RINGS OF SQUARES AROUND ORTHOLOGIC TRIANGLES

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Abstract. We explore some properties of the geometric configuration when a ring of six squares with the same orientation are erected on the segments BD, DC, CE, EA, AF and FB connecting the vertices of two orthologic triangles ABC and DEF. The special case when DEF is the pedal triangle of a variable point P with respect to the triangle ABC was studied earlier by Bottema [1], Deaux [5], Erhmann and Lamoen [4], and Sashalmi and Hoffmann [8]. We extend their results and discover several new properties of this interesting configuration.

1. Introduction – Bottema's Theorem

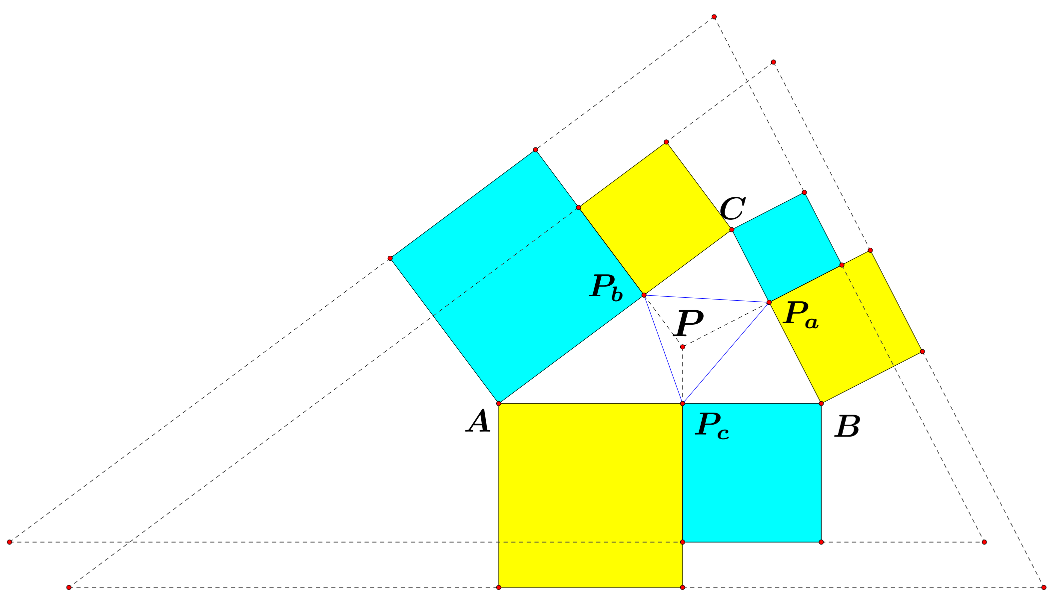

Figure 1. Bottema's Theorem on sums of areas of squares.

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The orthogonal projections P_a, P_b and P_c of a point P onto the sidelines BC, CA and AB of the triangle ABC are vertices of its pedal triangle. In [1], Bottema made the remarkable observation that

$$|BP_a|^2 + |CP_b|^2 + |AP_c|^2 = |P_aC|^2 + |P_bA|^2 + |P_cB|^2.$$

This equation has an interpretation in terms of area which is illustrated in Fig. 1. Rather than using geometric squares, other similar figures may be used as in [8].

Fig. 1 also shows two congruent triangles homothetic with the triangle ABC that are studied in [4] and [8].

The primary purpose of this paper is to extend Bottema's Theorem (see Fig. 2).

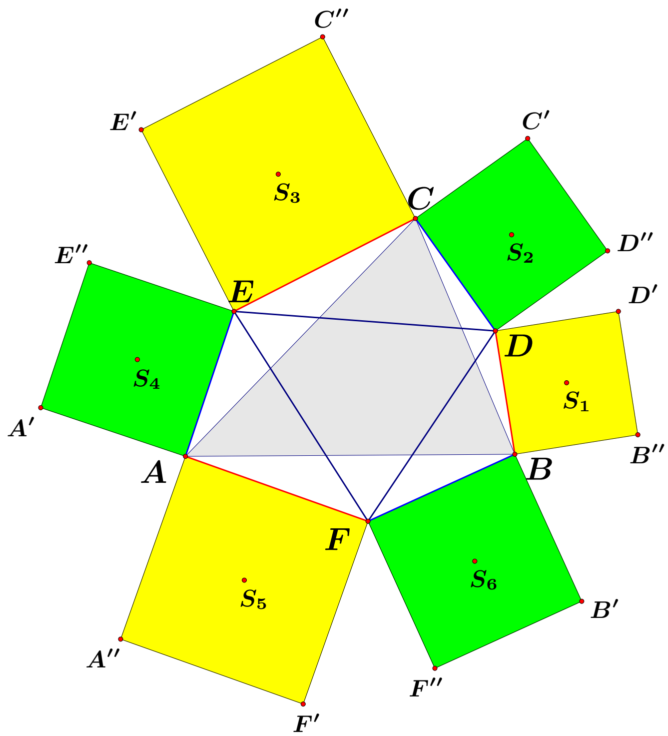

FIGURE 2. Notation for a ring of six squares around two triangles.

2. Connection with orthology

The origin of our generalization comes from asking if it is possible to replace the pedal triangle $P_aP_bP_c$ in Bottema's Theorem with some other triangles. In other words, if ABC and DEF are triangles in the plane, when will the following equality hold?

$$|BD|^2 + |CE|^2 + |AF|^2 = |DC|^2 + |EA|^2 + |FB|^2 \tag{1}$$

The straightforward analytic attempt to answer this question gives the following simple characterization of the equality (1).

Throughout, triangles will be non-degenerate.

Theorem 1. *The relation (1) holds for triangles ABC and DEF if and only if they are orthologic.*

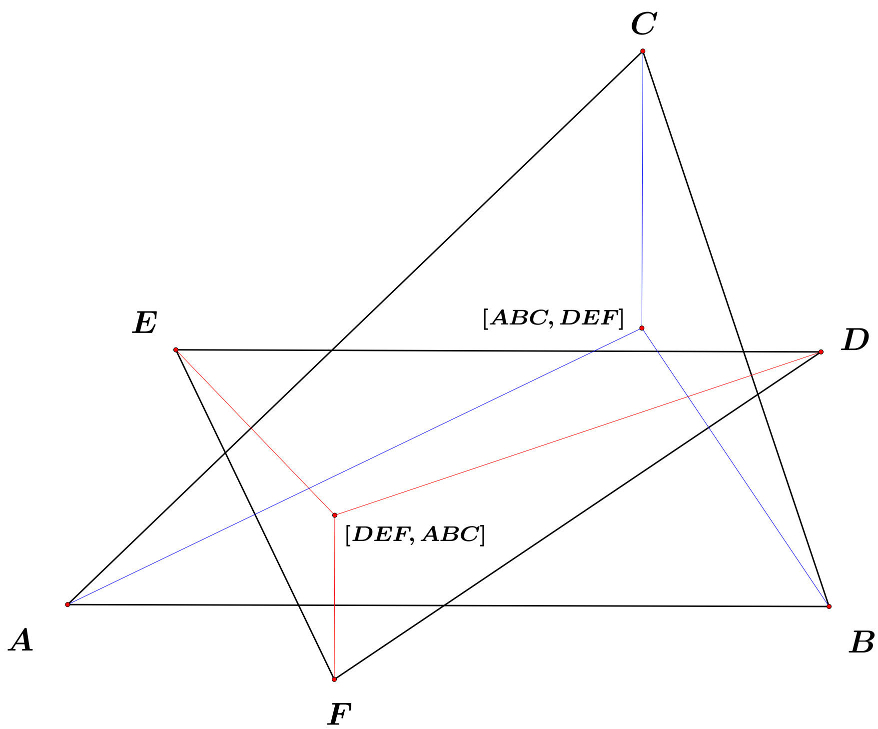

FIGURE 3. The triangles ABC and DEF are orthologic.

Recall that triangles ABC and DEF are *orthologic* provided the perpendiculars at vertices of ABC onto sides EF, FD and DE of DEF are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC, DEF]$. It is well-known that this relation is reflexive and symmetric. Hence, the perpendiculars from vertices of DEF onto the sides BC, CA, and AB are concurrent at the point $[DEF, ABC]$. These points are called the *first* and *second orthology centers* of the (orthologic) triangles ABC and DEF.

It is obvious that a triangle and the pedal triangle of any point are orthologic so that Theorem 1 extends Bottema's Theorem and the results in [8] (Theorem 3 and the first part of Theorem 5).

Proof of Theorem 1. The proofs in this paper will all be analytic.

In the rectangular coordinate system in the plane, we shall assume throughout that $A(0, 0)$, $B(1, 0)$, $C(u, v)$, $D(d, \delta)$, $E(e, \varepsilon)$ and $F(f, \varphi)$ for real numbers

u, v, d, δ, e, ε, f and φ. The lines will be treated as ordered triples of coefficients $(a,\, b,\, c)$ of their (linear) equations $a\,x + b\,y + c = 0$. Hence, the perpendiculars from the vertices of DEF onto the corresponding sidelines of ABC are $(u-1,\, v,\, d(1-u) - v\,\delta)$, $(u,\, v,\, -(u\,e + v\,\varepsilon))$ and $(1,\, 0,\, -f)$. They will be concurrent provided the determinant $v\,\Delta = v((u-1)\,d - ue + f + v\,(\delta - \varepsilon))$ of the matrix from them as rows is equal to zero. In other words, $\Delta = 0$ is a necessary and sufficient condition for ABC and DEF to be orthologic.

On the other hand, the difference of the right and the left side of (1) is $2\,\Delta$ which clearly implies that (1) holds if and only if ABC and DEF are orthologic triangles. $\square$

3. The triangles $S_1S_3S_5$ and $S_2S_4S_6$

We continue our study of the ring of six squares with the Theorem 2 about two triangles associated with the configuration. Like Theorem 1, this theorem detects when two triangles are orthologic. Recall that $S_1, \ldots, S_6$ are the centers of the squares in Fig. 2. Note that a similar result holds when the squares are folded inwards, and the proof is omitted.

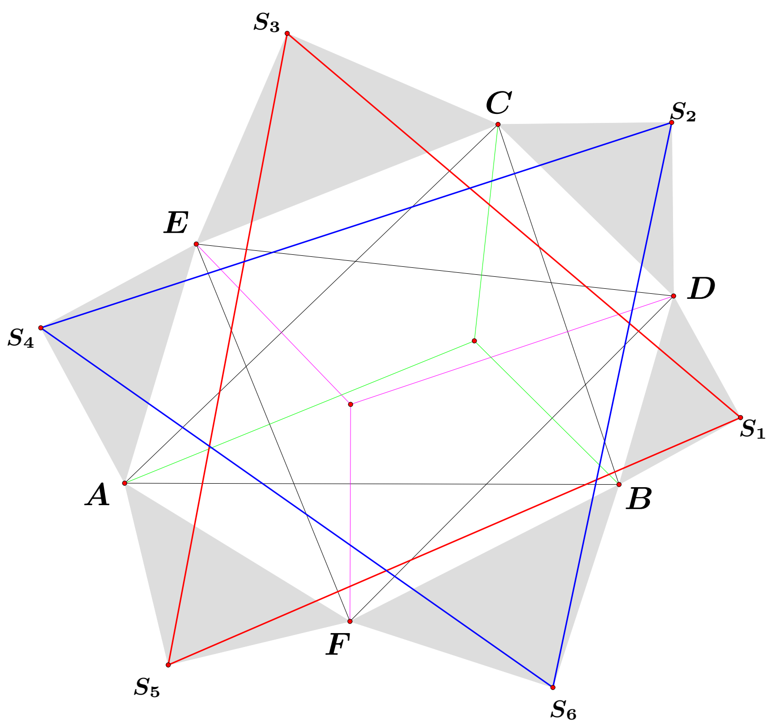

Figure 4. $|S_1S_3S_5| = |S_2S_4S_6|$ iff ABC and DEF are orthologic.

Theorem 2. *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof of Theorem 2. The vertices V and U of the square $DEVU$ have co-ordinates $(e+\varepsilon-\delta,\, \varepsilon+d-e)$ and $(d+\varepsilon-\delta,\, \delta+d-e)$. From this we infer easily coordinates of all points in Fig. 2.

$$A'(-\varepsilon,\, e), \qquad A''(\varphi,\, -f), \qquad B'(1-\varphi,\, f-1), \qquad B''(1+\delta,\, 1-d),$$
$$C'(v+u-\delta,\, v-u+d), \quad C''(u-v+\varepsilon,\, u+v-e), \quad D'(d+\delta,\, 1-d+\delta),$$
$$D''(d+v-\delta,\, d-u+\delta), \qquad E'(e-v+\varepsilon,\, u-e+\varepsilon), \qquad E''(e-\varepsilon,\, e+\varepsilon),$$
$$F'(f+\varphi,\, \varphi-f), \qquad F''(f-\varphi,\, f-1+\varphi), \qquad S_1((1+d+\delta)/2,\, (1-d+\delta)/2),$$
$$S_2((d+u+v-\delta)/2,\, (d-u+v+\delta)/2), \qquad S_3((u-v+e+\varepsilon)/2,\, (u+v-e+\varepsilon)/2),$$
$$S_4((e-\varepsilon)/2,\, (e+\varepsilon)/2), \quad S_5((f+\varphi)/2,\, (\varphi-f)/2), \quad S_6((f+1-\varphi)/2,\, (f-1+\varphi)/2).$$

Let P^x and P^y be the $x-$ and $y-$ coordinates of the point P. Since the area $|DEF|$ is a half of the determinant of the matrix with the rows $(D^x,\, D^y,\, 1)$, $(E^x,\, E^y,\, 1)$ and $(F^x,\, F^y,\, 1)$, the difference $|S_2S_4S_6| - |S_1S_3S_5|$ is $\frac{\Delta}{4}$. We conclude that the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

4. The first family of pairs of triangles

The triangles $S_1S_3S_5$ and $S_2S_4S_6$ are just one pair from a whole family of triangle pairs which all have the same property with a single notable exception.

For any real number t different from -1 and 0, let $S_1^t, \ldots, S_6^t$ denote points that divide the segments AS_1, AS_2, BS_3, BS_4, CS_5 and CS_6 in the ratio $t:1$. Let $\rho(P,\, \theta)$ denote the rotation about the point P through an angle θ. Let G_σ and G_τ be the centroids of ABC and DEF.

The following result is curios (See Figure 5) because the particular value $t=2$ gives a pair of congruent triangles regardless of the position of the triangles ABC and DEF.

Theorem 3. *The triangle $S_2^2S_4^2S_6^2$ is the image of the triangle $S_1^2S_3^2S_5^2$ under the rotation $\rho\left(G_\sigma,\, \frac{\pi}{2}\right)$. The radical axis of their circumcircles goes through the centroid G_σ.*

Proof of Theorem 3. Since the point that divides the segment DE in the ratio $2:1$ has coordinates $\left(\frac{d+2\,e}{3},\, \frac{\delta+2\,\varepsilon}{3}\right)$, it follows that

$$S_1^2\left(\frac{1+d+\delta}{3},\, \frac{1-d+\delta}{3}\right) \text{ and } S_2^2\left(\frac{d+u+v-\delta}{3},\, \frac{d-u+v+\delta}{3}\right).$$

Since $G_\sigma\left(\frac{1+u}{3},\, \frac{v}{3}\right)$, it is easy to check that S_2^2 is the vertex of a (negatively oriented) square on $G_\sigma S_1^2$. The arguments for the pairs $(S_3^2,\, S_4^2,)$ and $(S_5^2,\, S_6^2,)$ are analogous.

Finally, the proof of the claim about the radical axis starts with the observation that since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are congruent it suffices to show

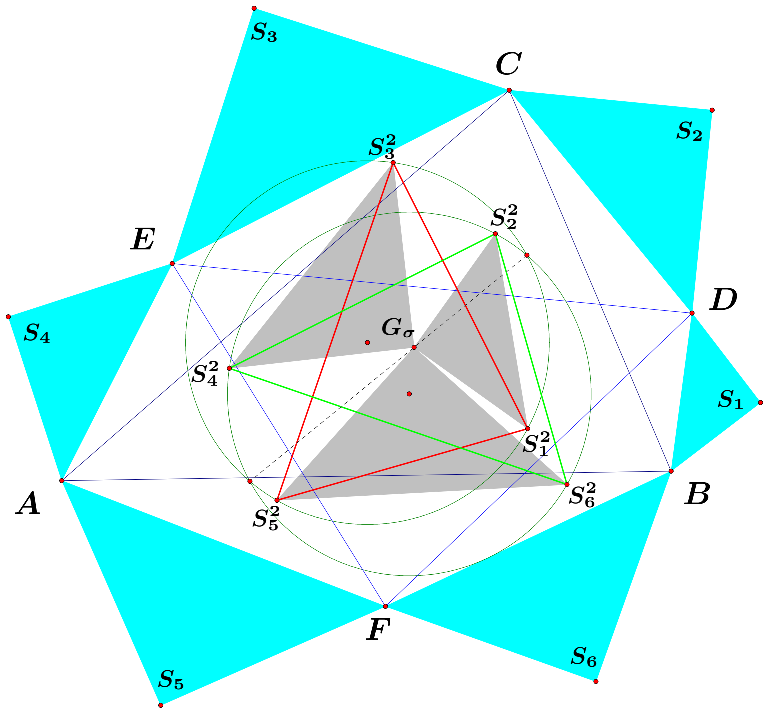

FIGURE 5. The triangles $S_1^2 S_3^2 S_5^2$ and $S_2^2 S_4^2 S_6^2$ are congruent.

that $|G_\sigma O_{odd}|^2 = |G_\sigma O_{even}|^2$, where O_{odd} and O_{even} are their circumcenters. This routine task was accomplished with the assistance of a computer algebra system. $\square$

The following result resembles Theorem 2 (see Figure 6) and shows that each pair of triangles from the first family could be used to detect if the triangles ABC and DEF are orthologic.

Theorem 4. *For any real number t different from -1, 0 and 2, the triangles $S_1^t S_3^t S_5^t$ and $S_2^t S_4^t S_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof of Theorem 3. Since the point that divides the segment DE in the ratio $t:1$ has coordinates $\left(\frac{d+te}{t+1}, \frac{\delta+t\varepsilon}{t+1}\right)$, it follows that the points S_i^t have the coordinates

$$S_1^t\left(\frac{t\,(1+d+\delta)}{2(t+1)}, \frac{t\,(1-d+\delta)}{2(t+1)}\right), \quad S_2^t\left(\frac{t\,(d+u+v-\delta)}{2(t+1)}, \frac{t\,(d-u+v+\delta)}{2(t+1)}\right),$$

$$S_3^t\left(\frac{2+t(u-v+e+\varepsilon)}{2(t+1)}, \frac{t\,(u+v-e+\varepsilon)}{2(t+1)}\right), \quad S_4^t\left(\frac{2+t(e-\varepsilon)}{2(t+1)}, \frac{t\,(e+\varepsilon)}{2(t+1)}\right),$$

$$S_5^t\left(\frac{2\,u+t(f+\varphi)}{2(t+1)}, \frac{2\,v-t(f-\varphi)}{2(t+1)}\right), S_6^t\left(\frac{2\,u+t(1+f-\varphi)}{2(t+1)}, \frac{2\,v-t(1-f-\phi)}{2(t+1)}\right).$$

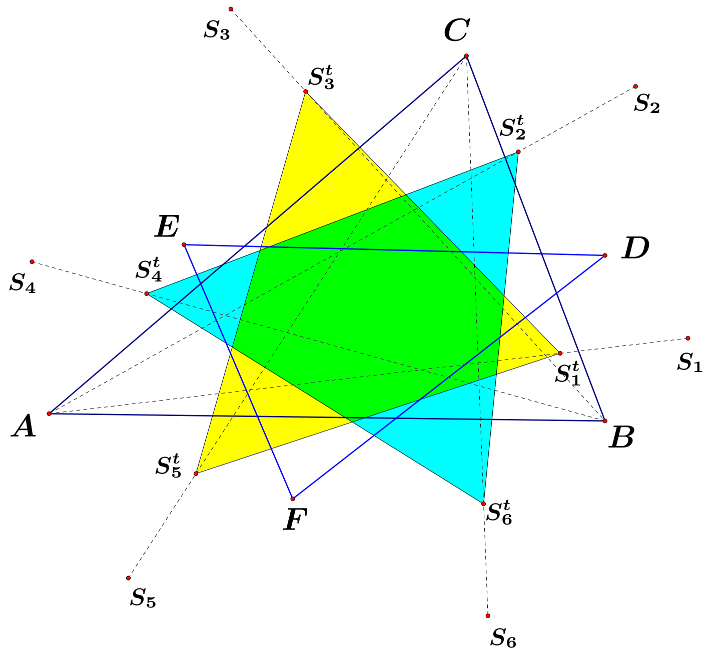

FIGURE 6. $|S_1^t S_3^t S_5^t| = |S_2^t S_4^t S_6^t|$ iff ABC and DEF are orthologic.

As in the proof of Theorem 2, we find that the difference of areas of the triangles $S_2^t S_4^t S_6^t$ and $S_1^t S_3^t S_5^t$ is $\frac{t(2-t)\Delta}{4(t+1)^2}$. Hence, for $t \neq -1$, 0, 2, the triangles $S_1^t S_3^t S_5^t$ and $S_2^t S_4^t S_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

5. The second family of pairs of triangles

The first family of pairs of triangles was constructed on lines joining the centers of the squares with the vertices A, B and C. In order to get the second analogous family we shall use instead lines joining midpoints of sides with the centers of the squares (see Figure 7). A slight advantage of the second family is that it has no exceptional cases.

Let A_g, B_g and C_g denote the midpoints of the segments BC, CA and AB. For any real number s different from -1, let $T_1^s, \ldots, T_6^s$ denote points that divide the segments A_gS_1, A_gS_2, B_gS_3, B_gS_4, C_gS_5 and C_gS_6 in the ratio $s:1$. Notice that $T_1^sT_2^sA_g$, $T_3^sT_4^sB_g$ and $T_5^sT_6^sC_g$ are isosceles triangles with the right angles at the vertices A_g, B_g and C_g.

Theorem 5. *For any real number s different from -1 and 0, the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ have equal area if and only if the triangles ABC and DEF are orthologic.*

Proof of Theorem 5. As in the proof of Theorem 4, we find that the difference of areas of the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ is $\frac{s\,\Delta}{4(s+1)}$. Hence, for $s \neq -1$, 0, the

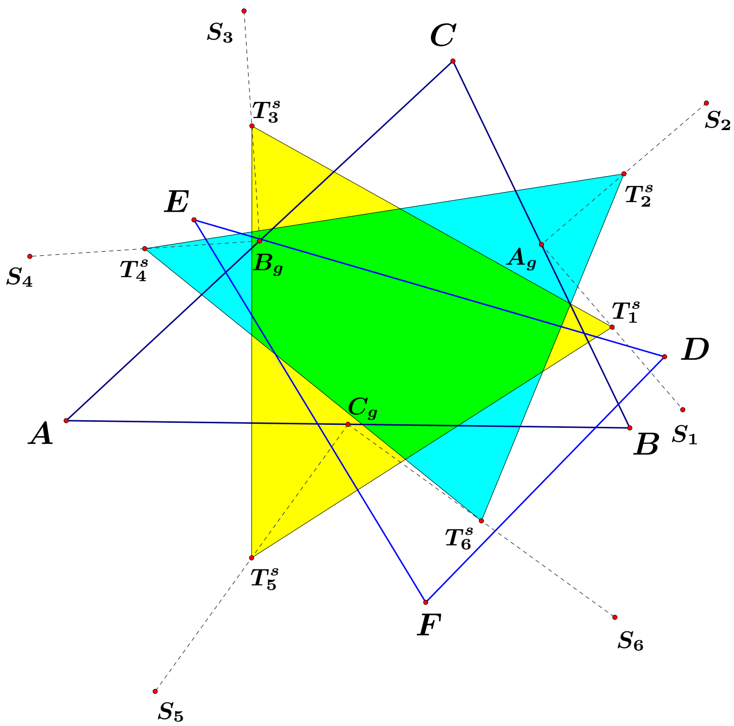

Figure 7. $|T_1^sT_3^sT_5^s| = |T_2^sT_4^sT_6^s|$ iff ABC and DEF are orthologic.

triangles $T_1^tT_3^tT_5^t$ and $T_2^tT_4^tT_6^t$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

6. The third family of pairs of triangles

When we look for reasons why the previous two families served our purpose of detecting orthology it is clear that the vertices of a triangle homothetic with ABC should be used. This leads us to consider a family of pairs of triangles that depend on two real parameters and a point (the center of homothety).

For any real numbers s and t different from -1 and any point P the points X, Y and Z divide the segments PA, PB and PC in the ratio $s : 1$ while the points $U_i^{(s,t)}$ for $i = 1, \dots, 6$ divide the segments XS_1, XS_2, YS_3, YS_4, ZS_5 and ZS_6 in the ratio $t : 1$.

The above results (Theorems 4 and 5) are special cases of the following theorem (see Figure 8).

Theorem 6. *For any point P and any real numbers $s \neq -1$ and $t \neq -1$, $\frac{2s}{s+1}$, the triangles $U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}$ and $U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}$ have equal areas if and only if the triangles ABC and DEF are orthologic.*

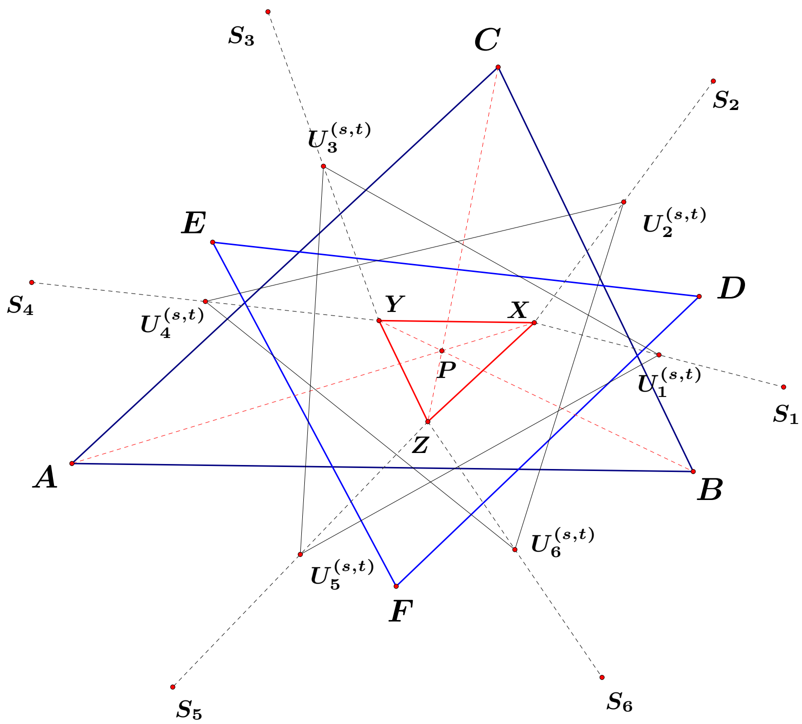

FIGURE 8. $|U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}| = |U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}|$ iff ABC and DEF are orthologic.

Proof of Theorem 6. Let the point P has the coordinates $(p,\, q)$. The point $U_1^{(s,t)}$ has coordinates $\left(\frac{(1+d+\delta)st+(1+d+\delta)t+2\,p}{2(s+1)(t+1)},\, \frac{(1-d+\delta)st+(1-d+\delta)t+2\,q}{2(s+1)(t+1)}\right)$. The other points $U_i^{(s,t)}$ for $i = 2, \ldots, 6$ have similar coordinates. As in the proof of Theorem 4, we find that the difference of areas of the triangles $U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}$ and $U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}$ is $\frac{t(2s-t(s+1))\,\Delta}{4(s+1)(t+1)^2}$. Hence, for $s \neq -1$ and $t \neq -1, \frac{2s}{s+1}$, the triangles $U_1^{(s,t)}U_3^{(s,t)}U_5^{(s,t)}$ and $U_2^{(s,t)}U_4^{(s,t)}U_6^{(s,t)}$ have equal area if and only if the triangles ABC and DEF are orthologic. $\square$

7. THE TRIANGLES $A_0B_0C_0$ AND $D_0E_0F_0$

In this section we shall see that the midpoints of the sides of the hexagon $S_1S_2S_3S_4S_5S_6$ also have some interesting properties.

Let A_0, B_0, C_0, D_0, E_0 and F_0 be the midpoints of the segments S_1S_2, S_3S_4, S_5S_6, S_4S_5, S_6S_1 and S_2S_3. Notice that the triangles $A_0B_0C_0$ and $D_0E_0F_0$ have as centroid the midpoint of the segment $G_\sigma G_\tau$.

Recall that triangles ABC and XYZ are *homologic* provided the lines AX, BY, and CZ are concurrent. In stead of homologic many authors use *perspective*.

Theorem 7. *(a) The triangles ABC and $A_0B_0C_0$ are orthologic if and only if the triangles ABC and DEF are orthologic.*

(b) The triangles DEF and $D_0E_0F_0$ are orthologic if and only if the triangles ABC and DEF are orthologic.

(c) If the triangles ABC and DEF are orthologic, then the triangles $A_0B_0C_0$ and $D_0E_0F_0$ are homologic.

Proof of Theorem 7. Let $D_1(d_1,\, \delta_1)$, $E_1(e_1,\, \varepsilon_1)$ and $F_1(f_1,\, \varphi_1)$. Recall from [2] that the triangles DEF and $D_1E_1F_1$ are orthologic if and only if $\Delta_0 = 0$, where

$$\Delta_0 = \Delta_0(DEF,\, D_1E_1F_1) = \begin{vmatrix} d & d_1 & 1 \\ e & e_1 & 1 \\ f & f_1 & 1 \end{vmatrix} + \begin{vmatrix} \delta & \delta_1 & 1 \\ \varepsilon & \varepsilon_1 & 1 \\ \varphi & \varphi_1 & 1 \end{vmatrix}.$$

Then *(a)* and *(b)* follow from the relations

$$\Delta_0(ABC,\, A_0B_0C_0) = -\frac{\Delta}{2} \quad \text{and} \quad \Delta_0(DEF,\, D_0E_0F_0) = \frac{\Delta}{2}.$$

The line DD_1 is $(\delta - \delta_1, d_1 - d, \delta_1\, d - d_1\, \delta)$, so that the triangles DEF and $D_1E_1F_1$ are homologic if and only if $\Gamma_0 = 0$, where

$$\Gamma_0 = \Gamma_0(DEF,\, D_1E_1F_1) = \begin{vmatrix} \delta - \delta_1 & d_1 - d & \delta_1\, d - d_1\, \delta \\ \varepsilon - \varepsilon_1 & e_1 - e & \varepsilon_1\, e - e_1\, \varepsilon \\ \varphi - \varphi_1 & f_1 - f & \varphi_1\, f - f_1\, \varphi \end{vmatrix}.$$

Part *(c)* follows from the observation that $\Gamma_0(A_0B_0C_0,\, D_0E_0F_0)$ contains Δ as a factor. $\square$

8. Triangles from centroids

Let G_1, G_2, G_3 and G_4 denote the centroids of the triangles $G_{12A}G_{34B}G_{56C}$, $G_{12D}G_{34E}G_{56F}$, $G_{45A}G_{61B}G_{23C}$ and $G_{45D}G_{61E}G_{23F}$ where G_{12A}, G_{12D}, G_{34B}, G_{34E}, G_{56C}, G_{56F}, G_{45A}, G_{45D}, G_{61B}, G_{61E}, G_{23C} and G_{23F} are centroids of the triangles S_1S_2A, S_1S_2D, S_3S_4B, S_3S_4E, S_5S_6C, S_5S_6F, S_4S_5A, S_4S_5D, S_6S_1B, S_6S_1E, S_2S_3C and S_2S_3F.

Theorem 8. *The points G_1 and G_2 are the points G_3 and G_4 respectively. The points G_1 and G_2 divide the segments $G_\sigma G_\tau$ and $G_\tau G_\sigma$ in the ratio* $1 : 2$.

Proof of Theorem 8. The centroids G_{12A}, G_{34B} and G_{56C} have the coordinates $\left(\frac{2\,d+1+v+u}{6},\, \frac{2\,\delta+1+v-u}{6}\right)$, $\left(\frac{2\,e+2+u-v}{6},\, \frac{2\,\varepsilon+v+u}{6}\right)$ and $\left(\frac{2\,f+2\,u+1}{6},\, \frac{2\,\varphi+2\,v-1}{6}\right)$. It follows that G_1 is $\left(\frac{d+e+f+2\,u+2}{9},\, \frac{\delta+\varepsilon+\varphi+2\,v}{9}\right)$. Similarly, G_2 is $\left(\frac{2\,d+2\,e+2\,f+1+u}{9},\, \frac{2\,\delta+2\,\varepsilon+2\,\varphi+v}{9}\right)$. It is now easy to check that $G_3 = G_1$ and $G_4 = G_2$.

Let G_1' divide the segment $G_\sigma G_\tau$ in the ratio $1 : 2$. Since $G_\tau\left(\frac{d+e+f}{3},\, \frac{\delta+\varepsilon+\varphi}{3}\right)$ and $G_\sigma\left(\frac{1+u}{3},\, \frac{v}{3}\right)$, we have $(G_1')^x = \frac{2\,(G_\sigma)^x+(G_\tau)^x}{3} = \frac{(2+2\,u)+(d+e+f)}{9} = (G_1)^x$. Of course, in the same way we see that $(G_1')^y = (G_1)^y$ and that G_2 divides $G_\tau G_\sigma$ in the same ratio $1 : 2$. $\square$

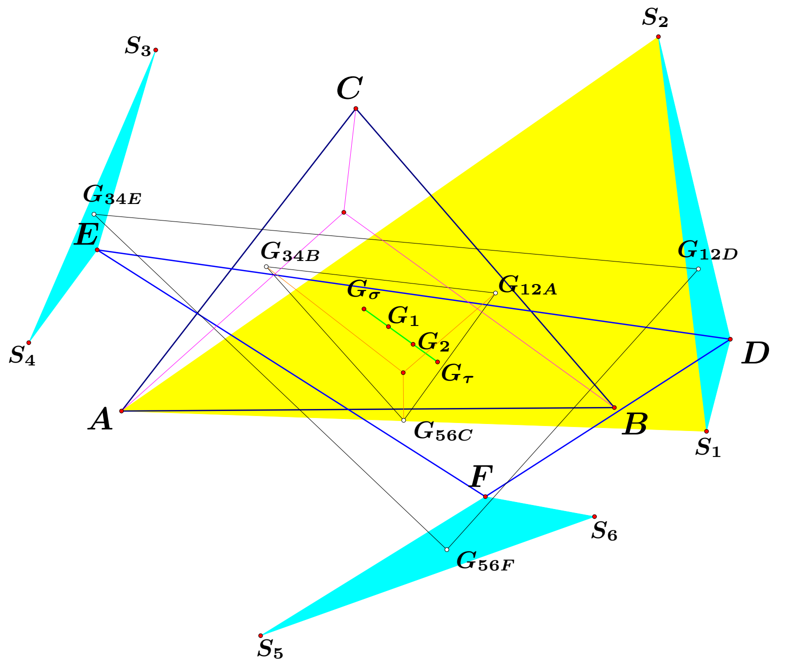

FIGURE 9. G_1 and G_2 divide $G_\sigma G_\tau$ in four equal parts and ABC is orthologic with $G_{12A}G_{34B}G_{56C}$ iff it is orthologic with DEF.

Theorem 9. *The following statements are equivalent:*

(a) The triangles ABC and $G_{12A}G_{34B}G_{56C}$ are orthologic.
(b) The triangles ABC and $G_{12D}G_{34E}G_{56F}$ are orthologic.
(c) The triangles DEF and $G_{45A}G_{61B}G_{23C}$ are orthologic.
(d) The triangles DEF and $G_{45D}G_{61E}G_{23F}$ are orthologic.
(e) The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.
(f) The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.
(g) The triangles ABC and DEF are orthologic.

Proof of Theorem 9. The equivalence of *(a)* and *(g)* follows from the relation

$$\Delta_0(ABC,\, G_{12A}G_{34B}G_{56C}) = \frac{\Delta}{3}.$$

The equivalence of *(g)* with *(b)*, *(c)*, *(d)*, *(e)* and *(f)* one can prove in the same way. □

9. FOUR TRIANGLES ON VERTICES OF SQUARES

In this section we consider four triangles $A'B'C'$, $D'E'F'$, $A''B''C''$, $D''E''F''$ which have twelve outer vertices of the squares as vertices. The sum of areas of the first two is equal to the sum of areas of the last two. The same relation holds if we replace the word "area" by the phrase "sum of the squares of the sides".

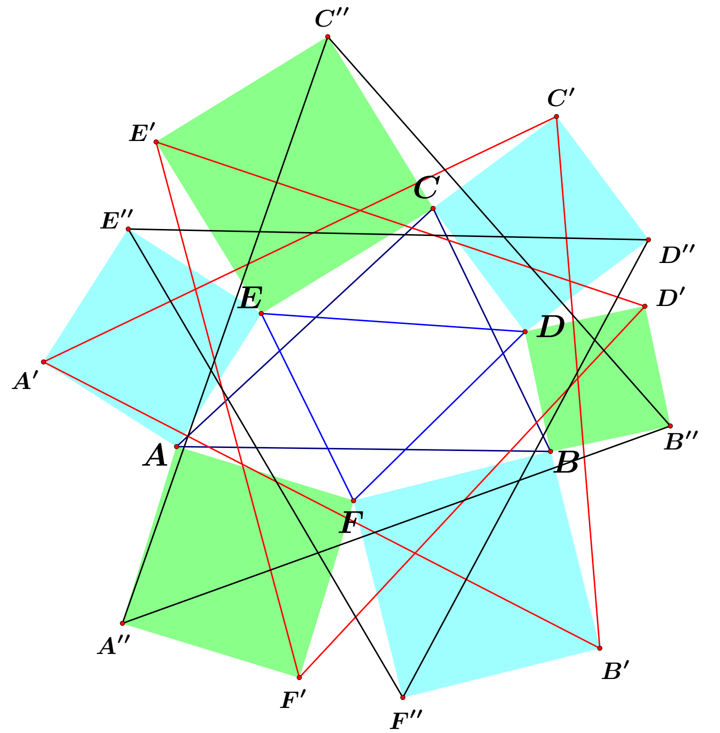

FIGURE 10. Four triangles $A'B'C'$, $D'E'F'$, $A''B''C''$ and $D''E''F''$.

For a triangle XYZ let $|XYZ|$ and $s_2(XYZ)$ denote its (oriented) area and the sum $|YZ|^2 + |ZX|^2 + |XY|^2$ of squares of lengths of its sides.

Theorem 10. *(a) The following equality for areas of triangles holds:*

$$|A'B'C'| + |D'E'F'| = |A''B''C''| + |D''E''F''|.$$

(b) The following equality also holds:

$$s_2(A'B'C') + s_2(D'E'F') = s_2(A''B''C'') + s_2(D''E''F'').$$

Proof of Theorem 10. The areas $|A'B'C'|$, $|D'E'F'|$, $|A''B''C''|$ and $|D''E''F''|$ are

$$\tfrac{1}{2}\left[(\varepsilon - \varphi + 1)\, d + (u + v - 1 - \delta + \varphi)\, e - (u + v - \delta + \varepsilon)\, f - \delta + (v - u + 1)\, \varepsilon + (u - v)\, \varphi + 2\, v\right],$$

$$\tfrac{1}{2}\left[(u - v + 2\, \varepsilon - 2\, \varphi)\, d - (1 + 2\, \delta - 2\, \varphi)\, e + (1 - u + v + 2\, \delta - 2\, \varepsilon)\, f + (u + v)\, \delta - \varepsilon + (1 - u - v)\, \varphi + v\right],$$

$$\tfrac{1}{2}\left[(u - v + \varepsilon - \varphi)\, d - (1 + \delta - \varphi)\, e + (1 - u + v + \delta - \varepsilon)\, f + (u + v)\, \delta - \varepsilon + (1 - u - v)\, \varphi + 2\, v\right],$$

$$\tfrac{1}{2}\left[(1 + 2\, \varepsilon - 2\, \varphi)\, d + (u + v - 1 - 2\, \delta + 2\, \varphi)\, e - (u + v - 2\, \delta + 2\, \varepsilon)\, f - \delta + (1 - u + v)\, \varepsilon + (u - v)\, \varphi + v\right],$$

By looking vertically at each term we see easily that (a) is true.

On the other hand, $s_2(A'B'C')$, $s_2(D'E'F')$, $s_2(A''B''C'')$ and $s_2(D''E''F'')$ are

$$2\left[2(u^2 - u + v^2 + 1) - (2\, d - e - f)(u - v) - (2\, \delta - \varepsilon - \varphi)(u + v) + d^2 + e^2 + f^2 + \delta^2 + \varepsilon^2 + \varphi^2 - de - df - ef - \delta\varepsilon - \delta\varphi - \varepsilon\varphi + d + e - 2f + \delta + \varepsilon - 2\varphi\right],$$

$$2\left[u^2-u+v^2+1+(d-2e+f)(u+v)+(\delta-2\varepsilon+\varphi)(v-u)+2(d^2+e^2+\right.$$
$$\left.f^2+\delta^2+\varepsilon^2+\varphi^2-de-df-ef-\delta\varepsilon-\delta\varphi-\varepsilon\varphi)-2d+e+f+2\delta-\varepsilon-\varphi\right],$$

$$2\left[2(u^2-u+v^2+1)+(d-2e+f)(u+v)+(\delta-2\varepsilon+\varphi)(v-u)+d^2+e^2+\right.$$
$$\left.f^2+\delta^2+\varepsilon^2+\varphi^2-de-df-ef-\delta\varepsilon-\delta\varphi-\varepsilon\varphi-2d+e+f+2\delta-\varepsilon-\varphi\right],$$

$$2\left[u^2-u+v^2+1+(2d-e-f)(v-u)-(2\delta-\varepsilon-\varphi)(v+u)+2(d^2+e^2+\right.$$
$$\left.f^2+\delta^2+\varepsilon^2+\varphi^2-de-df-ef-\delta\varepsilon-\delta\varphi-\varepsilon\varphi)-d+e-2f+\delta+\varepsilon-2\varphi\right],$$

Again, by direct inspection, we see that (b) holds. □

Let A_1', B_1' and C_1' denote centers of squares of the same orientation built on the segments $B'C'$, $C'A'$ and $A'B'$. The points D_1', E_1', F_1', A_1'', B_1'', C_1'', D_1'', E_1'' and F_1'' are defined analogously. Notice that $(A'B'C', A_1'B_1'C_1')$, $(A''B''C'', A_1''B_1''C_1'')$, $(D'E'F', D_1'E_1'F_1')$ and $(D''E''F'', D_1''E_1''F_1'')$ are four pairs of both orthologic and homologic triangles.

The following theorem claims that the four triangles from these centers of squares retain the same property regarding sums of areas and sums of squares of lengths of sides.

Theorem 11. *(a) The following equality for areas of triangles holds:*
$$|A_1'B_1'C_1'|+|D_1'E_1'F_1'|=|A_1''B_1''C_1''|+|D_1''E_1''F_1''|.$$
(b) The following equality also holds:
$$s_2(A_1'B_1'C_1')+s_2(D_1'E_1'F_1')=s_2(A_1''B_1''C_1'')+s_2(D_1''E_1''F_1'').$$

Notice that in the above theorem we can take instead of the centers any points that have the same position with respect to the squares erected on the sides of the triangles $A'B'C'$, $D'E'F'$, $A''B''C''$ and $D''E''F''$. Also, there are obvious extensions of the previous two theorems from two triangles to the statements about two n-gons for any integer $n > 3$.

Of course, it is possible to continue the above sequences of triangles and define for every integer $k \geq 0$ the triangles $A_k'B_k'C_k'$, $A_k''B_k''C_k''$, $D_k'E_k'F_k'$ and $D_k''E_k''F_k''$. The sequences start with $A'B'C'$, $A''B''C''$, $D'E'F'$ and $D''E''F''$. Each member is homologic, orthologic, and shares the centroid with all previous members and for each k an analogue of Theorem 11 is true.

Proof of Theorem 11. The (a) part follows from Theorem 10 and the observation that the expressions $\frac{8\,|A_1'B_1'C_1'|-s_2(A'B'C')}{4}$, $\frac{8\,|D_1'E_1'F_1'|-s_2(D'E'F')}{4}$, $\frac{8\,|A_1''B_1''C_1''|-s_2(A''B''C'')}{4}$ and $\frac{8\,|D_1''E_1''F_1''|-s_2(D''E''F'')}{4}$ are respectively equal to

$$(e-f)(u+v)+(\varepsilon-\varphi)(u-v)+(1+\varepsilon-\varphi)d-(1+\delta-\varphi)e+(\delta-\varepsilon)f-\delta+\varepsilon+2v,$$

$$(d-f)(u-v)+(\delta-\varphi)(u+v)+2(\varepsilon-\varphi)d-(1+2\delta-2\varphi)e+(1+2\delta-2\varepsilon)f-\varepsilon+\varphi+v,$$

$$(d-f)(u-v)+(\delta-\varphi)(u+v)+(\varepsilon-\varphi)d-(1+\delta-\varphi)e+(1+\delta-\varepsilon)f-\varepsilon+\varphi+2v,$$

$$(e-f)(u+v)+(\varepsilon-\varphi)(v-u)+(1+2\varepsilon-2\varphi)d-(1+2\delta-2\varphi)e+2(\delta-\varepsilon)f-\delta+\varepsilon+v.$$

On the other hand, the (b) part follows also from Theorem 10 and the observation that $\frac{s_2(A_1'B_1'C_1')-s_2(A'B'C')}{3}$, $\frac{s_2(D_1'E_1'F_1')-s_2(D'E'F')}{3}$, $\frac{s_2(A_1''B_1''C_1'')-s_2(A''B''C'')}{3}$ and $\frac{s_2(D_1''E_1''F_1'')-s_2(D''E''F'')}{3}$ are respectively equal to the same above expressions. $\square$

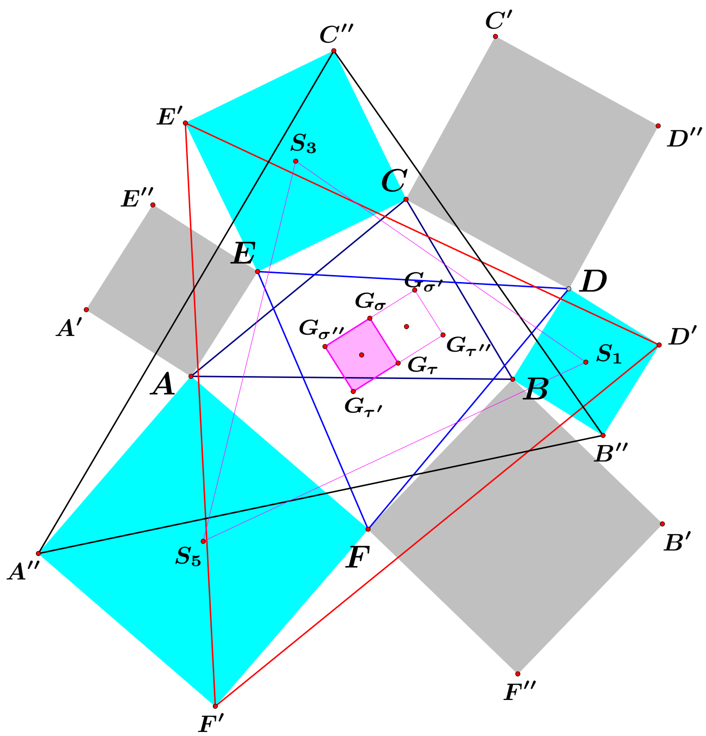

FIGURE 11. $G_\sigma G_\tau G_{\tau'} G_{\sigma''}$ and $G_\sigma G_\tau G_{\tau''} G_{\sigma'}$ are squares.

10. The centroids of the four triangles

Let $G_{\sigma'}$, $G_{\tau'}$, $G_{\sigma''}$, $G_{\tau''}$, G_o and G_e be shorter notation for the centroids $G_{A'B'C'}$, $G_{D'E'F'}$, $G_{A''B''C''}$, $G_{D''E''F''}$, $G_{S_1S_3S_5}$ and $G_{S_2S_4S_6}$. The following theorem shows that these centroids are the vertices of three squares associated with the ring of six squares.

Theorem 12. *(a) The centroids $G_{\sigma''}$, $G_{\tau'}$, G_τ and G_σ are vertices of a square.*

(b) The centroids $G_{\sigma'}$ and $G_{\tau''}$ are reflections of the centroids $G_{\sigma''}$ and $G_{\tau'}$ in the line $G_\sigma G_\tau$. Hence, the centroids $G_{\tau''}$, $G_{\sigma'}$, G_σ and G_τ are also vertices of a square.

(c) The centroids G_e and G_o are the centers of the squares in (a) and (b), respectively. Hence, the centroids G_σ, G_e, G_τ and G_o are also vertices of a square.

Proof of Theorem 12. (a) The centroid $G_{\sigma''}$ is $\left(\frac{\delta+\varepsilon+\varphi+u-v+1}{3}, \frac{u+v+1-d-e-f}{3}\right)$ while the centroids G_σ and G_τ are $\left(\frac{u+1}{3}, \frac{v}{3}\right)$ and $\left(\frac{d+e+f}{3}, \frac{\delta+\varepsilon+\varphi}{3}\right)$. In other words, $G_{\sigma''}$ is the vertex above G_σ of a (negatively oriented) square built on the segment $G_\sigma G_\tau$. Its fourth vertex is $G_{\tau'}$.

(b) The proof is very similar to the proof of (a). The only difference is that the positively oriented square on $G_\sigma G_\tau$ appears.

(c) The centroid G_e is $\left(\frac{d+e+f-\delta-\varepsilon-\varphi+u+v+1}{6}, \frac{d+e+f+\delta+\varepsilon+\varphi-u+v-1}{6}\right)$ and these are precisely the coordinates of the center of the (negatively oriented) square built on the segment $G_\sigma G_\tau$ (look at the coordinates of the point S_2 and apply the rule to the points G_σ and G_τ instead of the points D and C). The argument for the centroid G_o is similar. □

11. Remarkable midpoints

Let A^*, B^*, C^*, D^*, E^*, F^*, D_g, E_g and F_g denote the midpoints of the segments $A'A''$, $B'B''$, $C'C''$, $D'D''$, $E'E''$, $F'F''$, EF, FD and DE. Notice that the points A^*, B^*, C^*, D^*, E^* and F^* are the centers of squares built on the segments S_4S_5, S_6S_1, S_2S_3, S_1S_2, S_3S_4 and S_5S_6, respectively. Also, the triangles $A^*B^*C^*$ and $D^*E^*F^*$ share the centroids with the triangles ABC and DEF.

Theorem 13. *The triangles ABC and DEF are orthologic if and only if the triangles $A^*B^*C^*$ and $D^*E^*F^*$ are orthologic.*

Proof of Theorem 13. Since the coordinates of A^*, B^*, C^*, D^*, E^* and F^* are $\left(\frac{\varphi-\varepsilon}{2}, \frac{e-f}{2}\right)$, $\left(1+\frac{\delta-\varphi}{2}, \frac{f-d}{2}\right)$, $\left(u-\frac{\delta-\varepsilon}{2}, v+\frac{d-e}{2}\right)$, $\left(d+\frac{v}{2}, \delta-\frac{u-1}{2}\right)$, $\left(e-\frac{v}{2}, \varepsilon+\frac{u}{2}\right)$, and $\left(f, \varphi-\frac{1}{2}\right)$, we easily get $4\,\Delta_0(A^*B^*C^*, D^*E^*F^*) = -\Delta$. □

Theorem 14. *The triangles ABC and $A^*B^*C^*$ are homologic if and only if the triangles ABC and DEF are orthologic or the points D, E and F are collinear. Also, the triangles DEF and $D^*E^*F^*$ are homologic if and only if the triangles ABC and DEF are orthologic or the points A, B and C are collinear.*

Proof of Theorem 14. The first part follows from the relation

$$4\,\Gamma_0(ABC, A^*B^*C^*) + |DEF|\,\Delta = 0,$$

while the second is a consequence of $4\,\Gamma_0(DEF, D^*E^*F^*) + |ABC|\,\Delta = 0$. □

Theorem 15. *The lines AA^*, BB^*, CC^*, DD^*, EE^* and FF^* are perpendicular to the sidelines EF, FD, DE, BC, CA and AB, respectively. Moreover, $|AA^*| = \frac{|EF|}{2}$, $|BB^*| = \frac{|FD|}{2}$, $|CC^*| = \frac{|DE|}{2}$, $|DD^*| = \frac{|BC|}{2}$, $|EE^*| = \frac{|CA|}{2}$ and $|FF^*| = \frac{|AB|}{2}$.*

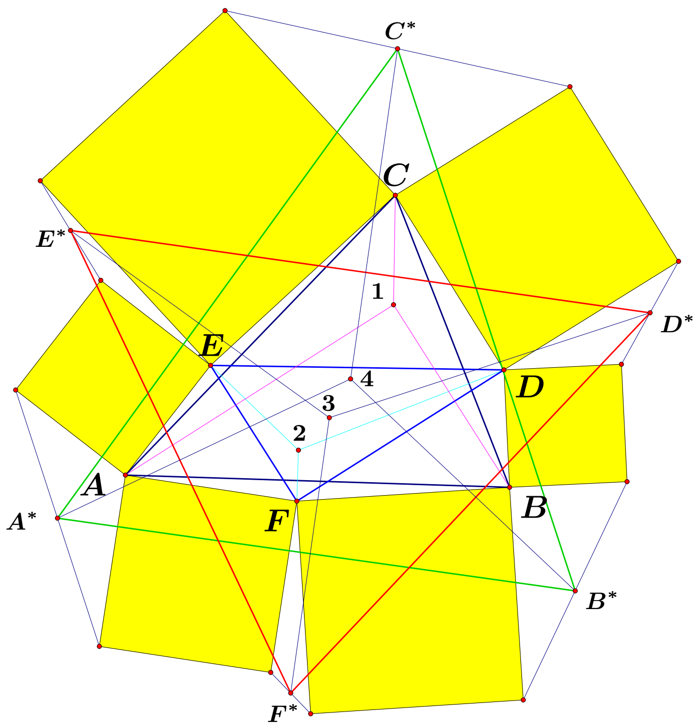

FIGURE 12. The triangles ABC and DEF are orthologic iff the triangles $A^*B^*C^*$ and $D^*E^*F^*$ are orthologic.

Proof of Theorem 15. The lines EF and AA^* are $(\varepsilon - \varphi,\, f - e,\, e\,\varphi - f\,\varepsilon)$ and $(e - f,\, \varepsilon - \varphi,\, 0)$. Clearly, they are perpendicular because the condition for lines $(a,\, b,\, c)$ and $(p,\, q,\, r)$ to be perpendicular is $a\,p + b\,q = 0$. On the other hand, the formula for the Euclidean distance gives $|AA^*| = \frac{1}{2}\sqrt{(e-f)^2 + (\varepsilon - \varphi)^2} = \frac{|EF|}{2}$. The arguments for the other corresponding pairs of points are similar. □

Theorem 16. *If the triangles ABC and DEF are orthologic then the homology centers of $(ABC,\, A^*B^*C^*)$ and $(DEF,\, D^*E^*F^*)$ are the orthology centers $[ABC, DEF]$ and $[DEF, ABC]$.*

Proof of Theorem 16. This follows from the first part of Theorem 15. □

Theorem 17. *The triangles ABC and DEF are orthologic if and only if the triangles ABC and $D^*E^*F^*$ and/or the triangles DEF and $A^*B^*C^*$ are orthologic.*

Proof of Theorem 17. This follows from the relations $\Delta_0(ABC,\, D^*E^*F^*) = -\Delta$ and $\Delta_0(DEF,\, A^*B^*C^*) = \Delta$. □

Let A_a, B_a and C_a denote the vertices of the anticomplementary triangle of the triangle ABC.

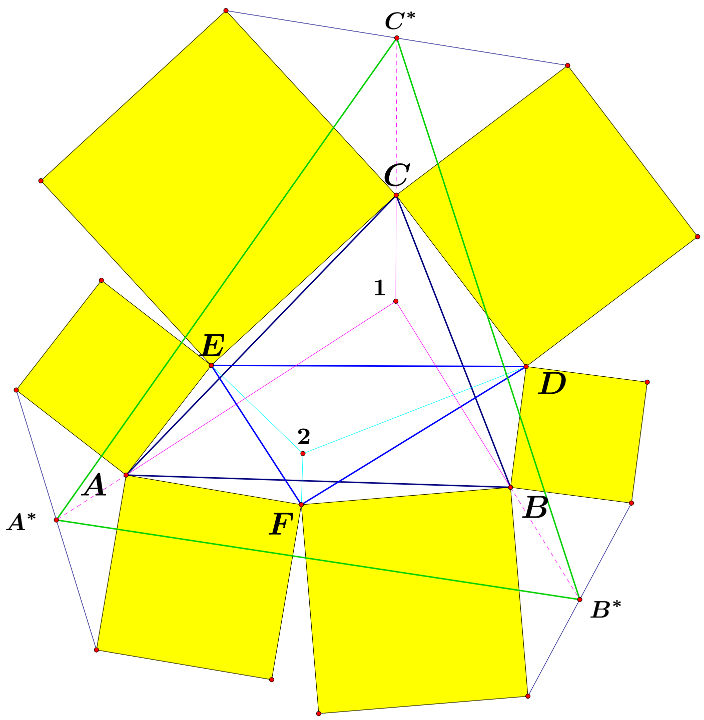

FIGURE 13. If the triangles ABC and DEF are orthologic, then the triangles ABC and $A^*B^*C^*$ are homologic.

Theorem 18. *If the triangles ABC and DEF are orthologic, then the following are pairs of homologic triangles: $(A^*B^*C^*,\, A_aB_aC_a)$, $(D^*E^*F^*,\, D_aE_aF_a)$, $(A^*B^*C^*,\, A_gB_gC_g)$ and $(D^*E^*F^*,\, D_gE_gF_g)$.*

Proof of Theorem 18. Since $A_a(u+1,\, v)$, $B_a(u-1,\, v)$ and $C_a(1-u,\, -v)$, we get $4\,\Gamma_0(A^*B^*C^*,\, A_aB_aC_a) = \Delta\, M$, where the factor M is $(\varepsilon - \varphi + 2u + 2)d + (\varphi - \delta + 2u - 4)e + (\delta - \varepsilon - 4u + 2)f + 2v(\delta + \varepsilon - 2\varphi + 6)$. It follows that if the triangles ABC and DEF are orthologic, then the triangles $A^*B^*C^*$ and $A_aB_aC_a$ are homologic. The other three pairs are treated similarly. $\square$

12. Another set of midpoints

Let A_m, D_m, B_m, E_m, C_m and F_m be midpoints of the segments BD, DC, CE, EA, AF and FB.

Theorem 19. *(a) The common centroid of the triangles $A_mB_mC_m$ and $D_mE_mF_m$ is the midpoint of the segment $G_\sigma G_\tau$.*

(b) The triangles ABC and DEF are orthologic if and only if the triangles $A_mB_mC_m$ and $D_mE_mF_m$ have equal sums of squares of lengths of sides.

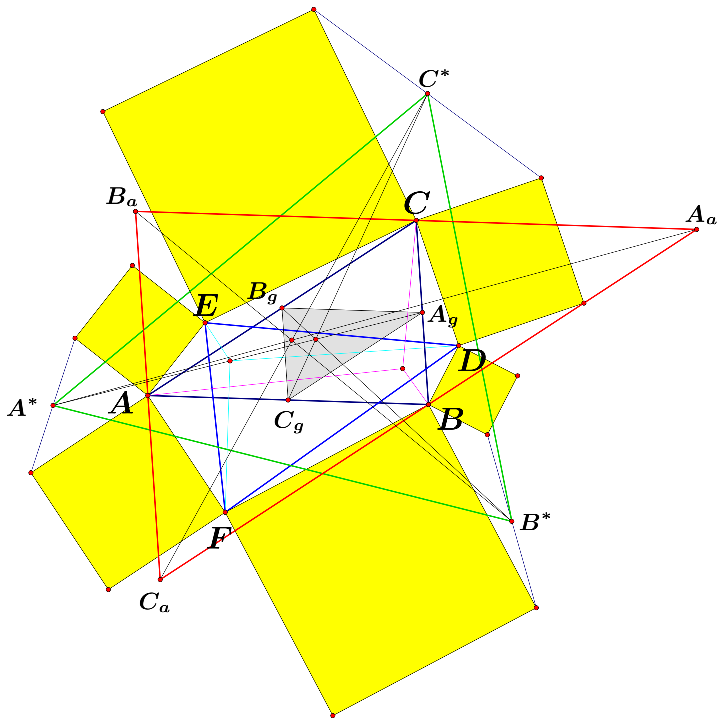

FIGURE 14. If the triangles ABC and DEF are orthologic, then the triangle $A^*B^*C^*$ is homologic with $A_aB_aC_a$ and $A_gB_gC_g$.

Proof of Theorem 19. (a) The triangles $A_mB_mC_m$ and $D_mE_mF_m$ have the point $\left(\frac{d+e+f+u+1}{6}, \frac{\delta+\varepsilon+\varphi+v}{6}\right)$ as a common centroid and these are precisely the coordinates of the midpoint of the segment $G_\sigma G_\tau$.

(b) This follows from the relation $s_2(D_mE_mF_m) - s_2(A_mB_mC_m) = \frac{3}{2}\,\Delta$. $\square$

Let s be a real number different from 0 and -1. Let the points A_s, B_s and C_s divide the segments BD, CE and AF in the ratio $s:1$ and let the points D_s, E_s and F_s divide the segments DC, EA and FB in the ratio $1:s$. The above theorem is a special case (for $s=1$) of the following result.

Theorem 20. *(a) The common centroid of the triangles $A_sB_sC_s$ and $D_sE_sF_s$ divides the segment $G_\sigma G_\tau$ in the ratio $s:1$.*

(b) The triangles ABC and DEF are orthologic if and only if the triangles $A_sB_sC_s$ and $D_sE_sF_s$ have equal sums of squares of lengths of sides.

(c) When the triangles ABC and DEF are orthologic then neither the triangle $A_sB_sC_s$ nor the triangle $D_sE_sF_s$ is orthologic with the triangle ABC.

Proof of Theorem 20. (a) The points A_s, B_s, C_s, D_s, E_s and F_s have the coordinates $\left(\frac{1+sd}{s+1}, \frac{s\delta}{s+1}\right)$, $\left(\frac{u+se}{s+1}, \frac{v+s\varepsilon}{s+1}\right)$, $\left(\frac{sf}{s+1}, \frac{s\varphi}{s+1}\right)$, $\left(\frac{sd+u}{s+1}, \frac{s\delta+v}{s+1}\right)$, $\left(\frac{se}{s+1}, \frac{s\varepsilon}{s+1}\right)$ and $\left(\frac{sf+1}{s+1}, \frac{s\varphi}{s+1}\right)$ so that the triangles $A_sB_sC_s$ and $D_sE_sF_s$ have the point $\left(\frac{(d+e+f)s+1+u}{3(s+1)}, \frac{(\delta+\varepsilon+\varphi)s+v}{3(s+1)}\right)$

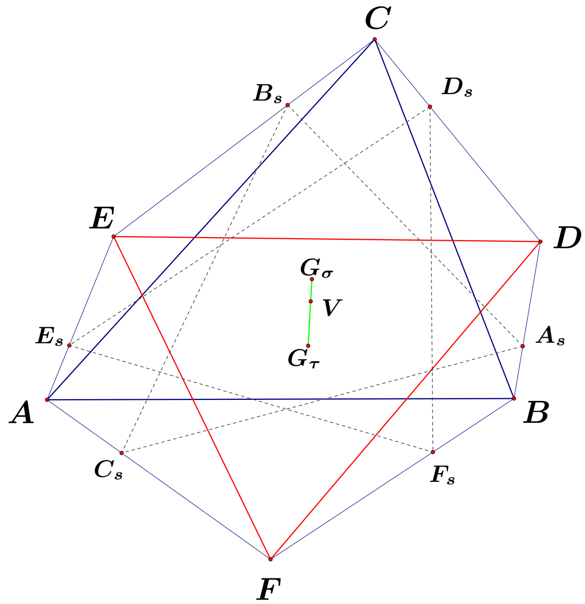

FIGURE 15. The common centroid V of the triangles $A_sB_sC_s$ and $D_sE_sF_s$ divides $G_\sigma G_\tau$ in the ratio $s : 1$.

as a common centroid and these are precisely the coordinates of the point which divides the segment $G_\sigma G_\tau$ in the ratio $s : 1$.

(b) This follows from the relation $s_2(D_sE_sF_s) - s_2(A_sB_sC_s) = \frac{6s}{(s+1)^2}\,\Delta$.

(c) We know that the triangles ABC and DEF are orthologic if and only if $f = (1-u)\,d + ue + v\,(\varepsilon - \delta)$. If we compute the expressions $\Delta_0(A_sB_sC_s,\, ABC)$ and $\Delta_0(D_sE_sF_s,\, ABC)$ for this value of f, we get the quotients $\frac{u-u^2-v^2-1}{s+1}$ and $\frac{u^2-u+v^2\,1}{s+1}$ which could never be equal to zero (for real values of u and v). $\square$

13. Improvement of Ehrmann–Lamoen results

Let $K_aK_bK_c$ be a triangle from intersections of parallels to the lines BC, CA and AB through the points B'', C'' and A''. Similarly, $L_aL_bL_c$, $M_aM_bM_c$, $N_aN_bN_c$, $P_aP_bP_c$ and $Q_aQ_bQ_c$ are constructed in the same way through the triples of points (C', A', B'), (D'', E'', F''), (D', E', F'), (S_1, S_3, S_5) and (S_2, S_4, S_6), respectively. Some of these triangles have been considered in the case when the triangle DEF is the pedal triangle $P_aP_bP_c$ of the point P. Work has been done by Ehrmann and Lamoen in [4] and also by Hoffmann and Sashalmi in [8]. In this section we shall see that natural analogues of their results hold in more general situations.

Theorem 21. *(a) The triangles $K_aK_bK_c$, $L_aL_bL_c$, $M_aM_bM_c$, $N_aN_bN_c$, $P_aP_bP_c$ and $Q_aQ_bQ_c$ are each homothetic with the triangle ABC.*

(b) The quadrangles $K_aL_aM_aN_a$, $K_bL_bM_bN_b$ and $K_cL_cM_cN_c$ are parallelograms.

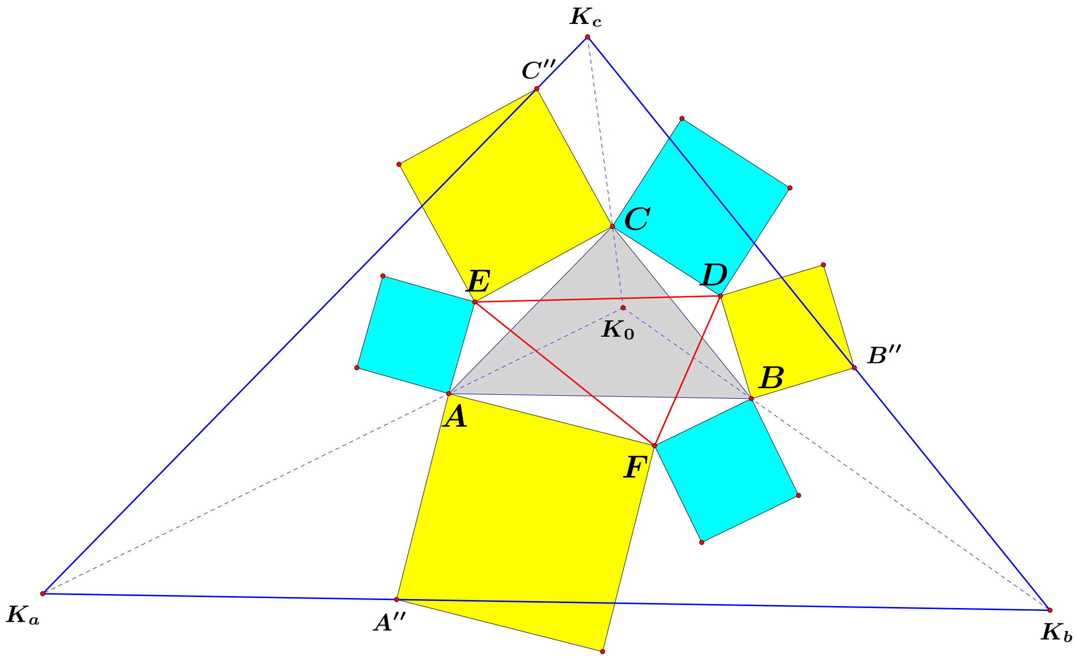

FIGURE 16. The triangle $K_a K_b K_c$ from parallels to BC, CA, AB through B'', C'', A'' is homothetic to ABC from the center K_0.

(c) The centers J_a, J_b and J_c of these parallelograms are the vertices of a triangle that is also homothetic with the triangle ABC.

Proof of Theorem 21. (a) The parallels to the lines AB and CA through the points A'' and C'' are $(0,\, 1,\, f)$ and $(v,\, -u,\, u^2+v^2-eu-\varepsilon v)$ so that their intersection K_a is $\left(\frac{u(e-f)+\varepsilon v-u^2-v^2}{v},\, -f\right)$. In a similar way we find that the points K_b and K_c have coordinates $\left(\frac{(u-1)(d-f)+\delta v-u+v+1}{v},\, -f\right)$ and $(p_c,\, q_c)$ with

$$p_c = \frac{u(u-1)(d-e)+v(u\delta-(u-1)\varepsilon)+u^3+uv^2-2\,u^2+uv-v^2+u}{v}$$

and $q_c = (u-1)d - ue + v\delta - v\varepsilon + u^2 - u + v^2 + v + 1$. The lines AK_a, BK_b and CK_c intersect in the point $K_0(p_0,\, q_0)$ where $p_0 = \frac{-ue-v\varepsilon+uf+v^2+u^2}{(u-1)d+v\delta-ue-v\varepsilon+f+u^2-u+v^2+1}$ and $q_0 = \frac{vf}{(u-1)d+v\delta-ue-v\varepsilon+f+u^2-u+v^2+1}$. Moreover, the pairs of lines $(BC,\, K_bK_c)$, $(CA,\, K_cK_a)$ and $(AB,\, K_aK_b)$ are parallel so that we conclude that the triangle $K_aK_bK_c$ is homothetic with the triangle ABC.

(b) The simplest method is to prove that the midpoints of the segments K_xM_x and L_xN_x coincide for $x = a,\, b,\, c$.

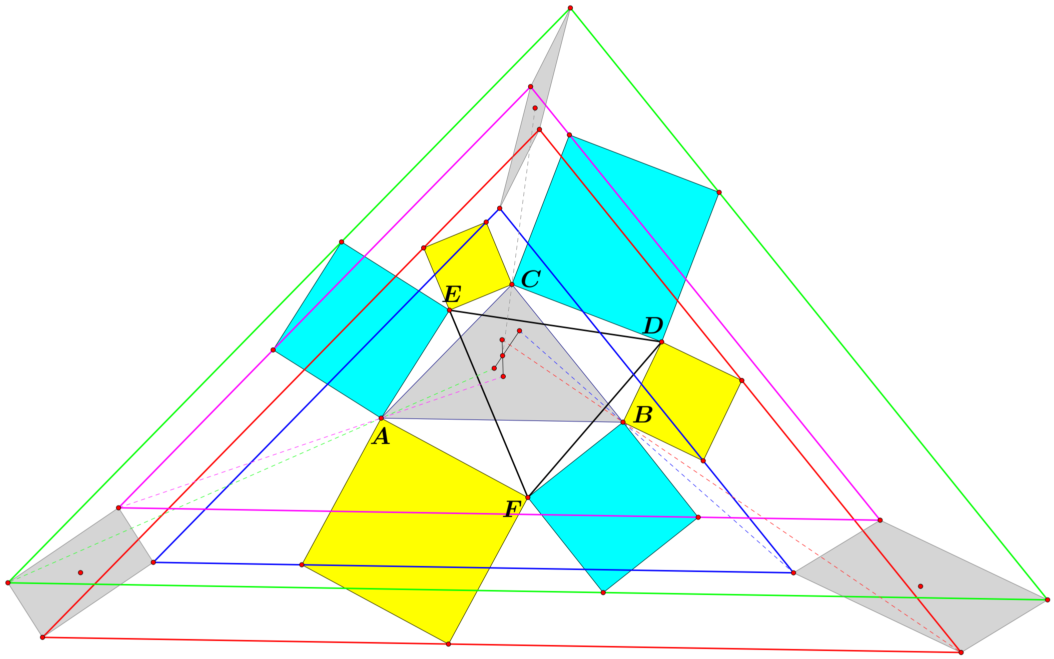

FIGURE 17. The triangles $K_aK_bK_c$, $L_aL_bL_c$, $M_aM_bM_c$ and $N_aN_bN_c$ together with three parallelograms.

(c) The points J_a, J_b and J_c have the coordinates $\left(\frac{u(\varphi-\varepsilon)+ev-u^2-u-v^2}{2v}, \frac{\varphi-1}{2}\right)$, $\left(\frac{(u-1)(\varphi-\delta)+u^2-3u+v^2+v(1+d)+2}{2v}, \frac{\varphi-1}{2}\right)$ and (j_c, k_c) where

$$j_c = \frac{u(u-1)(\delta-\varepsilon)+uvd+(1-u)ve+2u^3+2uv^2-3u^2+uv-v^2+u}{2v}$$

and $k_c = \frac{vd+(1-u)\delta-ve+u\varepsilon-2u+v+1+2u^2+2v^2}{2}$. Then we proceed as in the proof of (a). $\square$

Let J_0, K_0, L_0, M_0, N_0, P_0 and Q_0 be centers of the above homotheties. Notice that J_0 is the intersection of the lines K_0M_0 and L_0N_0.

Theorem 22. *(a) The symmedian point K of the triangle ABC lies on the line K_0L_0.*

(b) The points P_0 and Q_0 coincide with the points N_0 and M_0.

(c) The equalities $2 \cdot \overrightarrow{P_vQ_v} = \overrightarrow{K_vL_v}$ hold for $v = a$, b, c.

Proof of Theorem 22. (a) From the proof of Theorem 21 we know the coordinates of K_0. The same method gives the coordinates $\frac{u(f-1)-ue-v\varepsilon}{(u-1)d+v\delta-ue-v\varepsilon+f-u^2+u-v^2-1}$ and $\frac{v(f-1)}{(u-1)d+v\delta-ue-v\varepsilon+f-u^2+u-v^2-1}$ of the center L_0. Hence, the line K_0L_0 is the triple

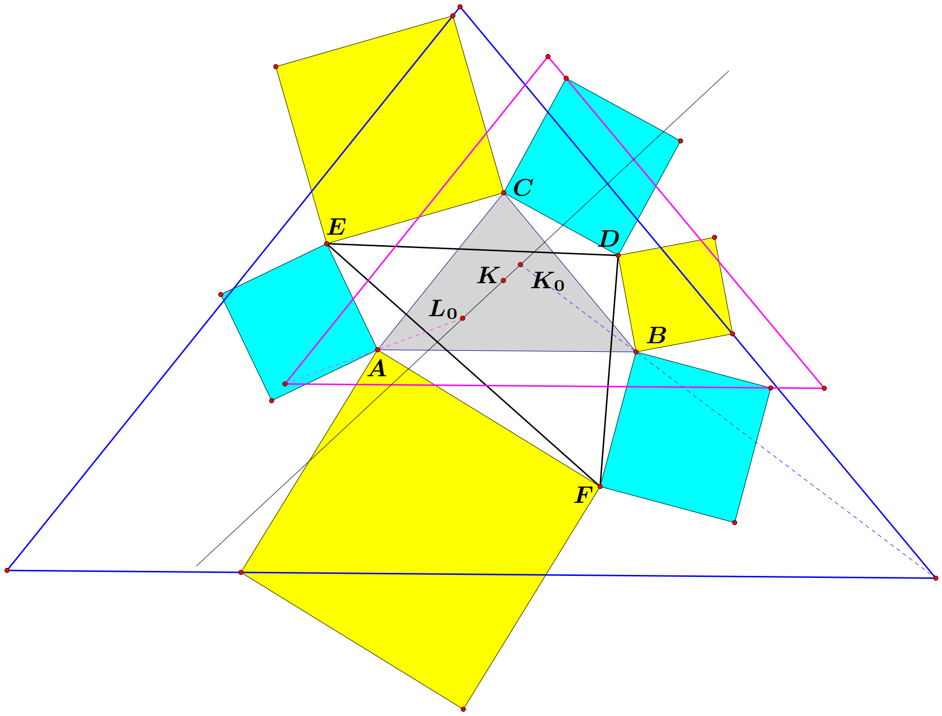

FIGURE 18. The line K_0L_0 goes through the symmedian point K of the triangle ABC.

$((u-1)d + v\delta - ue - v\varepsilon - (2u^2 - 2u + 2v^2 + 1)f + u^2 - u + v^2 + 1,\ (u^2 + u + v^2)(d - du - v\delta) - (u^2 - 3u + v^2 + 2)(ue + v\varepsilon) + (u^2 - u + v^2)\,(\,(2u-1)f + u^2 - u + v^2 + 1),\ v\,(ue + v\varepsilon + (u^2 + v^2)f - u^2 - v^2))$. The equation of this line is satisfied by the coordinates $\left(\frac{u^2+u+v^2}{2(u^2-u+v^2+1)}, \frac{v}{2(u^2-u+v^2+1)}\right)$ of the symmedian point K of the triangle ABC.

(b) That the center P_0 coincides with the center N_0 follows easily from the fact that $(A,\, N_a,\, P_a)$ and $(B,\, N_b,\, P_b)$ are triples of collinear points.

(c) Since $Q_a^y = \frac{f+\varphi-1}{2}$, $P_a^y = \frac{\varphi-f}{2}$, $L_a^y = f-1$ and $K_a^y = -f$, we see that

$$2 \cdot (Q_a^y - P_a^y) = L_a^y - K_a^y.$$

Similarly, $2 \cdot (Q_a^x - P_a^x) = L_a^x - K_a^x$. This proves the equality $2 \cdot \overrightarrow{P_aQ_a} = \overrightarrow{K_aL_a}$. □

Theorem 23. *The triangles $K_aK_bK_c$ and $L_aL_bL_c$ are congruent if and only if the triangles ABC and DEF are orthologic.*

Proof of Theorem 23. Since the triangles $K_aK_bK_c$ and $L_aL_bL_c$ are both homothetic to the triangle ABC, we conclude that they will be congruent if and only if $|K_a K_b| = |L_a L_b|$. Hence, the theorem follows from the equality

$$|K_a K_b|^2 - |L_a L_b|^2 = \frac{\left[(2u-1)^2 + (2v+1)^2 + 2\right] \Delta}{v^2}.$$

□

Let O and ω denote the circumcenter and the Brocard angle of the triangle ABC.

Theorem 24. *If the triangles ABC and DEF are orthologic then the following statements are true.*

(a) The symmedian point K of the triangle ABC is the midpoint of the segment $K_0 L_0$.

(b) The triangles $M_a M_b M_c$ and $N_a N_b N_c$ are congruent.

(c) The triangles $P_a P_b P_c$ and $Q_a Q_b Q_c$ are congruent.

(d) The common ratio of the homotheties of the triangles $K_a K_b K_c$ and $L_a L_b L_c$ with the triangle ABC is $(1 + \cot \omega) : 1$.

(e) The vector of the translations $K_a K_b K_c \mapsto L_a L_b L_c$ and $N_a N_b N_c \mapsto M_a M_b M_c$ is the image of the vector $2 \cdot \overrightarrow{O[DEF,\, ABC]}$ under the rotation $\rho\left(O,\, \frac{\pi}{2}\right)$.

(f) The vector of the translation $P_a P_b P_c \mapsto Q_a Q_b Q_c$ is the image of the vector $\overrightarrow{O[DEF,\, ABC]}$ under the rotation $\rho\left(O,\, \frac{\pi}{2}\right)$.

Proof of Theorem 24. (a) Let $\xi = u^2 - u + v^2$. Let the triangles ABC and DEF be such that the centers K_0 and L_0 are well-defined. In other words, let $M,\, N \neq 0$, where $M,\, N = (u-1)d + v\delta - ue - v\varepsilon + f \pm (\xi + 1)$. Let Z_0 be the midpoint of the segment $K_0 L_0$. Then $|Z_0 K|^2 = \frac{\Delta^2\, P}{4(\xi+1)^2\, M^2\, N^2}$, where

$$P = \frac{Q\, S^2}{(\xi+u)^2(\xi+3u+1)^2} + \frac{4v^2(\xi+1)^2\, T^2}{(\xi+u)(\xi+3u+1)},$$

$$\begin{aligned} S = {} & (ue + v\varepsilon)(\xi^2 + \xi - 3u(u-1)) + (\xi + u) \\ & [(\xi + 3u + 1)((u-1)d + v\delta) + ((1 - 2\xi)u - \xi - 1)f - (\xi+1)(\xi + u - 1)], \end{aligned}$$

$Q = \xi^2 + (4u+1)\xi + u(3u+1)$ and $T = ue + v\varepsilon + (\xi + u)(f - 1)$. Hence, when the triangles ABC and DEF are orthologic (i. e., $\Delta = 0$), then $K = Z_0$. The converse is not true because the factors S and T can be simultaneously equal to zero. For example, this happens for the points $A(0,0)$, $B(1,0)$, $C\left(\frac{1}{3},\, 1\right)$, $D(2,5)$, $E\left(4,\, -\frac{32}{9}\right)$ and $F(3,\, -1)$. An interesting problem is to give geometric description for the conditions $S = 0$ and $T = 0$.

(b) This follows from the equality

$$|N_a N_b|^2 - |M_a M_b|^2 = \frac{4(vd + (1-u)\delta - ve + u\varepsilon - \varphi + \xi + 1)\,\Delta}{v^2}.$$

(c) This follows similarly from the equality

$$|P_aP_b|^2 - |Q_aQ_b|^2 = \frac{(vd + (1-u)\delta - ve + u\varepsilon - \varphi + \xi + v + 1)\,\Delta}{v^2}.$$

(d) The ratio $\frac{|K_aK_b|}{|AB|}$ is $\frac{|\Delta + u^2 - u + v^2 + v + 1|}{v}$. Hence, when the triangles ABC and DEF are orthologic, then $\Delta = 0$ and this ratio is

$$\frac{u^2 - u + v^2 + v + 1}{v} = 1 + \frac{|BC|^2 + |CA|^2 + |AB|^2}{4 \cdot |ABC|} = 1 + \cot\omega.$$

(e) The tip of the vector $\overrightarrow{K_aL_a}$ (translated to the origin) is at the point $V(\xi - 2(ue + v\varepsilon - uf),\, 2f - 1)$. The intersection of the perpendiculars through the points D and E onto the sidelines BC and CA is the point

$$U\left((1-u)d - v\delta + ue + v\varepsilon,\ \frac{uv\delta + (u-1)(du - ue - v\varepsilon)}{v}\right).$$

When the triangles ABC and DEF are orthologic this point will be the second orthology center $[DEF,\, ABC]$. Since the circumcenter O has the coordinates $\left(\frac{1}{2},\, \frac{\xi}{2v}\right)$, the tip of the vector $2 \cdot \overrightarrow{OU}$ is at the point

$$W^*\left(2((1-u)d - v\delta + ue + v\varepsilon) - 1,\ \frac{2((u-1)(ud - ue - v\varepsilon) + uv\delta) - \xi}{v}\right).$$

Its rotation about the circumcenter by the angle $\frac{\pi}{2}$ has the tip in the point $W(-(W^*)^y,\, (W^*)^x)$. The relations $U^x - W^x = \frac{2u\Delta}{v}$ and $U^y - W^y = 2\Delta$ now confirm that the claim *(e)* holds.

(f) The proof for this part is similar to the proof of the part *(e)*. □

14. New results for the pedal triangle

Let a, b, c and S denote the lengths of sides and the area of the triangle ABC. In this section we shall assume that DEF is the pedal triangle of the point P with respect to ABC. Our goal is to present several new properties of Bottema's original configuration. It is particularly useful for the characterizations of the Brocard axis.

Theorem 25. *There is a unique central point P with the property that the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are congruent. The first trilinear coordinate of this point P is $a((b^2 + c^2 + 2\,S)a^2 - b^4 - c^4 - 2\,S(b^2 + c^2))$. It lies on the Brocard axis and divides the segment OK in the ratio $(-\cot\omega) : (1 + \cot\omega)$ and is also the image of K under the homothety $h(O, -\cot\omega)$.*

Proof of Theorem 25. Let $P(p, q)$. The orthogonal projections P_a, P_b and P_c of the point P onto the sidelines BC, CA and AB have the coordinates

$$\left(\frac{(u-1)^2p + v(u-1)q + v^2}{\xi - u + 1},\ \frac{v((u-1)p + vq - u + 1)}{\xi - u + 1}\right),$$

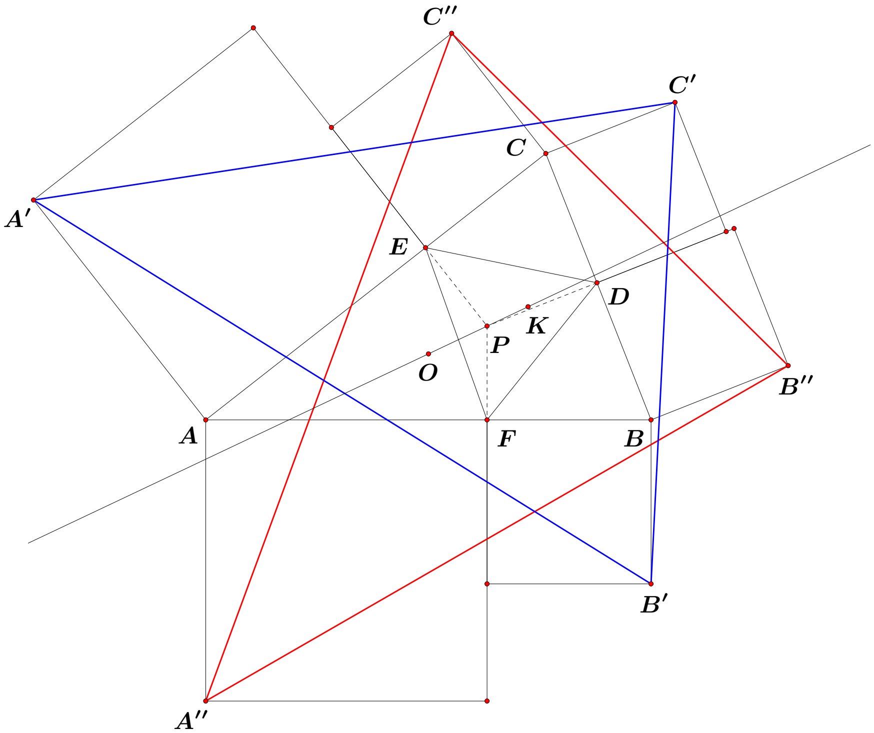

FIGURE 19. $s_2(A'B'C') = s_2(A''B''C'')$ iff P is on the Brocard axis.

$\left(\frac{u(up+vq)}{\xi+u}, \frac{v(up+vq)}{\xi+u}\right)$ and $(p, 0)$.

Since the triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal area, it is easy to prove using the Heron formula that they will be congruent if and only if two of their corresponding sides have equal length. In other words, we must find the solution of the equations

$$|S_3S_5|^2 - |S_4S_6|^2 = \frac{v\xi p}{\xi+u} - \frac{v^2 q}{\xi+u} + \frac{\xi+u-1}{2} = 0,$$

$$|S_5S_1|^2 - |S_6S_2|^2 = \frac{v\xi p}{\xi-u+1} + \frac{v^2 q}{\xi-u+1} - \frac{\xi^2 - (2(u-v)-1)\,\xi + u\,(u-1)}{2(\xi-u+1)} = 0.$$

As this is a linear system it is clear that there is only one solution. The required point is $P\left(\frac{1-2u+v}{2v}, \frac{\xi^2+(v+1)\xi-v^2}{2v^2}\right)$. Let $s = -\frac{1}{1+\frac{v}{\xi+1}} = \frac{-\cot\omega}{1+\cot\omega}$. The point P divides the segment OK in the ratio $s:1$, where $O\left(\frac{1}{2}, \frac{\xi}{2v}\right)$ and $K\left(\frac{\xi+2u}{2(\xi+1)}, \frac{v}{2(\xi+1)}\right)$. □

Theorem 26. *The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have the same centroid if and only if the point P is the circumcenter of the triangle ABC.*

Proof of Theorem 26. We easily get that $|G_oG_e|^2 = \frac{M^2+N^2}{9(\xi-u+1)(\xi+u)(1+4\xi)}$, where $N = v(1+\xi)(2p-1)$ and $M = 3\xi(2u-1)p + v(1+4\xi)q - \xi(2\xi+3u-1)$. Hence, G_o

$= G_e$ if and only if $N = 0$ and $M = 0$. In other words, the centroids of the triangles $S_1S_3S_5$ and $S_2S_4S_6$ coincide if and only if $p = \frac{1}{2}$ and $q = \frac{\xi}{2v}$ (i. e., if and only if the point P is the circumcenter O of the triangle ABC). □

Recall that the Brocard axis of the triangle ABC is the line joining its circumcenter with the symmedian point.

Theorem 27. *For the pedal triangle DEF of a point P with respect to the triangle ABC the following statements are equivalent:*

(a) The triangles $A_0B_0C_0$ and $D_0E_0F_0$ are orthologic.

(b) The triangles ABC and $G_{45A}G_{61B}G_{23C}$ are orthologic.

(c) The triangles ABC and $G_{45D}G_{61E}G_{23F}$ are orthologic.

(d) The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.

(e) The triangles $G_{12D}G_{34E}G_{56F}$ and $G_{45A}G_{61B}G_{23C}$ are orthologic.

(f) The triangles $A'B'C'$ and $A''B''C''$ have the same area.

(g) The triangles $A'B'C'$ and $A''B''C''$ have the same sums of squares of lengths of sides.

(h) The triangles $D'E'F'$ and $D''E''F''$ have the same area.

(i) The triangles $D'E'F'$ and $D''E''F''$ have the same sums of squares of lengths of sides.

(j) The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal sums of squares of lengths of sides.

(k) For any real number $t \neq -1, 0, 2$, the triangles $S_1^tS_3^tS_5^t$ and $S_2^tS_4^tS_6^t$ have equal sums of squares of lengths of sides.

(l) For any real number $s \neq -1, 0$, the triangles $T_1^sT_3^sT_5^s$ and $T_2^sT_4^sT_6^s$ have equal sums of squares of lengths of sides.

(m) The triangles $A_sB_sC_s$ and $D_sE_sF_s$ have the same area.

(n) The point P lies on the Brocard axis of the triangle ABC.

Proof of Theorem 27. (a) The orthology criterion $\Delta_0(A_0B_0C_0,\, D_0E_0F_0)$ is equal to the quotient $\frac{-v\,M}{8(\xi+u)(\xi-u+1)}$, with M the following linear polynomial in p and q.

$$M = 2\,\left(\xi^2 + \xi - v^2\right) p + 2\,v\,(2\,u - 1)\,q - (\xi + u)\,(\xi + u - 1)\,.$$

In fact, $M = 0$ is the equation of the Brocard axis because the coordinates $\left(\frac{1}{2},\, \frac{\xi}{2v}\right)$ and $\left(\frac{\xi+2u}{2(\xi+1)},\, \frac{v}{2(\xi+1)}\right)$ of the circumcenter O and the symmedian point K satisfy this equation. Hence, the statements (a) and (n) are equivalent.

(f) It follows from the equality $|A''B''C''| - |A'B'C'| = \frac{v\,M}{2(\xi+u)(\xi-u+1)}$ that the statements (f) and (n) are equivalent.

(i) It follows from the equality $s_2(D'E'F') - s_2(D''E''F'') = \frac{v\,M}{2(\xi+u)(\xi-u+1)}$ that the statements (i) and (n) are equivalent. □

It is well-known that $\cot\omega = \frac{a^2+b^2+c^2}{4S}$ so that we shall assume that the degenerate triangles do not have well-defined Brocard angle. It follows that the statement "The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal Brocard angles" could be added to the list of the previous theorem provided we exclude the points for which the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are degenerate. The following result explains when this happens. Let $K_{-\omega}$ denote the point described in Theorem 25.

Theorem 28. *The following statements are equivalent:*
(a) The points S_1, S_3 and S_5 are collinear.
(b) The points S_2, S_4 and S_6 are collinear.
(c) The point P is on the circle with the center $K_{-\omega}$ and the radius equal to the circumradius R of the triangle ABC times the number $\sqrt{(1+\cot\omega)^2+1}$.

Proof of Theorem 28. Let M be the following quadratic polynomial in p and q:
$v^2(p^2+q^2) + v\,(2\,u - v - 1)\,p - \left(\xi^2 + (v+1)\,\xi - v^2\right) q - (\xi+u)\,(\xi - u + v + 1)\,.$
The points S_1, S_3 and S_5 are collinear if and only if

$$0 = \begin{vmatrix} S_1^x & S_1^y & 1 \\ S_3^x & S_3^y & 1 \\ S_5^x & S_5^y & 1 \end{vmatrix} = \frac{v\,M}{2(u-1-\xi)(u+\xi)}.$$

The equivalence of (a) and (c) follows from the fact that $M = 0$ is the equation of the circle described in (c). Indeed, we see directly that the coordinates of its center are $\left(\frac{1-2u+v}{v}, \frac{\xi^2+(v+1)\xi-v^2}{2v^2}\right)$ so that this center is the point $K_{-\omega}$ while the square of its radius is $\frac{(\xi-u+1)(\xi+u)\left((\xi+v+1)^2+v^2\right)}{4v^4} = \frac{(\xi-u+1)(\xi+u)}{4v^2}\cdot\frac{(\xi+v+1)^2+v^2}{v^2} = R^2\cdot\beta^2$, where β is equal to the number $\sqrt{(1+\cot\omega)^2+1}$ because $\cot\omega = \frac{\xi+1}{v}$.

The equivalence of (b) and (c) is proved in the same way. □

Theorem 29. *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ always have different sums of squares of lengths of sides.*

Proof of Theorem 29. The difference $s_2(A_0B_0C_0) - s_2(D_0E_0F_0)$ is equal to the quotient $\frac{3v^3\,N}{4(\xi-u+1)(u+\xi)}$, where N denotes the following quadratic polynomial in variables p and q:

$$\left(p-\frac{1}{2}\right)^2 + \left(q - \frac{\xi}{2v}\right)^2 + \frac{3(\xi-u+1)(\xi+u)}{4v^2}.$$

However, this polynomial has no real roots. □

Theorem 30. *The triangles $A_0B_0C_0$ and $D_0E_0F_0$ have the same areas if and only if the point P lies on the circle θ_0 with the center at the symmedian point K of the triangle ABC and the radius $R\sqrt{4-3\tan^2\omega}$, where R and ω have their usual meanings associated with triangle ABC.*

Proof of Theorem 30. The difference $|D_0E_0F_0| - |A_0B_0C_0|$ is $\frac{v^2(\xi+1)^2\,M}{16(\xi-u+1)(\xi+u)}$, where M denotes the following quadratic polynomial in variables p and q:

$$\left(p - \frac{\xi+2\,u}{2(\xi+1)}\right)^2 + \left(q - \frac{v}{2(\xi+1)}\right)^2 - \frac{(\xi+u)\,(\xi-u+1)\,(4(\xi+1)^2 - 3\,v^2)}{4\,(\xi+1)^2\,v^2}.$$

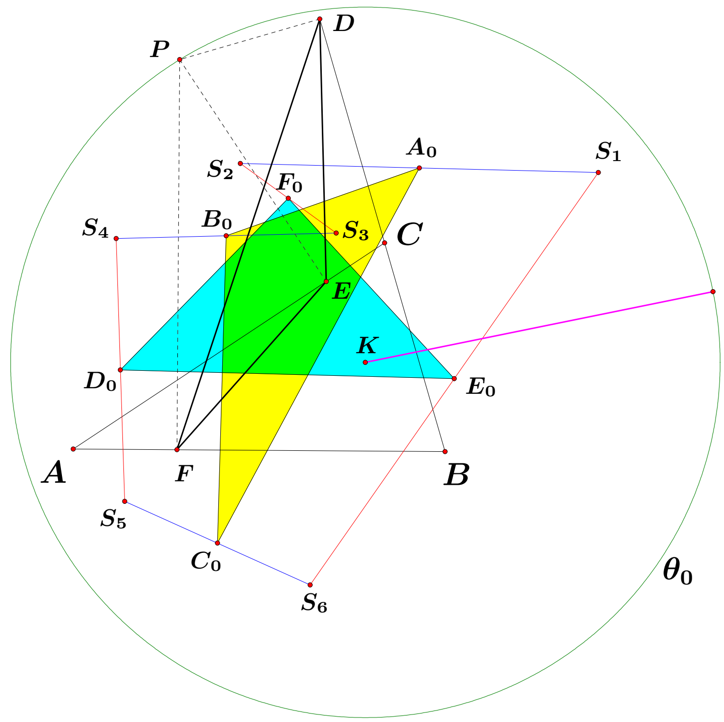

FIGURE 20. $|A_0B_0C_0| = |D_0E_0F_0|$ iff P is on the circle θ_0.

The third term is clearly equal to $-R^2(4 - 3\tan^2\omega)$. Hence, $M = 0$ is the equation of the circle whose center is the symmedian point of the triangle ABC with the coordinates $\left(\frac{\xi+2\,u}{2(\xi+1)}, \frac{v}{2(\xi+1)}\right)$ and the radius $R\sqrt{4 - 3\tan^2\omega}$. $\square$

Notice that the lines AA^*, BB^* and CC^* intersect in the isogonal conjugate of the point P with respect to the triangle ABC.

Theorem 31. *The triangles $A^*B^*C^*$ and $D^*E^*F^*$ have the same sums of squares of lengths of sides if and only if the point P lies on the circle θ_0.*

Proof of Theorem 31. The proof is almost identical to the proof of the previous theorem since the difference $s_2(D^*E^*F^*) - s_2(A^*B^*C^*)$ is equal to $\frac{v^2(\xi+1)^2\,M}{2(\xi-u+1)(\xi+u)}$. $\square$

Theorem 32. *For any point P the triangles $A^*B^*C^*$ and $D^*E^*F^*$ always have different areas.*

Proof of Theorem 32. The proof is similar to the proof of Theorem 29 since the difference $|D^*E^*F^*| - |A^*B^*C^*|$ is equal to $\frac{v^3\,N}{8(\xi-u+1)(\xi+u)}$. $\square$

15. New results for the antipedal triangle

Recall that the antipedal triangle $P_a^* P_b^* P_c^*$ of a point P not on the side lines of the triangle ABC has as vertices the intersections of the perpendiculars erected at A, B and C to PA, PB and PC respectively. Note that the triangle $P_a^* P_b^* P_c^*$ is orthologic with the triangle ABC so that Bottema's Theorem holds also for antipedal triangles.

Our final result is an analogue of Theorem 27 for the antipedal triangle of a point. It gives a nice connection of a Bottema configuration with the Kiepert hyperbola (i. e., the rectangular hyperbola which passes through the vertices, the centroid and the orthocenter [3]).

In the next theorem we shall assume that DEF is the antipedal triangle of the point P with respect to ABC. Of course, the point P must not be on the side lines BC, CA and AB.

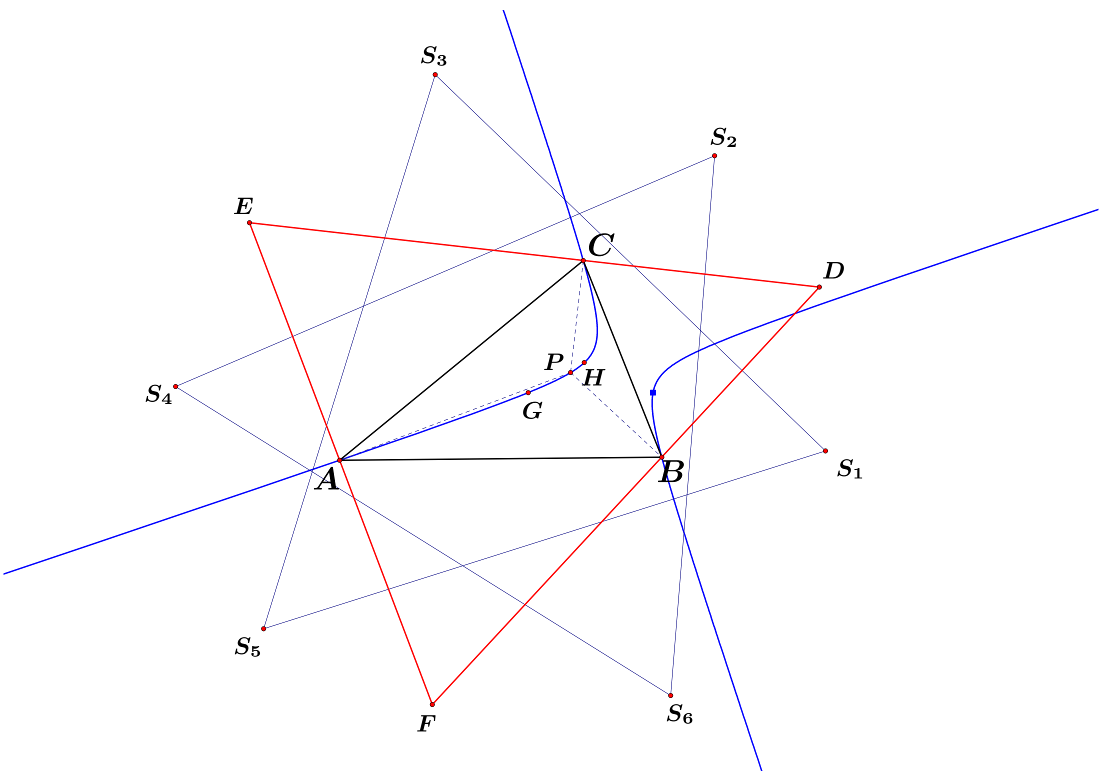

Figure 21. $s_2(S_1S_3S_5) = s_2(S_2S_4S_6)$ when P is on the Kiepert hyperbola.

Theorem 33. *The following statements are equivalent:*

(a) The triangles $A_0B_0C_0$ and $D_0E_0F_0$ are orthologic.
(b) The triangles ABC and $G_{45A}G_{61B}G_{23C}$ are orthologic.
(c) The triangles ABC and $G_{45D}G_{61E}G_{23F}$ are orthologic.
(d) The triangles $G_{12A}G_{34B}G_{56C}$ and $G_{45D}G_{61E}G_{23F}$ are orthologic.

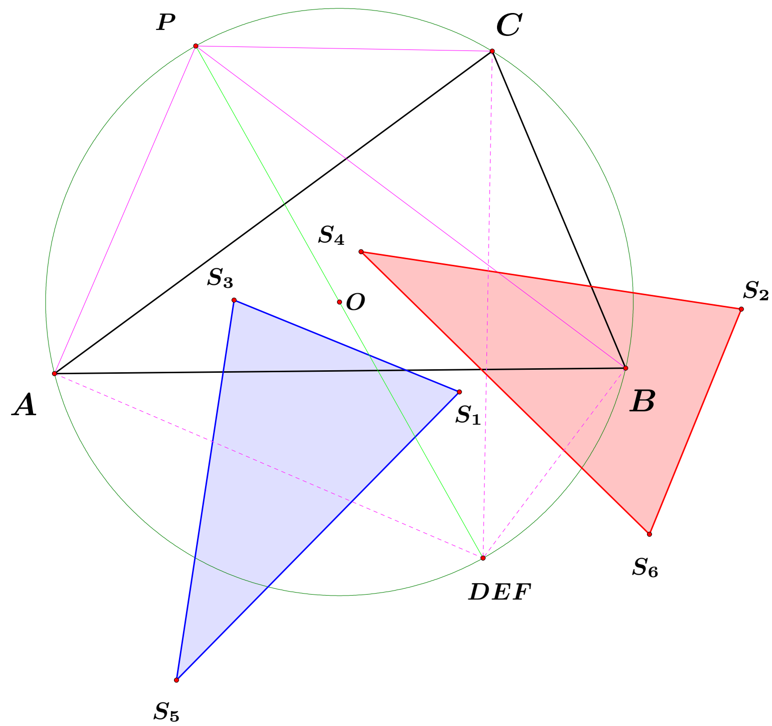

FIGURE 22. $s_2(S_1S_3S_5) = s_2(S_2S_4S_6)$ also when P is on the circumcircle.

(e) The triangles $G_{12D}G_{34E}G_{56F}$ *and* $G_{45A}G_{61B}G_{23C}$ *are orthologic.*
(f) The triangles $A'B'C'$ *and* $A''B''C''$ *have the same area.*
(g) The triangles $A'B'C'$ *and* $A''B''C''$ *have the same sums of squares of lengths of sides.*
(h) The triangles $D'E'F'$ *and* $D''E''F''$ *have the same area.*
(i) The triangles $D'E'F'$ *and* $D''E''F''$ *have the same sums of squares of lengths of sides.*
(j) The triangles $S_1S_3S_5$ *and* $S_2S_4S_6$ *have equal sums of squares of lengths of sides.*
(k) For any real number $t \neq -1, 0, 2$, *the triangles* $S_1^tS_3^tS_5^t$ *and* $S_2^tS_4^tS_6^t$ *have equal sums of squares of lengths of sides.*
(l) For any real number $s \neq -1, 0$, *the triangles* $T_1^sT_3^sT_5^s$ *and* $T_2^sT_4^sT_6^s$ *have equal sums of squares of lengths of sides.*
(m) The triangles $A_sB_sC_s$ *and* $D_sE_sF_s$ *have the same area.*
(n) The point P *lies either on the Kiepert hyperbola of the triangle* ABC *or on its circumcircle.*

Proof of Theorem 33. (g) $s_2(A''B''C'') - s_2(A'B'C') = \frac{2v\,M\,N}{q(vp-uq)(v(p-1)-(u-1)q)}$, with

$$M = \left(p - \frac{1}{2}\right)^2 + \left(q - \frac{\xi}{2v}\right)^2 - \frac{\xi^2 + v^2}{4v^2},$$

$$N = v\,(2u-1)\left(p^2 - q^2 - p\right) - 2\,\left(u^2 - u - v^2 + 1\right) pq + \left(u^2 + u - v^2\right) q.$$

In fact, $M = 0$ is the equation of the circumcircle of the triangle ABC while $N = 0$ is the equation of its Kiepert hyperbola because the coordinates of the vertices A, B and C and the coordinates $\left(u,\, \frac{u(1-u)}{v}\right)$ and $\left(\frac{u+1}{3},\, \frac{v}{3}\right)$ of the orthocenter H and the centroid G satisfy this equation. Hence, the statements (g) and (n) are equivalent.

(j) It follows from the equality $s_2(S_2S_4S_6) - s_2(S_1S_3S_5) = \frac{v\,M\,N}{q(vp-uq)(v(p-1)-(u-1)q)}$ that the statements (j) and (n) are equivalent.

(m) It follows from the equality

$$|D_sE_sF_s| - |A_sB_sC_s| = \frac{s\,v\,M\,N}{2(s+1)^2\,q(vp-uq)(v(p-1)-(u-1)q)}$$

that the statements (m) and (n) are equivalent. $\square$

Of course, as in the case of the pedal triangles, we can add the statement "The triangles $S_1S_3S_5$ and $S_2S_4S_6$ have equal Brocard angles." to the list in Theorem 33 but the points on the circle described in Theorem 28 must be excluded from consideration.

Notice that when the point P is on the circumcircle of ABC then much more could be said about the properties of the six squares built on segments BD, DC, CE, EA, AF and FB. A considerable simplification arises from the fact that the antipedal triangle DEF reduces to the antipodal point Q of the point P. For example, the triangles $S_1S_3S_5$ and $S_2S_4S_6$ are the images under the rotations $\rho(U,\, \frac{\pi}{4})$ and $\rho(V,\, -\frac{\pi}{4})$ of the triangle $A_\diamond B_\diamond C_\diamond = h(O,\, \frac{\sqrt{2}}{2})(ABC)$ (the image of ABC in the homothety with the circumcenter O as the center and the factor $\frac{\sqrt{2}}{2}$). The points U and V are constructed as follows.

Let the circumcircle $\sigma_\diamond$ of the triangle $A_\diamond B_\diamond C_\diamond$ intersect the segment OQ in the point R, let ℓ be the perpendicular bisector of the segment QR and let T be the midpoint of the segment OQ. Then the point U is the intersection of the line ℓ with $\rho(T,\, \frac{\pi}{4})(PQ)$ while the point V is the intersection of the line ℓ with $\rho(T,\, -\frac{\pi}{4})(PQ)$.

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