SUMS OF GENERALIZED FIBONACCI NUMBERS

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Abstract. In this paper we first give formulae for sums of a fixed number of consecutive generalized Fibonacci numbers from the same residue class. Analogous alternating sums are also studied as well as various derived sums when terms are multiplied by binomial coefficients or members of some simple integer sequences.

1. Introduction

The generalized Fibonacci sequence $\{w_n\} = \{w_n(a,\, b;\, p,\, q)\}$ is often defined by

$$w_0 = a, \quad w_1 = b, \quad w_n = p\, w_{n-1} - q\, w_{n-2} \qquad (n \geq 2),$$

where a, b, p and q are arbitrary complex numbers, with $q \neq 0$. The numbers w_n have been studied by Horadam (see, e. g. [3]). A useful and interesting special cases are $\{U_n\} = \{w_n(0,\, 1;\, p,\, q)\}$ and $\{V_n\} = \{w_n(2,\, p;\, p,\, q)\}$ that were investigated by Lucas [5].

2. Sums of generalized Fibonacci numbers

We first want to find the formula for the sum $\sum_{i=0}^{n} w_{r+t\,i}(a,\, b;\, p,\, q)$ when $n \geq 0$, $r \geq 0$ and $t > 0$ are integers.

Let α and β be the roots of $x^2 - p\,x + q = 0$. Then $\alpha = \frac{p+\Delta}{2}$ and $\beta = \frac{p-\Delta}{2}$, where $\Delta = \sqrt{p^2 - 4\,q}$. Moreover, $\alpha - \beta = \Delta$, $\alpha + \beta = p$, $\alpha\,\beta = q$ and the Binet forms of w_n, U_n and V_n are

$$w_n = \frac{(b - a\beta)\,\alpha^n + (a\alpha - b)\,\beta^n}{\alpha - \beta}, \quad U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n,$$

if $\alpha \neq \beta$, and

$$w_n = \widetilde{w}_n = \alpha^{n-1}(a + n\,(b - a\,\alpha)),\ U_n = \widetilde{U}_n = n\,\alpha^{n-1},\ V_n = \widetilde{V}_n = 2\,\alpha^n,$$

if $\alpha = \beta$.

Let $\gamma = b - a\,\alpha$ and $\delta = b - a\,\beta$. For any integer k, let $A_k = \alpha^k - 1$ and $B_k = \beta^k - 1$. Let $\Psi_1 = \sum_{i=0}^{n} w_{r+t\,i}(a,\, b;\, p,\, q)$.

AMS 2000 Math. Subject Classification: Primary 11B39, 11Y55, 05A19.

Key words and phrases. generalized Fibonacci numbers, integer sequences, sum, alternating sum, binomials, Maple V.

Theorem 1. *(a) When $\Delta = 0$ and $A_t = 0$, then*

$$\Psi_1 = \frac{\alpha^{r-1}\,(n+1)\,[2(r\,\gamma + a\,\alpha) + t\,n\,\gamma]}{2}.$$

(b) When $\Delta = 0$ and $A_t \neq 0$, then the sum Ψ_1 is

$$\frac{\alpha^{r-1}\left[A_{t(n+1)}(((t\,n + r)\gamma + a\,\alpha)A_t + t\,n\,\gamma) - t(n+1)\gamma\,\alpha^t\,A_{t\,n}\right]}{A_t^2}.$$

(c) When $\Delta \neq 0$ and $A_t = 0$ and $B_t = 0$, then

$$\Psi_1 = \frac{(n+1)\,(\delta\,\alpha^r - \gamma\beta^r)}{\Delta}.$$

(d) When $\Delta \neq 0$, $A_t = 0$ and $B_t \neq 0$, then

$$\Psi_1 = \frac{\alpha^r\,(n+1)\delta}{\Delta} - \frac{\beta^r\,\gamma\,B_{t(n+1)}}{\Delta\,B_t}.$$

(e) When $\Delta \neq 0$, $A_t \neq 0$ and $B_t = 0$, then

$$\Psi_1 = \frac{\alpha^r\,\delta\,A_{t(n+1)}}{\Delta\,A_t} - \frac{\beta^r\,(n+1)\gamma}{\Delta}.$$

(f) When $\Delta \neq 0$, $A_t \neq 0$ and $B_t \neq 0$, then

$$\Psi_1 = \frac{\alpha^r\,A_{t(n+1)}\,\delta}{\Delta\,A_t} - \frac{\beta^r\,B_{t(n+1)}\,\gamma}{\Delta\,B_t}.$$

Note that $\alpha^n = \frac{V_n + \Delta\,U_n}{2}$ and $\beta^n = \frac{V_n - \Delta\,U_n}{2}$ for $\Delta \neq 0$ and $\alpha^n = \beta^n = \frac{\widetilde{U}_{n+1}}{n+1} = \frac{\widetilde{V}_n}{2}$ for $\Delta = 0$. Hence, it is clear that each of the above expressions for the sum Ψ_1 could be transformed into an expression in Lucas numbers U_n and V_n (or $\widetilde{U}_n$ and $\widetilde{V}_n$). In most cases these formulae are more complicated then the ones given above. This applies also to other sums that we consider in this paper.

We shall now prove only parts (a) and (f) of Theorem 1. The proofs of the other parts are similar. This style of proving only few cases from a large number will be retained in the rest of the paper. The proofs are mostly by induction so that the main problem was to discover the formulae which we did with the help from magic possibilities of experimentation in Maple V.

Proof of (a). The proof will be by induction on n.

For $\alpha^t = 1$ and $n = 0$, the sum Ψ_1 is $a\,\alpha^r(1-r) + \alpha^{r-1}\,r\,b$ while the right hand side is $\frac{\alpha^{r-1}(2\,r(b - a\,\alpha) + 2\,a\,\alpha)}{2}$. In other words, for the initial value of n the relation (a) holds.

Assume that (a) is true for $n = m$. Then

$$\sum_{i=0}^{m+1} \widetilde{w}_{r+ti} = \sum_{i=0}^{m} \widetilde{w}_{r+ti} + \widetilde{w}_{r+t(m+1)} =$$
$$\frac{\alpha^{r-1}\,(m+1)\,[2(r\,\gamma + a\alpha) + t\,m\,\gamma]}{2} + \alpha^{r-1}\,[(m+1)\,\gamma\,t + (r\,\gamma + a\alpha)]$$
$$= \frac{\alpha^{r-1}\,(m+2)\,[2(r\,\gamma + a\alpha) + t\,(m+1)\,\gamma]}{2}.$$

Hence, the relation (a) holds also for $n = m + 1$. □

Proof of (f). The proof will again be by induction on n.

For $n = 0$, the sum Ψ_1 is $\frac{\alpha^r\,\delta - \beta^r\,\gamma}{\Delta}$ while the right hand side clearly has the same value. Hence, for the initial value of n the relation (f) holds.

Assume that (f) is true for $n = m$. Then

$$\sum_{i=0}^{m+1} w_{r+ti} = \left(\sum_{i=0}^{m} w_{r+ti}\right) + w_{r+t(m+1)} =$$
$$\left(\frac{\alpha^r\,A_{t(m+1)}\,\delta}{\Delta\,A_t} - \frac{\beta^r\,B_{t(m+1)}\,\gamma}{\Delta\,B_t}\right) + \left(\frac{\alpha^r\,\delta\,\alpha^{t(m+1)}}{\Delta} - \frac{\beta^r\gamma\,\beta^{t(m+1)}}{\Delta}\right)$$
$$= \frac{\alpha^r\,A_{t(m+2)}\,\delta}{\Delta\,A_t} - \frac{\beta^r\,B_{t(m+2)}\,\gamma}{\Delta\,B_t}.$$

Hence, the relation (f) holds also for $n = m + 1$. □

Of course, it is clear that Theorem 1 includes many interesting sums for all kinds of integer sequences. In particular, it is related to the results in the articles [1] and [2]. Recall that sequences of Fibonacci (F_n), Lucas (L_n), Pell (P_n), Pell-Lucas (Q_n), Jacobsthal (J_n), Jacobsthal-Lucas (j_n) listed respectively as A000045, A00032, A000129, A002203, A001045, A014551 in [7] are $w_n(0,\,1;\,1,\,-1)$, $w_n(2,\,1;\,1,\,-1)$, $w_n(0,\,2;\,2,\,-1)$, $w_n(2,\,2;\,2,\,-1)$, $w_n(0,\,1;\,1,\,-2)$, $w_n(2,\,1;\,1,\,-2)$ for $n \geq 0$.

3. The alternating sum and its generalization

For any integer k, let $\ell_k = (-1)^k$, $C_k = \alpha^k + 1$ and $D_k = \beta^k + 1$. Let Ψ_2 denote the sum $\sum_{i=0}^{n} \ell_i\, w_{r+t\,i}(a,\,b;\,p,\,q)$. By $\Psi_1^{(a)}$ we mean the sum Ψ_1 in the subcase (a) of the theorem where this sum is studied. The similar notation is used throughout.

Theorem 2. *(a) When* $\Delta = 0$ *and* $C_t = 0$*, then* $\Psi_2 = \Psi_1^{(a)}$.

(b) When $\Delta = 0$ *and* $C_t \neq 0$*, then the sum* Ψ_2 *is*

$$\frac{\alpha^{r-1}\left[\ell_n\, C_{t(n+1)}(((t\,n+r)\gamma + a\,\alpha)C_t + t\,\gamma) + (r\,\gamma + a\,\alpha)\,C_t - t\,\gamma\right]}{C_t^2}.$$

(c) When $\Delta \neq 0$ *and* $C_t = 0$ *and* $D_t = 0$*, then* $\Psi_2 = \Psi_1^{(c)}$.

(d) When $\Delta \neq 0$*,* $C_t = 0$ *and* $D_t \neq 0$*, then*

$$\Psi_2 = \frac{\alpha^r\,(n+1)\delta}{\Delta} - \frac{\beta^r\,\gamma\left(\ell_n\,\beta^{t(n+1)} + 1\right)}{\Delta\, D_t}.$$

(e) When $\Delta \neq 0$*,* $C_t \neq 0$ *and* $D_t = 0$*, then*

$$\Psi_2 = \frac{\alpha^r\,\delta\left(\ell_n\,\alpha^{t(n+1)} + 1\right)}{\Delta\, C_t} - \frac{\beta^r\,(n+1)\gamma}{\Delta}.$$

(f) When $\Delta \neq 0$*,* $C_t \neq 0$ *and* $D_t \neq 0$*, then*

$$\Psi_2 = \frac{\alpha^r\left(1 + \ell_n\,\alpha^{t(n+1)}\right)\delta}{\Delta\, C_t} - \frac{\beta^r\left(1 + \ell_n\,\beta^{t(n+1)}\right)\gamma}{\Delta\, D_t}.$$

Proof of (f). The proof will again be by induction on n.

For $n = 0$, the sum Ψ_2 is $\frac{\alpha^r\,\delta - \beta^r\,\gamma}{\Delta}$ while the right hand side clearly has the same value. Hence, for the initial value of n the relation (f) holds.

Assume that (f) is true for $n = m$. Then

$$\sum_{i=0}^{m+1} \ell_i\, w_{r+ti} = \left(\sum_{i=0}^{m} \ell_i\, w_{r+ti}\right) + \ell_{m+1}\, w_{r+t(m+1)} =$$
$$\left(\frac{\alpha^r\left(1 + \ell_m\,\alpha^{t(m+1)}\right)\delta}{\Delta\, C_t} - \frac{\beta^r\left(1 + \ell_m\,\beta^{t(m+1)}\right)\gamma}{\Delta\, D_t}\right)$$
$$+\,\ell_{m+1}\left(\frac{\alpha^r\delta\,\alpha^{t(m+1)}}{\Delta} - \frac{\beta^r\gamma\,\beta^{t(m+1)}}{\Delta}\right)$$
$$= \frac{\alpha^r\left(1 + \ell_{m+1}\,\alpha^{t(m+2)}\right)\delta}{\Delta\, C_t} - \frac{\beta^r\left(1 + \ell_{m+1}\,\beta^{t(m+2)}\right)\gamma}{\Delta\, D_t}.$$

Hence, the relation (f) holds also for $n = m + 1$. □

An obvious generalization of the sum Ψ_2 is the following sum Ψ_3 defined as $\sum_{i=0}^{n} k^i\, w_{r+t\,i}(a,\, b;\, p,\, q)$, for any complex number k. Let $E_t = \alpha^t\, k - 1$ and $F_t = \beta^t\, k - 1$.

Theorem 3. *(a) When* $\Delta = 0$ *and* $E_t = 0$*, then* $\Psi_3 = \Psi_1^{(a)}$.

(b) When $\Delta = 0$ *and* $E_t \neq 0$*, then* Ψ_3 *is*

$$\frac{\alpha^{r-1}\left[(E_t((r+t\,n)\gamma + a\,\alpha) - t\,\gamma)(\alpha^t\,k)^{n+1} - E_t((r-t)\gamma + a\,\alpha) + t\,\gamma\right]}{E_t^2}.$$

(c) When $\Delta \neq 0$ *and* $E_t = 0$ *and* $F_t = 0$*, then* $\Psi_3 = \Psi_1^{(c)}$.

(d) When $\Delta \neq 0$*,* $E_t = 0$ *and* $F_t \neq 0$*, then*

$$\Psi_3 = \frac{\alpha^r\,(n+1)\delta}{\Delta} - \frac{\beta^r\,\gamma\left((\beta^t\,k)^{n+1} - 1\right)}{\Delta\,F_t}.$$

(e) When $\Delta \neq 0$*,* $E_t \neq 0$ *and* $F_t = 0$*, then*

$$\Psi_3 = \frac{\alpha^r\,\delta\left((\alpha^t\,k)^{n+1} - 1\right)}{\Delta\,E_t} - \frac{\beta^r\,(n+1)\gamma}{\Delta}.$$

(f) When $\Delta \neq 0$*,* $E_t \neq 0$ *and* $F_t \neq 0$*, then*

$$\Psi_3 = \frac{\alpha^r\left((\alpha^t\,k)^{n+1} - 1\right)\delta}{\Delta\,E_t} - \frac{\beta^r\left((\beta^t\,k)^{n+1} - 1\right)\gamma}{\Delta\,F_t}.$$

Proof of (f). The proof will again be by induction on n.

For $n = 0$, the sum Ψ_3 is $\frac{\alpha^r\,\delta - \beta^r\,\gamma}{\Delta}$ while the right hand side clearly has the same value. Hence, for the initial value of n the relation (f) holds.

Assume that (f) is true for $n = m$. Then

$$\begin{aligned}\sum_{i=0}^{m+1} k^i\,w_{r+ti} &= \left(\sum_{i=0}^{m} k^i\,w_{r+ti}\right) + k^{m+1}\,w_{r+t(m+1)} = \\ &\left(\frac{\alpha^r\left((\alpha^t\,k)^{m+1} - 1\right)\delta}{\Delta\,E_t} - \frac{\beta^r\left((\beta^t\,k)^{m+1} - 1\right)\gamma}{\Delta\,F_t}\right) \\ &+ k^{m+1}\left(\frac{\alpha^r\,\delta\,\alpha^{t(m+1)}}{\Delta} - \frac{\beta^r\gamma\,\beta^{t(m+1)}}{\Delta}\right) \\ &= \frac{\alpha^r\left((\alpha^t\,k)^{m+2} - 1\right)\delta}{\Delta\,E_t} - \frac{\beta^r\left((\beta^t\,k)^{m+2} - 1\right)\gamma}{\Delta\,F_t}.\end{aligned}$$

Hence, the relation (f) holds also for $n = m + 1$. □

4. Sum with binomial coefficients

Let Ψ_4 denote the sum $\sum_{i=0}^{n} \binom{n}{i} w_{r+t\,i}(a,\,b;\,p,\,q)$.

Theorem 4. *(a) When* $\Delta = 0$*, then*

$$\Psi_4 = \begin{cases} \alpha^{r-1}\, [r\,\gamma + a\,\alpha]\,, & \text{if } n = 0, \\ \alpha^{r-1}\left[C_t(r\,\gamma + a\,\alpha) + t\,\alpha^t\,\gamma\right], & \text{if } n = 1, \\ \alpha^{r-1}\,C_t^{n-1}\left[C_t(r\,\gamma + a\,\alpha) + t\,n\,\alpha^t\,\gamma\right], & \text{if } n \geq 2. \end{cases}$$

(b) When $\Delta \neq 0$*, then*

$$\Psi_4 = \frac{\alpha^r\,\delta\,C_t^n - \beta^r\,\gamma\,D_t^n}{\Delta}.$$

Proof of (b). Since $w_{r+t\,i}(a,\, b;\, p,\, q) = \frac{\delta\,\alpha^r\,(\alpha^t)^i - \gamma\,\beta^r\,(\beta^t)^i}{\Delta}$, we have

$$\sum_{i=0}^{n} \binom{n}{i} w_{r+t\,i}(a,\, b;\, p,\, q) =$$

$$\frac{\delta\,\alpha^r}{\Delta}\left(\sum_{i=0}^{n}\binom{n}{i}(\alpha^t)^i\right) - \frac{\gamma\,\beta^r}{\Delta}\left(\sum_{i=0}^{n}\binom{n}{i}(\beta^t)^i\right)$$

$$= \frac{\delta\,\alpha^r}{\Delta}\left(\alpha^t+1\right)^n - \frac{\gamma\,\beta^r}{\Delta}\left(\beta^t+1\right)^n = \frac{\alpha^r\,\delta\,C_t^n - \beta^r\,\gamma\,D_t^n}{\Delta}.$$

□

5. The alternating sum with binomial coefficients

Let Ψ_5 denote the sum $\sum_{i=0}^{n} \ell_i \binom{n}{i} w_{r+t\,i}(a,\, b;\, p,\, q)$.

Theorem 5. *(a) When* $\Delta = 0$*, then*

$$\Psi_5 = \begin{cases} \alpha^{r-1}\, [r\,\gamma + a\,\alpha]\,, & \text{if } n = 0, \\ -\alpha^{r-1}\left[A_t(r\,\gamma + a\,\alpha) + t\,\alpha^t\,\gamma\right], & \text{if } n = 1, \\ \ell_n\,\alpha^{r-1}\,A_t^{n-1}\left[A_t(r\,\gamma + a\,\alpha) + t\,n\,\alpha^t\,\gamma\right], & \text{if } n \geq 2. \end{cases}$$

(b) When $\Delta \neq 0$*, then*

$$\Psi_5 = \frac{\ell_n\,\left[\alpha^r\,\delta\,A_t^n - \beta^r\,\gamma\,B_t^n\right]}{\Delta}.$$

Let Ψ_6 denote the sum $\sum_{i=0}^{n} k^i \binom{n}{i} w_{r+t\,i}(a,\, b;\, p,\, q)$ for any complex number k. Let $G_t = \alpha^t\,k + 1$ and $H_t = \beta^t\,k + 1$.

Theorem 6. *(a) When* $\Delta = 0$*, then*

$$\Psi_6 = \begin{cases} \alpha^{r-1}\, [r\,\gamma + a\,\alpha]\,, & \text{if } n = 0, \\ \alpha^{r-1}\left[G_t(r\,\gamma + a\,\alpha) + \alpha^t\,k\,t\,\gamma\right], & \text{if } n = 1, \\ \alpha^{r-1}\,G_t^{n-1}\left[G_t(r\,\gamma + a\,\alpha) + n\,\alpha^t\,k\,t\,\gamma\right], & \text{if } n \geq 2. \end{cases}$$

(b) When $\Delta \neq 0$, *then*

$$\Psi_6 = \frac{\alpha^r \, \delta \, G_t^n - \beta^r \, \gamma \, H_t^n}{\Delta}.$$

6. Terms multiplied by natural numbers

Let Ψ_7 denote the sum $\sum_{i=0}^{n} (i+1) w_{r+t\,i}(a,\, b;\, p,\, q)$.

Theorem 7. *(a) When* $\Delta = 0$ *and* $A_t = 0$, *then*

$$\Psi_7 = \frac{\alpha^{r-1}(n+2)(n+1)(3(r\gamma + a\alpha) + 2\,n\gamma\,t)}{6}.$$

(b) When $\Delta = 0$ *and* $A_t \neq 0$, *then* $\Psi_7 = \frac{\alpha^{r-1}(\alpha^{t(n+1)}\, M + N)}{A_t^3}$, *with*

$$N = ((r - 2\,t)\gamma + a\alpha)\, A_t - 2\,\gamma\, t,$$

$$M = (n+1)\, ((tn + r)\gamma + a\alpha)\, {A_t}^2 - ((2\,tn + r)\gamma + a\alpha)\, A_t + 2\, t\, \gamma.$$

(c) When $\Delta \neq 0$ *and* $A_t = 0$ *and* $B_t = 0$, *then*

$$\Psi_7 = \frac{(n+1)(n+2)\,(\delta\,\alpha^r - \gamma\beta^r)}{2\,\Delta}.$$

(d) When $\Delta \neq 0$, $A_t = 0$ *and* $B_t \neq 0$, *then*

$$\Psi_7 = \frac{\alpha^r (n+1)(n+2)\delta}{2\,\Delta} - \frac{\beta^r\, \gamma((n+1)B_{t(n+2)} - (n+2)B_{t(n+1)})}{\Delta\, B_t^2}.$$

(e) When $\Delta \neq 0$, $A_t \neq 0$ *and* $B_t = 0$, *then*

$$\Psi_7 = \frac{\alpha^r\, \delta((n+1)A_{t(n+2)} - (n+2)A_{t(n+1)})}{\Delta\, A_t^2} - \frac{\beta^r (n+1)(n+2)\gamma}{2\,\Delta}.$$

(f) When $\Delta \neq 0$, $A_t \neq 0$ *and* $B_t \neq 0$, *then* $\Psi_7 = M(n) - N(n)$, *where*

$$M(n) = \frac{\alpha^r\, ((n+1)A_{t(n+2)} - (n+2)A_{t(n+1)})\, \delta}{\Delta\, A_t^2},$$

$$N(n) = \frac{\beta^r\, ((n+1)B_{t(n+2)} - (n+2)B_{t(n+1)})\, \gamma}{\Delta\, B_t^2}.$$

Proof of (f). The proof will again be by induction on n.

For $n = 0$, the sum Ψ_7 is $\frac{\alpha^r\,\delta - \beta^r\,\gamma}{\Delta}$ while the right hand side clearly has the same value because $A_{2t} - 2\,A_t = A_t^2$ and $B_{2t} - 2\,B_t = B_t^2$. Hence, for the initial value of n the relation (f) holds.

Assume that (f) is true for $n = m$. Then

$$\sum_{i=0}^{m+1} (i+1)\, w_{r+ti} = \left(\sum_{i=0}^{m} (i+1)\, w_{r+ti} \right) + (m+2) w_{r+t(m+1)}$$
$$= (M(m) - N(m)) + (m+2) \left(\frac{\alpha^r \delta\, \alpha^{t(m+1)}}{\Delta} - \frac{\beta^r \gamma\, \beta^{t(m+1)}}{\Delta} \right)$$
$$= M(m+1) - N(m+1).$$

Hence, the relation (f) holds also for $n = m+1$. □

7. Alternating terms multiplied by natural numbers

Let Ψ_8 denote the sum $\sum_{i=0}^{n} (-1)^i\, (i+1) w_{r+t\,i}(a,\, b;\, p,\, q)$.

Theorem 8. *(a) When $\Delta = 0$ and $C_t = 0$, then $\Psi_8 = \Psi_7^{(a)}$.*

(b) When $\Delta = 0$ and $C_t \neq 0$, then $\Psi_8 = \frac{\alpha^{r-1}(\ell_n\, \alpha^{t(n+1)}\, M + N)}{C_t^3}$, with

$$N = C_t(r\,\gamma + a\,\alpha) - 2(C_t - 1)t,$$

$$M = C_t((n+1)C_t + 1)(r\,\gamma + a\,\alpha) + (n(n+1)C_t^2 + 2\,n\,C_t + 2)t.$$

(c) When $\Delta \neq 0$ and $C_t = 0$ and $D_t = 0$, then $\Psi_8 = \Psi_7^{(c)}$.

(d) When $\Delta \neq 0$, $C_t = 0$ and $D_t \neq 0$, then

$$\Psi_8 = \frac{\alpha^r (n+1)(n+2)\delta}{2\,\Delta} - \frac{\beta^r\, \gamma(\ell_n\, \beta^{t(n+1)}((n+1)D_t + 1) + 1)}{\Delta\, D_t^2}.$$

(e) When $\Delta \neq 0$, $C_t \neq 0$ and $D_t = 0$, then

$$\Psi_8 = \frac{\alpha^r\, \delta(\ell_n\, \alpha^{t(n+1)}((n+1)C_t + 1) + 1)}{\Delta\, C_t^2} - \frac{\beta^r (n+1)(n+2)\gamma}{2\,\Delta}.$$

(f) When $\Delta \neq 0$, $C_t \neq 0$ and $D_t \neq 0$, then $\Psi_8 = M - N$, where

$$M = \frac{\alpha^r\, \delta \left[1 + \ell_n\, \alpha^{t(n+1)}((n+1)C_t + 1)\right]}{C_t^2\, \Delta},$$

$$N = \frac{\beta^r\, \gamma \left[1 + \ell_n\, \beta^{t(n+1)}((n+1)D_t + 1)\right]}{D_t^2\, \Delta}.$$

Let Ψ_9 denote the sum $\sum_{i=0}^{n} k^i\, (i+1)\, w_{r+t\,i}(a,\, b;\, p,\, q)$ for any complex number k.

Theorem 9. *(a) When* $\Delta = 0$ *and* $E_t = 0$*, then* $\Psi_9 = \Psi_7^{(a)}$.

(b) When $\Delta = 0$ *and* $E_t \neq 0$*, then* $\Psi_9 = \alpha^{r-1}\left[\frac{M(\gamma r + a\,\alpha)}{E_t^2} + \frac{N\,\alpha^t\,k\,t\,\gamma}{E_t^3}\right]$*, where* M *and* N *are* $(n+1)(\alpha^t\,k)^{n+2} - (n+2)(\alpha^t\,k)^{n+1} + 1$ *and*

$$n(n+1)(\alpha^t\,k)^{n+2} - 2\,n(n+2)(\alpha^t\,k)^{n+1} + (n+1)(n+2)(\alpha^t\,k)^n - 2.$$

(c) When $\Delta \neq 0$ *and* $E_t = 0$ *and* $F_t = 0$*, then* $\Psi_9 = \Psi_7^{(c)}$.

(d) When $\Delta \neq 0$*,* $E_t = 0$ *and* $F_t \neq 0$*, then* Ψ_9 *is equal to*

$$\frac{\alpha^r(n+1)(n+2)\delta}{2\,\Delta} - \frac{\beta^r\,\gamma((n+1)(\beta^t\,k)^{n+2} - (n+2)(\beta^t\,k)^{n+1} + 1)}{\Delta\,F_t^2}.$$

(e) When $\Delta \neq 0$*,* $E_t \neq 0$ *and* $F_t = 0$*, then* Ψ_9 *is equal to*

$$\frac{\alpha^r\,\delta((n+1)(\alpha^t\,k)^{n+2} - (n+2)(\alpha^t\,k)^{n+1} + 1)}{\Delta\,E_t^2} - \frac{\beta^r(n+1)(n+2)\gamma}{2\,\Delta}.$$

(f) When $\Delta \neq 0$*,* $E_t \neq 0$ *and* $F_t \neq 0$*, then* $\Psi_9 = M - N$*, where*

$$M = \frac{\alpha^r\,\delta\left[(n+1)(\alpha^t\,k)^{n+2} - (n+2)(\alpha^t\,k)^{n+1} + 1\right]}{E_t^2\,\Delta},$$

$$N = \frac{\beta^r\,\gamma\left[(n+1)(\beta^t\,k)^{n+2} - (n+2)(\beta^t\,k)^{n+1} + 1\right]}{F_t^2\,\Delta}.$$

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