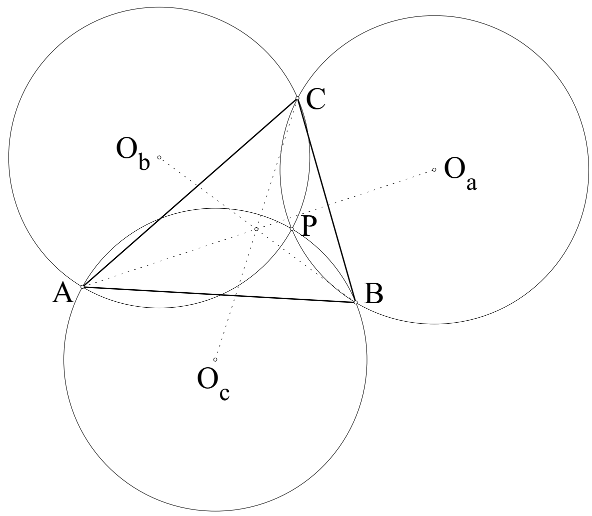


Figure 14: Lines AO_a, BO_b, and CO_c are concurrent, where O_a, O_b, and O_c are circumcenters of triangles BCP, CAP, and ABP.

Another description also due to Neuberg is based on the notion of the power of a point with respect to a circle that we recall now (see Figure 15).

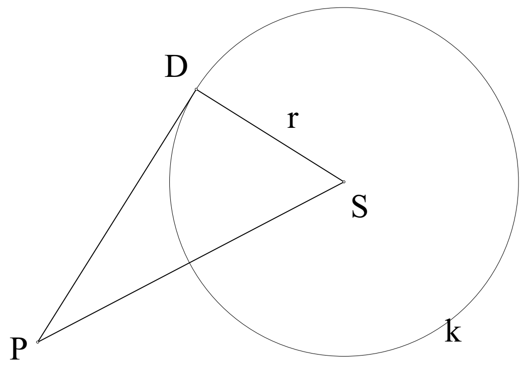


Figure 15: Power of a point with respect to a circle.

Suppose that P is a point and k is a circle in the plane and that S is the centre of this circle and r is it's radius. Then the power of the point P with respect to the circle k is the number $w_k(P) = |PS|^2 - r^2$, that is, the difference of squares of distance from P to S and the radius r of k.

For points P and Q, let kPQ denote the circle with centre P and radius $|PQ|$.

Neuberg has proved in [17] that his cubic is the locus cuN of all points P such that $M(P) = 0$, where

$$M(P) = w_{kAB}(P)\, w_{kBC}(P)\, w_{kCA}(P) - w_{kAC}(P)\, w_{kBA}(P)\, w_{kCB}(P).$$

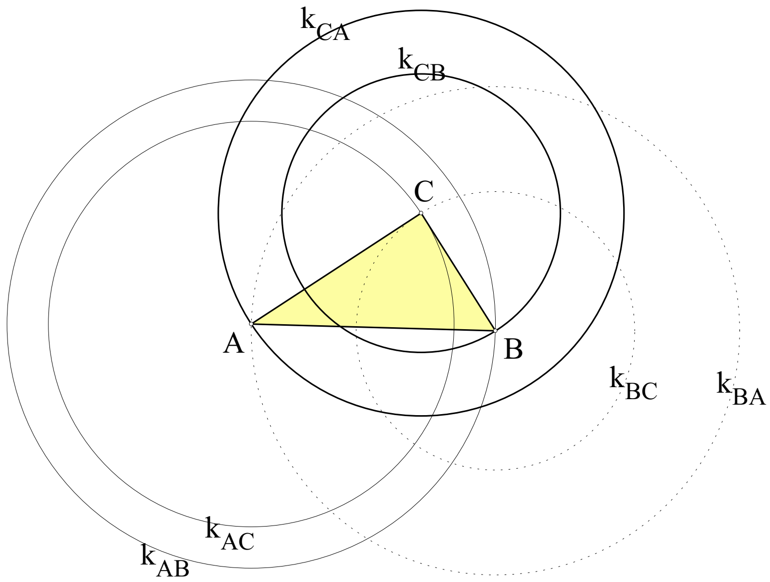


Figure 16: Six circles from Neuberg's characterization.

We shall prove that N and cuN are the same curve by showing that they have identical equations in the chosen coordinate system. The equation for N is $R = 0$. Let us determine the equation for cuN by computing the powers of the point P with respect to the six circles above and substituting them into the expression $M(P)$. Let $w = p^2 + q^2$. One can easily find that

$$w_{kAB}(P) = w - b^2,\ w_{kBC}(P) = w - 2\,cp,\ w_{kCA}(P) = w - 2\,up - 2\,vq - c^2 + 2\,cu,$$
$$w_{kAC}(P) = w - c^2,\ w_{kBA}(P) = w - 2\,cp - b^2 + 2\,cu,\ w_{kCB}(P) = w - 2\,up - 2\,vq,$$

and that $M(P)$ is in fact the product $-2\,c\,R$. Since c is a constant, we conclude that curves N and cuN have identical equations and our claim is proved.

4. LOCUS RECOGNITION IN AREAL COORDINATES

In this section we shall give a solution for the Superior Locus Problem using analytic geometry in areal coordinates. The advantage of this approach is that all expressions are symmetric. However, it is more likely that the reader is unfamiliar with basics of analytic geometry in areal coordinates that we use in arguments below.

We shall identify a point and a line with the row matrix formed by its areal coordinates and its coefficients. For points or lines P, Q, and R, let $P \cdot Q$ be the scalar product of P and Q, let $[P, Q, R]$ be the 3×3-matrix with rows P, Q, and R and let $d[P, Q, R]$ be it's determinant.

Recall [12, p.6] that $M[f + r, g + s, h + t]$ is the midpoint of points $P[[f, g, h]]$ and $Q[[r, s, t]]$, that [12, p.8] a point $X[x, y, z]$ lies on the line PQ if and only if $d[P, Q, X] = 0$, that [23] the lines $p[f, g, h]$ and $q[r, s, t]$ are perpendicular if and only if

$$\alpha\,(g - h)\,(s - t) + \beta\,(f - h)\,(r - t) + \gamma\,(f - g)\,(r - s) = 0,$$

and parallel if and only if $f\,(s-t)+g\,(t-r)+h\,(r-s)=0$, and that [12, p.10] the lines p, q, and r are concurrent if and only if $d[p,\,q,\,r]=0$.

Now we a ready to begin our proof. First observe that $A[1,0,0]$, $B[0,1,0]$, and $C[0,0,1]$. Let $P[x,y,z]$. In order to find areal coordinates of the centroid G_a of the triangle BCP, we determine the midpoint $A_m[0,1,1]$ of BC and the midpoint $Q[x,\,x+2y+z,\,z]$ of BP. Then $G_a[x,\,x+2y+z,\,x+y+2z]$ is the intersection of lines A_mP and CQ. The centroids G_b and G_c of the triangles CAP and ABP are relatives of G_a and the centroid of ABC is $G[1,\,1,\,1]$.

The equation of an altitude of a triangle can be derived using the facts that it is perpendicular to a side line and that it passes through a vertex. In this way we discover that $H[\beta\,\gamma]$ is the orthocenter of ABC, that areal coordinates of the orthocenter H_a of the triangle BCP are

$$x[H_a]=(\gamma x+\beta y+\gamma y)(\beta x+\beta z+\gamma z),\qquad y[H_a]=(\alpha x-\gamma z)(\gamma x+\beta y+\gamma y),$$
$$z[H_a]=(\alpha x-\beta y)(\beta x+\beta z+\gamma z),$$

and that orthocenters H_b and H_c of triangles CAP and ABP are relatives of H_a.

Joining these centroids and orthocenters give us the Euler line $eu[\alpha\,(\beta-\gamma)]$ of ABC and the Euler line $eu_a[f,\,g,\,h]$ of BCP, where

$$\begin{aligned}f=\alpha\,(\beta-\gamma)\,x^3-\left(\beta^2-\alpha\,\beta+2\,\alpha\,\gamma\right)x^2y+\left(\gamma^2-\alpha\,\gamma+2\,\alpha\,\beta\right)x^2z-\\ \left(2\,\beta^2-\beta\,\gamma+1\right)xy^2+2\,(\gamma-\beta)\,(\gamma+\beta)\,xyz+\left(2\gamma^2-\beta\,\gamma+1\right)xz^2-\\ (\gamma+\beta)\,(2\,\beta-\gamma)\,y^2z-(\gamma+\beta)\,(\beta-2\,\gamma)\,yz^2,\end{aligned}$$

$$\begin{aligned}g=\beta\,(\gamma-\alpha)\,x^3+2\,\beta\,(\beta+\gamma)\,x^2y+\left(\gamma^2+4\,\beta\,\gamma-1\right)x^2z+\beta\,(\gamma+\beta)\,xy^2+\\ 2\,(\gamma+\beta)\,(2\,\beta+\gamma)\,xyz+2\,(\gamma+\beta)\,\gamma\,xz^2+(\gamma+\beta)^2\,y^2z+2\,(\gamma+\beta)^2\,yz^2,\end{aligned}$$

$$\begin{aligned}h=\gamma\,(\alpha-\beta)\,x^3-\left(\beta^2+4\,\beta\,\gamma-1\right)x^2y-2\,(\gamma+\beta)\,\gamma\,x^2z-2\,\beta\,(\gamma+\beta)\,xy^2-\\ 2\,(\beta+2\,\gamma)\,(\gamma+\beta)\,xyz-(\gamma+\beta)\,\gamma\,xz^2-2\,(\gamma+\beta)^2\,y^2z-(\gamma+\beta)^2\,yz^2,\end{aligned}$$

while Euler lines eu_b and eu_c of triangles CAP and ABP are relatives of eu_a.

Then $d[eu_a,\,eu_b,\,eu_c]$ is the product $K\,L^4\,M$, where $K=\mathbb{S}\,(\beta+\gamma)\,y\,z$, $L=\mathbb{S}\,x$, and

$$M=\mathbb{S}\,(1-3\,\beta\,\gamma)\,x\,[(\alpha+\beta)\,y^2-(\alpha+\gamma)\,z^2].$$

Here $K=0$ is the equation of the circumcircle of ABC, $L=0$ is the equation of the line at infinity, and $M=0$ is the equation of the Neuberg cubic in areal coordinates [17]. This concludes our proof. In order to get the proof of the Appendix to the Superior Locus Problem it suffices to observe that up to a sign $d[eu,\,eu_a,\,eu_b]$, $d[eu,\,eu_a,\,eu_c]$, and $d[eu,\,eu_b,\,eu_c]$ are equal to the product $K\,L\,M$.

5. PARALLELS AT VERTEXES TO EULER LINES

As an easy consequence of our knowledge of equations of Euler lines eu_a, eu_b, and eu_c is the following result which provides a new characterization of the Neuberg cubic.

Theorem 5.1. *A point P is either on the Neuberg cubic or on the circumcircle of the triangle ABC if and only if parallels at vertexes of ABC to the Euler lines of triangles BCP, CAP, and ABP are concurrent.*

Let pa_a, pa_b, and pa_c be parallels at A, B, and C to eu_a, eu_b, and eu_c. Then $pa_a[0,f,g]$, where

$$f=(1-3\,\alpha\,\beta)\,x^2+(1+3\,\beta^2)\,x\,y+(3\,\beta\,\gamma-1)\,x\,z+3\,\beta\,(\beta+\gamma)\,y\,z,$$
$$g=(1-3\,\alpha\,\gamma)\,x^2+(3\,\beta\,\gamma-1)\,x\,y+(3\,\gamma^2+1)\,x\,z+3\,\gamma\,(\beta+\gamma)\,y\,z,$$

and pa_b and pa_c are relatives of pa_a. Since the determinant $d[pa_a,\, pa_b,\, pa_c]$ is the product $3\,K\,L^4\,M$ our theorem is proved.

Remark 5.1. Instead of parallels at the vertexes of ABC we can get the same conclusion for parallels at vertexes of some triangles related to ABC. For example, we can take the complementary triangle $A_mB_mC_m$ whose vertexes are midpoints of sides or the anticomplementary triangle $A_aB_aC_a$ whose vertexes are intersections of parallels through vertexes to sides. But, for the orthic triangle $A_oB_oC_o$ whose vertexes are feet of altitudes the theorem is not true.

6. PARALLEL EULER LINES

In this section we shall consider the related locus problem which asks to find all points P in the plane such that Euler lines eu_a, eu_b, and eu_c of the triangles BCP, CAP, and ABP are parallel. Since parallel lines are concurrent and for points P on the circumcirle the above Euler lines intersect at the circumcenter, it is clear that the required locus is a subset of the Neuberg cubic of ABC.

The Appendix to the Superior Locus Problem implies that eu_a, eu_b, and eu_c are parallel if and only if they are all parallel to the Euler line eu of ABC. Hence, the required locus is the intersection $L_a \cap L_b \cap L_c$, where L_i for $i = a,\, b,\, c$ denotes a locus of all points P such that eu and eu_i are parallel. Of course, we must consider only scalene triangles here.

In order to determine L_i, observe that eu and eu_i are parallel if and only if they are concurrent with the line $\ell_\infty = [1,\, 1,\, 1]$ at infinity, that is, if and only if the determinant $d[eu,\, eu_i,\, \ell_\infty]$ vanishes ($i = a,\, b,\, c$). It follows that

$$L_a = 3\,(\beta^2 - \gamma^2)\,y\,z + (1 - 3\,\gamma^2)\,z\,x + (3\,\beta^2 - 1)\,x\,y,$$

$L_b = L_a^\sigma$, and $L_c = L_a^\tau$. These are conics circumscribed to the triangle ABC (see [12, p.38]) which intersect at vertexes and the point

$$Ze\left[\frac{1}{1 - 6\,\alpha^2 + 9\,\alpha^2\beta^2 + 9\,\alpha^2\gamma^2 - 9\,\beta^2\gamma^2}\right].$$

Our locus consists only of the point Ze. This central point of the triangle ABC is not on Kimberling's list [11].

Applying the procedure described in [12, p.43], we can easily find that the center of the conic L_a is the midpoint A_m of the side BC.

Let us now use the method of [12, p.49] to decide on the shape of the conic L_a. It is well known that a conic will be either an ellipse, a hyperbola, or a parabola provided it has two imaginary, two real, or just one real intersection with the line at infinity.

A point $P[x,\, y,\, z]$ will be at the required intersection if the expression $H(x,\, y)$ obtained by a substitution of $z = -x - y$ into the equation of L_a vanishes.

When $\beta = \gamma$, then

$$H(x,\, y) = (3\,\beta^2 - 1)\,x\,(x + 2\,y),$$

so that L_a is a hyperbola (because the angle B can not be $\frac{\pi}{3}$ since ABC is scalene nor can it be $\frac{2\,\pi}{3}$ since sum of angles must be π).

When β and γ are different, then $H(x,\, y)/(3\,\gamma^2 - 3\,\beta^2)$ has the form

$$\left(y + \frac{x\left(3\,\gamma^2 - 1\right)}{3\,\gamma^2 - 3\,\beta^2}\right)^2 - \frac{x^2\left(3\,\beta^2 - 1\right)\left(3\,\gamma^2 - 1\right)}{9\,\left(\gamma - \beta\right)^2\left(\gamma + \beta\right)^2}.$$

From this expression we can immediately conclude that L_a is a parabola when either B or C is either $\frac{\pi}{3}$ or $\frac{2\,\pi}{3}$, a hyperbola when either B or C is greater than $\frac{\pi}{3}$ and smaller than $\frac{2\,\pi}{3}$, and in all other cases it is an ellipse.

7. LOCUS RECOGNITION IN DISTANCE COORDINATES

For the points P and Q, let P_Q denote the distance from P to Q and let P^Q denote the square of P_Q. For a scalene triangle ABC in the plane, let E_{ABC} denote the function which associates to the point P the determinant of the matrix

$$\begin{bmatrix} 1 & A^P & B^C \\ 1 & B^P & C^A \\ 1 & C^P & A^B \end{bmatrix}.$$

Lemma 7.1. *The point P lies on the Euler line of the scalene triangle ABC if and only if $E_{ABC}(P) = 0$.*

Necessity. Consider the scalene triangle ABC in the Cartesian coordinate system and assume that $A(0, 0)$, $B(c, 0)$, and $C(u, v)$. The Euler line GO is the line connecting the centroid $G(\frac{c+u}{3}, \frac{v}{3})$ and the circumcenter $O(\frac{c}{2}, \frac{b^2-c\,u}{2\,v})$ It follows that GO has the equation $eGO = 0$, where

$$eGO = \left(3\,u^2 - 3\,cu + v^2\right) x + v\,(2\,u - c)\,y + u\left(c^2 - b^2\right).$$

If $2\,u \neq c$, then we can solve $eGO = 0$ for y which leads to the conclusion that a point P on GO has coordinates x_0 and y_0, where x_0 is any real number and

$$y_0 = \frac{x_0\,(3\,c\,u - 3\,u^2 - v^2) + u\,(b^2 - c^2)}{v\,\,(2\,u - c)}.$$

If $2\,u = c$, then a point P on GO has coordinates $\frac{c}{2}$ and y_0, where y_0 is any real number. In both cases it is easy to check that $E_{ABC}(P) = 0$. □

Sufficiency. Conversely, if $P(x, y)$ is any point in the plane, an easy computation shows that $E_{ABC}(P) = 2\,c\,eGO$. It follows that P is on GO when $E_{ABC}(P) = 0$. □

We let $E_{ABC}(Q) = 0$, $E_{BCP}(Q) = 0$, $E_{CAP}(Q) = 0$, and $E_{ABP}(Q) = 0$ be shortly denoted by e_1, e_2, e_3, and e_4, where ABC is any scalene triangle (if e_1 is mentioned), P is any point in the plane outside the set V, and Q is any point in the plane. Notice that the sum of any three of these equations gives the fourth which provides an alternative proof of the Appendix to the Superior Locus Problem.

Theorem 7.1. *Let ABC be an equilateral triangle and let W_e denote the union of the lines BC, CA, and AB and the three vertexes A_v, B_v, and C_v of equilateral triangles constructed externally on the sides of ABC. Then for every point P in the plane outside the set W_e, the Euler lines of the triangles BCP, CAP, and ABP are concurrent.*

Proof. Let P be a point outside of the set W_e. It follows from the above lemma that the Euler lines of the triangles BCP, CAP, and ABP are concurrent if and only if either they are parallel or there is a point Q such that equations e_2, e_3, and e_4 hold.

If the point P is neither on kBC (the circle with the center at B and with the radius B_C) nor on kAC, then we can solve e_3 for x and e_2 for y. Since the substitution of these values into e_4 gives identity, we conclude that for every such point P the point Q exists.

If P is on $kBC \setminus \{A, C, A_v, C_v\}$, then $P^A \neq C^A$ and $P^C \neq B^C$, so that after substituting $B^P = B^C$ into equations e_2, e_3, and e_4, we get $A^Q = P^Q$ from the second and $C^Q = P^Q$ from the first, while the third holds.

If P is on $kAC \setminus \{B, C, B_v, C_v\}$, then $B^P \neq A^B$ and $C^P \neq A^C$, so that after substituting $A^P = B^A$ into equations e_2, e_3, and e_4, we get $B^Q = P^Q$ from the third and $C^Q = P^Q$ from the second, while the first holds.

Hence, in both cases the point Q exists and all points of the plane not in W_e are in the locus. □

Theorem 7.2. *Let ABC be a scalene triangle and let W_s denote the union of the lines BC, CA, and AB and the vertexes A_v, B_v, C_v, A_u, B_u, and C_u of the equilateral triangles constructed externally and internally on the sides of ABC. Then for a point P in the plane outside the set W_s, the Euler lines of the triangles BCP, CAP, and ABP are concurrent if and only if P lies on the circumcircle or on the Neuberg cubic of the triangle ABC.*

Proof. We have seen in the section 6 that Euler lines of triangles BCP, CAP, and ABP are parallel to the Euler line of ABC if and only if P is the point Ze on the Neuberg cubic of ABC. Hence, it remains to consider the case when at least one among lines eu_a, eu_b, and eu_c intersects the line eu. Without loss of generality, we can assume that eu_a intersect eu at a point Q.

If $b \neq c$ and $B^P \neq C^P$, then we can solve e_1 for A^Q and e_2 for P^Q. By substituting these values into e_3 and e_4 up to a sign in both cases we get

$$\frac{J\,(B^Q - C^Q)}{(A^B - A^C)(B^P - C^P)},$$

where J is the determinant of the matrix

$$\begin{bmatrix} 1 & A^P + B^C & A^P B^C \\ 1 & B^P + C^A & B^P C^A \\ 1 & C^P + A^B & C^P A^B \end{bmatrix}.$$

We conclude that the point Q will also lie on the lines eu_b and eu_c if and only if either $J = 0$ which is Neuberg's condition for a point to be on his cubic [17] or that $B^Q = C^Q$. Then from e_1 and e_2 we get $A^Q = C^Q$ and $P^Q = C^Q$ so that Q is equally distant from A, B, C, and P. Hence, P is on the circumcircle and Q is the circumcenter.

Suppose now that $b = c$. The equation e_1 is now $(B^C - B^A)(B^Q - C^Q) = 0$. Since the triangle ABC is scalene, we must have $a \neq c$, so that $B^Q = C^Q$ (that is, Q lies on the perpendicular bisector of BC). Observe that for $b = c$, the determinant J which is the equation of the Neuberg cubic in distance coordinates has the form $J_{bc} = (B^C - B^A)(A^P - A^B)(C^P - B^P)$, Also, the equation e_2 has the form $(C^Q - P^Q)(C^P - B^P) = 0$. This holds when either $C^Q = P^Q$ or $C^P = B^P$. In the second case $J_{bc} = 0$ and P lies on the Neuberg cubic of ABC. In the first case (when $C^Q = P^Q$), both equations e_3 and e_4 are equivalent to the equation $(A^P - A^B)(A^Q - C^Q) = 0$. This equation holds if and only if $A^P = A^B$ (that is, P is on the Neuberg cubic of ABC because J_{bc} again vanishes) or $A^Q = C^Q$ (that is, Q is the circumcenter and P is on the circumcircle).

Finally, suppose that $b \neq c$ and $B^P = C^P$. The equation e_2 has now the form $(C^Q - B^Q)(C^P - C^B) = 0$ and the determinant J becomes

$$J_{qr} = (A^B - A^C)(C^P - C^B)(C^P - A^P).$$

When $C^P = C^B$, then $J_{qr} = 0$ so that P is on the Neuberg cubic of ABC. On the other hand, when $C^Q = B^Q$, then from e_1 it follows that $A^Q = C^Q$ and the equations e_3 and e_4 are both equivalent to the equation $(C^Q - P^Q)(C^P - A^P) = 0$. In the same way as above we conclude that P is either on the circumcircle or on the Neuberg cubic of ABC. □

8. BASIC PROPERTIES OF THE CURVE N

The most fascinating property of the curve N is that isogonal conjugate F of a point P from $N \setminus V$ also belongs to $N \setminus V$.

Recall that the point F is the intersection of lines AD and BE, where D and E are reflections of P with respect to bisectors of angles at A and B, respectively. Hence, the normal coordinates of P and F are reciprocal, that is, if $P\langle x\rangle$ then $F\langle \frac{1}{x}\rangle$.

In other words, the equation of the Neuberg cubic in normal coordinates remains invariant under the replacement of variables with their reciprocal values.

This is true because in normal coordinates the equation of N is $M^\star = 0$, where

$$M^\star = \mathbb{S}\,(\alpha_\kappa - 2\,\beta_\kappa\,\gamma_\kappa)\,x\,(y^2 - z^2).$$

Indeed, if $P\langle x\rangle$, then $P[a\,x]$ so that we get $M^\star$ from M by replacing x, y, and z with $a\,x$, $b\,y$, and $c\,z$ and applying the formulas from section 2.

An easy task is to determine points of intersection of the curve N with lines BC, CA, and AB. Since these lines are $BC[1,\,0,\,0]$, $CA[0,\,1,\,0]$, and $AB[0,\,0,\,1]$ we can easily find these intersections. We discover that N intersects BC at vertexes B and C, and the point $E_a[0,\,f,\,g]$, the line CA at vertexes C and A and the point E_b, and the line AB at vertexes A and B and the point E_c, where $f = 1 - 3\,\gamma\,\alpha$, $g = 1 - 3\,\alpha\,\beta$, and E_b and E_c are relatives of E_a.

The line $AE_a[0, 3\,\alpha\,\beta - 1, 1 - 3\,\gamma\,\alpha]$ and its relatives BE_a and CE_c are parallel to the Euler line eu of the triangle ABC. Moreover, the area of the triangle $E_aE_bE_c$ is twice the area of the triangle ABC.

It is obvious that the Euler line of the triangle ABC plays an important role in the properties of the curve N. This is also evident from the following observations.

Let $f = a^2\,k_a - k_b\,k_c$, $g = f^\sigma$, and $h = f^\tau$. One can easily show that the Neuberg cubic in areals has an equivalent representation in the form $M_\star = 0$, where

$$M_\star = f\,x\,(c^2y^2 - b^2z^2) + g\,y\,(a^2z^2 - c^2x^2) + h\,z\,(b^2x^2 - a^2y^2).$$

Let $P[x,\,y,\,z]$ be any point in the plane. Since the Euler line eu of the reference triangle ABC is $[h-g,\,f-h,\,g-f]$, the parallel pa to eu through P is $[h\,y - g\,z, f\,z - h\,x, g\,x - f\,y]$. The line pa intersects the curve $M_\star$ in three points $P_\infty[f,\,g,\,h]$ and $P_i[fy - gx + (hx - fz)\,k_i,\ (hy - gz)\,k_i,\ hy - gz]$ $(i = 1,\,2)$, and k_i are roots of the quadratic polynomial $Q = p\,Z^2 + q\,Z + r$ with coefficients

$$p = c^2\,(fz - hx)\,(fy - gx)\,, \qquad r = b^2\,(fz - hx)\,(fy - gx)\,,$$

$$q = a^2\,(hy - gz)^2 - b^2\,(fz - hx)^2 - c^2\,(gx - fy)^2.$$

The discriminant $q^2 - 4\,p\,r$ of Q is the product of equations of lines

$$\ell[cg - bh,\, ah - cf,\, bf - ag], \qquad \ell_a[bh - cg,\, ah + cf,\, -bf - ag],$$

$\ell_b = \ell_a^\sigma$, and $\ell_c = \ell_a^\tau$. These are parallels to the Euler line through the incenter $I[a,\,b,\,c]$ and the excenters A_e, B_e, and C_e. We conclude that lines ℓ, ℓ_a, ℓ_b, and ℓ_c are the only parallels to the Euler line which touch the Neuberg cubic and that they decompose the plane into regions R_0 and R_2 such that each parallel to the Euler line has no real intersections with N if and only if it lies in R_0 and it has two real intersections with N if and only if it lies in R_2. Moreover, the two intersections for parallels in the region R_2 are isogonal conjugates.

The last observation can make one believe that $N \setminus \{I,\,A_e,\,B_e,\,C_e\}$ is the locus of all points P such that the line joining P with its isogonal conjugate F is parallel with the Euler line of ABC. This is indeed true. The proof is a simple exercise in showing that the equation $M = 0$ is equivalent to the condition that the Euler line $[\alpha\,(\beta - \gamma)]$ is parallel to the line $PF[x\,(c^2y^2 - b^2z^2)]$ joining a point $P[x]$ with its isogonal conjugate $F[a^2/x]$. The points I, A_e, B_e, and C_e are the only fixed points of the isogonal conjugation and therefore must be excluded because for these points $F = P$ and the line PF is undetermined.

The Euler line of the triangle ABC itself intersects the curve N only at the orthocenter H and the circumcenter $O[1 - \beta\gamma]$.

9. THIRTYFIVE POINTS OF THE CURVE N

Our goal now is to prove that some important points associated to the triangle ABC belong to the curve N. The method of proof is to identify coordinates of these points with respect to the chosen coordinate system and to check that they are roots of the polynomial which represents N. Without the use of some computer program (for example Derive, Maple, or Mathematica) some of these are rather difficult exercises in simplification of algebraic expressions. On the other hand, these programs can verify our claims in seconds and we challenge the reader to work them out.

The following 35 points associated with the triangle ABC lie on the curve N, (that is, their coordinates satisfy one of the equations $R = 0$, $J = 0$, $M^{\star} = 0$, or $M_{\star} = 0$):

(a) The vertexes $A[1, 0, 0]$, $B[0, 1, 0]$, and $C[0, 0, 1]$.

(b) The orthocenter $H[\beta\gamma]$.

(c) The circumcenter $O[1 - \beta\gamma]$.

Recall that O and H are isogonal conjugates of each other.

(d) The six vertexes A_u, B_u, C_u, A_v, B_v and C_v of equilateral triangles constructed on the sides of the triangle ABC both inside and outside. Perhaps the simplest proof for A_v is to observe that it has two equal distance coordinates so that J has two identical rows and is therefore zero. The other approach is to show that $A_v[-2\sqrt{3}\,a^2, \sqrt{3}\,k_c + \delta, \sqrt{3}\,k_b + \delta]$ and that these areals of A_v satisfy $M_{\star} = 0$.

(e) The two isogonic centers $I_v[1/(\sqrt{3}\,k_a + \delta)]$ and $I_u[1/(\sqrt{3}\,k_a - \delta)]$ which are by definition intersections of concurrent lines AA_v, BB_v, CC_v and AA_u, BB_u, CC_u, respectively. If all angles of the triangle ABC are less than 120^o, then the point I_v is also known as Fermat point or Toricelli point of ABC, so that this improves Tabov's statement (from Sharygin's book) that the Toricelli point lies in the locus Z. These points are listed as points X_{13} and X_{14} in [11] with normal coordinates $I_v\langle csc(A + \pi/3)\rangle$ and $I_u\langle csc(A - \pi/3)\rangle$.

(f) The six isogonal conjugates A_j, B_j, C_j, A_i, B_i, and C_i of the points A_v, B_v, C_v, A_u, B_u, and C_u, respectively. They have interesting distance coordinates that we describe now. Let

$$q = \sqrt{k + \sqrt{3}\,\delta}, \quad w = \sqrt{k - \sqrt{3}\,\delta}, \quad q_a = 3\,k_a + \sqrt{3}\,\delta, \quad w_a = |\,3\,k_a - \sqrt{3}\,\delta\,|.$$

Then we can easily compute the following $A_j(3\sqrt{3}\,m_a\,q/q_a,\, c\,q_c/q_a,\, b\,q_b/q_a)$ and $A_i(3\sqrt{3}\,m_a\,w/w_a,\, c\,w_c/w_a,\, b\,w_b/w_a)$, while B_j and C_j are relatives of A_j and also B_i and C_i are relatives of A_i.

(g) The two isodynamic centers $D_v[a^2\,(\sqrt{3}\,k_a + \delta)]$ and $D_u[a^2\,(\sqrt{3}\,k_a - \delta)]$ which are the intersections of concurrent lines AA_j, BB_j, CC_j and AA_i, BB_i, CC_i, respectively. The points D_v and D_u are two points of the intersection of the three circles of Apollonious. Observe that D_v and D_u are isogonal conjugates of I_v and I_u. Isodynamic points are listed as points X_{15} and X_{16} in [11] with normal coordinates $D_v\langle sin(A + \pi/3)\rangle$ and $D_u\langle sin(A - \pi/3)\rangle$.

(h) The incenter (the center of the circle inscribed into ABC) I and the three excenters (centers of three circles that touch the sides of ABC from outside) A_e, B_e, and C_e. Their normal coordinates are $I\langle 1\rangle$, $A_e\langle -1, 1, 1\rangle$, $B_e\langle 1, -1, 1\rangle$, and $C_e\langle 1, 1, -1\rangle$ while distance coordinates for I and A_e are given by $I(\sqrt{m_a\,t_a/t})$ and $A_e(\sqrt{m_a\,t/t_a}, \sqrt{m_b\,t_c/t_a}, \sqrt{m_c\,t_b/t_a})$, and B_e and C_e are relatives of A_e.

(i) The reflections of vertexes A, B, and C with respect to sides BC, CA, and AB (the vertexes of the "three images" triangle) $A_r[2\,\delta^2\,a^2,\, t_a\,g_c,\, t_a\,g_b]$ and its relatives B_r and C_r, where $g_a = \delta^2 - (k_a + 2\,m_a)^2$. These points have very simple distance coordinates. For example, $A_r(\delta/a, c, b)$. That the point A_r belongs to the locus Z could be seen as follows. The line BC is a common Euler line for triangles CAA_r

and ABA_r, while Euler lines of triangles ABC and BCA_r are images of each other under the reflection at this line so that the four Euler lines intersect on it.

(j) The isogonal conjugates A_q, B_q, and C_q of the points A_r, B_r, and C_r. These points can be also described as follows. Let the perpendicular bisector of the side BC intersect the sides CA and AB at the points Z_{ab} and Z_{ac}, respectively. We similarly define points Z_{ba}, Z_{bc}, Z_{ca} and Z_{cb}. Then the points A_q, B_q, and C_q are the reflections of the points A, B, and C with respect to the lines $Z_{bc}Z_{cb}$, $Z_{ca}Z_{ac}$, and $Z_{ab}Z_{ba}$, respectively. The areals of A_q are $A_q[t_a\, g_c\, g_b,\, 2\, b^2\, \delta^2 g_b,\, 2\, c^2\, \delta^2 g_c]$ and distance coordinates $A_q(m_a\, \delta/(a\, k_a),\, c^2 k_c/(a\, k_a),\, b^2 k_b/(a\, k_a))$.

(k) The points E_a, E_b, and E_c in which the parallels through the vertexes A, B, and C to the Euler line of ABC intersect the sides BC, CA, and AB, respectively. We have determined areals of those points earlier. Distance coordinates of E_a are $E_a(\delta\, e/f,\, a\, h/f,\, a\, g/f)$, where $e = 3\, m^2 - \mathbb{S}a^4\, k_a$, $f = a^2\, k_a - k_b\, k_c$, $g = f^\sigma$, and $h = f^\tau$.

(l) The point $W[a^2\, g\, h,\, b^2\, h\, f,\, c^2\, f\, g]$ of intersection different from A, B, and C of the curve N with the circumcircle of the triangle ABC. It can be easily checked that the Wallace-Simson line of the point W is perpendicular to the Euler line of ABC. Hence, we can get the point W with the following simple procedure. Let V denote the intersection other than the vertex A of the circumcircle with the perpendicular from A to the Euler line eu. Then the point W is the intersection other than V of the circumcircle with the perpendicular to the side BC through the point V (see [10]).

In many respects the point W is among the most important points of the curve N. It can also be described as the isogonal conjugate of the point in which the Euler line meets the line at infinity. The curve N has as it's real asymptote the line through the point W parallel to the Euler line of ABC. Tripolar coordinates of the point W are $W(f\, m_a/(\delta\, e))$.

10. TANGENTS AND NORMALS OF THE CURVE N

In this section we shall describe some properties of tangents and normals to the curve N at the 35 points discussed in the previous section.

Let K be a curve in the plane. The points P, Q, and R are called K-*tangent* provided they belong to K and there is a point X on K such that the tangents at P, Q, and R on the curve K are concurrent at X. The point X is called K-*pole* of (P, Q, R).

(a) The following eight triples of points are N-tangent: (A, B, C), (E_a, E_b, E_c), (A_i, B_i, C_i), (A_j, B_j, C_j), (A_q, B_q, C_q), (A_r, B_r, C_r), (A_u, B_u, C_u), (A_v, B_v, C_v). The N-pole of (A, B, C) is the point W. The N-poles of other triples from this list are all different points of the curve N some with very complicated coordinates. Many more examples of N-tangent triples can be found among intersections of lines determined by certain pairs of points among the 35 points from the previous section. For example, let X, Y, and Z denote third intersections of lines A_rO, B_rO, and C_rO with the curve N. Then the points X, Y, and Z are N-tangent. Similarly, the third intersections of lines A_qH, B_qH, and C_qH with the curve N are N-tangent.

(b) The normals to the curve N at the vertexes A, B, and C are concurrent at the point $F[a^2/(d_a\, z_a)]$ on the circumcircle which is diametrically opposite to the point W. The point F belongs to the curve N if and only if the triangle ABC is isosceles. It is a singular focus of the cubic N. Distance coordinates of F are $F(m_a\, |d_a|\, z_a/e)$. It follows that for each triangle ABC without equal sides, the segments AF, BF, and CF are sides of a triangle.

(c) The normals to the curve N at the points E_a, E_b, and E_c are rarely concurrent. But, the intersections of these normals determine the triangle which is similar with the triangle ABC.

(d) The tangents to the curve N at the circumcenter O and the orthocenter H intersect at the isogonal conjugate of the center O_9 of the nine-points circle or Feuerbach circle of the triangle ABC (the central point $X_{54}\langle sec(B-C)\rangle$ in [11]).

(e) The tangents to the curve N at the incenter I and the excenters A_e, B_e, and C_e are parallel to the Euler line of ABC. These are the only points of the cubic N with this property.

11. FACTORIZATION OF THE NEUBERG CUBIC

In this section we shall consider cases when due to special properties of the triangle ABC its Neuberg cubic degenerates into curves with simpler description (circles and lines).

(a) If the base triangle ABC is equilateral, then all coefficients of the equation for the curve N vanish. This implies that the locus Z is the set difference $\mathbb{R}^2 \setminus V$ (the entire plane except the points that are always excluded).

Other interesting cases occur when we look for the possibility of representing the equation R of the curve N as the product of a quadratic and a linear expressions. The following two factorizations are the only two possibilities.

(b) If the base triangle ABC is scalene and isosceles, then the curve N is the union of the Euler line and the circle through the vertexes opposite equal sides with the center at the vertex of their intersection.

(c) If the base triangle ABC is scalene and has at some vertex an angle of either $\pi/3$ or $2\pi/3$ radians, then the curve N is the union of the bisector of the exterior angle of this vertex and the circle through the other two vertexes and the circumcenter and the orthocenter.

For the last case, observe that two vertexes of a triangle are concyclic with the orthocenter and the circumcenter if and only if the angle at the remaining vertex is either $\pi/3$ or $2\pi/3$ radians.

12. INTERSECTIONS OF EULER LINES

We know that for every point P in the locus Z the Euler lines of triangles BCP, CAP, and ABP are concurrent at the point Q of the Euler line of the triangle ABC. We shall say that the point Q is *E-related* to the point P. In this final section we shall present some results that give information on E-related points of various points of Z.

(a) The circumcenter O of the triangle ABC is E-related to itself and to all points from $K \setminus V$ (that is, to all points on the circumcircle different from the vertexes A, B, and C). Indeed, Euler lines of triangles BCO, CAO, and ABO are side bisectors and these intersect at O. On the other hand, we have already observed that for P from $K \setminus V$ triangles BCP, CAP, and ABP share circumcenter with ABC.

(b) The center O_9 of the nine-point circle of the triangle ABC is E-related to the orthocenter H of ABC. Indeed, since A is the orthocenter of the triangle BCH and $[\beta\,\gamma,\, 1+\gamma\,\alpha,\, 1+\alpha\,\beta]$ is its centroid, the Euler line of BCH is $[0,\, -(1+\alpha\,\beta),\, 1+\alpha\,\gamma]$. The Euler lines of CAH and ABH are its relatives. Those Euler lines are concurrent at the nine-point center $O_9[1+\beta\,\gamma]$ (the central point X_5 in [11]).

(c) The Schiffler central point X_{21} in [11] by definition is E-related to the incenter I.

(d) The centroid G is E-related to both isogonal points I_u and I_v.

(e) If X and Y denote points on the Euler line of ABC which are E-related to isodynamic points D_v and D_u, respectively, then (X, Y, G, O) is a harmonic quadruple of points.

(f) The intersections of the Euler line of the triangle ABC with the sides BC, CA, and AB are E-related to the points A_r, B_r, and C_r, respectively (see (i) in section 9).

These observations lead to the question if every point on the Euler line of ABC is E-related to a point from N. The following example shows that this question has negative answer.

Let us use Cartesian coordinates and a triangle ABC, where $A(0, 0)$, $B(3, 0)$, and $C(-1, 1)$. The circumcenter S of the tangential triangle of ABC (the central point X_{26} in [11]) lies on the Euler line of ABC and has coordinates $S(\frac{7}{16}, -\frac{21}{80})$. Let $P(p, q)$ be a point such that the Euler lines eu_c and eu_b of triangles ABP and CAP intersect at the point S. In other words, substitution of coordinates of S into equations of eu_c and eu_b gives zero. Hence, $p_1 = 0$ and $p_2 = 0$, where

$$p_1 = 5\,p\,(16\,p + 27)\,(p - 3) + (42\,p - 63)\,q + (80\,p - 35)\,q^2,$$

$$p_2 = 2\,p\,(40\,p + 57)\,(p - 3) + \left(314 + 112\,p - 80\,p^2\right)q + (80\,p - 98)\,q^2 - 80\,q^3.$$

Since p_1 is a quadratic trinomial in q, we can solve for q and substitute these values into p_2 to discover that the only real values for p are -1, 0, $\frac{7}{16}$, 3. However, none of these lead to solutions other than vertexes.

Points X and Y from the locus Z are called *F-related* provided the same point is E-related to both X and Y. For example, by (d), the isogonal points I_u and I_v are F-related.

(g) The following are pairs of F-related points: (A_i, A_j), (B_i, B_j), (C_i, C_j).

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