DETERMINANTS AND PERMANENTS OF MATRICES FROM FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper we shall continue to study from [4], for $k = -1$ and $k = 5$, the infinite sequences of triples $A = (F_{2n+1},\ F_{2n+3},\ F_{2n+5})$, $B = (F_{2n+1},\ 5F_{2n+3},\ F_{2n+5})$, $C = (L_{2n+1},\ L_{2n+3},\ L_{2n+5})$, $D = (L_{2n+1},\ 5L_{2n+3},\ L_{2n+5})$ with the property that the product of any two different components of them increased by k are squares. The sequences A and B are built from the Fibonacci numbers F_n while the sequences C and D from the Lucas numbers L_n. We show many interesting properties of various matrices with rows from these sequences and give methods how to compute some of their generalized determinants and permanents. We also study numerous tetrahedra with vertices from these sequences concentrating on their volumes and centroids. Some of our theorems have versions for the associated sequences $\widetilde{A} = (F_{2n+4}, F_{2n+3}, F_{2n+2})$, $\widetilde{B} = (L_{2n+4}, F_{2n+3}, L_{2n+2})$, $\widetilde{C} = (L_{2n+4}, L_{2n+3}, L_{2n+2})$ and $\widetilde{D} = (5F_{2n+4}, L_{2n+3},\ 5F_{2n+2})$.

1. Introduction

For integers a, b and c, let us write $a \overset{b}{\sim} c$ provided $a + b = c^2$. For the triples $A = (a, b, c)$, $D = (d, e, f)$ and $\widetilde{A} = (\widetilde{a}, \widetilde{b}, \widetilde{c})$ the notation $A \overset{D}{\sim} \widetilde{A}$ means that $b\,c \overset{d}{\sim} \widetilde{a}$, $c\,a \overset{e}{\sim} \widetilde{b}$ and $a\,b \overset{f}{\sim} \widetilde{c}$. When $D = (k, k, k)$, let us write $A \overset{k}{\sim} \widetilde{A}$ for $A \overset{D}{\sim} \widetilde{A}$. Hence, A is the $D(k)$-triple (see [1]) if and only if there is a triple $\widetilde{A}$ such that $A \overset{k}{\sim} \widetilde{A}$.

In the paper [4] we constructed infinite sequences $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ and $\beta = \{\beta(n)\}_{n=0}^{\infty}$ of $D(-1)$-triples and $\gamma = \{\gamma(n)\}_{n=0}^{\infty}$ and $\delta = \{\delta(n)\}_{n=0}^{\infty}$ of $D(5)$-triples. Here, $\alpha(n) = A = (F_{2n+1},\ F_{2n+3},\ F_{2n+5})$, $\beta(n) = B = (F_{2n+1},\ 5F_{2n+3},\ F_{2n+5})$ and $\gamma(n) = C = (L_{2n+1},\ L_{2n+3},\ L_{2n+5})$, $\delta(n) = D = (L_{2n+1},\ 5L_{2n+3},\ L_{2n+5})$, where the Fibonacci and Lucas sequences of natural numbers F_n and L_n are defined by the recurrence relations $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ and $L_0 = 2$, $L_1 = 1$,

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$L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. For an integer k, let us use π_k, ϱ_k, $\mathfrak{p}_k$ and $\mathfrak{r}_k$ for F_{2n+k}, L_{2n+k}, F_{4n+k} and L_{4n+k}.

The numbers F_k make the integer sequence $A000045$ from [8] while the numbers L_k make $A000032$.

The goal of this article is to further explore the properties of the sequences α, β, γ and δ. Each member of these sequences is an Euler $D(-1)$- or $D(5)$-triple (see [2] and [3]) so that many of their properties follow from the properties of the general (pencils of) Euler triples. It is therefore interesting to look for those properties in which at least two of the sequences appear. The paper [5] presented several results of this kind giving many squares from the components, various sums and products of the sequences α, β, γ and δ. On the other hand, the reference [6] considers various matrices with rows from these sequences and explore many of their properties. In this article we continue with the (generalized) determinants and permanents of these matrices and the computations of volumes of numerous tetrahedra built on these sequences.

Some of our theorems have versions for the associated sequences $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\delta}$, where $\widetilde{\alpha}(n) = \widetilde{A} = (\pi_4, \pi_3, \pi_2)$, $\widetilde{\beta}(n) = \widetilde{B} = (\varrho_4, \pi_3, \varrho_2)$, $\widetilde{\gamma}(n) = \widetilde{C} = (\varrho_4, \varrho_3, \varrho_2)$, $\widetilde{\delta}(n) = \widetilde{D} = (5\pi_4, \varrho_3, 5\pi_2)$ satisfy $A \overset{-1}{\sim} \widetilde{A}$, $B \overset{-1}{\sim} \widetilde{B}$, $C \overset{5}{\sim} \widetilde{C}$ and $D \overset{5}{\sim} \widetilde{D}$.

2. The generalized determinants and permanents

For $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{Z}^3$, let

$$a \cdot b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

$$a : b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Note that $a \cdot b$ is the determinant of the rectangular 2×3 matrix with rows a and b (see [7]). When we replace in the above definition the determinants $|\ |$ with the permanents $||\ ||$, we shall get the definition of the functionals $\cdot\cdot$ and $::$.

We begin with a result that lists some cases when the values of $\cdot$ and $:$ are constant.

Theorem 1. *The following relations hold for the triples $A,\ldots,\widetilde{D}$:*

$$A\cdot C=\widetilde{A}\cdot\widetilde{C}=-\widetilde{A}:\widetilde{C}=-2,\qquad A:C=10,$$

$$B\cdot D=-\widetilde{B}:\widetilde{D}=14,\qquad \widetilde{B}\cdot\widetilde{D}=6,\qquad B:D=26.$$

Proof. Since $A=(\pi_1,\pi_3,\pi_5)$ and $C=(\varrho_1,\varrho_3,\varrho_5)$, the product $A\cdot C$ is

$$\begin{vmatrix}\pi_3 & \pi_5\\ \varrho_3 & \varrho_5\end{vmatrix}+\begin{vmatrix}\pi_5 & \pi_1\\ \varrho_5 & \varrho_1\end{vmatrix}+\begin{vmatrix}\pi_1 & \pi_3\\ \varrho_1 & \varrho_3\end{vmatrix}=2-6+2=-2.$$

□

The analogous values of $\cdot\cdot$ and $::$ are either zero or multiples of only three Fibonacci numbers $\mathfrak{p}_7$, $\mathfrak{p}_6$ and $\mathfrak{p}_5$.

Theorem 2. *The following relations hold for the triples $A,\ldots,\widetilde{D}$:*

$$\frac{A\cdot\cdot C}{8}=\frac{A::C}{4}=\frac{B\cdot\cdot D}{32}=\frac{B::D}{28}=\frac{\widetilde{B}\cdot\cdot\widetilde{D}}{20}=\mathfrak{p}_6,$$

$$\widetilde{B}::\widetilde{D}=0,\qquad \widetilde{A}\cdot\cdot\widetilde{C}=4\,\mathfrak{p}_7,\qquad \widetilde{A}::\widetilde{C}=4\,\mathfrak{p}_5.$$

Proof. Since $A=(\pi_1,\pi_3,\pi_5)$ and $C=(\varrho_1,\varrho_3,\varrho_5)$, the value $A\cdot\cdot C$ is

$$\left\|\begin{matrix}\pi_3 & \pi_5\\ \varrho_3 & \varrho_5\end{matrix}\right\|+\left\|\begin{matrix}\pi_5 & \pi_1\\ \varrho_5 & \varrho_1\end{matrix}\right\|+\left\|\begin{matrix}\pi_1 & \pi_3\\ \varrho_1 & \varrho_3\end{matrix}\right\|=2(\mathfrak{p}_8+\mathfrak{p}_6+\mathfrak{p}_4)=8\mathfrak{p}_6.$$

□

The pairs (A,B) and (C,D) show similar relationships as the above for the pairs (A,C) and (B,D). The values of the functionals $\cdot$ and $:$ in these pairs are multiples of $\mathfrak{p}_6$ while those of $\cdot\cdot$ and $::$ involve the Lucas numbers $\mathfrak{r}_6$.

Theorem 3. *The following relations hold for the triples $A,\ldots,D$:*

$$\frac{A\cdot B}{4}=\frac{A:B}{4}=\frac{C\cdot D}{20}=\frac{C:D}{20}=-\mathfrak{p}_6,$$

$$\frac{A\cdot\cdot B}{2\,\mathfrak{r}_6+5}=\frac{5\,A::B}{8\,\mathfrak{r}_6+11}=\frac{C\cdot\cdot D}{5(2\,\mathfrak{r}_6-5)}=\frac{C::D}{8\,\mathfrak{r}_6-11}=2.$$

Proof. Since $A = (\pi_1, \pi_3, \pi_5)$ and $B = (\pi_1, 5\pi_3, \pi_5)$, the value $A\cdot B$ is

$$\begin{vmatrix} \pi_3 & \pi_5 \\ 5\pi_3 & \pi_5 \end{vmatrix} + \begin{vmatrix} \pi_5 & \pi_1 \\ \pi_5 & \pi_1 \end{vmatrix} + \begin{vmatrix} \pi_1 & \pi_3 \\ \pi_1 & 5\pi_3 \end{vmatrix} = -4\pi_3\pi_5 + 0 + 4\pi_1\pi_3 = -4\mathfrak{p}_6.$$

□

The following interesting double identity

$$4\,A \cdot\cdot B - 5\,A :: B = C :: D - \frac{4}{5}\,C \cdot\cdot D = 18$$

is an easy consequence of the previous theorem.

For the associated pairs $(\widetilde{A}, \widetilde{B})$ and $(\widetilde{C}, \widetilde{D})$ the values of $\cdot$, $:$, $\cdot\cdot$ and $::$ use the products of π_i and ϱ_j and the Lucas numbers $\mathfrak{r}_5$.

Theorem 4. *The following relations hold for the triples* $\widetilde{A}, \dots, \widetilde{D}$*:*

$$\frac{\widetilde{A}\cdot\widetilde{B}}{\pi_1\pi_4} = \frac{5\,\widetilde{A}:\widetilde{B}}{\mathfrak{r}_5 - 6} = \frac{\widetilde{C}\cdot\widetilde{D}}{\varrho_1\varrho_4} = \frac{\widetilde{C}:\widetilde{D}}{\mathfrak{r}_5 + 6} = -2,$$

$$\frac{\widetilde{A}\cdot\cdot\widetilde{B}}{\pi_3\varrho_5} = \frac{\widetilde{A}::\widetilde{B}}{\pi_3\varrho_2} = \frac{\widetilde{C}\cdot\cdot\widetilde{D}}{5\,\pi_5\varrho_3} = \frac{\widetilde{C}::\widetilde{D}}{5\,\pi_2\varrho_3} = 2.$$

Proof. Since $\widetilde{A} = (\pi_4, \pi_3, \pi_2)$ and $\widetilde{B} = (\varrho_4, \pi_3, \varrho_2)$, the value $\widetilde{A}\cdot\widetilde{B}$ is $\pi_3(\varrho_2 - \pi_2) + \pi_2\varrho_4 - \pi_4\varrho_2 + \pi_3(\pi_4 - \varrho_4) = -2\,\pi_1\pi_4$. □

The formulas $\widetilde{C}\cdot\widetilde{D} - 5\,\widetilde{A}\cdot\widetilde{B} = 16$ and $5\,\widetilde{A}:\widetilde{B} - \widetilde{C}:\widetilde{D} = 24$ and the double identity $5\,\widetilde{A}\cdot\cdot\widetilde{B} - \widetilde{C}\cdot\cdot\widetilde{D} = 5\,\widetilde{A}::\widetilde{B} - \widetilde{C}::\widetilde{D} = 20$ are corollaries.

The values of $\cdot$ and $:$ for the mixed pairs $(A, \widetilde{C})$, $(\widetilde{A}, C)$, etc. could be figured out from the following relations.

Theorem 5. *The following hold for the triples* $A, \dots, \widetilde{D}$*:*

$$A\cdot\widetilde{C} + \widetilde{A}\cdot C = 4, \qquad A\cdot\widetilde{C} - \widetilde{A}\cdot C = 4\,\mathfrak{p}_4, \qquad B\cdot\widetilde{D} + \widetilde{B}\cdot D = -20,$$

$$B\cdot\widetilde{D} - \widetilde{B}\cdot D = -4\,\mathfrak{r}_6, \quad A:\widetilde{C} + \widetilde{A}:C = 4, \quad \widetilde{A}:C - A:\widetilde{C} = -4\,\mathfrak{r}_7,$$

$$B:\widetilde{D} + \widetilde{B}:D = -12, \qquad \widetilde{B}:D - B:\widetilde{D} = -20\,\mathfrak{r}_6.$$

Proof. Since $\widetilde{A} = (\pi_4, \pi_3, \pi_2)$ and $\widetilde{C} = (\varrho_4, \varrho_3, \varrho_2)$, the value $A\cdot\widetilde{C}$ is

$$\begin{vmatrix} \pi_3 & \pi_5 \\ \varrho_3 & \varrho_2 \end{vmatrix} + \begin{vmatrix} \pi_5 & \pi_1 \\ \varrho_2 & \varrho_4 \end{vmatrix} + \begin{vmatrix} \pi_1 & \pi_3 \\ \varrho_4 & \varrho_3 \end{vmatrix} = 2\,\pi_1\varrho_3$$

and the value $\widetilde{A}\cdot C$ is

$$\begin{vmatrix} \pi_3 & \pi_2 \\ \varrho_3 & \varrho_5 \end{vmatrix} + \begin{vmatrix} \pi_2 & \pi_4 \\ \varrho_5 & \varrho_1 \end{vmatrix} + \begin{vmatrix} \pi_4 & \pi_3 \\ \varrho_1 & \varrho_3 \end{vmatrix} = -2\pi_3\varrho_1.$$

Hence, $A\cdot\widetilde{C} + \widetilde{A}\cdot C = 4$ and $A\cdot\widetilde{C} - \widetilde{A}\cdot C = 4\,\mathfrak{p}_4$.

The remaining six identities are proved similarly. □

Similarly, the values of ·· and :: for the mixed pairs $(A, \widetilde{C})$, $(\widetilde{A}, C)$, etc. could also be figured out from the following relations.

Theorem 6. *The following hold for the triples* $A, \dots, \widetilde{D}$*:*

$$A\cdot\cdot\widetilde{C} + \widetilde{A}\cdot\cdot C = 12\,\mathfrak{p}_7, \quad A\cdot\cdot\widetilde{C} - \widetilde{A}\cdot\cdot C = 8, \quad \widetilde{A}::C = A::\widetilde{C} = 2\,\mathfrak{p}_4,$$

$$\frac{B\cdot\cdot\widetilde{D}}{38} = \frac{\widetilde{B}\cdot\cdot D}{38} = \frac{\widetilde{B}::D}{18} = \frac{B::\widetilde{D}}{18} = \mathfrak{p}_6.$$

Proof. Since $\widetilde{A} = (\pi_4, \pi_3, \pi_2)$ and $\widetilde{C} = (\varrho_4, \varrho_3, \varrho_2)$, the value $A\cdot\cdot\widetilde{C}$ is

$$\left\|\begin{matrix} \pi_3 & \pi_5 \\ \varrho_3 & \varrho_2 \end{matrix}\right\| + \left\|\begin{matrix} \pi_5 & \pi_1 \\ \varrho_2 & \varrho_4 \end{matrix}\right\| + \left\|\begin{matrix} \pi_1 & \pi_3 \\ \varrho_4 & \varrho_3 \end{matrix}\right\| = 6\,\mathfrak{p}_7 + 4$$

and the value $\widetilde{A}\cdot\cdot C$ is

$$\left\|\begin{matrix} \pi_3 & \pi_2 \\ \varrho_3 & \varrho_5 \end{matrix}\right\| + \left\|\begin{matrix} \pi_2 & \pi_4 \\ \varrho_5 & \varrho_1 \end{matrix}\right\| + \left\|\begin{matrix} \pi_4 & \pi_3 \\ \varrho_1 & \varrho_3 \end{matrix}\right\| = 6\,\mathfrak{p}_7 - 4.$$

Hence, $A\cdot\cdot\widetilde{C} + \widetilde{A}\cdot\cdot C = 12\,\mathfrak{p}_7$ and $A\cdot\cdot\widetilde{C} - \widetilde{A}\cdot\cdot C = 8$.

The remaining six identities are proved similarly. □

Theorem 7. *The following relations hold for the triples* $A, \dots, \widetilde{D}$*:*

$$A\cdot\widetilde{A} = 2\,\pi_2^2, \quad 5\,B\cdot\widetilde{B} = D\cdot\widetilde{D} = -10\,\mathfrak{p}_6, \quad C\cdot\widetilde{C} = 2\,\varrho_2^2,$$

$$5\,A:\widetilde{A} = C:\widetilde{C} = -10\,\pi_7, \qquad 5\,B:\widetilde{B} = D:\widetilde{D} = -50\,\mathfrak{p}_6.$$

Proof. Since $A = (\pi_1, \pi_3, \pi_5)$ and $\widetilde{A} = (\pi_4, \pi_3, \pi_2)$, the value $A\cdot\widetilde{A}$ is

$$\begin{vmatrix} \pi_3 & \pi_5 \\ \pi_3 & \pi_2 \end{vmatrix} + \begin{vmatrix} \pi_5 & \pi_1 \\ \pi_2 & \pi_4 \end{vmatrix} + \begin{vmatrix} \pi_1 & \pi_3 \\ \pi_4 & \pi_3 \end{vmatrix} = 2\,\pi_2^2.$$

□

Notice that $C\cdot\widetilde{C} - 5A\cdot\widetilde{A} = 8$.

Theorem 8. *The following relations hold for the triples* $A, \dots, \widetilde{D}$*:*

$$A \cdot\cdot \widetilde{A} = 6\pi_3\pi_4, \qquad 5B \cdot\cdot \widetilde{B} = 2(19\mathfrak{r}_6 + 33), \qquad C \cdot\cdot \widetilde{C} = 6\varrho_3\varrho_4,$$
$$D \cdot\cdot \widetilde{D} = 2(19\mathfrak{r}_6 - 33), \qquad A :: \widetilde{A} = 2\pi_1\pi_3, \qquad 5\,B :: \widetilde{B} = 2(9\mathfrak{r}_6 + 23),$$
$$C :: \widetilde{C} = 2\varrho_1\varrho_3, \qquad D :: \widetilde{D} = 2(9\mathfrak{r}_6 - 23).$$

Proof. Since $A = (\pi_1, \pi_3, \pi_5)$ and $\widetilde{A} = (\pi_4, \pi_3, \pi_2)$, the value $A \cdot\cdot \widetilde{A}$ is

$$\left\| \begin{matrix} \pi_3 & \pi_5 \\ \pi_3 & \pi_2 \end{matrix} \right\| + \left\| \begin{matrix} \pi_5 & \pi_1 \\ \pi_2 & \pi_4 \end{matrix} \right\| + \left\| \begin{matrix} \pi_1 & \pi_3 \\ \pi_4 & \pi_3 \end{matrix} \right\| = 6\,\pi_3\pi_4.$$

□

The formulas $5B \cdot\cdot \widetilde{B} - D \cdot\cdot \widetilde{D} = 132$ and $5B :: \widetilde{B} - D :: \widetilde{D} = 92$ and the identities $5A \cdot\cdot \widetilde{A} - C \cdot\cdot \widetilde{C} = 5A :: \widetilde{A} - C :: \widetilde{C} = 12$ are the corollaries.

3. Volumes of some tetrahedra

Let T, $\widetilde{T}$, T_A and $\widetilde{T}_A$ denote the tetrahedra $ABCD$, $\widetilde{A}\widetilde{B}\widetilde{C}\widetilde{D}$, $\widetilde{A}BCD$ and $A\widetilde{B}\widetilde{C}\widetilde{D}$. The tetrahedra T_B, T_C, T_D, $\widetilde{T}_B$, $\widetilde{T}_C$ and $\widetilde{T}_D$ are defined similarly.

The following result shows that A, B, C and D are coplanar points and that the tetrahedron $\widetilde{T}$ has a nice oriented volume.

Theorem 9. *For every natural number* n*, the points* A*,* B*,* C *and* D *are in the plane* $\pi_4\,x - \pi_0\,z = 3$*. The tetrahedron* $\widetilde{T}$ *has the oriented volume* $\frac{8}{3}\,\pi_2$ *and the centroid* (π_6, π_4, π_4)*.*

Proof. Recall that three points $P(a, b, c)$, $Q(d, e, f)$ and $R(g, h, i)$ in general position in the space $\mathbb{R}^3$ determine the plane

$$M_1^*\,x + M_2^*\,y + M_3^*\,z = M,$$

where M, M_1^*, M_2^* and M_3^* are the determinants

$$\left| \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \right|, \quad \left| \begin{matrix} 1 & b & c \\ 1 & e & f \\ 1 & h & i \end{matrix} \right|, \quad \left| \begin{matrix} a & 1 & c \\ d & 1 & f \\ g & 1 & i \end{matrix} \right|, \quad \left| \begin{matrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{matrix} \right|.$$

In our case, for $P = A$, $Q = B$ and $R = C$, these determinants are $M = 24\pi_3$, $M_1^* = 8\pi_3\pi_4$, $M_2^* = 0$ and $M_3^* = -8\pi_3\pi_0$. Hence, the points A, B and C determine the plane $\pi_4\,x - \pi_0\,z = 3$. The point D also lies in it. In order to prove the statement about the volume of the tetrahedron $\widetilde{T}$, we use the formula

$$|P_1P_2P_3P_4| = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

for the oriented volume of the tetrahedron $P_1P_2P_3P_4$, where the points P_i have the coordinates $(x_i,\, y_i,\, z_i)$ $(i = 1,\, 2,\, 3,\, 4)$. Also, the centroid of this tetrahedron is the point $\left(\frac{\sum x_i}{4}, \frac{\sum y_i}{4}, \frac{\sum z_i}{4}\right)$. □

Let $G(T)$ denote the centroid of the tetrahedron T.

Theorem 10. *The following relations hold for the triples $A, \dots, \widetilde{D}$:*

$$|T_A|+|T_B|+|T_C|+|T_D| = -\tfrac{16}{3}\pi_0, \quad |\widetilde{T}_A|+|\widetilde{T}_B|+|\widetilde{T}_C|+|\widetilde{T}_D| = -\tfrac{16}{3}\pi_1,$$

$$|G(T_A)G(T_B)G(T_C)G(T_D)| = -|G(\widetilde{T}_A)G(\widetilde{T}_B)G(\widetilde{T}_C)G(\widetilde{T}_D)| = \tfrac{1}{6}\pi_1.$$

Proof. While the volumes $|\widetilde{T}_A|$ $|\widetilde{T}_B|$ $|\widetilde{T}_C|$ and $|\widetilde{T}_D|$ are quite complicated, the sums $|\widetilde{T}_A| + |\widetilde{T}_C|$ and $|\widetilde{T}_B| + |\widetilde{T}_D|$ are $8\pi_2$ and $-\frac{8}{3}\pi_5$. Hence, $|\widetilde{T}_A| + |\widetilde{T}_B| + |\widetilde{T}_C| + |\widetilde{T}_D| = -\frac{16}{3}\pi_1$. The other identities in this theorem are proved similarly. □

Let $G(G(T_A)G(T_B)G(T_C)G(T_D))$ and $G(G(\widetilde{T}_A)G(\widetilde{T}_B)G(\widetilde{T}_C)G(\widetilde{T}_D))$ be shortened to G_T and $G_{\widetilde{T}}$.

Theorem 11. *The points G_T and $G_{\widetilde{T}}$ divide the segment $G(T)G(\widetilde{T})$ in the ratios $1:3$ and $3:1$.*

Proof. Recall that $G(P_1P_2P_3P_4)$ has $\frac{\sum x_i}{4}$, $\frac{\sum y_i}{4}$ and $\frac{\sum z_i}{4}$ as coordinates. Hence, the centroids $G(T)$, $G(\widetilde{T})$, G_T and $G_{\widetilde{T}}$ are the triples $(\pi_2, 3\pi_4, \pi_6)$, (π_6, π_4, π_4), $\left(\frac{3\pi_6 - \varrho_0}{8}, \frac{5\pi_4}{2}, \frac{3\varrho_6 + \pi_0}{8}\right)$ and $\left(\frac{5\pi_5 + \pi_0}{4}, \frac{3\pi_4}{2}, \frac{3\varrho_7 - \varrho_0}{20}\right)$. Now that we know the coordinates of the four centroids it is easy to check the claims in this theorem. □

Let T_{AB} denote the tetrahedron $AB\widetilde{A}\widetilde{B}$. The tetrahedra T_{AC}, T_{CD} and T_{BD} are defined similarly. The (oriented) volumes of these tetrahedra are given in the next result. Their centroids lie in the plane

$$10\,\pi_3(\pi_5 + \pi_0)\,x - (4\,\mathfrak{r}_5 + 1)\,y - 10\,\pi_2\pi_4\,z = 5\,\pi_4.$$

Theorem 12. *The following relations hold for the triples $A,\dots,\widetilde{D}$:*

$$|T_{AB}| = \tfrac{8}{15}\pi_3(2\,\mathfrak{r}_5+3), \qquad |T_{CD}| = \tfrac{8}{3}\varrho_3(2\,\mathfrak{r}_5-3),$$
$$11\,|T_{AC}| = -|T_{BD}| = \tfrac{88}{3}\pi_2.$$

Proof. The volume $|T_{AB}|$ is

$$\frac{1}{6}\begin{vmatrix} \pi_1 & \pi_3 & \pi_5 & 1 \\ \pi_1 & 5\pi_3 & \pi_5 & 1 \\ \pi_4 & \pi_3 & \pi_2 & 1 \\ \varrho_4 & \pi_3 & \varrho_2 & 1 \end{vmatrix} = \frac{2\pi_3(2\pi_2\varrho_2+2\pi_3\varrho_4+\pi_1\pi_2-\pi_4\pi_5)}{3}.$$

The long parenthesis simplifies to $\frac{4}{5}(2\,\mathfrak{r}_5+3)$ that gives the above value. □

The first two formulas imply $5|T_{AB}|\varrho_3 - |T_{CD}|\pi_3 = 16\mathfrak{p}_6$.

4. Tetrahedra from products ↓ and ↑

This section uses the binary operations ↓ and ↑ defined by

$$(a,b,c)\downarrow(d,e,f) = (b\,f-c\,e, c\,d-a\,f, a\,e-b\,d),$$
$$(a,b,c)\uparrow(d,e,f) = (b\,f+c\,e, c\,d+a\,f, a\,e+b\,d).$$

Note that restricted on the standard Euclidean 3-space $\mathbb{R}^3$ the product ↓ is the familiar vector cross-product.

Notice that the products $A\downarrow C$, $B\downarrow D$, $\widetilde{A}\downarrow\widetilde{C}$ and $\widetilde{B}\downarrow\widetilde{D}$ have the constant values $(2,-6,2)$, $(10,-6,10)$, $(2,-2,2)$ and $(-2,10,-2)$. Hence, many expressions that include them become somewhat simpler.

Let $T_{\downarrow\uparrow}$ and $T_{\uparrow\downarrow}$ denote the tetrahedra $(A\!\downarrow\! C)(A\!\downarrow\! D)(B\!\uparrow\! C)(B\!\uparrow\! D)$ and $(A\!\uparrow\! C)(A\!\uparrow\! D)(B\!\downarrow\! C)(B\!\downarrow\! D)$. We now show that they have equal volumes and the centroids at nice distance.

Theorem 13. $|T_{\downarrow\uparrow}| = |T_{\uparrow\downarrow}| = \frac{32}{3}\pi_1^2\varrho_3^2\,\mathfrak{p}_{10}$ *and* $|G(T_{\downarrow\uparrow})G(T_{\uparrow\downarrow})| = 4\,\pi_3\varrho_1$.

Proof. Since $A\downarrow C = (2,-6,2)$, $A\downarrow D = (6-4\mathfrak{p}_8,-6,6+4\mathfrak{p}_4)$, $B\uparrow C = (6\mathfrak{p}_8+4, 2\mathfrak{p}_6, 6\mathfrak{p}_4-4)$ and $B\uparrow D = (10\mathfrak{p}_8, 2\mathfrak{p}_6, 10\mathfrak{p}_4)$, the volume $|T_{\downarrow\uparrow}|$ is

$$\frac{1}{6}\begin{vmatrix} 2 & -6 & 2 & 1 \\ 6-4\mathfrak{p}_8 & -6 & 6+4\mathfrak{p}_4 & 1 \\ 6\mathfrak{p}_8+4 & 2\mathfrak{p}_6 & 6\mathfrak{p}_4-4 & 1 \\ 10\mathfrak{p}_8 & 2\mathfrak{p}_6 & 10\mathfrak{p}_4 & 1 \end{vmatrix}$$

that simplifies to the above value. The calculation for the volume of $T_{\uparrow\downarrow}$ is similar and for the distance $|G(T_{\downarrow\uparrow})G(T_{\uparrow\downarrow})|$ is routine since we know the coordinates of the vertices. $\square$

Let $\widetilde{T}_{\downarrow}$ and $\widetilde{T}_{\uparrow}$ denote the tetrahedra $(\widetilde{A}\!\downarrow\!\widetilde{C})(\widetilde{A}\!\downarrow\!\widetilde{D})(\widetilde{B}\!\downarrow\!\widetilde{C})(\widetilde{B}\!\downarrow\!\widetilde{D})$ and $(\widetilde{A}\!\uparrow\!\widetilde{C})(\widetilde{A}\!\uparrow\!\widetilde{D})(\widetilde{B}\!\uparrow\!\widetilde{C})(\widetilde{B}\!\uparrow\!\widetilde{D})$. We similarly define the tetrahedra $T_{\downarrow}$ and $T_{\uparrow}$. Note that $T_{\downarrow}$ is a parallelogram in the plane $y = -6$ with the centroid $(6,-6,6)$, the sides $4\pi_3\sqrt{7\mathfrak{r}_6 - 4}$ and $\frac{4}{5}\varrho_3\sqrt{7\mathfrak{r}_6 + 4}$ and area $48\sqrt{2}\,\mathfrak{p}_6$ while the centroid of $\widetilde{T}_{\downarrow}$ is the point $(0,2,-2)$. Similarly, $T_{\uparrow}$ is a parallelogram in the plane $y = 2\mathfrak{p}_6$ with the centroid $(6\mathfrak{p}_8, 2\mathfrak{p}_6, 6\mathfrak{p}_4)$, the sides $4\pi_3\sqrt{7\mathfrak{r}_6 - 4}$ and $\frac{4}{5}\varrho_3\sqrt{7\mathfrak{r}_6 + 4}$ and area $48\sqrt{2}\,\mathfrak{p}_6$ while the centroid of $\widetilde{T}_{\uparrow}$ is the point $2(\mathfrak{p}_6, \mathfrak{p}_8, \mathfrak{p}_8)$.

Theorem 14. *$|\widetilde{T}_{\downarrow}| = -\frac{32}{3}\,\mathfrak{p}_6$ and $|\widetilde{T}_{\uparrow}| = -\frac{32}{3}\,\mathfrak{p}_6\,\mathfrak{p}_4$.*

Proof. Since $\widetilde{A}\downarrow\widetilde{C} = (2,-2,-2)$, $\widetilde{A}\downarrow\widetilde{D} = (2\mathfrak{p}_4, 0, -2\pi_4\varrho_2)$, $\widetilde{B}\downarrow\widetilde{C} = (-2\mathfrak{p}_4, 0, 2\pi_2\varrho_4)$ and $\widetilde{B}\downarrow\widetilde{D} = (-2,10,-2)$, the volume $|\widetilde{T}_{\downarrow}|$ is

$$\frac{1}{6}\begin{vmatrix} 2 & -2 & -2 & 1 \\ 2\mathfrak{p}_4 & 0 & -2\pi_4\varrho_2 & 1 \\ -2\mathfrak{p}_4 & 0 & 2\pi_2\varrho_4 & 1 \\ -2 & 10 & -2 & 1 \end{vmatrix}$$

that simplifies to the above value. The calculation for the volume of $\widetilde{T}_{\uparrow}$ is a bit more complicated. $\square$

5. Tetrahedra from products $\odot$, $\triangleright$ and $\triangleleft$

Let us introduce three binary operations $\odot$, $\triangleright$ and $\triangleleft$ on the set $\mathbb{Z}^3$ of triples of integers by the rules $(a,\,b,\,c)\odot(u,\,v,\,w) = (a\,u,\,b\,v,\,c\,w)$, $(a,\,b,\,c)\triangleright(u,\,v,\,w) = (a\,v,\,b\,w,\,c\,u)$, and

$$(a,\,b,\,c)\triangleleft(u,\,v,\,w) = (a\,w,\,b\,u,\,c\,v).$$

Notice that $|A\odot C, A\odot D| = 4\,\mathfrak{p}_6$, $B\odot C = A\odot D$, and the point $B\odot D$ divides the segment $(A\odot C)(A\odot D)$ in the ratio $-6:5$. The third coordinates of the points $A\triangleright C$, $A\triangleright D$, $B\triangleright C$ and $B\triangleright D$ are $\pi_5\varrho_1$. Similarly, the first coordinates of the points $A\triangleleft C$, $A\triangleleft D$, $B\triangleleft C$ and $B\triangleleft D$ are $\pi_1\varrho_5$. Finally, the second coordinates of the points $\widetilde{A}\odot\widetilde{C}$, $\widetilde{A}\odot\widetilde{D}$, $\widetilde{B}\odot\widetilde{C}$ and $\widetilde{B}\odot\widetilde{D}$ are $\mathfrak{p}_6$.

Let $\widetilde{T}_{\triangleright}$ denote the tetrahedron with the vertices $\widetilde{A} \triangleright \widetilde{C}$, $\widetilde{A} \triangleright \widetilde{D}$, $\widetilde{B} \triangleright \widetilde{C}$ and $\widetilde{B} \triangleright \widetilde{D}$. The tetrahedron $\widetilde{T}_{\triangleleft}$ is defined similarly.

Theorem 15. $|\widetilde{T}_{\triangleright}| = -|\widetilde{T}_{\triangleleft}| = \frac{8}{3}\,\mathfrak{p}_6^2\,\mathfrak{p}_2$ *and* $|G(\widetilde{T}_{\triangleright}), G(\widetilde{T}_{\triangleleft})| = \sqrt{2}\,\pi_3\,\varrho_4$.

Proof. Since $\widetilde{A} \triangleright \widetilde{C} = (\pi_4\varrho_3, \pi_3\varrho_2, \pi_2\varrho_4)$, $\widetilde{A} \triangleright \widetilde{D} = (\pi_4\varrho_3, 5\pi_2\pi_3, 5\pi_2\pi_4)$, $\widetilde{B} \triangleright \widetilde{C} = (\varrho_3\varrho_4, \pi_3\varrho_2, \varrho_2\varrho_4)$ and $\widetilde{B} \triangleright \widetilde{D} = (\varrho_3\varrho_4, 5\pi_2\pi_3, 5\pi_4\varrho_2)$, the volume $|\widetilde{T}_{\triangleright}|$ is

$$\frac{1}{6}\begin{vmatrix} \pi_4\varrho_3 & \pi_3\varrho_2 & \pi_2\varrho_4 & 1 \\ \pi_4\varrho_3 & 5\pi_2\pi_3 & 5\pi_2\pi_4 & 1 \\ \varrho_3\varrho_4 & \pi_3\varrho_2 & \varrho_2\varrho_4 & 1 \\ \varrho_3\varrho_4 & 5\pi_2\pi_3 & 5\pi_4\varrho_2 & 1 \end{vmatrix}$$

that simplifies to the above value. The calculation for the volume of $\widetilde{T}_{\triangleleft}$ is analogous. ☐

Let $T_{\odot\triangleright}$ denote the tetrahedron with the vertices $A \odot C$, $A \odot D$, $B \triangleright C$ and $B \triangleright D$. The tetrahedra $T_{\triangleright\odot}$, $T_{\odot\triangleleft}$, $T_{\triangleleft\odot}$, $T_{\triangleright\triangleleft}$ and $T_{\triangleleft\triangleright}$ are defined similarly.

Theorem 16. $5\,|T_{\odot\triangleright}| = |T_{\triangleright\odot}| = 5\,|T_{\odot\triangleleft}| = |T_{\triangleleft\odot}| = \frac{200}{3}\,\mathfrak{p}_6^2\,\pi_5\,\pi_1$,

$$|G(T_{\odot\triangleright}), G(T_{\triangleright\odot})| = 2\,\pi_3\,\varrho_1, \qquad |G(T_{\odot\triangleleft}), G(T_{\triangleleft\odot})| = 2\,\pi_3\,\varrho_5,$$

$$|T_{\triangleright\triangleleft}| = 8\,\mathfrak{p}_6\pi_5\pi_1\varrho_3\varrho_0, \quad |T_{\triangleleft\triangleright}| = 8\,\mathfrak{p}_6\pi_5\pi_1\varrho_3\varrho_6, \quad |G(T_{\triangleright\triangleleft}), G(T_{\triangleleft\triangleright})| = 10\,\pi_3^2.$$

Proof. Since $A \odot C = (\mathfrak{p}_2, \mathfrak{p}_6, \mathfrak{p}_{10})$, $A \odot D = (\mathfrak{p}_2, 5\mathfrak{p}_6, \mathfrak{p}_{10})$, $B \triangleright C = (\pi_1\,\varrho_3, 5\pi_3\varrho_5, \pi_5\varrho_1)$ and $B \triangleright D = (5\pi_1\varrho_3, 5\pi_3\varrho_5, \pi_5\varrho_1)$, the volume $|T_{\odot\triangleright}|$ is

$$\frac{1}{6}\begin{vmatrix} \mathfrak{p}_2 & \mathfrak{p}_6 & \mathfrak{p}_{10} & 1 \\ \mathfrak{p}_2 & 5\mathfrak{p}_6 & \mathfrak{p}_{10} & 1 \\ \varrho_3 & 5\pi_3\varrho_5 & \pi_5\varrho_1 & 1 \\ 5\pi_1\varrho_3 & 5\pi_3\varrho_5 & \pi_5\varrho_1 & 1 \end{vmatrix}$$

that simplifies to the above value. The calculation for the other volumes is similar. ☐

Let $T_1 = T_{A,B,C}$ denote the tetrahedron with the vertices A, B, C and $A \odot B \odot C$. The tetrahedra $T_2 = T_{A,B,D}$, $T_3 = T_{C,D,A}$ and $T_4 = T_{C,D,B}$ are defined similarly.

Theorem 17.

$$|T_1| = |T_2| = 4\,\pi_0\pi_3\pi_4\mathfrak{r}_7, \qquad |G(T_1), G(T_2)| = \varrho_2\varrho_3\varrho_4,$$

$$|T_3| = |T_4| = -20\,\pi_0\pi_3\pi_4\mathfrak{p}_7, \qquad |G(T_3), G(T_4)| = \pi_3(5\mathfrak{r}_6 - 9).$$

Proof. Since $A \odot B \odot C = (\pi_1\,\mathfrak{p}_2, 5\pi_3\,\mathfrak{p}_6, \pi_5\,\mathfrak{p}_{10})$, the volume $|T_1|$ is

$$\frac{1}{6}\begin{vmatrix} \pi_1 & \pi_3 & \pi_5 & 1 \\ \pi_1 & 5\pi_3 & \pi_5 & 1 \\ \varrho_1 & \varrho_3 & \varrho_5 & 1 \\ \pi_1\,\mathfrak{p}_2 & 5\pi_3\,\mathfrak{p}_6 & \pi_5\,\mathfrak{p}_{10} & 1 \end{vmatrix}$$

that simplifies to the above value. The calculation for the other volumes is similar. □

Let $T_{\triangleright\downarrow}$ denote the tetrahedron with the vertices $A \triangleright C$, $A \triangleright D$, $B \downarrow C$ and $B \downarrow D$. The tetrahedra $T_{\downarrow\triangleright}$, $T_{\triangleright\uparrow}$, $T_{\uparrow\triangleright}$, $T_{\triangleleft\downarrow}$, $T_{\downarrow\triangleleft}$, $T_{\triangleleft\uparrow}$ and $T_{\uparrow\triangleleft}$, are defined similarly.

Theorem 18.

$$|T_{\triangleright\downarrow}| = \tfrac{8}{3}\pi_1^2\varrho_3^2(\mathfrak{p}_8+7), \quad |T_{\downarrow\triangleright}| = \tfrac{8}{3}\pi_1^2\varrho_3^2(5\mathfrak{p}_8+11), \quad |T_{\triangleright\uparrow}| = \tfrac{8}{3}\pi_1^2\varrho_2\varrho_3\mathfrak{p}_6,$$

$$|T_{\uparrow\triangleright}| = \tfrac{8}{3}\,\pi_1^2\,\varrho_3\,\varrho_8\,\mathfrak{p}_6, \quad |T_{\triangleleft\downarrow}| = \tfrac{8}{3}\,\pi_5^2\,\varrho_3^2(\mathfrak{p}_4+5), \quad |T_{\downarrow\triangleleft}| = \tfrac{8}{3}\,\pi_3\,\pi_5^2\,\varrho_3^2\,\varrho_4,$$

$$|T_{\triangleleft\uparrow}| = \tfrac{8}{3}\pi_5^2\varrho_3^2(5\mathfrak{p}_4+1), \qquad |T_{\uparrow\triangleleft}| = \tfrac{8}{3}\,\pi_3\,\pi_5^2\,\varrho_3^2\,\varrho_{-2},$$

$$|G(T_{\triangleright\downarrow})G(T_{\downarrow\triangleright})| = |G(T_{\triangleright\uparrow})G(T_{\uparrow\triangleright})| = \sqrt{6}\pi_3\sqrt{\mathfrak{r}_9 + 5\mathfrak{r}_6 - 4},$$

$$|G(T_{\triangleleft\downarrow})G(T_{\downarrow\triangleleft})| = |G(T_{\triangleleft\uparrow})G(T_{\uparrow\triangleleft})| = \sqrt{3}\pi_3\sqrt{5\mathfrak{p}_9 + \mathfrak{r}_0 - 8}.$$

Proof. Since $A \triangleright C = (\pi_1\varrho_3, \pi_3\varrho_5, \pi_5\varrho_1)$, $A \triangleright D = (5\pi_1\varrho_3, \pi_3\varrho_5, \pi_5\varrho_1)$, $B \downarrow C = (2(3+2\mathfrak{p}_8), -6, 2(3-2\mathfrak{p}_4))$ and $B \downarrow D = (10, -6, 10)$, the volume $|T_{\triangleright\downarrow}|$ is

$$\frac{1}{6}\begin{vmatrix} \pi_1\varrho_3 & \pi_3\varrho_5 & \pi_5\varrho_1 & 1 \\ 5\pi_1\varrho_3 & \pi_3\varrho_5 & \pi_5\varrho_1 & 1 \\ 2(3+2\mathfrak{p}_8) & -6 & 2(3-2\mathfrak{p}_4) & 1 \\ 10 & -6 & 10 & 1 \end{vmatrix}$$

that simplifies to the above value. The calculations for the other volumes and distances are similar. □

Let $T_{.}$ be the tetrahedron with vertices $(A\cdot B, A\cdot C, A\cdot D)$, $(B\cdot C, B\cdot D, B\cdot A)$, $(C\cdot D, C\cdot A, C\cdot B)$ and $(D\cdot A, D\cdot B, D\cdot C)$. The tetrahedra $T_{:}$, $\widetilde{T}_{.}$ and $\widetilde{T}_{:}$ are defined similarly.

Theorem 19.

$$|T_{.}| = \tfrac{128}{3}\mathfrak{p}_8(9 - 16\mathfrak{p}_6), \quad |T_{:}| = \tfrac{128}{3}\mathfrak{p}_8(81 - 16\mathfrak{p}_6),$$

$$|\widetilde{T}_{.}| = \tfrac{64}{15}\pi_2\pi_5(19 - 4\mathfrak{r}_5), \quad |\widetilde{T}_{:}| = \tfrac{64}{75}(\mathfrak{r}_7 + 6)(8\mathfrak{r}_5 + 87),$$

The tetrahedra $T_{.}$ and $T_{:}$ have the point $(-4\mathfrak{p}_4, 0, 4\mathfrak{p}_4)$ as a common centroid. The centroids $2\pi_0\pi_3(-1,0,1)$ and $\tfrac{2}{5}(\mathfrak{r}_3 + 6)(-1,0,1)$ of the tetrahedra $\widetilde{T}_{.}$ and $\widetilde{T}_{:}$ have the constant distance $4\sqrt{2}$.

Proof. Since $-A\cdot B = B\cdot A = 4\mathfrak{p}_6$, $-A\cdot C = C\cdot A = 2$, $-A\cdot D = D\cdot A = 2(2\mathfrak{r}_6 - 3)$, $B\cdot C = -C\cdot B = 2(2\mathfrak{r}_6 + 3)$, $B\cdot D = -D\cdot B = 14$ and $-C\cdot D = D\cdot C = 20\mathfrak{p}_6$, the volume $|T_{.}|$ is

$$\frac{1}{6}\begin{vmatrix} -4\mathfrak{p}_6 & -2 & -2(2\mathfrak{r}_6 - 3) & 1 \\ 2(2\mathfrak{r}_6 + 3) & 14 & 4\mathfrak{p}_6 & 1 \\ -20\mathfrak{p}_6 & 2 & -2(2\mathfrak{r}_6 + 3) & 1 \\ 2(2\mathfrak{r}_6 - 3) & -14 & 20\mathfrak{p}_6 & 1 \end{vmatrix}$$

that simplifies to the above value. The calculations for the other volumes and distances are similar. □

References

[1] E. Brown, *Sets in which $xy + k$ is always a square*, Mathematics of Computation, **45** (1985), 613-620.

[2] Z. Čerin, *On pencils of Euler triples, I*, (preprint).

[3] Z. Čerin, *On pencils of Euler triples, II*, (preprint).

[4] Z. Čerin, *On Diophantine triples from Fibonacci and Lucas numbers*, (preprint).

[5] Z. Čerin, *Squares in Euler triples from Fibonacci and Lucas numbers*, (preprint).

[6] Z. Čerin, *Matrices with rows in Euler triples from Fibonacci and Lucas numbers*, (preprint).

[7] M. Radić, *A definition of determinant of rectangular matrix*, Glasnik Mat. 1 (21) (1966), 17-22.

[8] N. Sloane, *On-Line Encyclopedia of Integer Sequences*, `http://www.research. att.com/~njas/sequences/`.

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