ON THÉBAULT'S PROBLEM 3887

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Abstract. The famous Thébault's configuration of the triangle ABC depends on a variable point D on its sideline BC and consists of eight circles touching the lines AD and BC and its circumcircle. These circles are best considered in four pairs that are related to the four circles touching the sidelines BC, CA and AB (the incircle and the three excircles). We use the analythic geometry to determine the coordinates of the centers P, Q, S, T, U, V, X and Y of the eight Thébault's circles with respect to a parametrization of the triangle ABC with the inradius r and the cotangents f and g of the angles $\frac{B}{2}$ and $\frac{C}{2}$. The position of the point D is described by the cotangent of the half of the angle between the lines AD and BC. The coordinates of many points in this configuration are simple rational functions in r, f, g and k that makes most computations simple especially when done by a computer. In this approach, the proof of the original Thébault's problem about the incenter I dividing the segment QP in the ratio k^2 is straightforward. A large number of other interesting properties of this gem of the triangle geometry are explored by analythic methods.

1. Introduction

In [27], the authors say that the following result is usually called *Thébault's theorem* (see the portion of the Fig. 1 above the line BC).

Teorem 1. *Let $u(I,r)$ be the incircle of a triangle $\triangle ABC$ (u is the name, I is the center and r is the radius), and D any point on the line BC. Let $k_1(P,r_1)$ and $k_2(Q,r_2)$ be two circles touching the lines AD and BC and the circumcircle $o(O,R)$ of ABC. Then the three centers P, Q and I are collinear and the following relations hold:*

$$PI : IQ = \tau^2, \tag{1}$$

$$r_1 + r_2\,\tau^2 = r(1+\tau^2), \tag{2}$$

where $2\,\theta = \angle ADB$ and $\tau = \tan\,\theta$.

The primary goal of this paper is to give correct versions of the above "theorem". Its formulation is wrong because the requirement "touching the lines AD and BC and the circumcircle $o(O,R)$" is not restrictive enough. This is obvious from the part of the Figure 1 under the line BC since the centers Y, U and I are not collinear. On the other hand, the relation (2) does not hold for all positions of the point D on the line BC.

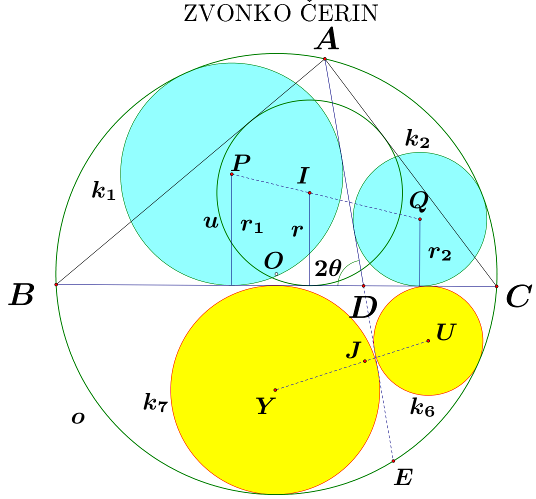

FIGURE 1. Thébault's theorem.

The Problem 3887 in the American Mathematical Monthly by Victor Thébault [25] addresses an unusual result in elementary geometry that is easier to formulate and prove within the analytic geometry rather than in the synthetic geometry. The synthetic approach is traditionally considered as more valuable while the inferior analytic method is always a kind of brute force with lengthly computations.

We need the following notation to have shorter expressions. Let $d = f - g$, $z = f + g$, $\zeta = f g$, $h = \zeta - 1$, $\bar{h} = \zeta + 1$, $f_\pm = f \pm k$, $g_\pm = k \pm g$, $f^\pm = f^2 \pm 1$, $g^\pm = g^2 \pm 1$, $\varphi_\pm = f\,k \pm 1$, $\psi_\pm = g\,k \pm 1$, $K = k^2 + 1$ and $L = k^2 - 1$. Let $\lambda(a, b)$ replace $(\lambda\, a, \lambda\, b)$.

Let ABC be a triangle in the plane. Let $\beta = \angle CBA$ and $\gamma = \angle ACB$. Let $f = \cot\left(\frac{\beta}{2}\right)$ and $g = \cot\left(\frac{\gamma}{2}\right)$ and let $u(I, r)$ be the incircle of the triangle ABC. We shall use the rectangular coordinate system that has the point B as the origin and the point C is on the positive part of the x-axis while the point A is above it. For a point P, let x_P and y_P denote its x- and y-coordinate with respect to this system. Then the vertices A, B and C of the triangle ABC have the coordinates $\frac{rg}{h}(f^-,\, 2f)$, $(0,\, 0)$ and $(r\,z,\, 0)$, where the positive real numbers r, f and g satisfy $h > 0$. The position of a variable point D on the line BC is determined by the positive real number $k = \cot\left(\frac{\delta}{2}\right)$, where δ is the angle between the lines AD and BC. Hence, $D = D_k = D\left(\frac{r\,g\,f_+\,\varphi_-}{h\,k},\, 0\right)$.

2. THÉBAULT'S THEOREM

We shall first determine the coordinates of the centers of Thébault's circles (see Theorem 2). With this important information the proof of the (complete) Thébault's theorem (see Theorems 3, 4 and 5 and the Figure 2) is indeed very simple and straightforward. Of course, our approach is similar to [3] and [21]. However, our choice of the parametrization gives simpler expressions and allows more extensive study of the Thébault's configuration.

Teorem 2. *The points P, Q, S, T, U, V, X and Y with coordinates* $\frac{r\varphi_-}{k}\left(1, \frac{\psi_+}{hk}\right)$, $rf_+\left(1, -\frac{g_-}{h}\right)$, $\frac{rgf_+}{k}\left(1, \frac{fg_-}{hk}\right)$, $-rg\varphi_-\left(1, \frac{f\psi_+}{h}\right)$, $\frac{rg\varphi_-}{hk}\left(z, \frac{g_-}{k}\right)$, $\frac{rgf_+}{h}(z, \psi_+)$, $\frac{rf_+}{hk}\left(-z, \frac{f\psi_+}{k}\right)$ *and* $\frac{r\varphi_-}{h}(z, fg_-)$ *are the centers and* $r_1 = |y_P|, \ldots, r_8 = |y_Y|$ *are the radii of the eight circles* k_i *(*$i = 1, \ldots, 8$*) that touch the lines* BC *and* AD *and the circumcircle* $o(O, R)$.

Proof. Let $P(p, q)$ be the center of the circle that touches the lines BC and AD and the circle o. Then

$$|PP''| = |q|, \tag{3}$$

and

$$|PO|^2 = (R \pm q)^2, \tag{4}$$

where P'' is the orthogonal projection of the point P on the line AD. If $\mathfrak{u} = Lp - 2\,k\,q$, $\mathfrak{v} = Lq + 2\,k\,p$, $\mathfrak{w} = h\,K^2$, then $\frac{4\,r\,g\,k\,f_+\,\varphi_- + hL\mathfrak{u}}{\mathfrak{w}}$ and $\frac{2\,r\,g\,Lf_+\,\varphi_- - 2\,h\,k\mathfrak{u}}{\mathfrak{w}}$ are $x_{P''}$ and $y_{P''}$. Hence, $|PP''| = \left|\frac{h\mathfrak{v} - 2\,r\,g\,f_+\,\varphi_-}{\mathfrak{w}}\right|$. On the other hand, $R = \frac{rf^+g^+}{4\,h}$ and O has the coordinates $\frac{r}{4h}(2z, z^2 - h^2)$. It is now easy to see (perhaps with a little help from Maple V) that the above eight cases of pairs (p, q) are all solutions of the equations (3) and (4). □

While it is easy to find the coordinates of the centers $P, \ldots, Y$ of the eight Thébault circles and their radii $|y_P|, \ldots, |y_Y|$, it is difficult to describe them precisely by purely geometric means because when the point D changes position on the line BC these circles are changing considerably so that it is hard to tell one from the other. For the points P, Q, S and T this was done in [3, Section 3] by use of oriented configurations.

For a real number $\lambda \neq -1$ and different points M and N, the λ-*point* of the segment MN is a unique point F on the line MN such that the ratio of oriented distances $|MF|$ and $|FN|$ is equal to λ. We can extend this definition to the case when $M = N$ taking that the λ-point is the point M for every real number $\lambda \neq -1$. Recall that the coordinates of the λ-point are $\left(\frac{x_M + \lambda x_N}{\lambda + 1}, \frac{y_M + \lambda y_N}{\lambda + 1}\right)$.

Let $k_a(I_a, r_a)$, $k_b(I_b, r_b)$ and $k_c(I_c, r_c)$ be the excircles of the triangle ABC. Then I, I_a, I_b and I_c have the coordinates $r(f, 1)$, $rg(1, -f)$, $\frac{rgz}{h}(f, 1)$ and $\frac{rz}{h}(-1, f)$. Also, $r_a = rfg$, $r_b = \frac{rgz}{h}$ and $r_c = \frac{rfz}{h}$.

The part of the following result for the segment QP is the correct form of Thébault's theorem while the part for the segment TS is the correct form of the Thébault's external theorem (see [27, Remark 2]). In [21], Shail calls Theorem 3 the full Thébault theorem.

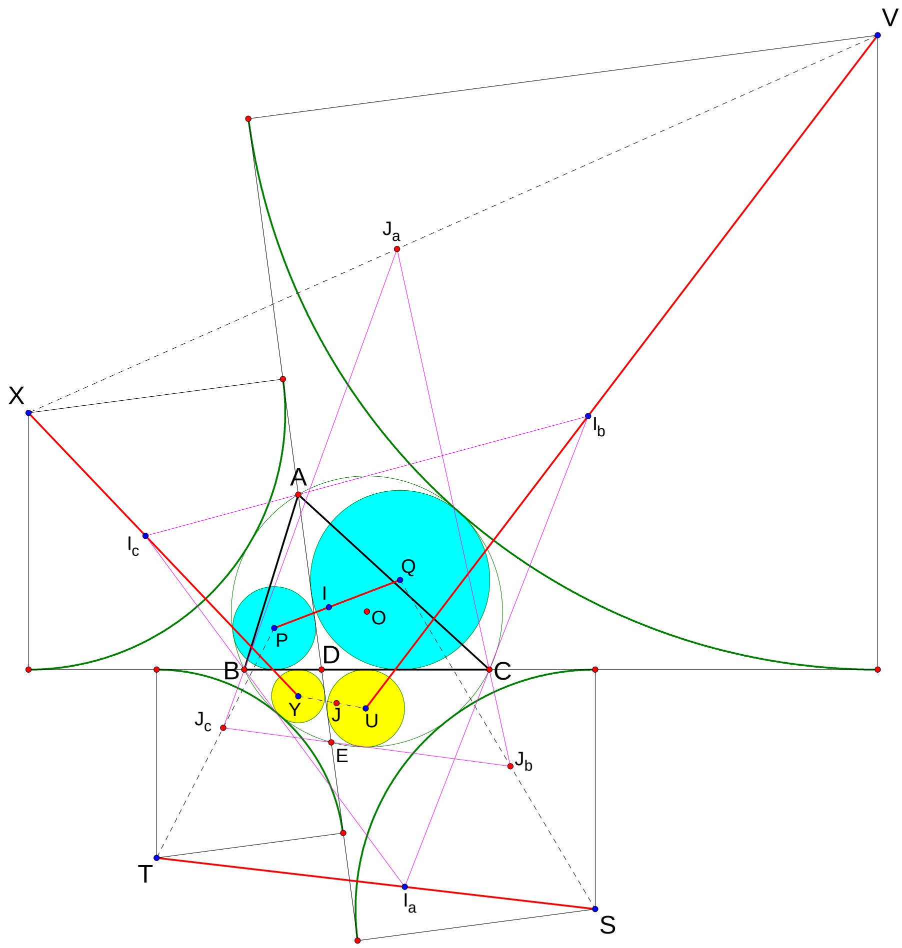

FIGURE 2. Theorems 3 and 4 together.

Teorem 3. *The points I, I_a, I_b and I_c are the k^2-points of the segments QP, TS, VU and YX.*

Proof. Since

$$\frac{x_Q + k^2 x_P}{K} = \frac{r\,f_+ + k^2 \frac{r\varphi_-}{k}}{K} = rf = x_I$$

and

$$\frac{y_Q + k^2 y_P}{K} = \frac{-\frac{rf_+ g_-}{h} + k^2 \frac{r\varphi_-\,\psi_+}{hk^2}}{K} = r = y_I,$$

it follows that I is the k^2-point of the segment QP. The other cases have similar proofs. □

Corollary 1. *The abscises of the centers of Thébault's circles satisfy:*

(5) $$x_Q + k^2\, x_P = Krf, \qquad x_T + k^2\, x_S = Krg,$$

(6) $$x_V + k^2\, x_U = Kr_b f, \qquad f(x_Y + k^2\, x_X) = -Kr_c.$$

Corollary 2. *The ordinates of the centers of Thébault's circles satisfy:*

$$y_Q + k^2\, y_P = K r, \qquad y_T + k^2\, y_S = -K r_a, \tag{7}$$

$$y_V + k^2\, y_U = K r_b, \qquad y_Y + k^2\, y_X = K r_c. \tag{8}$$

Note that only when the point D is on the segment BC it holds $y_P = r_1$, $y_Q = r_2$, $y_S = -r_3$ and $y_T = -r_4$ so that from (7) we get (2) since $k = \frac{1}{\tau}$. The second relation in (7) gives us the analogous formula $r_3 + r_4\, \tau^2 = r_a(1+\tau^2)$ for the Thébault's external theorem.

On the other hand, when the point D is on the left from the point B, the ordinate y_P of the center P is negative so that the relation (7) gives $r_2 - k^2\, r_1 = (1+k^2) r$. Moreover, when the point D is on the right from the point C, the ordinate y_Q is negative so that the relation (7) implies the third part $k^2\, r_1 - r_2 = (1+k^2) r$ of the correct version of the formula (2).

As was already noticed in [23], the eight Thébault's circles are also connected with the triangle EBC, where the point E is the second intersection (besides the point A) of the line AD and the circumcircle o. Its coordinates are $\frac{r\, f_+\, \varphi_-}{h\, K^2}\left(\psi_+^2 - g_-^2,\, 2\, \psi_+\, g_-\right)$. One can easily find that its incenter J and the excenters J_b, J_c and J_e have the coordinates $\frac{r\, z\, \varphi_-}{h\, K}\,(\psi_+,\, g_-)$, $\frac{r\, f_+}{K}\,(\psi_+,\, g_-)$, $\frac{r\, \varphi_-}{K}\,(g_-,\, -\psi_+)$ and $\frac{r\, z\, f_+}{h\, K}\,(-g_-,\, \psi_+)$. It is important to note here that as the parameter k changes the actual role of these points changes so that from the excenters they can become other excenters or the incenter and vice verse.

Teorem 4. *The four points J, J_b, J_c and J_e are the k^2-points of the segments YU, QS, TP and VX.*

Proof. Since

$$\frac{x_Y + k^2 x_U}{K} = \frac{r z \varphi_-}{hK} + \frac{r z g k \varphi_-}{hK} = \frac{r z \varphi_- \psi_+}{hK} = x_J$$

and

$$\frac{y_Y + k^2 y_U}{K} = \frac{r f \varphi_- g_-}{hK} + \frac{r g \varphi_- g_-}{hK} = \frac{r z \varphi_- g_-}{hK} = y_J,$$

it follows that J is the k^2-point of the segment YU. The other cases have similar proofs. □

The approach in [23] also suggests that the other two triangles ABE and ACE and their incenters and the excenters should play a similar role. We denote those centers by $\mathfrak{I}$, $\mathfrak{I}_a$, $\mathfrak{I}_b$, $\mathfrak{I}_e$ and $\mathfrak{J}$, $\mathfrak{J}_a$, $\mathfrak{J}_c$, $\mathfrak{J}_e$. Their coordinates are $\frac{r\, \varphi_-}{h\, K}\,(h\, k + d, z\, k - h)$, $-\frac{r\, g\, \varphi_-}{h\, K}\,(h - z\, k, h\, k + z)$, $\frac{r\, f_+\, g}{h\, K}\,(h\, k + z, z\, k - h)$, $\frac{r\, f_+}{h\, K}\,(h - z\, k, h\, k + z)$, $\frac{r}{h\, K}\,(\zeta\, z\, k^2 - g^+\, k + f\, h,$ $g_-(h\, k - z))$, $\frac{r}{h\, K}\,(g\, h\, k^2 - f^2\, g^+\, k - z, f\, g_-(z\, k + h))$, $-\frac{r}{h\, K}\,(z\, k^2 + f^2$ $g^+\, k - g\, h, f\, \psi_+(h\, k - z))$ and $\frac{r}{h\, K}\,(f\, h\, k^2 + g^+\, k + \zeta\, z, \psi_+(z\, k + h))$.

Teorem 5. *(i) The points* $\mathfrak{J}$, $\mathfrak{J}_a$, $\mathfrak{J}_b$ *and* $\mathfrak{J}_e$ *are the* k^2*-points of the segments* YP, TU, VS *and* QX.

(ii) The points $\mathfrak{J}$, $\mathfrak{J}_a$, $\mathfrak{J}_c$ *and* $\mathfrak{J}_e$ *are the* k^2*-points of the segments* QU, YS, TX *and* VP.

Proof. Since

$$\frac{x_Y + k^2\, x_P}{K} = \frac{rz\varphi_-}{hK} + \frac{r\,\varphi_-\,k}{K} = \frac{r\,\varphi_-(h\,k+z)}{hK} = x_{\mathfrak{J}}$$

and

$$\frac{y_Y + k^2\, y_P}{K} = \frac{r\,f\,\varphi_-\,g_-}{hK} + \frac{r\,\varphi_-\,\psi_+}{hK} = \frac{r\,\varphi_-(z\,k-h)}{hK} = y_{\mathfrak{J}},$$

it follows that $\mathfrak{J}$ is the k^2-point of the segment YP. The other cases have similar proofs. □

Now we could say that the Theorems 3, 4 and 5 together represent the complete Thébault theorem.

The rather simple coordinates of the incenters and the excenters of the triangles ABC, BCE, ABE and ACE allow us to prove easily the following results that Johnson in [9, p. 193] calls the "Japanese Theorem" (see also [16]).

Teorem 6. *(i) The following quadrangles* $I\mathfrak{J}J\mathfrak{J}$, $I_a\mathfrak{J}_bJ_e\mathfrak{J}_c$, $I_b\mathfrak{J}_aJ_c\mathfrak{J}_e$ *and* $I_c\mathfrak{J}_eJ_b\mathfrak{J}_a$ *are the rectangles.*

(ii) Their areas satisfy: $|I\mathfrak{J}J\mathfrak{J}|\,|I_a\mathfrak{J}_bJ_e\mathfrak{J}_c| = |I_b\mathfrak{J}_aJ_c\mathfrak{J}_e|\,|I_c\mathfrak{J}_eJ_b\mathfrak{J}_a|$.

(ii) Their centers are vertices of a parallelogram with the center at the circumcenter O *of the triangle* ABC.

Proof. Since $|I\mathfrak{J}|^2 = |J\mathfrak{J}|^2 = \frac{r^2(f^+)^2(g_-)^2}{h^2\,K}$ and $|I\mathfrak{J}|^2 = |J\mathfrak{J}|^2 = \frac{r^2(g^+)^2(\varphi_-)^2}{h^2\,K}$, it follows that $I\mathfrak{J}J\mathfrak{J}$ is a parallelogram. On the other hand, since the lines $I\mathfrak{J}$ and $I\mathfrak{J}$ have the equations $k\,x - y = r\,\varphi_-$ and $x + k\,y = r\,f_+$, we conclude that they are perpendicular and $I\mathfrak{J}J\mathfrak{J}$ is a rectangle.

Since the area of a rectangle is the product of the lengths of its adjacent sides, we see that $|I\mathfrak{J}J\mathfrak{J}| = \frac{r^2\,f^+\,g^+\,|g_-\,\varphi_-|}{h^2\,K}$. Similarly,

$$|I_a\mathfrak{J}_bJ_e\mathfrak{J}_c| = \frac{r^2\,\zeta\,f^+\,g^+\,f_+\,\psi_+}{h^2\,K}, \quad |I_b\mathfrak{J}_aJ_c\mathfrak{J}_e| = \frac{r^2\,g\,f^+\,g^+\,\psi_+\,|\varphi_-|}{h^2\,K},$$

and $|I_c\mathfrak{J}_eJ_b\mathfrak{J}_a| = \frac{r^2\,f\,f^+\,g^+\,f_+\,|g_-|}{h^2\,K}$. The identity in (ii) is now obvious.

Finally, it is easy to check that the circumcenter O is the midpoint of the segments $G_{I\mathfrak{J}J\mathfrak{J}}G_{I_a\mathfrak{J}_bJ_e\mathfrak{J}_c}$ and $G_{I_b\mathfrak{J}_aJ_c\mathfrak{J}_e}G_{I_c\mathfrak{J}_eJ_b\mathfrak{J}_a}$ joining the centers (i. e., the centroids) of these rectangles. □

Note that the inradii j, $\mathfrak{r}$ and $\mathfrak{j}$ and the exradii j_b, j_c, j_e, $\mathfrak{r}_a$, $\mathfrak{r}_b$, $\mathfrak{r}_e$, $\mathfrak{j}_a$, $\mathfrak{j}_c$ and $\mathfrak{j}_e$ of the triangles BCE, ABE and ACE are the absolute values of the quotients $\frac{r\,g_-\,z\,\varphi_-}{h\,K}$, $\frac{r\,\varphi_-(\bar{h}-d\,k)}{h\,K}$, $\frac{r\,g_-(\bar{h}\,k+d)}{h\,K}$, $\frac{r\,f_+\,g_-}{K}$, $\frac{r\,\varphi_-\,\psi_+}{K}$, $\frac{r\,f_+\,\psi_+\,z}{h\,K}$, $\frac{r\,\varphi_-\,g(\bar{h}\,k+d)}{h\,K}$, $\frac{r\,f_+\,g(\bar{h}-d\,k)}{h\,K}$, $\frac{r\,f_+(\bar{h}\,k+d)}{h\,K}$, $\frac{r\,f\,g_-(\bar{h}-d\,k)}{h\,K}$, $\frac{r\,f\,\psi_+(\bar{h}\,k+d)}{h\,K}$ and

$\frac{r\,\psi_+(\bar{h}-d\,k)}{h\,K}$. Now, at least under the assumption that D is on the segment BC, we can easily check the following identities:

$$r + j = \mathfrak{r} + \mathfrak{j}, \quad r_a + j_e = \mathfrak{r}_b + \mathfrak{j}_c, \quad r_b + j_c = \mathfrak{r}_a + \mathfrak{j}_e, \quad r_c + j_b = \mathfrak{r}_e + \mathfrak{j}_a.$$

The first is the relation (2.2) in [16].

3. Some conics as loci and envelopes

In order to find the locus of Thébault's center P, let us eliminate the parameter k from the equations $x_P = x$ and $y_P = y$. We get the equation $y = \frac{x(rz-x)}{rh}$ of the parabola μ with the circumcenter O as the focus and the horizontal line ε above the line BC at the distance R (the circumradius) as the directrix. Repeating this for the centers Q, S and T will always produce the same parabola μ. On the other hand, doing this for the centers U, V, X and Y, will give the equation $y = \frac{hx(x-rz)}{rz^2}$ of the parabola ν also with the circumcenter O as the focus and the horizontal line ε^* below the line BC at the distance R as the directrix.

Corollary 3. *The points P, Q, S and T are on the parabola μ and the points U, V, X and Y are on the parabola ν.*

The parabolas μ and ν intersect only in the points B and C and they enclose the region with the area $\frac{2}{3}aR$.

When the point D moves on the line BC, the many lines joining pairs of Thébault's centers provide families of lines that envelop some interesting conics of the triangle ABC.

For example, one interpretation of the Theorem 3 is that the lines PQ, ST, UV and XY envelop the points I, I_a, I_b and I_c (considered as degenerated ellipses), respectively.

On the other hand, it was noted in [3], the lines PS, QT, UX and VY envelop the parabola λ of focus A and directrix BC having the equation $y = \frac{h}{4r\zeta}\,x^2 - \frac{f^-}{2f}\,x + \frac{rg(f^+)^2}{4fh}$.

The parabolas λ, μ and ν are closely related in many respects: They have parallel directrices and axes and the distance between the foci of λ and μ and between the foci of λ and ν is equal to the distance between their directrices. It is not difficult to see that λ and μ touch in the $\frac{(b+c)^2-a^2}{a^2}$-point T_μ of the segment AO and that λ and ν touch in the $\frac{(b-c)^2-a^2}{a^2}$-point T_ν of the segment AO (when $b \neq c$).

When $b \neq c$, the lines PT and QS envelop the same hyperbola η with the equation $\zeta(2x - rz)^2 - (hy - 2r\zeta)^2 = r^2 d^2 \zeta$ ([3, Remark 7]).

The lines UY and VX envelop the same ellipsis χ with the equation $h^2\zeta(2x - rz)^2 + z^2(hy - 2r\zeta)^2 = r^2\bar{h}^2 z^2\zeta$. It can be shown that χ is symmetric with respect to the perpendicular bisector of BC, tangent to ν at B and C, tangent to lines $T_\nu I_b$ and $T_\nu I_c$ and to the perpendiculars to BC through I_b and I_c.

4. The line AD tangent to the circumcircle

We shall see that some positions of the point D on the line BC are particularly important. In the following two results we identify what happens when the line AD is the tangent to the circumcircle o in the point A. In this exceptional case many points of the configuration coincide. Of course, this can happen only when the angles B and C are different.

Let $P_o, \dots, Y_o$ denote the points in which the Thébault's circles touch the circumcircle o. Their coordinates are $\frac{r\varphi_-}{P_1}(P_2, 2h\psi_+)$, $\frac{rf_+}{Q_1}(Q_2, 2hg_-)$, $\frac{rgf_+}{S_1}(S_2, 2hfg_-)$, $\frac{rg\varphi_-}{T_1}(T_2, -2hf\psi_+)$, $\frac{rgz\varphi_-}{hU_1}(U_2, 2zg_-)$, $\frac{rgzf_+}{hV_1}(V_2, 2z\psi_+)$, $\frac{rzf_+}{hX_1}(X_2, 2zf\psi_+)$ and $\frac{rz\varphi_-}{hY_1}(Y_2, 2zfg_-)$, where $P_1, \dots, Y_1$ are $(h^2+d^2)k^2 -4dk+4$, $4k(k+d)+h^2+d^2$, $(h^2+d^2)k^2-4\zeta(dk-\zeta)$, $4\zeta k(\zeta k+d) +h^2+d^2$, $(\bar{h}^2+z^2)k^2-4g(\bar{h}k-g)$, $4gk(gk+\bar{h})+\bar{h}^2+z^2$, $(\bar{h}^2+z^2)k^2 +4f(\bar{h}k+f)$, $4fk(fk-\bar{h})+\bar{h}^2+z^2$ and $P_2, \dots, Y_2$ are $(h^2+dz)k-2z$, $2zk+h^2+dz$, $(h^2-dz)k+2z\zeta$, $2z\zeta k-h^2+dz$, $(\zeta^2+z^2-1)k-2gh$, $2ghk+\zeta^2+z^2-1$, $(\zeta^2-z^2-1)k+2fh$, $2fhk-\zeta^2+z^2+1$.

For eight points $P_1, \dots, P_8$, let $D(P_1, \dots, P_8)$ be the determinant

$$\begin{vmatrix} x_{P_1} & y_{P_1} & x_{P_2} & y_{P_2} \\ x_{P_3} & y_{P_3} & x_{P_4} & y_{P_4} \\ x_{P_5} & y_{P_5} & x_{P_6} & y_{P_6} \\ x_{P_7} & y_{P_7} & x_{P_8} & y_{P_8} \end{vmatrix}.$$

Teorem 7. *The following statements are equivalent: (i) $P = S$, (ii) $V = Y$, (iii) $P_o = A$, (iv) $S_o = A$, (v) $V_o = A$, (vi) $Y_o = A$, (vii) $I = J_c$, (viii) $I_a = J_b$, (xi) $I_b = J$, (x) $I_c = J_a$, (xi) $\mathfrak{I} = \mathfrak{I}_b$, (xii) $\mathfrak{I} = \mathfrak{I}_a$, (xiii) $\mathfrak{I} = \mathfrak{I}_e$, (xiv) $\mathfrak{I}_a = \mathfrak{I}_e$, (xv) the lines $\mathfrak{I}_b\mathfrak{I}_c$ and I_cJ_e are perpendicular, (xvi) $D(I, I_a, I_b, I_c, J, J_e, J_b, J_c) = 0$, (xv) $\mathfrak{I}_b \in AD$, (xvi) $\mathfrak{I}_e \in AD$, (xvii) the lines $\mathfrak{I}_b\mathfrak{I}_e$ and AD are perpendicular, (xviii) the lines I_cJ_a and AD are parallel, (xix) the lines I_aJ_c and AD are parallel and (xx) the angle B is smaller than the angle C and the lines AD and AO are perpendicular.*

Proof. Since $|PS|^2 = \frac{r^2K(\bar{h}-dk)^2}{k^4}$, we conclude that $P = S$ if and only if $k = \frac{\bar{h}}{d}$. However, the parameter k is positive, so that $f > g$ (i. e., the angle B is smaller than the angle C) and the point D divides the segment BC in the ratio $-\frac{|AB|^2}{|AC|^2}$ (i. e., the point D is the intersection of the tangent to the circumcircle at the vertex A with the line BC). This shows the equivalence of (i) and (xx). For the other parts, it suffices to note that the only factor that could be zero in the squares of distances of the points in this part is always the same $\bar{h} - dk$. □

The following companion result has similar proof. This time the common factor is $d + \bar{h}k$.

Teorem 8. *The following are equivalent: (i) $Q = T$, (ii) $U = X$, (iii) $Q_o = A$, (iv) $T_o = A$, (v) $U_o = A$, (vi) $X_o = A$, (vii) $I = J_b$, (viii) $I_a = J_c$, (xi) $I_b = J_a$, (x) $I_c = J$, (xi) $\mathfrak{I} = \mathfrak{I}_a$, (xii) $\mathfrak{I} = \mathfrak{I}_e$, (xiii) $\mathfrak{I} = \mathfrak{I}_c$, (xiv) $\mathfrak{I}_a = \mathfrak{I}_e$, (xv) the lines $\mathfrak{I}_b\mathfrak{I}_c$ and I_bJ_e are perpendicular, (xvi) $\mathfrak{I}_e \in AD$, (xvii) $\mathfrak{J}_c \in AD$, (xviii) the lines $\mathfrak{I}_e\mathfrak{J}_c$ and AD are perpendicular, (xix) the lines I_bJ_a and AD are parallel, (xx) the lines I_aJ_b and AD are parallel and (xxi) the angle B is larger than the angle C and the lines AD and AO are perpendicular.*

5. Identities for coordinates

Some of the basic algebraic identities among the products of the ordinates of the centers of Thébault's circles are given in the next result.

Teorem 9. *The following relations hold:*

$$\zeta^2 y_P y_Q = y_S y_T, \qquad k^2\zeta y_P y_Q = -y_V y_Y, \qquad g^2 y_U y_X = f^2 y_V y_Y, \tag{9}$$

$$k^2\zeta y_U y_X = -y_S y_T, \qquad y_P y_S = y_U y_X, \qquad k^4 y_P y_S = y_V y_Y, \tag{10}$$

$$y_Q y_T = y_V y_Y, \qquad y_Q y_T = k^4 y_U y_X, \qquad f_+^2 y_P y_T = -\varphi_-^2 y_V y_X, \tag{11}$$

$$g_-^2 y_P y_T = -\psi_+^2 y_U y_Y, \qquad \varphi_-^2 y_Q y_S = -f_+^2 y_U y_Y, \tag{12}$$

$$\psi_+^2 y_Q y_S = -g_-^2 y_V y_X, \;\; k^4 f^2 y_P y_U = -y_T y_Y, \;\; f^2 y_Q y_V = -y_S y_X, \tag{13}$$

$$f^2 y_P y_V = -y_T y_X, \;\; f^2 y_Q y_U = -y_S y_Y, \;\; k^4 g^2 y_P y_X = -y_T y_V, \tag{14}$$

$$g^2 y_Q y_Y = -k^4 y_S y_U, \;\; g^2 y_P y_Y = -y_T y_U, \;\; g^2 y_Q y_X = -y_S y_V. \tag{15}$$

Proof. Since $y_P = \frac{r\varphi_-\psi_+}{hk^2}$, $y_Q = -\frac{rf_+g_-}{h}$, $y_S = \frac{r\zeta f_+g_-}{hk^2}$ and $y_T = -\frac{r\zeta\varphi_-\psi_+}{h}$, it is easy to verify the first relation in (9). All other identities are proved similarly by direct inspection. □

Since the absolute values of $y_P, \dots, y_Y$ are the radii $r_1, \dots, r_8$ of Thébault's circles and the absolute value of the product is the product of the absolute values of the factors, from the above relations, we have the following results. The first identity in (17) is from [3, Corollary 5].

Corollary 4. *The radii of Thébault's circles satisfy:*

$$r_1 r_3 = r_5 r_7, \qquad r_2 r_4 = r_6 r_8, \qquad k^4 r_5 r_7 = r_2 r_4, \tag{16}$$

$$\frac{r_1 r_2}{r^2} = \frac{r_3 r_4}{r_a^2}, \quad \frac{r_5 r_6}{r_b^2} = \frac{r_7 r_8}{r_c^2}, \quad \frac{r_5 r_8}{j^2} = \frac{r_6 r_7}{j_e^2}, \quad \frac{r_2 r_3}{j_b^2} = \frac{r_1 r_4}{j_c^2}, \tag{17}$$

$$\frac{r_1 r_8}{\mathfrak{r}^2} = \frac{r_3 r_6}{\mathfrak{r}_b^2}, \quad \frac{r_4 r_5}{\mathfrak{r}_a^2} = \frac{r_2 r_7}{\mathfrak{r}_e^2}, \quad \frac{r_2 r_5}{\mathfrak{j}^2} = \frac{r_4 r_7}{\mathfrak{j}_c^2}, \quad \frac{r_3 r_8}{\mathfrak{j}_a^2} = \frac{r_1 r_6}{\mathfrak{j}_e^2}. \tag{18}$$

For the abscises many relations also hold. The following two are rather simple.

Teorem 10. *The following relations hold:*

$$x_P\, x_S\, x_V\, x_Y = x_Q\, x_T\, x_U\, x_X, \tag{19}$$

$$k^4\, x_P\, x_S\, x_U\, x_X = x_Q\, x_T\, x_V\, x_Y, \tag{20}$$

Proof. The products on the left and on the right sides of the relation (19) have as the common value the square of $\frac{r^2 f_+ \varphi_- g z}{hk}$. The common value in the relation (20) is minus the square of $\frac{r^2 f_+ \varphi_- g z}{h}$. □

We continue with the formulae that involve the radii of the incircle and the excircles.

Teorem 11. *The following relations hold:*

$$\frac{y_S}{r_a} + \frac{y_U}{r_b} + \frac{y_X}{r_c} = \frac{y_P}{r}, \tag{21}$$
$$\frac{y_T}{r_a} + \frac{y_V}{r_b} + \frac{y_Y}{r_c} = \frac{y_Q}{r}, \tag{22}$$
$$\frac{y_T}{j_b} + \frac{y_P}{j_c} + \frac{y_X}{j_e} = \frac{y_U}{j}, \tag{23}$$
$$\frac{y_Q}{j_b} + \frac{y_S}{j_c} + \frac{y_V}{j_e} = \frac{y_Y}{j}, \tag{24}$$
$$\frac{y_U}{\mathfrak{r}_a} + \frac{y_S}{\mathfrak{r}_b} + \frac{y_X}{\mathfrak{r}_e} = \frac{y_P}{\mathfrak{r}}, \tag{25}$$
$$\frac{y_T}{\mathfrak{r}_a} + \frac{y_V}{\mathfrak{r}_b} + \frac{y_Q}{\mathfrak{r}_e} = \frac{y_Y}{r}, \tag{26}$$
$$\frac{y_Y}{\mathfrak{j}_a} + \frac{y_T}{\mathfrak{j}_c} + \frac{y_V}{\mathfrak{j}_e} = \frac{y_Q}{\mathfrak{j}}, \tag{27}$$
$$\frac{y_S}{\mathfrak{j}_a} + \frac{y_X}{\mathfrak{j}_c} + \frac{y_P}{\mathfrak{j}_e} = \frac{y_U}{\mathfrak{j}}, \tag{28}$$

$$\frac{y_T\, y_S}{r_a} + \frac{y_V\, y_U}{r_b} + \frac{y_Y\, y_X}{r_c} = \frac{y_Q\, y_P}{r}, \tag{29}$$

$$\frac{y_Q\, y_S}{j_b} + \frac{y_T\, y_P}{j_c} + \frac{y_V\, y_X}{j_e} = \frac{y_Y\, y_U}{j}, \tag{30}$$

$$\frac{y_T\, y_U}{\mathfrak{r}_a} + \frac{y_V\, y_S}{\mathfrak{r}_b} + \frac{y_Q\, y_X}{\mathfrak{r}_e} = \frac{y_Y\, y_P}{\mathfrak{r}}, \tag{31}$$

$$\frac{y_Y\, y_S}{\mathfrak{j}_a} + \frac{y_T\, y_X}{\mathfrak{j}_c} + \frac{y_V\, y_P}{\mathfrak{j}_e} = \frac{y_Q\, y_U}{\mathfrak{j}}. \tag{32}$$

Proof. Since $\frac{y_S}{r_a} = \frac{f_+ g_-}{hk^2}$, $\frac{y_U}{r_b} = \frac{\varphi_- g_-}{zk^2}$ and $\frac{y_X}{r_c} = \frac{f_+ \psi_+}{zk^2}$, we get

$$\frac{y_S}{r_a} + \frac{y_U}{r_b} + \frac{y_X}{r_c} = \frac{\varphi_- \psi_+}{hk^2} = \frac{y_P}{r}.$$

This proves the relation (21). The other identities have similar proofs. □

It is interesting to note that any of the formulae (29)-(32) remains true if ordinates are replaced consistently by abscises. For example, the analogues of the formula (29) with abscises are the following three relations:

$$\frac{x_T\, y_S}{r_a} + \frac{x_V\, y_U}{r_b} + \frac{x_Y\, y_X}{r_c} = \frac{x_Q\, y_P}{r}, \tag{33}$$

$$\frac{y_T\, x_S}{r_a} + \frac{y_V\, x_U}{r_b} + \frac{y_Y\, x_X}{r_c} = \frac{y_Q\, x_P}{r}, \tag{34}$$

$$\frac{x_T\,x_S}{r_a}+\frac{x_V\,x_U}{r_b}+\frac{x_Y\,x_X}{r_c}=\frac{x_Q\,x_P}{r}. \tag{35}$$

Remark 1. The relations (21)–(28) hold also for the abscises in place of the ordinates.

Corollary 5. *The radii of Thébault's circles satisfy:*

$$\frac{r_3\,r_4}{r_a}-\frac{r_1\,r_2}{r}=\frac{r_5\,r_6}{r_b}+\frac{r_7\,r_8}{r_c}, \tag{36}$$

$$\frac{r_6\,r_7}{j_e}+\frac{r_5\,r_8}{j}=\frac{r_2\,r_3}{j_b}+\frac{r_1\,r_4}{j_c}, \tag{37}$$

$$\frac{r_4\,r_5}{\mathfrak{r}_a}-\frac{r_1\,r_8}{\mathfrak{r}}=\frac{r_3\,r_6}{\mathfrak{r}_b}+\frac{r_2\,r_7}{\mathfrak{r}_e}, \tag{38}$$

$$\frac{r_3\,r_8}{\mathfrak{j}_a}-\frac{r_2\,r_5}{\mathfrak{j}}=\frac{r_4\,r_7}{\mathfrak{j}_c}+\frac{r_1\,r_6}{\mathfrak{j}_e}, \tag{39}$$

and for the point D in the segment BC,

$$\frac{r_1+r_2}{r}+\frac{r_3+r_4}{r_a}=\frac{r_5-r_6}{r_b}+\frac{r_7-r_8}{r_c}, \tag{40}$$

$$\frac{r_6-r_7}{j_e}+\frac{r_5-r_8}{j}=\frac{r_2-r_3}{j_b}+\frac{r_1-r_4}{j_c}, \tag{41}$$

$$\frac{r_4+r_5}{\mathfrak{r}_a}+\frac{r_1-r_8}{\mathfrak{r}}=\frac{r_2+r_7}{\mathfrak{r}_e}-\frac{r_3+r_6}{\mathfrak{r}_b}, \tag{42}$$

$$\frac{r_3+r_8}{\mathfrak{j}_a}+\frac{r_2+r_5}{\mathfrak{j}}=\frac{r_7-r_4}{\mathfrak{j}_c}+\frac{r_1-r_6}{\mathfrak{j}_e}. \tag{43}$$

Proof. The identity (36) is a consequence of the relation (29). The ordinates of the centers of Thébault's circles are their radii up to a sign. These signs depend on the position of the point D on the line BC and are given in the next table.

D is in	y_P	y_Q	y_S	y_T	y_U	y_V	y_X	y_Y
$(-\infty,B)$	-	$+$	-	$+$	$+$	$+$	$+$	$+$
(B,C)	$+$	$+$	-	-	-	$+$	$+$	-
$(C,+\infty)$	$+$	-	$+$	-	$+$	$+$	$+$	$+$

Hence, from (29) we get (36) and from the sum of (21) and (22) we obtain (40). Of course, there are also the versions of (40) when D is in $(-\infty,B)$ and when it is in $(C,+\infty)$. $\square$

Let us close this group of identities with the following eight. The proofs are very similar to the ones above.

Teorem 12. *The following relations hold:*

$$\frac{y_S^2}{r_a}+\frac{y_U^2}{r_b}+\frac{y_X^2}{r_c}=\frac{y_P^2}{r}+\frac{4KR}{k^4}, \tag{44}$$

$$\frac{y_T^2}{r_a}+\frac{y_V^2}{r_b}+\frac{y_Y^2}{r_c}=\frac{y_Q^2}{r}+4k^2KR, \tag{45}$$

$$\frac{y_S^2}{j_b}+\frac{y_P^2}{j_c}+\frac{y_X^2}{j_e}=\frac{y_U^2}{j}+\frac{4KR}{k^4}, \tag{46}$$

$$\frac{y_Q^2}{j_b}+\frac{y_T^2}{j_c}+\frac{y_V^2}{j_e}=\frac{y_Y^2}{j}+4k^2KR, \tag{47}$$

$$\frac{y_U^2}{\mathfrak{r}_a}+\frac{y_S^2}{\mathfrak{r}_b}+\frac{y_P^2}{\mathfrak{r}_e}=\frac{y_X^2}{\mathfrak{r}}+\frac{4KR}{k^4}, \tag{48}$$

$$\frac{y_T^2}{\mathfrak{r}_a}+\frac{y_V^2}{\mathfrak{r}_b}+\frac{y_Q^2}{\mathfrak{r}_c}=\frac{y_Y^2}{\mathfrak{r}}+4k^2KR, \tag{49}$$

$$\frac{y_S^2}{\mathfrak{j}_a}+\frac{y_X^2}{\mathfrak{j}_c}+\frac{y_P^2}{\mathfrak{j}_e}=\frac{y_U^2}{\mathfrak{j}}+\frac{4KR}{k^4}, \tag{50}$$

$$\frac{y_Y^2}{\mathfrak{j}_a}+\frac{y_T^2}{\mathfrak{j}_c}+\frac{y_V^2}{\mathfrak{j}_e}=\frac{y_Q^2}{\mathfrak{j}}+4k^2KR. \tag{51}$$

Remark 2. For the abscises in the identities (44)–(51), the last terms are $\frac{4KR}{k^2}$ and $4KR$, respectively.

In the next group of formulae we prove that the products of squares of the Thébault's radii divided by fourth powers of the appropriate inradius or exradius also show considerable regularity.

Teorem 13. *The radii of Thébault's circles satisfy the identities:*

$$\frac{r_1^2r_2^2}{r^4}+\frac{r_7^2r_8^2}{r_c^4}=\frac{r_3^2r_4^2}{r_a^4}+\frac{r_5^2r_6^2}{r_b^4}, \qquad \frac{r_1^2r_2^2}{r^4}+\frac{r_5^2r_6^2}{r_b^4}=\frac{r_3^2r_4^2}{r_a^4}+\frac{r_7^2r_8^2}{r_c^4},$$

$$\frac{r_5^2r_8^2}{j^4}+\frac{r_2^2r_3^2}{j_b^4}=\frac{r_1^2r_4^2}{j_c^4}+\frac{r_6^2r_7^2}{j_e^4}, \qquad \frac{r_5^2r_8^2}{j^4}+\frac{r_1^2r_4^2}{j_c^4}=\frac{r_2^2r_3^2}{j_b^4}+\frac{r_6^2r_7^2}{j_e^4},$$

$$\frac{r_1^2r_8^2}{\mathfrak{r}^4}+\frac{r_2^2r_7^2}{\mathfrak{r}_e^4}=\frac{r_4^2r_5^2}{\mathfrak{r}_a^4}+\frac{r_3^2r_6^2}{\mathfrak{r}_b^4}, \qquad \frac{r_1^2r_8^2}{\mathfrak{r}^4}+\frac{r_4^2r_5^2}{\mathfrak{r}_a^4}=\frac{r_2^2r_7^2}{\mathfrak{r}_e^4}+\frac{r_3^2r_6^2}{\mathfrak{r}_b^4},$$

$$\frac{r_2^2r_5^2}{\mathfrak{j}^4}+\frac{r_1^2r_6^2}{\mathfrak{j}_e^4}=\frac{r_3^2r_8^2}{\mathfrak{j}_a^4}+\frac{r_4^2r_7^2}{\mathfrak{j}_c^4}, \qquad \frac{r_2^2r_5^2}{\mathfrak{j}^4}+\frac{r_3^2r_8^2}{\mathfrak{j}_a^4}=\frac{r_1^2r_6^2}{\mathfrak{j}_e^4}+\frac{r_4^2r_7^2}{\mathfrak{j}_c^4}.$$

Proof. Let $f^{2+} = f^4 + 1$ and $g^{2+} = g^4 + 1$. One can easily check that both sides in the first relation have the value
$$\frac{f_+^2\, g_-^2\, \varphi_-^2\, \psi_+^2 (f^{2+}\, g^{2+} - 4\,\zeta\, f^-\, g^- + 12\,\zeta^2)}{(h\,k\,z)^4}.$$
The other identities in this group have analogous proofs. $\square$

In the next result we show that a certain relationship among the radii of Thébault's circles can hold only when either the point D or the triangle ABC are rather special.

Teorem 14. *(i) The radii of the Thébault's circles satisfy the identity*
$$\frac{r_1^2 r_2^2}{r^4} + \frac{r_3^2 r_4^2}{r_a^4} = \frac{r_5^2 r_6^2}{r_b^4} + \frac{r_7^2 r_8^2}{r_c^4}$$
if and only if either $D = B$, $D = C$ or the angle A is right.

(ii) The radii of the Thébault's circles satisfy the identity
$$\frac{r_5^2 r_8^2}{j^4} + \frac{r_6^2 r_7^2}{j_e^4} = \frac{r_2^2 r_3^2}{j_b^4} + \frac{r_1^2 r_4^2}{j_c^4}$$
if and only if the angle A is right.

(iii) If the lines AD and AO are not perpendicular (see Theorems 7 and 8), then the radii of the Thébault's circles satisfy the identity
$$\frac{r_1^2 r_8^2}{\mathfrak{r}^4} + \frac{r_3^2 r_6^2}{\mathfrak{r}_b^4} = \frac{r_2^2 r_7^2}{\mathfrak{r}_e^4} + \frac{r_4^2 r_5^2}{\mathfrak{r}_a^4}$$
if and only if either $D = C$ or the point D is on the line AO.

Similarly, they satisfy the identity
$$\frac{r_2^2 r_5^2}{\mathfrak{j}^4} + \frac{r_4^2 r_7^2}{\mathfrak{j}_c^4} = \frac{r_1^2 r_6^2}{\mathfrak{j}_e^4} + \frac{r_3^2 r_8^2}{\mathfrak{j}_a^4}$$
if and only if either $D = B$ or the point D is on the line AO.

Proof. (i) This follows immediately from the identity
$$\left(\frac{r_5^2 r_6^2}{r_b^4} + \frac{r_7^2 r_8^2}{r_c^4}\right) - \left(\frac{r_1^2 r_2^2}{r^4} + \frac{r_3^2 r_4^2}{r_a^4}\right) = \frac{2f^+ g^+ f_+^2 g_-^2 \varphi_-^2 \psi_+^2 (h^2 - z^2)}{(hkz)^4}.$$
The other cases have similar proofs. $\square$

Here is an interesting inequality.

Teorem 15. *The ordinates of the centers of Thébault's circles satisfy the inequality:*
$$\frac{(y_S + y_T)^2}{r_a} + \frac{(y_U + y_V)^2}{r_b} + \frac{(y_X + y_Y)^2}{r_c} \geq 16\,R + \frac{(y_P + y_Q)^2}{r}. \tag{52}$$
The equality holds if and only if the line AD is perpendicular to the line BC. The same holds also for the abscises in place of the ordinates.

Proof. Since $\frac{(y_S+y_T)^2}{r_a}+\frac{(y_U+y_V)^2}{r_b}+\frac{(y_X+y_Y)^2}{r_c}-\frac{(y_P+y_Q)^2}{r} = \frac{4RK^2(k^2L+1)}{k^4}$ and the function $k \mapsto \frac{K^2(k^2L+1)}{k^4}$ has the minimum 16 for $k = 1$, we conclude that the inequality (52) holds.

It remains to note that the line AD is perpendicular to the line BC if and only if $k = 1$. □

Of course, there are three similar inequalities involving the inradii and the exradii of the triangles BCE, ABE and ACE. Also, these inequalities have the usual interpretations in terms of the radii of the Thébault's circles leading to the three versions depending on the position of the point D on the line BC.

6. Equal radii r_1 and r_2

In this section we shall explore when the pair r_1 and r_2 of the radii of the first and the second Thébault's circles are equal. In fact, the problem is to describe the positions of the point D on the line BC when $r_1 = r_2$ holds. It turns out that the equality happens for three values of the parameter k. The simpler value corresponds to the case when $r_1 = r_2 = r$ (see Theorem 16) and the two more complicated values to the case $r_1 = r_2$ and either $r_1 \neq r$ or $r_2 \neq r$ (see Theorem 17). In each situation many other geometric consequences hold. Some are characteristic for the equality of r_1 and r_2 (with r).

Let $k_{I'_a} = \frac{\sqrt{d^2+4}-d}{2}$ be the positive root of the polynomial $p_{I'_a} = L + dk$.

Let the perpendicular bisector of the segment BC intersect the circumcircle o in the points Z_1 and Z_2 such that Z_1 is above and Z_2 is below the line BC. Note that Z_1 is the midpoint of I_bI_c and the circle $k_{I_bI_c}$ goes through B, C and J_a. Similarly, Z_2 is the midpoint of J_bJ_c and the circle $k_{J_bJ_c}$ goes through B, C and I_a.

Teorem 16. *The following statements are equivalent: (i) the point D is the orthogonal projection I'_a of the excenter I_a onto the line BC, (ii) the parameter k is $k_{I'_a}$, (iii) the lines PQ and BC are parallel, (iv) the lines P_oQ_o and BC are parallel, (v) the line AD bisects the segment PQ, (vi) the segments PQ and $P''Q''$ share the midpoints, (vii) the line joining the incenter I and the midpoint of the segment BC is parallel to the line AD, (viii) the line joining the circumcenter O and the midpoint of either the segment $P'Q'$ or $P''Q''$ is perpendicular to the line PQ, (ix) the midpoint of the segment BC has the same power with respect to the circles k_1 and k_2, (x) the points P_o and Q_o are equidistant from the point Z_1 and/or Z_2 and (xi) the equalities $r_1 = r$ and $r_2 = r$ hold.*

Proof. Since the point I'_a has the coordinates $(rg, 0)$, we get that $|DI'_a|$ is equal $\frac{r\zeta\,|p_{I'_a}|}{hk}$. Hence, (i) and (ii) are equivalent.

The lines PQ and BC are parallel if and only if the points P and Q have equal ordinates. Since $y_P - y_Q = \frac{rKp_{I'_a}}{hk^2}$, we see that (ii) and (iii) are equivalent.

Similarly, since $y_{P_o} - y_{Q_o} = \frac{2rhKf^+g^+p_{I'_a}}{P_1Q_1}$, it follows that (ii) and (iv) are equivalent.

The midpoint of the segment PQ has the coordinates $\frac{r}{2k}\left(L+2fk, -\frac{p_4}{hk}\right)$, where p_4 is defined bellow. It is on the line AD whose equation is $2kx + Ly = \frac{2rgf_+\varphi_-}{h}$ if and only if $\frac{r^2\zeta K^2 p_{I'_a}}{2h^2k^3} = 0$. Hence, (ii) and (v) are equivalent.

The orthogonal projections P'' and Q'' of P and Q onto the line AD have $\frac{r\varphi_-}{hkK}\left(hk^2+2gk+\bar{h}, 2\psi_+k\right)$ and $\frac{rf_+}{hK}\left(\bar{h}k^2-2gk+h, -2g_-\right)$ as coordinates. It follows that the midpoints of the segments PQ and $P''Q''$ are $\frac{rK|p_{I'_a}|}{2hk^2}$ apart. Therefore, (ii) and (vi) are equivalent.

The line joining the incenter I and the midpoint of the segment BC has the equation $2x - dy = rz$. It will be parallel to the line AD if and only if $\frac{r^2\zeta p_{I'_a}}{hk} = 0$. This shows the equivalence of (ii) and (vii).

The line PQ has the equation $p_{I'_a}\,x + hk\,y = r\,f_+\,\varphi_-$. The line joining the circumcenter O and the midpoint of the segment $P'Q'$ has the equation $2(h^2-z^2)k\,y - 4\,h\,p_{I'_a}\,y = r(L+2\,f\,k)(h^2-z^2)$. They will be perpendicular if and only if $\frac{r^2\,K\,f^+\,g^+\,p_{I'_a}}{4\,h^2\,k^2} = 0$. The line joining O and the midpoint of the segment $P''Q''$ is more complicated but it will be perpendicular to the line PQ if and only if the same condition holds. This shows the equivalence of (ii) and (viii).

The power $w(A_g, k_2)$ of the midpoint A_g of the segment BC with respect to the circle k_2 is $|A_gQ|^2 - r_2^2$ or $\frac{r^2(d+2k)^2}{4}$. Similarly, $w(A_g, k_1)$ is $\frac{r^2(dk-2)^2}{4k^2}$. Their difference is $\frac{r^2Kp_{I'_a}}{k^2}$. Hence, (ix) and (ii) are equivalent.

The differences of squares $|QZ_1|^2 - |PZ_1|^2$ and $|PZ_2|^2 - |QZ_2|^2$ of distances are equal $\frac{r^2\,K\,(f^+)^2\,(g^+)^2\,p_{I'_a}}{[(h^2+d^2)k^2-4\,d\,k+4](4\,k^2+4\,d\,k+h^2+d^2)}$. It follows that (x) and (ii) are equivalent.

Finally, since $r_1^2 - r^2 = \frac{r^2Mp_{I'_a}}{h^2k^4}$ and $r_2^2 - r^2 = \frac{r^2Np_{I'_a}}{h^2}$ and the factors $M = (2\zeta-1)k^2 + dk - 1$ and $N = k^2 + dk - 2\zeta + 1$ are not both zero at any real number k, we conclude that (ii) and (xi) are equivalent. $\square$

Let $k_\pm = \frac{\sqrt{2\,N_\pm} \pm M - d}{4}$ be the positive roots of the quartic polynomial $p_4 = L(L+dk) - 2hk^2$, where $M = \sqrt{d^2+8h}$ and $N_\pm = d^2 \mp dM + 4\bar{h}$.

Teorem 17. *The following are equivalent: (i) the parameter k is either k_+ or k_-, (ii) the lines PQ and AD are parallel, (iii) the line P_oQ_o bisects the segment $P'Q'$, (iv) the line PQ bisects the segment $P'Q'$, (v) the segments PQ and $P'Q'$ share the midpoints and (vi) the lines AD and DI_a are perpendicular.*

Proof. Since $p_{I'_a} x + hky = r f_+ \varphi_-$ and $2kx + Ly = \frac{2rgf_+\varphi_-}{h}$ are the equations of the lines PQ and AD, they will be parallel if and only if $p_4 = 0$. This shows that (i) and (ii) are equivalent.

The orthogonal projections P' and Q' of the centers P and Q onto the line BC (the x-axis) have the abscises $\frac{r\varphi_-}{k}$ and rf_+. It follows that the midpoint of the segment $P'Q'$ lies on the line P_oQ_o (i. e., on the line $2hp_{I'_a}x - [2dL + (z^2 - \bar{h}^2 - 4)k]y = 2rhf_+\varphi_-$), provided

$$p_{I'_a}\left(\frac{rL}{2k} + rf\right) - rf_+\varphi_- = \frac{rp_4}{2k} = 0.$$

Hence, (i) and (iii) are equivalent.

This same calculation applies also in the proof that (i) and (iv) are equivalent because the line PQ has the equation $p_{I'_a}x + hky = rf_+\varphi_-$.

The midpoints of PQ and $P'Q'$ are $\frac{r|p_4|}{2hk^2}$ apart. We easily conclude that (i) and (v) are equivalent.

Finally, since $hkx - p_{I'_a}y = rf_+g\varphi_-$ is the equation of the line DI_a, we get that this line is perpendicular with the line AD if and only if $2hk^2 - p_{I'_a}L = -p_4 = 0$. Hence, the first and the last statements are equivalent. $\square$

Note that the condition (ii) in Theorem 17 implies $r_1 = r_2$. Hence, the correct version of Theorem 4 in [27] is the following result.

Corollary 6. *The following are equivalent: (i) the equality $r_1 = r_2$ holds, (ii) the parameter k is either $k_{I'_a}$, k_+ or k_-, (iii) the points P and Q are at equal distance from the midpoint of $P'Q'$ and/or $P''Q''$.*

Proof. Since $r_1 = |y_P|$ and $r_2 = |y_Q|$, it follows that $r_1 = r_2$ if and only if $y_P^2 - y_Q^2 = \frac{r^2Kp_{I'_a}p_4}{h^2k^4} = 0$. Let M' and M'' be the midpoints of $P'Q'$ and $P''Q''$. Then $|QM'|^2 - |PM'|^2 = |QM''|^2 - |PM''|^2 = \frac{r^2Kp_{I'_a}p_4}{h^2k^4}$. Hence, our claim follows from Theorems 16 and 17 because the parameter k is a positive real number. $\square$

7. Equal radii r_3 and r_4

In the next six theorems we state the companion results with the previous two theorems for the remaining three pairs (S, T), (U, V) and (X, Y) of related centers of Thébault's circles. The situation for these three pairs is a little bit different because the two more complicated values of the parameter k exist only when the angles B and C satisfy certain conditions.

In this section we consider the pair r_3 and r_4 of the radii of the third and the fourth Thébault's circles. We will omit the proofs because they are very similar to the corresponding proofs of the previous two theorems.

Let $k_{I'} = \frac{\sqrt{d^2+4\zeta^2}-d}{2\zeta}$ be the positive root of the quadratic polynomial $p_{I'} = \zeta L + dk$.

Teorem 18. *The following statements are equivalent: (i) the point D is the orthogonal projection I' of the incenter I onto the line BC, (ii) the parameter k is $k_{I'}$, (iii) the lines ST and BC are parallel, (iv) the lines S_oT_o and BC are parallel, (v) the line AD bisects the segment ST, (vi) the segments ST and $S''T''$ share the midpoints, (vii) the line joining the excenter I_a and the midpoint of the segment BC is parallel to the line AD and (viii) the equalities $r_3 = r_a$ and $r_4 = r_a$ are both true.*

Let $d^2 - 8h\zeta \geq 0$. Let $m_{\pm} = \frac{\sqrt{2\,N_{\pm}} \pm M - d}{4\zeta}$ be the positive roots of the quartic polynomial $q_4 = L(\zeta L + dk)$, where M and $N_{\pm}$ are the expressions $\sqrt{d^2 - 8h\zeta}$ and $d^2 \mp dM + 4\zeta^2$.

Teorem 19. *For a triangle ABC whose angles satisfy the inequality* $\cos(B - C) + 4(\cos(B + C) + \cos B + \cos C) \leq -3$, *the following are equivalent: (i) the parameter k is either m_+ or m_-, (ii) the lines ST and AD are parallel, (iii) the line S_oT_o bisects the segment $S'T'$, (iv) the segments ST and $S'T'$ share the midpoints and (v) the lines AD and DI are perpendicular.*

Note that the condition (ii) in Theorem 19 implies $r_3 = r_4$. Also, when both angles B and C are acute, then the polynomial q_4 is always positive because it is the sum $\frac{(2\zeta L+dk)^2}{4\zeta} + \frac{\left(8h\zeta-d^2\right)k^2}{4\zeta}$ with the second term positive. Indeed, the replacement of f and g in $8h\zeta - d^2$ with $1 + \varphi$ and $1 + \psi$ for φ, $\psi > 0$ gives a positive polynomial

$$8\,\varphi^2\psi^2 + 16\,\varphi^2\psi + 16\,\varphi\,\psi^2 + 7\,\varphi^2 + 26\,\varphi\,\psi + 7\,\psi^2 + 8\,\varphi + 8\,\psi.$$

8. Equal radii r_5 and r_6

In this section we consider similarly the pair r_5 and r_6 of the radii of the fifth and the sixth Thébault's circles.

Let $k_{I'_c} = \frac{\sqrt{\bar{h}^2+4g^2}-\bar{h}}{2\,g}$ be the positive root of the quadratic polynomial $p_{I'_c} = gL + \bar{h}k$.

Teorem 20. *The following statements are equivalent: (i) the point D is the orthogonal projection I'_c of the excenter I_c onto the line BC, (ii) the parameter k is $k_{I'_c}$, (iii) the lines UV and BC are parallel, (iv) the lines U_oV_o and BC are parallel, (v) the midpoint of the segment UV is on the perpendicular bisector of the segment BC, (vi) the segments UV and $U''V''$ share the midpoints, (vii) the line joining the excenter I_b and the midpoint of the segment BC is parallel to the line AD and (viii) the equalities $r_5 = r_b$ and $r_6 = r_b$ are both true.*

Let $\bar{h}^2 - 8gz \geq 0$. Let $n_{\pm} = \frac{\sqrt{2\,N_{\mp}} \pm M - \bar{h}}{4\,g}$ be the positive roots of the quartic polynomial $s_4 = L(gL + \bar{h}k + 2z) + 2z$, where M and $N_{\pm}$ are $\sqrt{\bar{h}^2 - 8gz}$ and $h^2 + 4g^2 \pm \bar{h}M$.

Teorem 21. *If in a triangle ABC its angles satisfy the inequality*

$$\cos(B - C) + 4(\cos(B + C) + \cos B - \cos C) \geq 3,$$

then the following are equivalent: (i) the parameter k is either n_+ or n_-, (ii) the lines UV and AD are parallel, (iii) the line U_oV_o bisects the segment $U'V'$, (iv) the segments UV and $U'V'$ share the midpoints and (v) the lines AD and DI_c are perpendicular.

Note that the condition (ii) in Theorem 20 implies $r_5 = r_6$.

9. Equal radii r_7 and r_8

In this section we consider similarly the pair r_7 and r_8 of the radii of the seventh and the last eighth Thébault's circles.

Let $k_{I_b'} = \frac{\bar{h} + \sqrt{\bar{h}^2 + 4f^2}}{2\,f}$ be the positive root of the quadratic polynomial $p_{I_b'} = fL - \bar{h}k$.

Teorem 22. *The following statements are equivalent: (i) the point D is the orthogonal projection I_b' of the excenter I_b onto the line BC, (ii) the parameter k is $k_{I_b'}$, (iii) the lines XY and BC are parallel, (iv) the lines X_oY_o and BC are parallel, (v) the midpoint of the segment XY is on the perpendicular bisector of the segment BC, (vi) the segments XY and $X''Y''$ share the midpoints, (vii) the line joining the excenter I_c and the midpoint of the segment BC is parallel to the line AD and (viii) the equalities $r_7 = r_c$ and $r_8 = r_c$ are both true.*

Let $\bar{h}^2 - 8fz \geq 0$. Let $p_{\pm} = \frac{\sqrt{2\,N_{\pm}} \pm M + \bar{h}}{4\,f}$ be the positive roots of the quartic polynomial $t_4 = L(fL - \bar{h}k + 2z) + 2z$, where M and $N_{\pm}$ are $\sqrt{\bar{h}^2 - 8fz}$ and $h^2 + 4f^2 \pm \bar{h}M$.

Teorem 23. *If in a triangle ABC its angles satisfy the inequality*

$$\cos(B - C) + 4(\cos(B + C) - \cos B + \cos C) \geq 3,$$

then the following are equivalent: (i) the parameter k is either p_+ or p_-, (ii) the lines XY and AD are parallel, (iii) the line X_oY_o bisects the segment $X'Y'$, (iv) the segments XY and $X'Y'$ share the midpoints and (v) the lines AD and DI_b are perpendicular.

Note that the condition (ii) in the above Theorem 23 implies $r_7 = r_8$.

Of course, we can also study the possibilities for equalities of r_i and r_j for other choices of i and j in the set $\{1, \dots, 8\}$. Let us mention only that the equalities $r_1 = r_7$ and $r_2 = r_8$ are impossible and that $r_3 = r_6$ if and only if $D = C$ and that $r_4 = r_8$ if and only if $D = B$.

10. When Thébault's circles touch?

The following two theorems explore when will some Thébault's circles touch each other. We shall prove only the first theorem and omit the proof of the second theorem because it is analogous.

Let k_0 be the positive root $(\sqrt{d^2+\bar{h}^2}-d)/\bar{h}$ of the polynomial $p_2 = \bar{h}L + 2dk$. Let $\mathfrak{w}$ denote the perpendicular bisector of the side BC in the triangle ABC.

Teorem 24. *For the circles k_1, k_2, k_3 and k_4 the following statements are equivalent: (i) $I \in k_1$, (ii) $I \in k_2$, (iii) $k_1 \cap k_2 = I$, (iv) $I_a \in k_3$, (v) $I_a \in k_4$, (vi) $k_3 \cap k_4 = I_a$, (vii) $r_2 = k^2 r_1$, (viii) $r_4 = k^2 r_3$, (ix) $|J_aJ_b| = |J_aJ_c|$, (x) $|OJ_b| = |OJ_c|$, (xi) $J \in \mathfrak{w}$, (xii) $J_a \in \mathfrak{w}$, (xiii) the lines BC and J_bJ_c are parallel, (xiv) the lines PQ and ST are parallel, (xv) the lines PQ and AD are perpendicular, (xvi) the lines ST and AD are perpendicular, (xvii) the triangles PTD and SQD have the same area, (xviii) either the point D, the point I or the point I_a has the same power with respect to the circles k_1 and k_2, (xix) either the point D, the point I or the point I_a has the same power with respect to the circles k_3 and k_4, (xx) the point D is the intersection of the lines AI and BC, (xxi) $D(\mathfrak{I}, \mathfrak{I}_e, \mathfrak{I}_a, \mathfrak{I}_b, \mathfrak{J}_a, \mathfrak{J}_e, \mathfrak{J}, \mathfrak{J}_c) = 0$, (xxii) the lines $\mathfrak{I}_a\mathfrak{J}_a$ and I_aJ_e are perpendicular, (xxiii) $|P'P''Q'Q''| = 0$, (xxiv) $|S'S''T'T''| = 0$, (xxv) the point D is in the segment BC and the sum of radii of the incircles and the excircles of the triangles ABC, ABE, BCE and ACE is the largest possible and (xxvi) the parameter k is equal k_0.*

Proof. We shall argue that each statement (i)–(xxv) is equivalent to (xxvi).

Since $I \in k_1$ is equivalent with $|PI| = |y_P|$ and $y_P^2 - |PI|^2 = \frac{r^2 p_2}{hk^2}$, we see that (i) is equivalent to (xxvi). Similarly, from $|QI|^2 - y_Q^2 = \frac{r^2 p_2}{h}$, $|SI_a|^2 - y_S^2 = \frac{r^2\zeta^2 p_2}{hk^2}$ and $y_T^2 - |TI_a|^2 = \frac{r^2\zeta^2 p_2}{h}$ it follows that (ii), (iv) and (v) are equivalent with (xxvi). It is obvious now that the same is true for (iii) and (vi).

The identities $k^4 y_P^2 - y_Q^2 = \frac{r^2 K p_2}{h}$ and $y_T^2 - k^4 y_S^2 = \frac{r^2\zeta^2 K p_2}{h}$ imply this for (vii) and (viii).

Since $|J_aJ_b|^2 - |J_aJ_c|^2 = \frac{r^2 f^+ g^+ p_2}{hK}$ and $|OJ_b|^2 - |OJ_c|^2 = \frac{r^2 f^+ g^+ p_2}{2hK}$, the same conclusion holds also for (ix) and (x).

The perpendicular bisector of the segment BC has the equation $2x = rz$. Since $rz - 2x_{J_a} = 2x_J - rz = \frac{rzp_2}{hK}$, we included (xi) and (xii) too.

The line J_bJ_c is parallel to the x-axis BC if and only if the centers J_b and J_c have equal ordinates. Since $y_{J_c} - y_{J_b} = \frac{rp_2}{K}$, it follows that (xiii) and (xxvi) are equivalent.

Since $(L + dk)x + hky = rf_+\varphi_-$ and $(\zeta L + dk)x - hky = rg^2 f_+\varphi_-$ are the equations of the lines PQ and ST, they are parallel provided

$(\zeta L + dk) + (L + dk) = \bar{h}L + 2dk = p_2 = 0$. In other words, (xiv) and (xxvi) are equivalent.

Similarly, since $2kx + hky = \frac{2rgf_{+}\varphi_{-}}{h}$ is the equation of the line AD, it follows that $-\frac{L+dk}{hk} = -\frac{1}{-\frac{2k}{L}}$ is the condition for the lines PQ and AD to be perpendicular. However, this identity reduces to $p_2 = 0$. Hence, (xv) and (xxvi) are equivalent. The proof for the statement (xvi) is analogous.

Using the well-known formula

$$|ABC| = \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \tag{53}$$

for the (oriented) area of any triangle ABC, we get that $|PTD| - |SQD|$ is $\frac{r^2K^2\zeta p_2}{2hk^3}$. Therefore, (xvii) and (xxi) are equivalent. Notice that $|PTA| = |SQA|$ if and only if $|AB| = |AC|$.

Since $w(D, k_1) - w(D, k_2) = \frac{r^2Kp_2}{hk^2}$, we conclude that (xviii) and (xxvi) are equivalent for the point D. Similar arguments holds for the points I and I_a and also for the three parts of the statement (xix).

Observe that the difference of the abscises of the point D_k and the intersection $\left(\frac{r\, f^{+}\, g}{\zeta+1}, 0\right)$ of the lines AI and BC is $\frac{r\,\zeta\, p_2}{(\zeta^2-1)k}$. Hence, (xx) and (xxvi) are equivalent.

Since $D(\mathfrak{I}, \mathfrak{I}_e, \mathfrak{I}_a, \mathfrak{I}_b, \mathfrak{J}_a, \mathfrak{J}_e, \mathfrak{J}, \mathfrak{J}_c) = \frac{r^4 (f^{+})^3\, g^{+}\, g^2\, p_2}{h^3\, K}$, we infer that (xxi) is equivalent with (xxvi).

The lines $\mathfrak{I}_a\mathfrak{J}_a$ and I_aJ_e have the equations $h(M\,x + N\,y) = r\,g\,\varphi_{-}\,F$ and $M_0\,x + N_0\,y = r\,g\,f_{+}\,z(\bar{h} - d\,k)$, where $N_0 = f\,g^{+}\,k^2 + d\,z\,k - g\,f^{+}$, $M_0 = g^2\,f^{+}\,k^2 + \bar{h}\,z\,k + f^2\,g^{+}$, $M = f^2\,g^{+}\,k^2 + d\,h\,k - g^2\,f^{+}$, $N = g\,f^{+}\,k^2 - (2\,\zeta^2 + f^2 + g^2)k + f\,g^{+}$ and $F = [z^3 + g(h^2 - z^2)]k + h^3 + h^2 - z^2$. These lines are perpendicular if and only if $M\,M_0 + N\,N_0 = 0$. Since $M\,M_0 + N\,N_0 = f^{+}\,g^{+}\,\zeta\,K\,p_2$, we conclude that (xxii) and (xxvi) are equivalent.

Since $|P'P''Q'Q''| = -\frac{r^2\,p_2}{h\,k}$ and $|S'S''T'T''| = \frac{r^2\,\zeta^2\,p_2}{h\,k}$, we see that the parts (xxiii) and (xxiv) are equivalent with (xxvi).

Finally, when $D \in BC$ then it is possible to get the radii of the incircles and the excircles of the triangles ABC, ABE, BCE and ACE and check that their sum is a function of k that has the maximal value precisely when $k = k_0$. Hence, (xxv) and (xxvi) are also equivalent. $\square$

Let ABC be a triangle such that $|AB| \neq |AC|$. Then $d \neq 0$. Let m_0 be the positive root $(\bar{h} + sgn(d)\sqrt{d^2 + \bar{h}^2})/d$ of the polynomial $s_2 = dL - 2\bar{h}k$.

Teorem 25. *For the circles k_5, k_6, k_7 and k_8 in a triangle ABC with $|AB| \neq |AC|$, the following statements are equivalent: (i) $I_b \in k_5$, (ii) $I_b \in k_6$, (iii) $k_5 \cap k_6 = I_b$, (iv) $I_c \in k_7$, (v) $I_c \in k_8$, (iv) $k_7 \cap k_8 = I_c$,* *(vii)* $r_6 = k^2 r_5$, *(viii)* $r_8 = k^2 r_7$, *(ix) the lines UV and AI are parallel, (x) the lines XY and AI are parallel, (xi) the lines UV and XY are parallel, (xii) the lines UV and AD are perpendicular, (xiii) the lines XY and AD are perpendicular, (xvi) the triangles UYD and XVD have the same area, (xv) either the point D, the point I_b or the point I_c has the same power with respect to the circles k_5 and k_6, (xvi) either the point D, the point I_b or the point I_c has the same power with respect to the circles k_7 and k_8, (xvii)* $J_b \in \mathfrak{w}$, *(xviii)* $J_c \in \mathfrak{w}$, *(xix) the point D is the intersection of the lines I_bI_c and BC, (xx)* $D(\mathfrak{I}, \mathfrak{I}_e, \mathfrak{I}_a, \mathfrak{I}_b, \mathfrak{I}_e, \mathfrak{I}_a, \mathfrak{I}_c, \mathfrak{I}) = 0$, *(xxi)* $|U'U''V'V''| = 0$, *(xxii)* $|X'X''Y'Y''| = 0$ *and (xxiii) the parameter k is m_0.*

The following theorem explains the conditions for other pairs of Thébault's circles to touch. We give a table that uses short notation for some statements about the form of the triangle and about the position of the point D. The proofs are omitted because they are similar to the proof of Theorem 24. For example, k_1 and k_3 will touch if and only if $|PS|^2 = (y_P \pm y_S)^2$. When we factor the difference of the left and the right sides of these identities we get the three possibilities from the table.

Let A_r, B_r and C_r denote the reflections of the vertices A, B and C in the sidelines BC, CA and AB. The triangle $A_rB_rC_r$ is called the three-images triangle.

Let q_1 and q_3 denote the following polynomials in k:

$$(d^2h^2 - 4\zeta^2)k^4 - 2d\bar{h}(\zeta^2+1)k^3 + M\,k^2 + 4d\bar{h}\zeta\,k - 4\zeta^2,$$

$$(z^2\bar{h}^2 - 4\zeta^2)k^4 + 2d\bar{h}(f^2+g^2)k^3 + N\,k^2 + 4d\bar{h}\zeta\,k - 4\zeta^2,$$

where $M = \zeta^4 + 6\zeta^2 + 4f^-g^-\zeta + 1$ and $N = f^4 + g^4 + 6\zeta^2 + 4f^-g^-\zeta$. Let $q_2 = k^4\,q_1\left(-\frac{1}{k}\right)$ and $q_4 = k^4\,q_3\left(-\frac{1}{k}\right)$. Note that q_1 and q_2 have at most three positive real roots while q_3 and q_4 have at most one positive real root.

Let b, b_+, b_-, B_+, B_-, C_+, C_-, D_B, D_C, t_A, t_B, t_C, r_B, r_C, q_1, q_2, q_3 and q_4 be the following statements "$B = C$", "$B > C$", "$B < C$", "$B > 90°$", "$B < 90°$", "$C > 90°$", "$C < 90°$", "$D = B$", "$D = C$", "the lines AD and AO are perpendicular", "the lines AD and BO are perpendicular", "the lines AD and CO are perpendicular", "the lines AD and CB_r are parallel", "the lines AD and BC_r are parallel" and "k is the positive real root of the polynomial q_j" (for $j = 1, 2, 3, 4$).

The pairs (k_1, k_2), (k_3, k_4) and (k_5, k_6), (k_7, k_8) have been covered by Theorems 24 and 25, respectively.

Teorem 26. *The Table 1 lists the necessary and sufficient conditions for pairs among the circles $k_1, \ldots, k_8$ to touch each other. For example, k_1 and k_3 touch if and only if either $D = C$ or $B < C$ and the lines AD and AO are perpendicular or the parameter k has additional at most three different values (the positive real roots of the polynomial q_1).*

	k_2	k_3	k_4	k_5	k_6	k_7	k_8
k_1	Th 24	D_C b_-t_A q_1	D_B	D_B C_+r_B	b_-t_A	B_-r_C	D_B b_-t_A t_C
k_2		D_C	b_+t_A q_2	b_+t_A t_B	C_-r_B	b_+t_A	b_+r_C
k_3			Th 24	D_C B_-r_C	b_-t_A t_C	C_+r_B	D_C b_-t_A
k_4				D_B b_+t_A	B_+r_C	b_+t_A t_B	D_B C_-r_B
k_5					Th 25	b_+t_A q_3	b D_B D_C
k_6						b	b_-t_A q_4
k_7							Th 25

Table 1. Conditions for Thébault's circles to touch

It would be interesting to get a purely geometric description of the positions of the point D which correspond to the positive real roots of the polynomials $q_1, \dots, q_4$.

11. Thébault's centers on lines

The following theorem explains when the centers of Thébault's circles lie on the perpendicular bisector $\mathfrak{w}$ of the segment BC.

Teorem 27. *(i) The center P is on the line $\mathfrak{w}$ if and only if the angle C is larger than the angle B, the relation $3b \neq a + c$ among the lengths of sides holds and D is the $\frac{a+b-3\,c}{3\,b-a-c}$-point of the segment BC.*

(ii) The center Q is on $\mathfrak{w}$ if and only if $B > C$, the relation $3b \neq a + c$ holds and D is the $\frac{a+b-3\,c}{3\,b-a-c}$-point of BC.

(iii) The center S is on $\mathfrak{w}$ if and only if $B > C$ and D is the $\frac{b-a-3\,c}{a-c+3\,b}$-point of BC.

(iv) The center T is on $\mathfrak{w}$ if and only if $C > B$ and D is the $\frac{b-a-3\,c}{a-c+3\,b}$-point of BC.

(v) The center U is on $\mathfrak{w}$ if and only if D is the $\frac{b-a+3\,c}{a+c+3\,b}$-point of BC.

(vi) The center Y is on $\mathfrak{w}$ if and only if D is the $\frac{a+b+3\,c}{c-a+3\,b}$-point of BC.

(vii) The centers V and X can never be on $\mathfrak{w}$.

Proof. (i) The point P is on the line $\mathfrak{w}$ if and only if

$$|PB|^2 - |PC|^2 = \frac{r^2 z[dk-2]}{k} = 0.$$

The unique positive real value of k when this can hold is $\frac{2}{d}$ for $f > g$. The corresponding point D is the $\frac{a+b-3c}{3b-a-c}$-point of the segment BC, where $a = rz$, $b = \frac{rfg^+}{h}$ and $c = \frac{rgf^+}{h}$.

The other parts have similar proofs. □

We can make a similar analysis when will the points $P, \ldots, Y$ lie on the line AO. Let us state only a simpler result for the centers P and Q. Moreover, we omit the discussion of the values when the denominators are zero. For example, when $B = C$, then P is never on the line AO.

Teorem 28. *(i) The point P is on the line AO if and only if either $k = \frac{\bar{h}}{d} > 0$ (see Theorem 7) or $k = \frac{2d}{\zeta^2+d^2-1} > 0$.*

(ii) The point Q is on the line AO if and only if either $k = -\frac{d}{\zeta+1} > 0$ (see Theorem 8) or $k = \frac{1-\zeta^2-d^2}{2d} > 0$.

Proof. (i) The line AO has the equation

$$h[(\bar{h}^2 - d^2)x - 2d\bar{h}y] - rgg^-(f^+)^2 = 0.$$

Substituting the coordinates of the point P for x and y, we get

$$\frac{r[dk - \bar{h}][(\zeta^2 + d^2 - 1)k - 2d]}{k^2} = 0.$$

Hence, the point P is on the line AO if and only if k is either $\frac{\bar{h}}{d}$ or $\frac{2d}{\zeta^2+d^2-1}$.

The part (ii) has a similar proof. □

The following analogous result for the line joining the circumcenter O with the Nagel point $\mathfrak{N}$ is also stated in a similar partial form to avoid listing many subcases. Note that $O = \mathfrak{N}$ iff $B = C = 30°$.

We define $u_P = \frac{\zeta-3}{d(\zeta-2)}$, $v_P = \frac{2\,d}{z^2-\zeta^2-3}$, $u_S = \frac{\zeta(3-\zeta)}{d}$, $v_S = \frac{2\,d\,\zeta}{(h-1)^2+z^2-1}$, $u_U = \frac{d\,g}{f\,h-2\,g}$, $v_U = \frac{2\,g(\zeta-3)}{h^2-d\,z-4}$, $u_X = \frac{d\,f}{g\,h-2\,f}$ and $v_X = \frac{2\,f(\zeta-3)}{4+d\,z-h^2}$.

Teorem 29. *(i) The point P is on the line $O\mathfrak{N}$ if and only if either $k = u_P > 0$ or $k = v_P > 0$.*

(ii) The point Q is on the line $O\mathfrak{N}$ if and only if either $k = -\frac{1}{u_P} > 0$ or $k = -\frac{1}{v_P} > 0$.

Proof. (i) The line $O\mathfrak{N}$ has the equation

$$h[(\zeta - 3)^2 - d^2)](x + y) = r(f\,L - 2\,g)(h^2 + d^2 - 2\,g^-).$$

Substituting the coordinates of the point P for x and y, we get

$$\frac{r^2[d(\zeta - 2)k - \zeta + 3][(\zeta^2 - z^2 + 3)k + 2d]}{4\,h^2\,k^2} = 0.$$

Hence, the point P is on the line $O\mathfrak{N}$ if and only if k is either u_P or v_P.

The part (ii) has a similar proof. □

The identical theorems hold for the pairs (S, T), (U, V) and (X, Y).

12. Central points as Thébault's centers

Since every Thébault's circle is touching the circumcircle and the lines BC and AD, it is now obvious that the circumcenter O is never the center of any Thébault's circles. The following result shows that the Nagel point $\mathfrak{N}$ of the triangle ABC is also rarely the center of Thébault's circles.

Teorem 30. *The point $\mathfrak{N}$ is never equal to any of the points P, Q, U, V, X or Y. The equality $\mathfrak{N} = S$ holds if and only if $f > \sqrt{2}$, $g = \frac{2}{f}$ and $k = \frac{2f}{f^2-2}$. The equality $\mathfrak{N} = T$ holds if and only if $f < \sqrt{2}$, $g = \frac{2}{f}$ and $k = \frac{2-f^2}{2f}$.*

Proof. Since the coordinates of $\mathfrak{N}$ are $\frac{r}{h}\,(f g^+ - 2g, 2)$, we can easily find that $|P\mathfrak{N}|^2 = \frac{r^2 MN}{h^2 k^4}$, where $M = (dk-1)^2 + k^2$ and $N = (k\zeta - 2k)^2 + 1$ are always positive. Hence, the center P is never the Nagel point. The arguments for the centers Q, U, V, X and Y are similar.

Analogously, $|S\mathfrak{N}|^2 = \frac{r^2 MN}{h^2 k^4}$, where $M = \zeta^2 + k^2$ is always positive and $N = [d^2 + (\zeta-2)^2]k^2 - 2d\zeta k + \zeta^2$ has the positive leading coefficient $d^2 + (\zeta - 2)^2$ and the discriminant $-4\zeta^2(\zeta-2)^2$. Hence, when $g = \frac{2}{f}$, then $|S\mathfrak{N}|^2 = \frac{r^2(k^2+4)[k(f^2-2)-2f]^2}{f^2 k^4}$. We infer that S will be $\mathfrak{N}$ for $k = \frac{2f}{f^2-2}$ and conclude, in addition, that $f > \sqrt{2}$ because k is always positive.

The argument for the center T is similar. □

13. Special relations

This section begins with two results that illustrate how some special relations among radii of Thébault's circles can hold only when the point D has some particular position.

Teorem 31. *(i) The relation $r^2(r_3^2 + r_4^2) = r_a^2(r_1^2 + r_2^2)$ holds if and only if either the lines AD and BC are perpendicular or the line AD goes through the incenter I.*

(ii) The relation $r_b^2(r_7^2 + r_8^2) = r_c^2(r_5^2 + r_6^2)$ holds if and only if either the lines AD and BC are perpendicular or the line AD goes through the excenters I_b and I_c.

Proof. We get this from relations $r^2(r_3^2 + r_4^2) - r_a^2(r_1^2 + r_2^2) = \frac{r^4\zeta^2 L K^2 p_2}{hk^4}$ and $r_c^2(r_5^2 + r_6^2) - r_b^2(r_7^2 + r_8^2) = \frac{r^4\zeta^2 z^3 L K^2 s_2}{(hk)^4}$ and the fact that for $k = 1$ the point D_k is the orthogonal projection of the vertex A onto the sideline BC. □

Teorem 32. *If the product of the tangents of the angles B and C in the triangle ABC is 2, then $r^2\, r_3^2\, r_4^2 + r_a^2\, r_1^2\, r_2^2 = r_b^2\, r_7^2\, r_8^2 + r_c^2\, r_5^2\, r_6^2$.*

Proof. Since $r^2\, r_3^2\, r_4^2 + r_a^2\, r_1^2\, r_2^2 - r_b^2\, r_7^2\, r_8^2 - r_c^2\, r_5^2\, r_6^2$ contains as a factor $\zeta^2 - z^2 + 1 = \frac{2(1+\cos(B))(1+\cos(C))(2\cos(B)\cos(C)-\sin(B)\sin(C))}{(\sin(B))^2(\sin(C))^2}$, it is clear that $\tan(B)\tan(C) = 2$ implies the above equality. □

In the next result we use again the coordinates of the centers of Thébault's circles. For $e, f \in \{x, y\}$, let $E(e, f)$ denote the identity $e_P + e_S - e_U - e_X = f_Q + f_T - f_V - f_Y$.

Teorem 33. *(i) The identities $E(x, y)$ and $E(x, x)$ are never true.*

(ii) The identities $E(y, y)$ and $E(y, x)$ hold if and only if the lines AD and BC are perpendicular.

Proof. (i) The difference of the left and the right sides of the identities $E(x, y)$ and $E(x, x)$ are $\frac{r\, f^+\, g^+ (k+1)(k^2-k+1)}{h\, k}$ and $\frac{r\, f^+\, g^+\, K}{h\, k}$. Since there is no positive value k for which these quotients vanish, it follows that they are never true.

(ii) The difference of the left and the right sides of the identities $E(y, x)$ and $E(y, y)$ are $\frac{r\, f^+\, g^+ (k-1)(k^2+k+1)}{h\, k^2}$ and $\frac{r\, f^+\, g^+\, K\, L}{h\, k^2}$. Since 1 is a unique positive value of k for which these quotients vanish, it follows that they hold if and only if the lines AD and BC are perpendicular. □

14. Lines containing many incenters and excenters

For any point M in the plane, let M' and M'' be the orthogonal projections of M onto the lines BC and AD.

We prove now that the lines $P'P''$, $Q'Q''$, $S'S''$, $T'T''$, $U'U''$, $V'V''$, $X'X''$ and $Y'Y''$ each contains four among incenters and/or excenters of the triangles ABC, BCE, ABE and ACE. In partial form this was observed in [23].

Teorem 34. *The following table gives the incidence relations of lines $P'P''$,..., $Y'Y''$ and the points I, I_a,..., $\mathfrak{I}_c$, $\mathfrak{I}_e$.*

$P'P''$	I	J_c	$\mathfrak{I}$	$\mathfrak{I}_e$
$Q'Q''$	I	J_b	$\mathfrak{I}_e$	$\mathfrak{I}$
$S'S''$	I_a	J_b	$\mathfrak{I}_b$	$\mathfrak{I}_a$
$T'T''$	I_a	J_c	$\mathfrak{I}_a$	$\mathfrak{I}_c$
$U'U''$	I_b	J	$\mathfrak{I}_a$	$\mathfrak{I}$
$V'V''$	I_b	J_a	$\mathfrak{I}_b$	$\mathfrak{I}_e$
$X'X''$	I_c	J_a	$\mathfrak{I}_e$	$\mathfrak{I}_c$
$Y'Y''$	I_c	J	$\mathfrak{I}$	$\mathfrak{I}_a$

Proof. Since the coordinates of P' and P'' are $\frac{r}{k}\,(\varphi_-, 0)$ and $\frac{r\,\varphi_-}{h\,k\,K}\,\big(h\,k^2 + 2\,g\,k + \bar{h}, 2\,\psi_+\,k\big)$, the line $P'P''$ has the equation $k\,x - y = r\,\varphi_-$. It is now easy to check that the coordinates of the points I, J_c, $\mathfrak{I}$ and $\mathfrak{I}_e$ satisfy this equation. The proofs for the other lines are analogous. □

From the above table it is possible to identify sixteen pairs of perpendicular lines among $P'P''$,…, $Y'Y''$ that intersect in the sixteen centers I,…, $\mathfrak{J}_e$. All other pairs of lines among $P'P''$,…, $Y'Y''$ are pairs of parallel lines.

For example, from the first two rows we conclude that the lines $P'P''$ and $Q'Q''$ intersect in I while from the first and the fourth row it follows that the lines $P'P''$ and $T'T''$ intersect in J_c. On the other hand, the line $P'P''$ is parallel to the lines $S'S''$, $U'U''$ and $X'X''$.

15. Circles with diameters on lines BC and AD

Let k_{MN} and s_{MN} denote the circle with the segment MN as a diameter and its center. In other words, s_{MN} is the midpoint of the segment MN.

Teorem 35. *(i) The line AD is parallel with the lines $s_{P'Q'}I$, $s_{S'T'}I_a$, $s_{U'V'}I_b$ and $s_{X'Y'}I_c$.*

(ii) The lines $s_{P''Q''}I$, $s_{S''T''}I_a$, $s_{U''V''}I_b$ and $s_{X''Y''}I_c$ are parallel with the line BC.

(iii) The intersection of the circles $k_{P'Q'}$ and $k_{P''Q''}$ is the incenter I and another point $\mathfrak{K}$ on the line PQ.

(iv) The circles $k_{S'T'}$ and $k_{S''T''}$ intersect in the point I_a and in another point K_a on the line ST.

(v) The intersection of the circles $k_{U'V'}$ and $k_{U''V''}$ is the excenter I_b and another point K_b on the line PQ.

(vi) The circles $k_{X'Y'}$ and $k_{X''Y''}$ intersect in the point I_c and in another point K_c on the line XY.

The following relation holds:

$$|P\mathfrak{K}| \cdot |SK_a| \cdot |VK_b| \cdot |YK_c| = |Q\mathfrak{K}| \cdot |TK_a| \cdot |UK_b| \cdot |XK_c|. \tag{54}$$

$\frac{|P\mathfrak{K}|^2}{r^2} + \frac{|SK_a|^2}{r_a^2} + \frac{|VK_b|^2}{r_b^2} + \frac{|YK_c|^2}{r_c^2} = \frac{|Q\mathfrak{K}|^2}{r^2} + \frac{|TK_a|^2}{r_a^2} + \frac{|UK_b|^2}{r_b^2} + \frac{|XK_c|^2}{r_c^2}$ *is true if and only if either $D = B$, $D = C$ or the lines AD and BC are perpendicular.*

Proof. (i) The midpoint M of the segment $P'Q'$ has the abscissa $\frac{r(L+2fk)}{2k}$ and the ordinate 0. Hence, the line IM is parallel to the line AD as their equations are $2kx + Ly = r(L + 2fk)$ and $2kx + Ly = 2rgf_+\varphi_-$. The remaining three claims have similar proofs.

(ii) The midpoint M of the segment $P''Q''$ has the abscissa $\frac{r(\bar{h}L+2gf^-k)}{2hk}$ and the ordinate r. Hence, the line IM is parallel to the line BC because the incenter also has the ordinate r. The remaining three claims have similar proofs.

(iii) Since P', Q', P'' and Q'' have the coordinates $(x_P, 0)$, $(x_Q, 0)$, $\frac{r\varphi_-}{hkK}(hK + 2\psi_+, 2k\psi_+)$ and $\frac{rf_+}{hK}\left(\bar{h}K - 2\psi_+, -2g_-\right)$, the second intersection of the circles $k_{P'Q'}$ and $k_{P''Q''}$ (besides the incenter I) is the point $\mathfrak{K}$ with the coordinates $\frac{rf_+\varphi_-}{M}(N, -g_-\psi_+)$, where M and N are

$(L+dk)^2+(hk)^2$ and $L+(fg^+-2g)k$. Its coordinates satisfy the equation of the line PQ (see the proof of Theorem 24).

The proofs of (iv), (v) and (vi) are similar.

The easiest way to check the identity (54) is to show that the squares of its left and right sides are equal.

Finally, the difference of the left and the right hand sides of the last identity is the quotient $\frac{2f^+g^+LKf_+\varphi_-g_-\psi_+}{h^2k^4z^2}$. Since the point D is B and C for k equal $\frac{1}{f}$ and g, we conclude that the last claim is true. $\square$

Of course, the above theorem has three additional versions for the triangles BCE, ABE and ACE. For example, the lines $s_{P'Y'}\mathfrak{I}$, $s_{S'V'}\mathfrak{I}_b$, $s_{T'U'}\mathfrak{I}_a$ and $s_{Q'X'}\mathfrak{I}_e$ are parallels of AD while the lines $s_{P''Y''}\mathfrak{I}$, $s_{S''V''}\mathfrak{I}_b$, $s_{T''U''}\mathfrak{I}_a$ and $s_{Q''X''}\mathfrak{I}_e$ are parallels of BC.

16. Three (of twelve) associated triangles

Let $\mathcal{I}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be the midpoints of the segments $s_{P'Q'}I$, $s_{S'T'}I_a$, $s_{U'V'}I_b$ and $s_{X'Y'}I_c$. Similarly, let $\mathbf{I}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the midpoints of the segments $s_{P''Q''}I$, $s_{S''T''}I_a$, $s_{U''V''}I_b$ and $s_{X''Y''}I_c$. Finally, let $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be the midpoints of the segments $\mathcal{I}\mathbf{I}$, $\mathcal{A}\mathbf{A}$, $\mathcal{B}\mathbf{B}$ and $\mathcal{C}\mathbf{C}$.

The few basic relationships among these points are described in the following result. Let s_a, s_b and s_c denote $\frac{b+c-a}{2}$, $\frac{c+a-b}{2}$ and $\frac{a+b-c}{2}$.

Teorem 36. *(i) The point $\mathcal{I}$ is the $\frac{s_b}{b}$-point of the segment $B\mathcal{B}$ and the $\frac{s_c}{c}$-point of the segment $C\mathcal{C}$.*

(ii) The point $\mathbf{I}$ is the $\frac{s_a}{a}$-point of the segment $A\mathbf{A}$.

(iii) The vertex A is the $\frac{s_c}{s_b}$-point of the segment $\mathbf{BC}$.

(iv) The vertex B is the $\frac{s_a}{s_c}$-point of the segment $\mathbf{CA}$.

(v) The vertex C is the $\frac{s_b}{s_a}$-point of the segment $\mathbf{AB}$.

(vi) The points $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are the 3-points of the segments DI, DI_a, DI_b and DI_c.

Proof. The points $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ have coordinates $-\frac{rg}{4k}\,(fL-4k, 2fk)$, $\frac{rgz}{4hk}\,(L+4fk, 2k)$ and $\frac{rz}{4hk}\,(fL-4k, 2fk)$. Similarly, the vertices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ have as coordinates the pairs $\frac{rg}{4hk}\big(f\bar{h}L+2(h+f^-)k, -4fk\big)$, $\frac{rg}{4hk}\,(dL+2(h+2f^2)k, 4kz)$ and $\frac{r}{4hk}\,(2(fh-2g)k-dfL, 4fkz)$. Also, the points $\mathbf{I}$ and $\mathcal{I}$ have the coordinates $\frac{r}{4hk}\big(\bar{h}L+2(2f\zeta-z)k, 4hk\big)$ and $\frac{r}{4k}\,(L+4fk, 2k)$. Since $\frac{s_b}{b}$ is equal to $\frac{h}{g^+}$, it follows that the $\frac{s_b}{b}$-point of the segment $B\mathcal{B}$ is the point $\mathcal{I}$. This proves the first claim in the part (i). All other claims in this theorem have similar routine verification. $\square$

Teorem 37. *The areas of the triangles satisfy the following relations:*

$$|\mathcal{ABC}| = |\mathbf{ABC}| = \tfrac{1}{2}\,|I_aI_bI_c|, \qquad |\mathbb{ABC}| = \tfrac{9}{16}\,|I_aI_bI_c|.$$

Proof. Using the formula (53), we find that $|\mathcal{ABC}| = |\mathbf{ABC}| = \frac{r^2f^+g^+\zeta z}{4h^2} = \frac{1}{2}\,|I_aI_bI_c|$.

On the other hand, since the points $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ have the coordinates $\frac{rg}{4hk}\,(fL+(f^-+3h)k, -3fk)$, $\frac{rg}{4hk}\left(fL+(4f^-+3\bar{h})k, 3kz\right)$ and $\frac{r}{4hk}\left(\zeta L+(f\bar{h}-4z)k, 3fkz\right)$, we similarly find that $|\mathbb{ABC}| = \frac{9r^2f^+g^+\zeta z}{32h^2}$. $\square$

17. Some orthologic triangles

For any real number $u \neq -1$, let Q_u, T_u, V_u and Y_u denote the u-points of the segments QP, TS, VU and YX. Let U_u, X_u, S_u and P_u denote the u-points of the segments YU, VX, QS and TP. Recall that the pedal triangle of the point M (with respect to the triangle ABC) has the orthogonal projections of M onto the lines BC, CA and AB as vertices. Let Ψ and Ξ denote the pedal triangle of the incenters I and J with respect to the triangles ABC and EBC.

Triangles ABC and DEF are *orthologic* provided the perpendiculars at the vertices of ABC onto the sides EF, FD and DE of DEF are concurrent. The point of concurrence of these perpendiculars is denoted by o_{ABC}^{DEF}. It is well-known that this relation is reflexive and symmetric. Hence, the perpendiculars from vertices of DEF onto the sides BC, CA, and AB are concurrent at the point o_{DEF}^{ABC}. These points are called the *first* and the *second orthology centers* of the (orthologic) triangles ABC and DEF. Replacing perpendiculars with parallels we get the analogous notion of *paralogic* triangles and centers p_{ABC}^{DEF} and p_{DEF}^{ABC}.

The quadruple $\{A, B, C, D\}$ of points in the plane is orthocentric provided every point is the orthocenter of the triangle on the remaining three points.

Let $\Delta_u = T_uV_uY_u$, $\Gamma_u = X_uS_uP_u$, $\Phi = I_aI_bI_c$ and $\Theta = J_aJ_bJ_c$. Let us notice that the orthocentric quadruples $\{I, I_a, I_b, I_c\}$ and $\{J, J_a, J_b, J_c\}$ are associated in the sense that the following holds:

Teorem 38. *For every point N in the plane,*

$$|NI|^2 + |NI_a|^2 + |NI_b|^2 + |NI_c|^2 = |NJ|^2 + |NJ_a|^2 + |NJ_b|^2 + |NJ_c|^2.$$

Proof. Let N has the coordinates (p, q). Both sides of the above identity have the value $4(p^2+q^2-rzp) + \frac{2r(h^2-z^2)}{h}q + \left(\frac{rf^+g^+}{h}\right)^2$. $\square$

In a similar way one can show that the orthocentric quadruples $\{\mathfrak{J}, \mathfrak{J}_a, \mathfrak{J}_b, \mathfrak{J}_e\}$ and $\{\mathfrak{J}, \mathfrak{J}_a, \mathfrak{J}_c, \mathfrak{J}_e\}$ are also associated to $\{I, I_a, I_b, I_c\}$.

Teorem 39. *The triangle Δ_u is orthologic with the triangle Φ. The triangle Γ_u is orthologic with the triangle Θ. Their areas satisfy*

$$\frac{|\Delta_u|}{|\Phi|} = \frac{|\Gamma_u|}{|\Theta|} = \frac{K^2\,u}{k^2(u+1)^2}.$$

Proof. Recall that the triangles ABC and XYZ are orthologic provided

$$\begin{vmatrix} x_A & x_X & 1 \\ x_B & x_Y & 1 \\ x_C & x_Z & 1 \end{vmatrix} + \begin{vmatrix} y_A & y_X & 1 \\ y_B & y_Y & 1 \\ y_C & y_Z & 1 \end{vmatrix} = 0. \tag{55}$$

We can easily find the coordinates of the vertices of the triangles Δ_u and Φ, substitute them into the the above determinants and make simplifications to conclude that the condition (55) holds for this pair of triangles. The same is true also for the pair (Γ_u, Θ).

Finally, using the formula (53), we get $|\Delta_u| = \frac{r^2 \zeta K^2 f^+ g^+ z u}{2(hk)^2(u+1)^2}$. Since $|\Phi| = \frac{r^2 \zeta f^+ g^+ z}{2h^2}$, the quotient $\frac{|\Delta_u|}{|\Phi|}$ is $\frac{K^2 u}{k^2(u+1)^2}$. For the pair (Γ_u, Θ) we get the same value. □

Let $k^2 \neq 1$. Let Q_v, T_v, V_v and Y_v be the $(-k^2)$-points of the segments QP, TS, VU and YX and let U_v, X_v, S_v and P_v be the $(-k^2)$-points of the segments YU, VX, QS and TP. Let $\Delta = T_v V_v Y_v$, $\Gamma = X_v S_v P_v$.

Teorem 40. *The quadruples $\{Q_v, T_v, V_v, Y_v\}$ and $\{U_v, X_v, S_v, P_v\}$ are orthocentric and for every point N in the plane the sums*

$$|NQ_v|^2 + |NT_v|^2 + |NV_v|^2 + |NY_v|^2$$

and $|NU_v|^2 + |NX_v|^2 + |NS_v|^2 + |NP_v|^2$ are equal. The triangles Δ and Γ have identical nine-point circles and are reversely similar to the extriangles Φ and Θ, respectively.

Proof. The points Q_v, T_v, V_v and Y_v have the pairs $\frac{r}{hL}\left(h(fL-2k), \bar{h}L + 2dk\right)$, $\frac{rg}{hL}\left(h(L+2fk), f(\bar{h}L+2dk)\right)$, $\frac{rg}{hL}\left(z(fL- \ 2k), dL - \bar{h}k\right)$ and $\frac{r}{hL}\left(-z(L+2fk), f(2\bar{h}k - dL)\right)$ as the coordinates. The perpendicular through the point T_v onto the line $V_v Y_v$ has the equation

$$(\bar{h}L + 2dk)x + (dL - 2\bar{h}k)y = \tfrac{rg(\bar{h}L+2dk)(f^- L - 4fk)}{hL} \tag{56}$$

and the perpendicular through the point V_v onto the line $T_v Y_v$ has the equation

$$(L + 2fk)x + (fL - 2k)y = \tfrac{2rg(fL-2k)f_+ \varphi_-}{hL}. \tag{57}$$

These perpendiculars intersect in the point Q_v. In other words, the linear system of the equations (56) and (57) has the coordinates of the point Q_v as a unique solution. It follows that $\{Q_v, T_v, V_v, Y_v\}$ is an orthocentric quadruple. We can similarly show that the quadruple $\{U_v, X_v, S_v, P_v\}$ is also orthocentric.

Let $N = (p, q)$. The both sums have the value $4[(p - \mathfrak{a})^2 + (q - \mathfrak{b})^2] + \frac{3\,r^2\,K^2\,(f^+)^2 (g^+)^2}{4\,h^2\,L^2}$, where $\mathfrak{a} = \frac{r(kz+h)(hk-z)}{2\,h\,L}$ and $\mathfrak{b} = \frac{r[(\bar{h}-d)^2\,k^2 - (\bar{h}+d)^2]}{4\,h\,L}$.

Since Q_v and U_v are the orthocenters of the triangles Δ and Γ, the easiest way to see that they have the same center of the nine-point circles is to find their centroids G_Δ and G_Γ and verify that the 3-points of $Q_v G_\Delta$ and $U_v G_\Gamma$ coincide. Their radii are also equal (check that this 3-point is equidistant from the midpoints of $T_v V_v$ and $X_v S_v$).

The triangles Δ and Φ are orthologic by Theorem 39. Hence, in order to see that they are reversely similar, by [2], it suffices to check that they are paralogic. Recall that the triangles ABC and XYZ are paralogic provided

$$\begin{vmatrix} x_A & y_X & 1 \\ x_B & y_Y & 1 \\ x_C & y_Z & 1 \end{vmatrix} + \begin{vmatrix} x_X & y_A & 1 \\ x_Y & y_B & 1 \\ x_Z & x_C & 1 \end{vmatrix} = 0. \tag{58}$$

Now we substitute the coordinates of the vertices of the triangles Δ and Φ into the above determinants and make simplifications to conclude that the condition (58) holds for this pair of triangles. The same is true also for the pair (Γ, Θ). $\square$

Recall that the Bevan point X_{40} of the triangle ABC [10] is $o_{I_aI_bI_c}^{ABC}$ (the orthology center of the triangles $I_aI_bI_c$ and ABC) and also the circumcenter of $I_aI_bI_c$. Its coordinates are $\frac{r}{2h}\left(2gh, z^2 - h\bar{h}\right)$.

Corollary 7. *The following are distances among the orthology and paralogy centers of the triangles Δ, Γ, Φ, Ψ, Θ and Ξ.*

$$|o_{\Delta}^{\Phi}\, p_{\Delta}^{\Phi}| = |o_{\Gamma}^{\Theta}\, p_{\Gamma}^{\Theta}| = \frac{4KR}{|L|},$$

$$|o_{\Phi}^{\Delta}\, p_{\Phi}^{\Delta}| = |o_{\Theta}^{\Gamma}\, p_{\Theta}^{\Gamma}| = 4R, \qquad |o_{\Psi}^{\Delta}\, p_{\Psi}^{\Delta}| = 2r, \qquad |o_{\Xi}^{\Gamma}\, p_{\Xi}^{\Gamma}| = 2\varrho.$$

More precisely, o_{Φ}^{Δ} and p_{Φ}^{Δ} are the antipodal points on the circle of radius $2R$ with the center at the Bevan point of the triangle ABC. Similarly, o_{Θ}^{Γ} and p_{Θ}^{Γ} are the antipodal points on the circle of radius $2R$ with the center at the Bevan point of the triangle EBC. Also, o_{Ψ}^{Δ} and p_{Ψ}^{Δ} are the antipodal points on the incircle of ABC and o_{Ξ}^{Γ} and p_{Ξ}^{Γ} are the antipodal points on the incircle of EBC. The locus of midpoints of $o_{\Delta}^{\Phi}p_{\Delta}^{\Phi}$ is a line and the locus of midpoints of $o_{\Gamma}^{\Theta}p_{\Gamma}^{\Theta}$ is a hyperbola.

Proof. We prove only the claims about o_{Ψ}^{Δ} and p_{Ψ}^{Δ} because for other centers the proofs are similar.

We find that the coordinates of these centers are $\frac{r}{f^+g^+K^2}(N_+, 2p_2^2)$ and $\frac{r}{f^+g^+K^2}(N_-, 2s_2^2)$, where $N_\pm = f^3g^+K^2 \pm 2f^-F + fG_\pm$, $F = (gL + 2k)(2gk - L)$, $G_+ = (3L+2)(L-2)g^2 + 16gkL - (K-4k)(K+4k)$ and $G_- = (4k-K)(K+4k)g^2 - 16gkL + (3L+2)(L-2)$. Now it is easy to check that $|o_{\Psi}^{\Delta}I| = r$ and that p_{Ψ}^{Δ} is the (-2)-point of the segment $o_{\Psi}^{\Delta}I$. Notice that from the ordinates of the points o_{Ψ}^{Δ} and p_{Ψ}^{Δ} we see that the statement $o_{\Psi}^{\Delta} \in BC$ could be added in Theorem 24 and $p_{\Psi}^{\Delta} \in BC$ in Theorem 25. $\square$

18. Lines connecting the touching points $P_o, \dots, Y_o$

The points where the eight Thébault's circles touch the circumcircle have many properties. Some are revelled in the next result.

Let $M_1, \dots, M_{24}$ denote the intersections of the lines P_oT_o, P_oV_o, P_oQ_o, S_oT_o, Q_oU_o, Q_oS_o, Q_oS_o, Q_oX_o, P_oQ_o, S_oT_o, P_oY_o, P_oT_o, P_oT_o,

P_oV_o, P_oY_o, S_oV_o, S_oY_o, U_oY_o, Q_oU_o, P_oY_o, U_oV_o, P_oQ_o, Q_oX_o and P_oV_o with the lines U_oY_o, S_oY_o, X_oY_o, U_oV_o, T_oX_o, V_oX_o, U_oY_o, T_oU_o, U_oV_o, X_oY_o, S_oV_o, V_oX_o, Q_oS_o, Q_oU_o, Q_oX_o, T_oU_o, T_oX_o, V_oX_o, S_oY_o, T_oU_o, X_oY_o, S_oT_o, S_oV_o and T_oX_o, respectively.

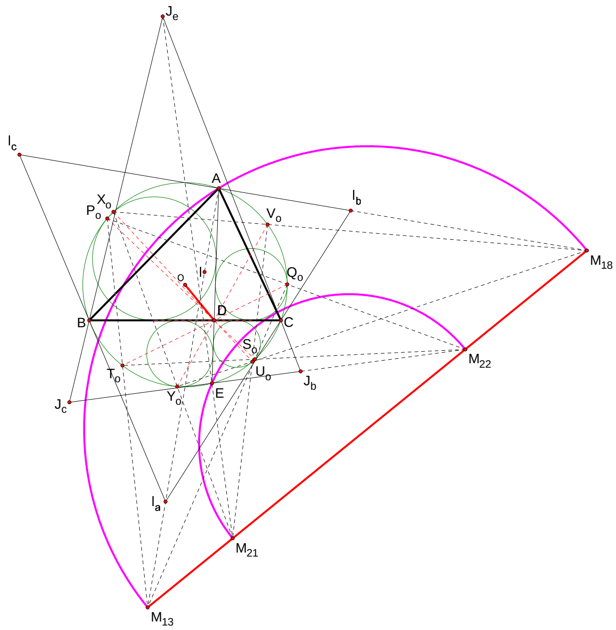

FIGURE 3. The points M_{13}, M_{18}, M_{22} and M_{21}.

Teorem 41. *The point D lies on the following lines: P_oS_o, Q_oT_o, U_oX_o and V_oY_o. The intersections $M_1, \dots, M_{24}$ are on the lines $\mathfrak{I}\mathfrak{I}_a$, $\mathfrak{I}\mathfrak{I}_b$, $\mathfrak{I}\mathfrak{I}_e$, $\mathfrak{I}_a\mathfrak{I}_b$, $\mathfrak{I}_a\mathfrak{I}_e$, $\mathfrak{I}_b\mathfrak{I}_e$, $\mathfrak{I}\mathfrak{I}_a, \dots$, $II_a, \dots$, J_cJ_e, respectively. The points M_2, M_5, M_8, M_{11}, M_{13}, M_{18}, M_{21} and M_{22} are on the line perpendicular to the line DO. The point D is collinear with the points M_1, M_6, M_9, M_{10}, M_{14}, M_{17}, M_{20} and M_{23} as well as with the points M_3, M_4, M_7, M_{12}, M_{15}, M_{16}, M_{19} and M_{24}. The point A is on the circles $k_{M_1M_6}$, $k_{M_7M_{12}}$ and $k_{M_{13}M_{18}}$, the point B is on the circles $k_{M_2M_5}$, $k_{M_{14}M_{17}}$ and $k_{M_{19}M_{24}}$, the point C is on the circles $k_{M_8M_{11}}$, $k_{M_{15}M_{16}}$ and $k_{M_{20}M_{23}}$ and the point E is on the circles $k_{M_3M_4}$, $k_{M_9M_{10}}$ and $k_{M_{21}M_{22}}$. Moreover, there are 32 triples of collinear points beginning with $\{M_4, M_2, M_1\}$ and ending with $\{M_{24}, M_{23}, M_{22}\}$ (one from each of the above three groups of eight points).*

Proof. When $\bar{h} \neq dk$, then the line P_oS_o has the equation

$$2h^2kx + [(h^2+d^2)k^2 - 2d\bar{h}k + 4\zeta]y = 2rghf_+\varphi_-.$$

The coordinates of the point D satisfy this equation. We prove similarly that D also lies on the lines Q_oT_o, U_oX_o and V_oY_o.

The intersection M_{13} has the coordinates $\frac{rg}{M}(N, -2fhs_2)$, where $M = 4d\zeta L + \bar{h}k(d^2 + h^2 - 4\zeta)$ and $N = 2dfzL + k[(f^-)^2 g^+ - 4\zeta f^+]$. It lies on the line II_a with the equation $\bar{h}x - dy = rgf^+$.

Similarly, the point M_{18} has the coordinates $\frac{rgz}{hM}(N, 2fzp_2)$, where $M = 4\bar{h}\zeta L + dk(d^2 + h^2 + 4\zeta)$ and $N = 2fh\bar{h}L + k[(f^-)^2 g^+ + 4\zeta f^+]$. It lies on the line $I_b I_c$ with the equation $dx + \bar{h}y = \frac{rgzf^+}{h}$. The line $M_{13}M_{18}$ is perpendicular to the line DO with the equation

$$k(h^2 - d^2)x - (4\zeta L + 2d\bar{h}k)y = \frac{(h^2 - d^2)rgf_+\varphi_-}{h}.$$

Moreover, the midpoint of $M_{13}M_{18}$ is equidistant from M_{13} and A.

The intersections M_{22} and M_{21} are treated similarly. Of course, they both lie on the line $M_{13}M_{18}$. □

19. Some homologic triangles

The triangles ABC and XYZ are homologic provided the lines AX, BY and CZ are concurrent. Their common intersection h_{ABC}^{XYZ} is called the center (of the homology). In terms of the coordinates the condition for homology is

$$\begin{vmatrix} x_A - x_X & x_B - x_Y & x_C - x_Z \\ y_X - y_A & y_Y - y_B & y_Z - y_C \\ x_A y_X - y_A x_X & x_B y_Y - y_B x_Y & x_C y_Z - y_C x_Z \end{vmatrix} = 0. \tag{59}$$

Let $\varphi = S_o U_o X_o$ and $\psi = T_o V_o Y_o$.

Teorem 42. *The triangle Φ is homologic to the triangles φ and ψ. The homology centers h_Φ^φ and h_Φ^ψ are the antipodal points on the circumcircle o. The lines $h_\Phi^\varphi h_\Phi^\psi$ and AD are perpendicular if and only if either $I \in AD$ or $I_b \in AD$.*

Proof. One can either show directly that the condition (59) holds for the pairs (Φ, φ) and (Φ, ψ) or check that the intersections $I_a S_o \cap I_b U_o$ and $I_a T_o \cap I_b V_o$ have the coordinates $\frac{r(g^- k + 2g)}{2hK}(f^- + 2fk, 2f - f^- k)$ and $\frac{r(2gk - g^-)}{2hK}(f^- k - 2f, 2fk + f^-)$ and that they lie on the lines $I_c X_o$ and $I_c Y_o$, respectively. The distance $|h_\Phi^\varphi h_\Phi^\psi|$ is $2R$ and the midpoint of the segment $h_\Phi^\varphi h_\Phi^\psi$ is the circumcenter O.

The lines $h_\Phi^\varphi h_\Phi^\psi$ and AD are perpendicular if and only if $\frac{r^2 \zeta p_2 s_2}{h^2 kK} = 0$. By Theorems 24 and 25, this happens if and only if either $I \in AD$ or $I_b \in AD$. □

Let $\tau = ABC$. Recall that the tangential triangle $\tau_t = A_t B_t C_t$ has the intersections of the tangents to the circumcircle o at the vertices of τ as vertices.

Teorem 43. *The tangential triangle τ_t is homologic to the triangles φ and ψ.*

Proof. Let $\tau_t = A_t B_t C_t$. These vertices have the coordinates $\frac{rz}{2(h^2-z^2)}\,(h^2 - z^2, 2hz)$, $\frac{r}{2f^- h}\,(f f^- g^- + 2(f^4+1)g, 2fhz)$ and $\frac{rg}{2g^- h}\ (h^2 - z^2, 2hz)$. One can now easily check that the condition (59) holds for the pairs (τ_t, φ) and (τ_t, ψ). $\square$

The above two theorems have more extensive versions that use the symmetry of the configuration. More precisely, the orthocentric quadrangle $I I_a I_b I_c$ is homologic to the quadrangles $P_0 S_0 U_0 X_0$ and $Q_0 T_0 V_0 Y_0$. Similarly, $J J_a J_b J_e$ is homologic to the quadrangles $U_0 S_0 P_0 X_0$ and $Y_0 Q_0 T_0 V_0$, $\mathfrak{I}\mathfrak{I}_a\mathfrak{I}_b\mathfrak{I}_e$ is homologic to the quadrangles $P_0 U_0 S_0 X_0$ and $Y_0 T_0 V_0 Q_0$ and $\mathfrak{J}\mathfrak{J}_a\mathfrak{J}_c\mathfrak{J}_e$ is homologic to the quadrangles $U_0 S_0 X_0 P_0$ and $Q_0 Y_0 T_0 V_0$. The centers of these homologies are antipodal points on the circumcircle and are at the distance $\frac{2Rk}{\sqrt{K}}$ and $\frac{2R}{\sqrt{K}}$ from the vertices A, B, E and C, respectively.

On the other hand, the triangles $U_o X_o P_o$ and $V_o Y_o Q_o$ are homologic to the tangential triangle of BCE, the triangles $S_o U_o P_o$ and $T_o V_o Q_o$ are homologic to the tangential triangle of ABE and the triangles $S_o X_o P_o$ and $T_o Y_o Q_o$ are homologic to the tangential triangle of ACE.

20. More on triangles $\mathcal{ABC}$, $\mathbf{ABC}$ and $\mathbb{ABC}$

In this section we explore additional properties of the triangles $\mathcal{ABC}$, $\mathbf{ABC}$ and $\mathbb{ABC}$ that have been introduces in section 15.

Teorem 44. *The triangles $\mathcal{ABC}$ and $\mathbf{ABC}$ are homologic if and only if either $D = I'$ or $D = AI \cap BC$. They are orthologic if and only if the lines AD and BC are perpendicular. They can never be paralogic.*

Proof. The condition (59) for the triangles $\mathcal{ABC}$ and $\mathbf{ABC}$ is

$$\frac{3\, r^4\, f^+\, g^+\, \zeta^2\, p_{I'}\, s_2\, z}{32\, h^4\, k^2} = 0.$$

Now it suffices to apply Theorems 18 and 25.

Similarly, the conditions (55) and (58) for these triangles are

$$\frac{3\, r^2\, f^+\, g^+\, \zeta\, L\, z}{8\, h^2\, k} = 0, \qquad -\frac{5\, r^2\, f^+\, g^+\, \zeta\, z}{4\, h^2} = 0.$$

The first holds if and only if $k = 1$, i. e., if and only if $D = A'$. The second does not depend on k and is never true so that the triangles $\mathcal{ABC}$ and $\mathbf{ABC}$ are not paralogic. $\square$

Teorem 45. *The triangles ABC and $\mathcal{ABC}$ are homologic if and only if $D = AI \cap BC$.*

Proof. The condition (59) for the triangles $\mathcal{ABC}$ and ABC is

$$-\frac{r^4\, \zeta^2\, p_2\, z^3}{4\, h^3\, k} = 0.$$

The claim of the theorem now follows from Theorem 24. $\square$

Teorem 46. *The triangles $\mathcal{ABC}$ and* $\mathbf{ABC}$ *are orthologic to* Φ *(the extriangle $I_a I_b I_c$) and/or* Ψ *(the pedal triangle $A_q B_q C_q$ of the incenter I) if and only if the lines AD and BC are perpendicular. These pairs of triangles are never paralogic.*

Proof. The conditions (55) and (58) for the pair $(\mathcal{ABC}, \Phi)$ are

$$\frac{r^2\, f^+\, g^+\, \zeta\, L\, z}{4\, h^2\, k} = 0, \qquad\qquad -\frac{3\, r^2\, f^+\, g^+\, \zeta\, z}{2\, h^2} = 0.$$

The first holds if and only if $k = 1$, i. e., if and only if the lines AD and BC are perpendicular. The second does not depend on k and is never true so that the triangles $\mathcal{ABC}$ and Φ are not paralogic. The similar argument holds for the pairs $(\mathcal{ABC}, \Psi)$, $(\mathbf{ABC}, \Phi)$ and $(\mathbf{ABC}, \Psi)$. $\square$

It follows from the part (vi) of Theorem 35 that the points $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are the images of the points I, I_a, I_b and I_c under the homothety $h\left(D, \frac{3}{4}\right)$. Since I is the orthocenter of the extriangle $I_a I_b I_c$, we infer that the quadruple $\{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{I}\}$ is orthocentric.

The variable triangle $\mathbb{ABC}$ has many additional nice properties that we now describe. They are all the consequence of the fact that it is homothetic with the extriangle for all positions of the point D.

(1) The triangles $\mathbb{ABC}$ and Φ are homologic and their homology center is the point D.
(2) The triangles $\mathbb{ABC}$ and Ψ are homologic and their homology center is the $-\frac{3(\zeta^2+d^2+3)}{2\,h}$-point of the segment joining the point D with the central point X_{57}, the isogonal conjugate of the Mittenpunkt X_9.
(3) The triangles $\mathbb{ABC}$ and ABC are orthologic. Moreover, $o^{\mathbb{ABC}}_{ABC} = I$ and $o^{ABC}_{\mathbb{ABC}}$ is the 3-point of the segment joining the point D with the Bevan point X_{40}.

The triangle $\mathbb{ABC}$ is also orthologic with other triangles associated with the triangle ABC. For example, with the anticomplementary triangle $A_a B_a C_a$ (on the reflections of the vertices in the midpoints of opposite sides), the Euler triangle $A_e B_e C_e$ (on the midpoints of the segments joining the vertices with the orthocenter), the complementary triangle $A_g B_g C_g$ (on the midpoints of the sides), the extriangle Φ, the cevian triangle $A_i B_i C_i$ of the incenter, the triangle $A_j B_j C_j$ (on the touching points of the excircles with the sides), the triangle $A_m B_m C_m$ (on the outer Gergonne points) and the pedal triangle Ψ of the incenter.

Some of the orthology centers for these pairs are interesting central points of the triangle ABC. For example, $o^{\mathbb{ABC}}_{A_i B_i C_i} = o^{\mathbb{ABC}}_{\Phi} = X_1 = I$ (the incenter), $o^{\mathbb{ABC}}_{A_a B_a C_a} = o^{\mathbb{ABC}}_{A_m B_m C_m} = X_8 = \mathfrak{N}$ (the Nagel point), $o^{\mathbb{ABC}}_{A_g B_g C_g}$ is the Spieker point X_{10} (the incenter of the complementary triangle), $o^{\mathbb{ABC}}_{A_e B_e C_e}$ is the intersection of the central lines $X_1 X_4$ and $X_2 X_{40}$, $o^{\mathbb{ABC}}_{A_j B_j C_j} = X_{72}$ and $o^{\mathbb{ABC}}_{\Psi} = X_{65}$.

On the other hand, $o_{\mathbb{ABC}}^{ABC} = o_{\mathbb{ABC}}^{A_aB_aC_a} = o_{\mathbb{ABC}}^{A_eB_eC_e} = o_{\mathbb{ABC}}^{A_gB_gC_g}$. Moreover, $o_{\mathbb{ABC}}^{\Phi} = o_{\mathbb{ABC}}^{A_mB_mC_m} = o_{\mathbb{ABC}}^{A_qB_qC_q}$ and $o_{\mathbb{ABC}}^{\Phi}$ is the 3-point of the segment joining the point D with the incenter I and $o_{\mathbb{ABC}}^{A_iB_iC_i}$ is the 3-point of the segment joining the point D with the circumcenter O.

21. Properties of quadrangles q_1, q_2, q_3 and q_4

Let us call the quadrangle *tame* provided it has equal sums of squares of opposite sides.

We shall now show that $q_1 = PQST$, $q_2 = PVSY$, $q_3 = UVXY$ and $q_4 = QUTX$ are tame quadrangles. There are many more such tame quadrangles from the Thébault's centers. Moreover, the quadrangles $P_oQ_oS_oT_o$ and $U_oV_oX_oY_o$ have equal symmetric products of four sides.

Teorem 47. *The quadrangles q_1, q_2, q_3 and q_4 are tame and*

$$|P_oQ_o| \cdot |S_oT_o| \cdot |U_oY_o| \cdot |V_oX_o| = |P_oT_o| \cdot |Q_oS_o| \cdot |U_oV_o| \cdot |X_oY_o|.$$

Proof. The formula $|MN|^2 = (x_M - x_N)^2 + (y_M - y_N)^2$ gives us easily $|ST|^2 = \frac{r^2K^2\zeta^2[\zeta^2K^+ + 2\zeta kL + (d^2+h^2-2\zeta^2)k^2]}{h^2k^4}$, $|PT|^2 = \frac{r^2K\varphi_-\psi_+(k^2\zeta^2+1)}{h^2k^4}$, $|QS|^2 = \frac{r^2Kf_+^2g_-^2(k^2+\zeta^2)}{h^2k^4}$ and $|PQ|^2 = \frac{r^2K^2[K^+ + 2dkL + (d^2+h^2-2)k^2]}{h^2k^4}$. From this one can derive the algebraic identity $|PQ|^2 + |ST|^2 = |PT|^2 + |QS|^2$ which proves that q_1 is a tame quadrangle. For the other quadrangles q_2, q_3 and q_4 the proof is similar. For the long identity, we actually prove that both sides have equal squares. □

Next, we find a situation when the quadrangles q_1, q_2, q_3 and q_4 are cyclic.

Teorem 48. *If the line AD and the sideline BC are perpendicular, then the quadrangles q_1, q_2, q_3 and q_4 are cyclic. Their circumcenters O_{q_1}, O_{q_2}, O_{q_3} and O_{q_4} are vertices of a square with the side $2\sqrt{2}\,R$ such that $O_{q_2}O_{q_4}$ is parallel to the sideline BC.*

Proof. Let us recall that $k = 1$ if and only if the lines AD and BC are perpendicular. Hence, the circumcenter of the triangle PQS has the coordinates $\frac{r}{h}\,(fg^-, -h^2)$ and is equidistant from the points P and T. It follows that q_1 is a cyclic quadrangle. Similarly, the circumcenter of the triangle UVX has the coordinates $\frac{r}{h}\,(fg^-, z^2)$ and is equidistant from the points U and Y so that the quadrangle q_3 is also cyclic. In fact, this argument shows that these quadrangles are non-degenerate and cyclic if and only if $k = 1$ (see [3, Remark 6] for $PQST$). For the quadrangles q_2 and q_4 these equivalences do not hold but for $k = 1$ they are also cyclic. The remaining claims have easy proofs by direct computation of coordinates and use of the distance formula. □

The centroids G_{q_1}, G_{q_2}, G_{q_3} and G_{q_4} of the quadrangles q_1, q_2, q_3 and q_4 are vertices of an interesting rectangle whose diagonals are never shorter than the diameter of the circumcircle of the triangle ABC.

Teorem 49. *The quadrangle $G_{q_1}G_{q_2}G_{q_3}G_{q_4}$ is a rectangle with sides $|G_{q_1}G_{q_2}| = R\,k\,\sqrt{K}$ and $|G_{q_2}G_{q_3}| = \frac{R\sqrt{K}}{k^2}$ and the diagonals $\frac{R\,K\sqrt{k^2L+1}}{k^2}$. Hence, $|G_{q_1}G_{q_3}| \geq 2\,R$.*

Proof. The centroids G_{q_1} and G_{q_1} have the coordinates

$$-\frac{r}{4hk^2}\left(hk(hL-2zk), \zeta^+L^2+(2h^2+d\bar{h}k)L+2h^2\right)$$

and

$$\frac{r}{4hk^2}\left(zk(zL+2hk), (f^2+g^2)L^2+(2z^2-d\bar{h}k)L+2z^2\right).$$

The coordinates of G_{q_2} and G_{q_4} are similar. It is now routine to check that $G_{q_1}G_{q_2}G_{q_3}G_{q_4}$ is a rectangle and to compute the lengths of its sides and diagonals and prove the above inequality. □

The following results explores when the diagonals of the quadrangle $G_{q_1}G_{q_2}G_{q_3}G_{q_4}$ have their minimal value $2\,R$.

Teorem 50. *The following are equivalent: (i) $|G_{q_1}G_{q_3}| = 2\,R$, (ii) the line $G_{q_1}G_{q_3}$ is perpendicular to the line BC, (iii) the line $G_{q_1}G_{q_3}$ is parallel to the line AD and (iv) the line AD is perpendicular to the line BC.*

Proof. The only singular value for the function $k \mapsto \frac{K^2(k^2L+1)}{k^4}$ is $k = 1$. This shows that (i) and (iv) are equivalent.

The line $G_{q_1}G_{q_3}$ is perpendicular to the line BC if and only if G_{q_1} and G_{q_3} have equal abscises. However, $x_{G_{q_3}} - x_{G_{q_1}} = \frac{rf^+g^+L}{4hk}$. Hence, again $k = 1$ and we conclude that (ii) and (iv) are equivalent.

Finally, the line $G_{q_1}G_{q_3}$ is parallel to the line AD if and only if they have equal slopes, i. e., if and only if $k = 1$. Therefore, (iii) and (iv) are also equivalent. □

The following three theorems consider the Newton lines of the quadrangles q_1 and q_3. Recall that the Newton line joins the midpoints of the diagonals of a quadrangle and its centroid.

Let $\zeta^+ = \zeta^2 + 1$.

Teorem 51. *The following are equivalent: (i) the Newton lines of the quadrangles q_1 and q_3 are parallel, (ii) the point D lies on the line joining the centroids G_{q_1} and G_{q_3} of the quadrangles q_1 and q_3 and (iii) the point D is the midpoint of the segment BC.*

Proof. The equations of the Newton lines of $PQST$ and $UVXY$ are

$$2[\zeta^+L+d\bar{h}k]x - 2h^2ky = r[z\zeta^+L+(h^3+d\bar{h}z)k]$$

and

$$2h[(f^2+g^2)L-d\bar{h}k]x - 2hkz^2y = rz[h(f^2+g^2)L-(z^3+dh\bar{h})k].$$

The condition for these lines to be parallel is $4f^+g^+h(2\zeta L+d\bar{h}k) = 0$. In order to prove the equivalence of (i) and (iii), it remains to notice that the distance between the point D and the midpoint of the segment BC is $\frac{r|2\zeta L+d\bar{h}k|}{2hk}$.

Let $K^+ = k^4+1$. The line $G_{q_1}G_{q_3}$ has the equation

$$4hk(K^+x - kLy) = r[2\zeta L^3 + (d\bar{h} + 2hz)kL^2 + 4hk^3z].$$

When we substitute $x = x_D$ and $y = y_D = 0$ and move the free term to the left, we get $rK^2(2\zeta L + d\bar{h}k)$. This shows that (ii) and (iii) are equivalent. □

Teorem 52. *The Newton lines of the quadrangles q_1, q_3, q_2 and q_4 go through the points Z_2, Z_1, R_1 and R_2, respectively.*

Proof. The coordinates of the point Z_2 are $\frac{r}{2}(z, -h)$. The equation of the Newton line of $PQST$ is

$$2[\zeta^+L + d\bar{h}k]x - 2h^2ky = r[z\zeta^+L + (h^3 + d\bar{h}z)k].$$

It is now easy to check that Z_2 is on it. The other claims in this theorem have similar proofs. □

Recall that the central point X_{69} is the symmedian point of the anticomplementary triangle. It is also the isotomic conjugate of the orthocenter.

Teorem 53. *The locus of intersections of Newton lines of the quadrangles q_1 and q_3 is the perpendicular to the line AX_{69} from the intersection of the line BC with the perpendicular in the vertex A to the line AO.*

Proof. The coordinates of the intersection M of the Newton lines of the quadrangles q_1 and q_3 are (see the proof of Theorem 50)

$$\tfrac{r}{2h(2\zeta L+d\bar{h}k)}\left(2ghzf_+\varphi_-, (4\zeta^2 + (\zeta^2+1)(f^2+g^2))L - d(h^2-z^2)\bar{h}k\right).$$

By eliminating the variable k from the equations $x = x_M$ and $y = y_M$, we get the equation $dh\bar{h}x + 2\zeta hy = rg^2(f^+)^2$ of the locus. Since the central point X_{69} has the coordinates

$$\tfrac{rf^2}{h((f^2+g^2)(\zeta^2+1)+\zeta f^-g^-)}\left(2f^-g(g^4+1) + f(g^-)^2, -2g^2(h^2-z^2)\right),$$

it is now easy to check that the locus is the line described in the statement of the theorem. □

Teorem 54. *The diagonals of the van Aubel pseudo-squares of the quadrangles $PQST$, $UVXY$, $PQUY$ and $STXV$ are on angle bisectors of the line AD and the perpendicular at the point D onto the line BC.*

Proof. The angle bisectors of the line AD and the perpendicular at the point D onto the line BC have the equations

$$(k-1)x - (k+1)y = (k-1)x_D, \tag{60}$$

and

$$(k+1)x + (k-1)y = (k+1)x_D. \tag{61}$$

The coordinates of the centers M and N of the negative squares on the segments QS and TP are $\frac{rf_+}{2hk^2}(u_M, v_M)$ and $\frac{r\varphi_-}{2hk^2}(u_N, v_N)$, where $u_M = (k-1)(k^2+g\zeta)+k(k+1)(\zeta-g)$, $v_M = (k+1)g_-(\zeta-k)$, $u_N = (k-1)(k^2g\zeta-1)+k(k+1)(\zeta+g)$ and $v_N = (k+1)\psi_+(1-\zeta k)$. It is now easy to check that these coordinates of both M and N satisfy the equation (60). The similar argument applies to the centers of the other negative and positive squares on sides of the quadrangles $PQST$, $UVXY$, $PQUY$ and $STXV$. □

Many other quadrangles from Thébault's centers $P,\dots,Y$ share the above properties with the quadrangles q_1, q_2, q_3 and q_4.

22. Lines concurrent in the points R_1, R_2, Z_1 and Z_2

Teorem 55. *The lines P_oP', Q_oQ', S_oS' and T_oT' concur in the point Z_2. The lines U_oU', V_oV', X_oX' and Y_oY' concur in the point Z_1.*

Proof. The line P_oP' has the equation $hkx + (2-dk)y = rh\varphi_-$. It is now easy to check that Z_2 is on the line P_oP'. The other claims in this theorem have similar proofs. □

Let the perpendicular bisector of the segment AD intersect the circumcircle o in the points R_1 and R_2 such that R_1 is closer to A than to B while R_2 is closer to B than to A. Hence,

$$|AR_1|^2 - |BR_1|^2 = |BR_2|^2 - |AR_2|^2 = \frac{4\,a\,R\,k}{K}.$$

Note that R_1 is the midpoint of $\mathfrak{I}_a\mathfrak{I}_e$ and R_2 is the midpoint of $\mathfrak{I}_a\mathfrak{I}_e$.

Teorem 56. *The lines P_oP'', S_oS'', V_oV'' and Y_oY'' concur in the point R_1. The lines Q_oQ'', T_oT'', U_oU'' and X_oX'' concur in the point R_2.*

Proof. The coordinates of the point R_1 are $\frac{r(kz+h)}{2hK}(hk+z, kz-h)$. The line P_oP'' has the equation

$$(dk+h-2)kx + (hk^2-dk+2)y = r\varphi_-(kz+h).$$

It is now easy to check that R_1 is on the line P_oP''. The other claims in this theorem have similar proofs. □

23. The points that envelop P_oQ_o, S_oT_o, U_oV_o and X_oY_o

In this section we show that the lines P_oQ_o, S_oT_o, U_oV_o and X_oY_o pass through the fixed points of the triangle ABC. The following is the part (a) of Proposition 9 in [3].

Teorem 57. *The central point X_{56} of the triangle ABC (i. e., the isogonal conjugate of the Nagel point X_8) lies on the line P_oQ_o.*

Proof. The coordinates of the point X_{56} are

$$\frac{r}{d^2+h^2+4}\left(f^- g(h-1)+f(f^2+3), h^2\right).$$

The line P_oQ_o has the equation

$$2(dk+L)hx+[(\bar{h}^2-z^2+4)k-2dL]y=2hrf_+\varphi_-.$$

It is now easy to check that X_{56} is on the line P_oQ_o. □

Of course, there are three related results where the central points X_{56} of the triangles BCE, ABE and ACE appear. Since the point E varies, these points are not fixed. They lie on the lines U_oY_o, P_oY_o and Q_oU_o, respectively.

Let N_a^*, N_b^* and N_c^* be the points on the lines AX_{55}, BX_{55} and CX_{55} with the coordinates $\frac{rg}{d^2+5\zeta^2-2\zeta+1}\left(g^2(3f^+-2)+f^-(2\zeta-1), -2fh^2\right)$, $\frac{rgz}{h(z^2+h^2+4g^2)}\left(f(g^+f^2+3g^-+2)+2f^-g, 2z^2\right)$ and $\frac{rz}{h(z^2+h^2+4f^2)}\left(f^-(g^2+2\zeta)-3f^++2, 2fz^2\right)$. Notice that N_a^*, N_b^* and N_c^* are isogonal conjugates of the associated Nagel points N_a, N_b and N_c with coordinates $-\frac{r}{h}\left(fg^++2g, 2g^2\right)$, $\frac{rf}{h}\left(g^++2\zeta, -2f\right)$ and $\frac{rf}{h}\left(2\zeta-g^+, 2g\zeta\right)$.

Teorem 58. *The lines S_oT_o, U_oV_o and X_oY_o pass through the points N_a^*, N_b^* and N_c^*, respectively.*

Proof. The line S_oT_o has the equation

$$2(dk+\zeta L)hx+[2d\zeta L-(5\zeta^2-f^2-g^2+1)k]y=2g^2hrf_+\varphi_-.$$

It is now easy to check that N_a^* is on the line S_oT_o. This is the part (b) of Proposition 9 in [3]. The remaining two claims are proved similarly. □

24. Perpendiculars passing through the point D

The point D is very important for the Thébault's configuration. This is supported by four similar results in this section about D being on some interesting perpendiculars to sides of the four orthocentric quadrangles from the incenters and the excenters.

Teorem 59. *If $k \neq k_0$, then the point D lies on the perpendicular from the intersection of the lines PQ and ST onto the line II_a. If $k \neq m_0$, then the point D lies on the perpendicular from the intersection of the lines UV and XY onto the line I_bI_c. These perpendiculars are perpendicular.*

Proof. Let $k \neq k_0$. The intersection M of the lines PQ and ST has the coordinates $\frac{rf_+\varphi_-}{hkp_2}\left(g^+hk, dg_-\psi_+\right)$. Hence, the perpendicular from M onto the line II_a has the equation $hk(dx+\bar{h}y)=rdgf_+\varphi_-$. It is now obvious that this perpendicular goes through the point D.

Let $k \neq m_0$. The intersection N of the lines UV and XY has the coordinates $-\frac{rf_+\varphi_-}{hks_2}\left(g^+zk, \bar{h}g_-\psi_+\right)$. Hence, the perpendicular from N onto the line I_bI_c has the equation $hk(\bar{h}x-dy)=rg\bar{h}f_+\varphi_-$. It is now clear that this perpendicular goes through the point D. □

Teorem 60. *Let $D \neq B, C$. The point D lies on the perpendicular from the intersection of the lines PT and QS onto the line $J_b J_c$. If, in addition, $|AB| \neq |AC|$, then the point D lies on the perpendicular from the intersection of the lines UY and VX onto the line JJ_a. These perpendiculars are perpendicular.*

Proof. The intersection M of the lines PT and QS has the coordinates $\frac{rg}{hhk}\,(f^{+}hk, -f s_2)$. Hence, the perpendicular from M onto the line $J_b J_c$ has the equation $hk(s_2 x - p_2 y) = rgs_2 f_+ \varphi_-$. It is now obvious that this perpendicular goes through the point D.

Let $|AB| \neq |AC|$ (i. e., let $d \neq 0$). The intersection N of the lines UY and VX has the coordinates $\frac{rg}{dhk}\,(f^{+}zk, fp_2)$. Hence, the perpendicular from N onto the line JJ_a has the equation $hk(p_2 x + s_2 y) = rgp_2 f_+ \varphi_-$. It is now clear that this perpendicular goes through the point D. □

Teorem 61. *Let $D \neq B, C$. The point D lies on the perpendicular from the intersection of the lines PY and QX onto the line $\mathfrak{I}_a \mathfrak{I}_b$. The point D lies on the perpendicular from the intersection of the lines UT and SV onto the line $\mathfrak{I}\mathfrak{I}_e$. These perpendiculars are perpendicular.*

Proof. The intersection M of the lines PY and QX has the coordinates $\frac{r}{h\,k}\,((g\,f^{-} - 2\,f)k, f(2\,g\,k - L))$. Hence, the perpendicular from M onto the line $\mathfrak{I}_a \mathfrak{I}_b$ has the equation

$$hk[(2\,g\,k - L)x + (g\,L + 2\,k)y] = rg(2\,g\,k - L)f_+ \varphi_-.$$

It is now obvious that this perpendicular goes through the point D.

The intersection N of the lines UT and SV has the coordinates $\frac{rg}{hk}\,((f^{-} + 2\,\zeta)k, f(g\,L + 2\,k))$. Hence, the perpendicular from N onto the line $\mathfrak{I}\mathfrak{I}_e$ has the equation

$$hk[(g\,L + 2\,k)x - (2\,g\,k - L)y] = rg(g\,L + 2\,k)f_+ \varphi_-.$$

It is now clear that the point D lies on this perpendicular. □

Teorem 62. *The point D, different from the vertex C, lies on the perpendicular from the intersection of the lines SY and QU onto the line $\mathfrak{I}_c \mathfrak{I}_e$. The point D lies on the perpendicular from the intersection of the lines PV and TX onto the line $\mathfrak{I}\mathfrak{I}_a$. These perpendiculars are perpendicular.*

Proof. The intersection M of the lines SY and QU has the coordinates $\frac{r\,f_+\,\varphi_-}{h\,k\,K}\,(g^{+}\,k, k^2 - g^2)$. Hence, the perpendicular from M onto the line $\mathfrak{I}_c \mathfrak{I}_e$ has the equation $hk(g_+\,x + \psi_-\,y) = r\,g\,f_+\,\varphi_-\,g_+$. It is now obvious that this perpendicular goes through the point D.

The intersection N of the lines PV and TX has the coordinates $-\frac{r\,f_+\,\varphi_-}{h\,k\,K}\,(g^{+}\,k, \psi_-\,\psi_+)$. Hence, the perpendicular from N onto the line $\mathfrak{I}\mathfrak{I}_a$ has the equation $hk(\psi_-\,x - g_+\,y) = r\,g\,f_+\,\varphi_-\,\psi_-$. It is now clear that the point D lies on this perpendicular. □

25. Certain pairs of perpendicular lines

Teorem 63. *The lines DI, DI_a, DI_b and DI_c are perpendicular to the lines ST, PQ, XY and UV, respectively.*

Proof. The lines DI and ST have the equations $hkx + p_{I'}y = rgf_+\varphi_-$ and $p_{I'}x - hky = rg^2 f_+\varphi_-$. It follows that they are perpendicular. The proofs for the remaining three pairs of lines are similar. □

Teorem 64. *The lines DJ, DJ_a, DJ_b and DJ_c are perpendicular to the lines VX, UY, QS and PT, respectively.*

Proof. The lines DJ and VX are perpendicular because they have the equations $kzx + (gL - d)y = rgzf_+\varphi_-$ and $(gL - d)x - kzy = -rgzf_+^2$. The proofs for the remaining three pairs of lines are analogous. □

In a similar way it is possible to prove the following:

Teorem 65. *(i) The lines $D\mathfrak{J}$, $D\mathfrak{J}_a$, $D\mathfrak{J}_b$ and $D\mathfrak{J}_e$ are perpendicular to the lines SV, XQ, PY and UT, respectively.*

(ii) The lines $D\mathfrak{J}$, $D\mathfrak{J}_a$, $D\mathfrak{J}_c$ and $D\mathfrak{J}_e$ are perpendicular to the lines XT, PV, UQ and SY, respectively.

26. Special relations for products of sides and diagonals

In this section, we consider some consequences of equalities among the products of lengths of sides and diagonals of some quadrangles from the eight centers of Thébault's circles.

Teorem 66. *If neither the angle B nor the angle C is right, then $|PQ||ST| = |UV||XY|$ holds if and only if the line AD is perpendicular either to the line AB or to the line AC.*

The equality $|PS||QT| = |UX||VY|$ holds if and only if either the angle A is right or $|AB| \neq |AC|$ and the line AD is perpendicular to the line AO.

The equality $|PT||QS| = |UY||VX|$ holds if and only if either $D = B$, $D = C$, $B = 90°$ or $C = 90°$.

The equality $|PU||QV| = |SX||TY|$ holds if and only if either $D = B$ or the angle B is right.

The equality $|PX||QY| = |SU||TV|$ holds if and only if either $D = C$ or the angle C is right.

Proof. The difference $|PQ|^2|ST|^2 - |UV|^2|XY|^2$ factors as the quotient $\frac{(rK)^4\zeta^2 f^+ g^+ (f^- L + 4fk)(g^- L + 4gk)}{h^4 k^6}$. When the angle B is not right, then the factor $f^- L + 4fk$ vanishes if and only if the line AD is perpendicular to the line AB. Similarly, when the angle C is not right, then the factor $g^- L + 4gk$ vanishes if and only if the line AD is perpendicular to the line AC.

The difference $|PS|^2|QT|^2 - |UX|^2|VY|^2$ simplifies to the quotient $\frac{r^4 K^2 f^+ g^+ (\bar{h} - dk)^2 (\bar{h}k + d)^2 (h^2 - z^2)}{(hk)^4}$. The factor $h^2 - z^2$ vanishes if and only if the angle A is right. When $|AB| \neq |AC|$, the factor $(\bar{h} - dk)^2(\bar{h}k + d)^2$ vanishes if and only if the line AD is perpendicular to the line AO.

The difference $|PT|^2|QS|^2 - |UY|^2|VX|^2$ is $\frac{r^4 f^+ f^- g^+ g^- (K f_+ g_+ \varphi_- \psi_+)^2}{h^4 k^6}$. Its numerator vanishes only for $k = \frac{1}{f}$ (when $D = B$), $k = g$ (when $D = C$), $f = 1$ (when $B = 90°$) and $g = 1$ (when $C = 90°$).

The last two claims have similar (somewhat simpler) proofs. $\square$

27. Diagonal points

The diagonal points in quadrangles are two intersections of pairs of opposite sidelines and the intersection of diagonals. In this section we consider these points for some quadrangles from the eight centers of Thébault's circles.

The only assumption in the following result is that $|AB| \neq |AC|$.

Teorem 67. *The intersections M_0 and N_0 of the lines PT and QS and of the lines UY and VX lie on the perpendicular to the line AD in the point A. The point D is on the circle $k_{M_0 N_0}$. When the lines AO and BC are not parallel, then its center lies on the line BC if and only if the circumcenter O is on the line AD.*

Proof. The coordinates of the points M_0 and N_0 are $\frac{rg}{h\bar{h}k}\left(f^+ hk, -f s_2\right)$ and $\frac{rg}{dhk}\left(f^+ kz, f p_2\right)$. It is now easy to check that they satisfy the equation $h(Lx - 2ky) = rg(\varphi_-^2 - f_+^2)$ of the perpendicular to the line AD in the point A.

The coordinates of the midpoint M of the segment $M_0 N_0$ are

$$\tfrac{rg}{2dh\bar{h}k}\left(2(f^+)^2 gk, f[(\bar{h} + d)k - \bar{h} + d][(\bar{h} - d)k + \bar{h} + d]\right).$$

Hence, $|MD|^2 = |MM_0|^2$. This implies that the point D is on the circle $k_{M_0 N_0}$.

Finally, when the lines AO and BC are not parallel, then the intersection N of these lines has the coordinates $\left(\frac{rgg^-(f^+)^2}{h(\bar{h}^2 - d^2)}, 0\right)$. It remains to observe that $|ND| = \frac{r\zeta|(\bar{h}+d)k - \bar{h} + d||(\bar{h} - d)k + \bar{h} + d|}{hk|\bar{h}^2 - d^2|}$. $\square$

In the following result we assume that $|AB| \neq |AC|$ and that the line PQ is not parallel to the line ST and that the line UV is not parallel to the line XY. In other words, the point D can not be the intersections of the line BC with the lines AI and $I_b I_c$.

Teorem 68. *The intersections M and N of the lines PQ and ST and of the lines UV and XY lie on the perpendicular to the line AD in the point E. The point D is on the circle k_{MN}. The following are equivalent: (i) the midpoint of the segment MN lies on the line BC, (ii) the lines MN_0 and NM_0 are perpendicular, (iii) the point D is on the line MN_0, (iv) the point D is on the line NM_0, (v) the relation $|MN|^2 + |M_0 N_0|^2 = |MM_0|^2 + |NN_0|^2$ holds and (vi) either $D = B$, $D = C$ or the circumcenter O is on the line AD.*

Proof. The coordinates of the points M and N are $\frac{rf_+\varphi_-}{hkp_2}\,(g^+hk, dg_-\psi_+)$ and $-\frac{rf_+\varphi_-}{hks_2}\left(g^+zk, \bar{h}g_-\psi_+\right)$. It is now easy to check that they satisfy the equation $h(Lx-2ky)=rf_+\varphi_-g^-$ of the perpendicular to the line AD in the point E.

The coordinates of the midpoint m of the segment MN are

$$\tfrac{rf_+\varphi_-}{2hkp_2s_2}\left(2f_+\varphi_-(g^+)^2k, g_-\psi_+[(\bar{h}+d)k-\bar{h}+d][(\bar{h}-d)k+\bar{h}+d]\right).$$

Hence, $|mD|^2=|mM|^2$. This implies that the point D is on the circle k_{MN}.

Finally, in order to prove the equivalence of the six statements, it suffices to notice that each condition described analytically involves as factors φ_-, g_-, $(\bar{h}+d)k-\bar{h}+d$ and $(\bar{h}-d)k+\bar{h}+d$. For example, the sum $|MN|^2+|M_0N_0|^2-|MM_0|^2-|NN_0|^2$ is equal

$$\frac{2r^2\zeta f_+\varphi_-g_-\psi_+[(\bar{h}+d)k-\bar{h}+d]^2[(\bar{h}-d)k+\bar{h}+d]^2}{d\bar{h}h^2k^2p_2s_2}.$$

$\square$

Teorem 69. *Let $k\neq\frac{h}{z}$. The intersections $\mathfrak{M}_0$ and $\mathfrak{N}_0$ of the lines PY and SV and of the lines TU and QX lie on the perpendicular to the line BC in the point C. The point D is on the circle $k_{\mathfrak{M}_0\mathfrak{N}_0}$.*

Proof. The points $\mathfrak{M}_0$ and $\mathfrak{N}_0$ have the abscises $r\,z$ (the same as that of the point C) and the ordinates $\frac{r\,f\,g_-\,\psi_+(h\,k+z)}{h\,k(z\,k-h)}$ and $\frac{r\,f\,g_-\,\psi_+(h-z\,k)}{h\,k(h\,k+z)}$.

The ordinate of the midpoint $\mathfrak{M}$ of the segment $\mathfrak{M}_0\mathfrak{N}_0$ is

$$\frac{r\,f\,g_-\,\psi_+[(h+z)k+z-h][(h-z)k+z+h]}{2\,h\,k(h\,k+z)(z\,k-h)}.$$

Hence, $|\mathfrak{M}D|^2=|\mathfrak{M}\mathfrak{M}_0|^2$. This implies that the point D is on the circle $k_{\mathfrak{M}_0\mathfrak{N}_0}$. $\square$

Teorem 70. *Let $k\neq\frac{z}{h}$. The intersections $\mathcal{M}_0$ and $\mathcal{N}_0$ of the lines PV and SY and of the lines QU and TX lie on the perpendicular to the line BC in the point B. The point D is on the circle $k_{\mathcal{M}_0\mathcal{N}_0}$.*

Proof. The points $\mathcal{M}_0$ and $\mathcal{N}_0$ have the abscises 0 (the same as that of the point B) and the ordinates $\frac{r\,g\,f_+\,\varphi_-(z-h\,k)}{h\,k(z\,k+h)}$ and $\frac{r\,g\,f_+\,\varphi_-(z\,k+h)}{h\,k(h\,k-z)}$.

The ordinate of the midpoint $\mathcal{M}$ of the segment $\mathcal{M}_0\mathcal{N}_0$ is

$$\frac{r\,g\,f_+\,\varphi_-[(h+z)k+h-z][(z-h)k+z+h]}{2\,h\,k(h\,k-z)(z\,k+h)}.$$

Hence, $|\mathcal{M}D|^2=|\mathcal{M}\mathcal{M}_0|^2$. This implies that the point D is on the circle $k_{\mathcal{M}_0\mathcal{N}_0}$. $\square$

28. The role of X_{40} and X_{20}

Two results in this section use the Longchamps point X_{20} and the Bevan point X_{40}. They give some consequences of certain positions of these central points with respect to the centers of Thébault circles.

Teorem 71. *(i) The relation* $\cos A = \cos B + \cos C$ *for the angles of the triangle* ABC *holds if and only if the reflection of the Bevan point* X_{40} *in the line* BC *lies on the line* ST.

(ii) The relation $\cos B + \cos C = 1$ *holds if and only if the reflection of the Bevan point* X_{40} *in the perpendicular bisector of the segment* BC *lies on the line* PQ.

(iii) The Bevan point X_{40} *never lies on the line* ST.

Proof. The Bevan point X_{40} has the coordinates $\frac{r}{2h}(2hg, 1+z^2-\zeta^2)$. Its reflection in the line BC (the x-axis!!) will be on the line ST with the equation $(\zeta L + dk)x - hky = r f_+ \varphi_- g^2$ if and only if

$$3\,\zeta^2 - 2\,\zeta - z^2 - 1 = 0.$$

When we substitute $f = \cot\frac{B}{2}$ and $g = \cot\frac{C}{2}$, this condition is seen equivalent with the identity $\cos A = \cos B + \cos C$. This proves the part (i). The proof of (ii) is similar. Finally, in order to prove (iii), when we substitute the coordinates of X_{40} into the above equation of the line ST and move all terms to the left side, we obtain $-\frac{r f^+ g^+ k}{2} = 0$ that is never true. $\square$

Teorem 72. *(i) Let* $p_2 \neq 0$. *The angle* A *in the triangle* ABC *is right if and only if the Longchamps point* X_{20} *is on the perpendicular to the line* AD *through the intersection* M *of the lines* PQ *and* ST.

(ii) Let $s_2 \neq 0$. *The angle* A *in the triangle* ABC *is right if and only if the Longchamps point* X_{20} *is on the perpendicular to the line* AD *through the intersection* N *of the lines* UV *and* XY.

Proof. (ii) When we substitute the coordinates $\frac{r}{h}(f g^-, z^2 - \zeta^2 - 1)$ of X_{20} into the equation $h(Lx - 2ky) = r f_+ \varphi_- g^-$ of the perpendicular to the line AD through the intersection N and move all terms to the left side, we obtain $rk(h^2 - z^2) = 0$ that is equivalent with the condition that the angle A is right. $\square$

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