Triangles with coordinates of vertices from Pell and Pell-Lucas numbers

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Abstract. We consider triangles in the plane with coordinates of points from the Pell and the Pell-Lucas sequences. It is possible to take for both coordinates consecutive either Pell numbers or Pell-Lucas numbers or mix these two kinds of numbers taking for the first coordinates Pell numbers and for the second coordinates Pell-Lucas numbers and vice verse. For these four infinite sequences of triangles we explore what geometric properties they share or how are they related to each other. We also calculate some of their quantities like area, Brocard angles, and distances of certain central points when these are rather simple expressions of Pell and Pell-Lucas numbers. Sometimes, these results give interesting relations among Pell and Pell-Lucas numbers.

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, \qquad P_1 = 2, \qquad P_n = 2\,P_{n-1} + P_{n-2} \ \text{ for } n \geqslant 2,$$

and

$$Q_0 = 2, \qquad Q_1 = 2, \qquad Q_n = 2\,Q_{n-1} + Q_{n-2} \ \text{ for } n \geqslant 2.$$

The numbers Q_k make the integer sequence $A002203$ from [6] while the numbers $\frac{1}{2}\,P_k$ make $A000129$.

Let k be a positive integer. Let Δ_k and Γ_k be the triangles with vertices

$$A_k = (P_k,\, P_{k+1}), \qquad B_k = (P_{k+1},\, P_{k+2}), \qquad C_k = (P_{k+2},\, P_{k+3})$$

and

$$X_k = (Q_k,\, Q_{k+1}), \qquad Y_k = (Q_{k+1},\, Q_{k+2}), \qquad Z_k = (Q_{k+2},\, Q_{k+3}),$$

respectively.

In this paper we shall explore some common properties of the triangles Δ_k and Γ_k. Analogous infinite series of triangles with coordinates from the Fibonacci and Lucas integer sequences was studied by the first author in [3].

There is a great similarity between these two papers in statements of some results and in methods of their proofs. Of course, there are also some new observations like the possibility to consider triangles with mixed coordinates of vertices and the involvement of the homology relation.

We begin with the following theorem which shows that these triangles share the property of orthology.

Recall that the triangles ABC and XYZ are called *orthologic* when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. The point of concurrence is $[ABC,\, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto the corresponding sides of ABC are concurrent at the point $[XYZ,\, ABC]$.

By replacing in the above definition perpendiculars with parallels we get the analogous notion of *paralogic* triangles and of points $\langle ABC,\, XYZ\rangle$ and $\langle XYZ,\, ABC\rangle$.

The triangle ABC is paralogic to its first Brocard triangle $A_bB_bC_b$ which has the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides as vertices (see [4] and [5]).

Theorem 1. *For all positive integers m and n, the following are pairs of orthologic triangles: $(\Delta_m,\, \Delta_n)$, $(\Delta_m,\, \Gamma_n)$, and $(\Gamma_m,\, \Gamma_n)$.*

Proof. It is well-known (see [1]) that the triangles ABC and XYZ with coordinates of points $(a_1,\, a_2)$, $(b_1,\, b_2)$, $(c_1,\, c_2)$, $(x_1,\, x_2)$, $(y_1,\, y_2)$, and $(z_1,\, z_2)$ are orthologic if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Note that $\alpha + \beta = 2$ and $\alpha\beta = -1$ so that the numbers α and β are solutions of the equation $x^2 - 2\,x - 1 = 0$. Since $P_j = \frac{2(\alpha^j - \beta^j)}{\alpha - \beta}$ and $Q_j = \alpha^j + \beta^j$ for every $j \geqslant 0$, when we substitute the coordinates of the vertices of Δ_m and Δ_n into the left hand side of the above criterion we get

$$\frac{4\,(\alpha - 1)(\beta - 1)(\alpha\,\beta + 1)(\alpha^n\,\beta^m - \alpha^m\,\beta^n)}{\alpha - \beta}.$$

For the pairs Δ_m, Γ_n and Γ_m, Γ_n we get

$$2\,(\alpha - 1)(\beta - 1)(\alpha\,\beta + 1)(\alpha^m\,\beta^n + \alpha^n\,\beta^m)$$

and

$$(\alpha - \beta)(\alpha - 1)(\beta - 1)(\alpha\,\beta + 1)(\alpha^m\,\beta^n - \alpha^n\,\beta^m).$$

From this the conclusion of the theorem is obvious because $\alpha\,\beta + 1 = 0$. $\square$

Theorem 2. *For all positive integers m the orthocenters $H(\Delta_m)$ and $H(\Gamma_m)$ of the triangles Δ_m and Γ_m and the orthology centers $[\Delta_m,\,\Gamma_m]$ and $[\Gamma_m,\,\Delta_m]$ satisfy*

$$\frac{|H(\Delta_m)[\Delta_m,\,\Gamma_m]|}{|H(\Gamma_m)[\Gamma_m,\,\Delta_m]|} = \frac{\sqrt{2}}{2}.$$

Proof. Let us use θ_b^a as a short notation for the expression $a + b\sqrt{2}$. Let $A = \alpha^m$ and $B = \beta^m$. Using the Binet formula for Pell and Pell-Lucas numbers it is easy to check that $H(\Delta_m)$ has the coordinates

$$\left(\frac{\theta_{12}^{17}\,A^3 + \theta_1^1\,A^2\,B + \theta_{-1}^1\,A\,B^2 + \theta_{-12}^{17}\,B^3}{2\,A\,B},\ \frac{-\theta_5^7\,A^3 + \theta_2^3\,A^2\,B + \theta_{-2}^3\,A\,B^2 - \theta_{-5}^7\,B^3}{2\,A\,B}\right).$$

Similarly, the orthocenter $H(\Gamma_m)$ has coordinates

$$\left(\frac{-\theta_{17}^{24}\,A^3 + \theta_1^2\,A^2\,B + \theta_{-1}^2\,A\,B^2 - \theta_{-17}^{24}\,B^3}{2\,A\,B},\ \frac{\theta_7^{10}\,A^3 + \theta_3^4\,A^2\,B + \theta_{-3}^4\,A\,B^2 + \theta_{-7}^{10}\,B^3}{2\,A\,B}\right).$$

For $[\Delta_m,\,\Gamma_m]$ this method gives

$$\left(\frac{-\theta_{12}^{17}\,A^3 + \theta_1^1\,A^2\,B + \theta_{-1}^1\,A\,B^2 - \theta_{-12}^{17}\,B^3}{2\,A\,B},\ \frac{\theta_5^7\,A^3 + \theta_2^3\,A^2\,B + \theta_{-2}^3\,A\,B^2 + \theta_{-5}^7\,B^3}{2\,A\,B}\right).$$

Finally, the second orthology center $[\Gamma_m,\,\Delta_m]$ has coordinates

$$\left(\frac{\theta_{17}^{24}\,A^3 + \theta_1^2\,A^2\,B + \theta_{-1}^2\,A\,B^2 + \theta_{-17}^{24}\,B^3}{2\,A\,B},\ \frac{-\theta_7^{10}\,A^3 + \theta_3^4\,A^2\,B + \theta_{-3}^4\,A\,B^2 - \theta_{-7}^{10}\,B^3}{2\,A\,B}\right)$$

The square of the distance between the points $H(\Gamma_m)$ and $[\Gamma_m,\,\Delta_m]$ is $\theta_{956}^{1352}\,A^6 + \theta_{-956}^{1352}\,B^6$ while the square of the distance between the points $H(\Delta_m)$ and $[\Delta_m,\,\Gamma_m]$ is exactly half of this value. $\square$

Theorem 3. *For all positive integers m the oriented areas $|\Delta_m|$ and $|\Gamma_m|$ of the triangles Δ_m and Γ_m are as follows:*

$$|\Delta_m| = 4\,(-1)^m \quad \textit{and} \quad |\Gamma_m| = 2\,|\Delta_{m+1}| = 8\,(-1)^{m+1}.$$

Proof. Let us again assume that $\alpha^m = A$ and $B = \beta^m$. Note that $\alpha\,\beta = -1$ so that $A\,B = (-1)^m$. Recall that the triangle with the vertices whose coordinates are $(x_1,\,x_2)$, $(y_1,\,y_2)$, and $(z_1,\,z_2)$ has the oriented area equal to

$$\frac{(z_1 - y_1)\,x_2 + (x_1 - z_1)\,y_2 + (y_1 - x_1)\,z_2}{2}.$$

By direct substitution and simplification we get that $|\Delta_m| = 4\,A\,B = 4\,(-1)^m$. On the other hand, for Γ_m we get $|\Gamma_m| = -8\,A\,B = 2\,|\Delta_{m+1}| = 8\,(-1)^{m+1}$. $\square$

At this point we can go back and keep coordinates of vertices according to their original definition and discover that the first claim in the above theorem is equivalent to the identity

$$P_m\,(P_{m+2} - P_{m+3}) + P_{m+1}\,(P_{m+2} + P_{m+3}) = P_{m+1}^2 + P_{m+2}^2 + 8(-1)^m,$$

while the second claim in the above theorem is equivalent to the identity

$$Q_m\,(Q_{m+2}-Q_{m+3})+Q_{m+1}\,(Q_{m+2}+Q_{m+3})=Q_{m+1}^2+Q_{m+2}^2-16(-1)^m.$$

Theorem 4. *For all natural numbers m the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ of the triangles Δ_m and Γ_m are at the distance $\frac{\sqrt{34\,P_{2m}+26\,Q_{2m}}}{3}$.*

Proof. With the notation from the proof of Theorem 2 we get that the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ have the coordinates

$$\left(\frac{\theta_5^6\,A+\theta_{-5}^6\,B}{6},\,\frac{\theta_{11}^{16}\,A+\theta_{-11}^{16}\,B}{6}\right)\quad\text{and}\quad\left(\frac{\theta_3^5\,A+\theta_{-3}^5\,B}{3},\,\frac{\theta_8^{11}\,A+\theta_{-8}^{11}\,B}{3}\right).$$

The square of their distance is $\frac{\theta_{17}^{26}\left(49\,A^2+\theta_{-442}^{627}\,B^2\right)}{441}$ which in turn is precisely $\frac{34\,P_{2m}+26\,Q_{2m}}{9}$. ∎

The following interesting identity represents an equivalent way in which we can state the above theorem.

$$\begin{gathered}34\,P_{2m}+26\,Q_{2m}-(P_m-Q_m)^2-(P_{m+3}-Q_{m+3})^2=\\ 2\,(P_{m+1}+P_{m+2}-Q_{m+1}-Q_{m+2})(P_m+P_{m+1}+P_{m+2}+P_{m+3}-Q_m-Q_{m+1}-Q_{m+2}-Q_{m+3}).\end{gathered}$$

Theorem 5. *For all positive integers m the circumcenters $O(\Delta_m)$ and $O(\Gamma_m)$ of the triangles Δ_m and Γ_m are at the distance $\frac{\sqrt{P_{2m+3}(10+Q_{4m+6})}}{2}$.*

Proof. With the notation from the proof of Theorem 2 we get that the circumcenters $O(\Delta_m)$ and $O(\Gamma_m)$ have

$$\left(\frac{-\theta_{4(A-B)(3A^2+2AB+3B^2)}^{(A+B)(17A^2-22AB+17B^2)}}{4AB},\,\frac{\theta_{(A-B)(5A^2+14AB+5B^2)}^{(A+B)(7A^2+6AB+7B^2)}}{4AB}\right)$$

and

$$\left(\frac{\theta_{(A-B)(17A^2+22AB+17B^2)}^{8(A+B)(3A^2-2AB+3B^2)}}{4AB},\,\frac{-\theta_{(A-B)(7A^2-6AB+7B^2)}^{2(A+B)(5A^2-14AB+5B^2)}}{4AB}\right)$$

as coordinates. The square of their distance is

$$\frac{\theta_{7(A^2-B^2)(199(A^2+B^2)+210)}^{10(A^2+B^2)(197(A^2+B^2)-186)}}{8}$$

which is equal to

$$\frac{985\,Q_{6m}+1393\,P_{6m}+55\,Q_{2m}+77\,P_{2m}}{4}$$

and therefore to $\frac{P_{2m+3}(10+Q_{4m+6})}{4}$. ∎

Theorem 6. *For every positive integer m, the triangles Γ_m and Δ_m are reversely similar and the sides of Γ_m are $\sqrt{2}$ times longer than the corresponding sides of Δ_m.*

Proof. It is well-known that two triangles are reversely similar if and only if they are ortologic and paralogic (see [2]). Since, by Theorem 1, we know that triangles Γ_m and Δ_m are orthologic, it remains to see that they are paralogic.

Recall that triangles ABC and XYZ with coordinates of points (a_1, a_2), (b_1, b_2), (c_1, c_2), (x_1, x_2), (y_1, y_2) and (z_1, z_2) are paralogic if and only if the expression $U - V$ is equal to zero where

$$U = \begin{vmatrix} a_1 & b_1 & c_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}, \qquad V = \begin{vmatrix} a_2 & b_2 & c_2 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix}.$$

In our situation when we represent coordinates of vertices of triangles Δ_m and Γ_m by the Binet formula in terms of α and β by substitution and easy simplification we get that $U - V = 0$ so that these triangles are indeed paralogic. In a similar way one can easily show that $|X_m Y_m|^2 = 2\,|A_m B_m|^2$. □

Theorem 7. *For every positive integer m, the triangles Γ_m and Δ_m are both orthologic and paralogic. The centers $[\Delta_m, \Gamma_m]$ and $\langle \Delta_m, \Gamma_m \rangle$ are antipodal points on the circumcircle of Δ_m. The centers $[\Gamma_m, \Delta_m]$ and $\langle \Gamma_m, \Delta_m \rangle$ are antipodal points on the circumcircle of Γ_m.*

Proof. The first claim has been established in the previous theorem. In order to prove the second claim we shall prove that the orthology center $[\Delta_m, \Gamma_m]$ lies on the circumcircle of Δ_m by showing that it has the same distance from its circumcenter $O(\Delta_m)$ as the vertex A_m and that the reflection of the point $\langle \Delta_m, \Gamma_m \rangle$ in the circumcenter $O(\Delta_m)$ agrees with the point $[\Delta_m, \Gamma_m]$ (because their distance is equal to zero!).

In the proof of Theorem 5 we found the coordinates of the point $O(\Delta_m)$ and in the proof of Theorem 2 of the center $[\Delta_m, \Gamma_m]$. The coordinates of the center $\langle \Delta_m, \Gamma_m \rangle$ are $\left(\frac{\theta^{4(A+B)}_{3(A-B)}}{2}, \frac{\theta^{10(A+B)}_{7(A-B)}}{2} \right)$. Now it is easy to establish that

$$|[\Delta_m, \Gamma_m] O(\Delta_m)|^2 - |O(\Delta_m)\, A_m|^2 = 0.$$

On the other hand, if W denotes the reflection of the point $\langle \Delta_m, \Gamma_m \rangle$ in the circumcenter $O(\Delta_m)$ (i. e., W divides the segment $\langle \Delta_m, \Gamma_m \rangle O(\Delta_m)$ in ratio -2), then $|W[\Delta_m, \Gamma_m]|^2 = 0$.

The third claim has a similar proof. □

Theorem 8. *The square of the diameter of the circumcircle of the triangle Δ_m is equal to $2\,(P_{2m+3})^2\, P_{2m+1}$.*

Proof. In the proof of the previous theorems we found the coordinates of the circumcenter $O(\Delta_m)$. Hence, the square of its distance from the vertex A_m is $169\,Q_{6m} + 31\,Q_{2m} + 239\,P_{6m} + 43\,P_{2m}$. However, this expression is in fact

$$\frac{P_{2m+3}^2\,P_{2m+1}}{2}.$$

□

Let k be a positive integer. Let Φ_k and Ψ_k be the triangles with vertices

$$D_k = (P_k,\, Q_{k+1}), \qquad E_k = (P_{k+1},\, Q_{k+2}), \qquad F_k = (P_{k+2},\, Q_{k+3})$$

and

$$U_k = (Q_k,\, P_{k+1}), \qquad V_k = (Q_{k+1},\, P_{k+2}), \qquad W_k = (Q_{k+2},\, P_{k+3}),$$

respectively.

In order to describe our next results, recall that triangles ABC and XYZ are *homologic* provided lines AX, BY, and CZ are concurrent. The point P in which they concur is their homology *center* and the line ℓ containing intersections of pairs of lines $(BC,\, YZ)$, $(CA,\, ZX)$, and $(AB,\, XY)$ is their homology *axis*.

In stead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*.

Theorem 9. *For all positive integers m the lines D_mU_{m+1}, E_mV_{m+1} and F_mW_{m+1} are parallel to the line $y = x$ so that the triangles Φ_m and Ψ_{m+1} are homologic. Their homology center is the point at infinity and their homology axis is the line $y = x$. They are never orthologic neither paralogic. The oriented area of both triangles is $4\,(-1)^m$.*

Proof. The lines D_mU_{m+1}, E_mV_{m+1} and F_mW_{m+1} have equations

$$x - y + \frac{\theta_{A-B}^{2(A+B)}}{2} = 0, \quad x - y + \frac{\theta_{3(A-B)}^{4(A+B)}}{2} = 0, \quad \text{and} \quad x - y + \frac{\theta_{7(A-B)}^{10(A+B)}}{2} = 0.$$

It follows that they are parallel to the line $y = x$.

Since

$$E_mF_m \cap V_{m+1}W_{m+1} = \left(\frac{\theta_{-8AB}^{12AB}}{\theta_{-12B}^{A+17B}},\, \frac{\theta_{-8AB}^{12AB}}{\theta_{-12B}^{A+17B}} \right),$$

$$F_mD_m \cap W_{m+1}U_{m+1} = \left(\frac{\theta_{-12AB}^{16AB}}{\theta_{12B}^{A-17B}},\, \frac{\theta_{-12AB}^{16AB}}{\theta_{12B}^{A-17B}} \right),$$

and

$$D_mE_m \cap U_{m+1}V_{m+1} = \left(\frac{\theta_{-4AB}^{4AB}}{\theta_{2B}^{A-3B}},\, \frac{\theta_{-4AB}^{4AB}}{\theta_{2B}^{A-3B}} \right),$$

we conclude that the homology axis of the triangles Φ_m and Ψ_{m+1} is the line $y = x$.

The above conditions for the triangles Φ_m and Ψ_{m+1} to be orthologic and paralogic are both equal to $16\,(-1)^{m+1} = 0$ which is not true for any value m. $\square$

The claims about the oriented areas of the triangles Φ_m and Ψ_{m+1} are equivalent to the following identities:

$$2\,Q_{m+2}\,P_{m+1} = (Q_{m+3} - Q_{m+2})P_m + (Q_{m+2} - Q_{m+1})P_{m+2} + 8(-1)^m,$$
$$2\,Q_{m+2}\,P_{m+3} + 8(-1)^m = (Q_{m+3} - Q_{m+2})P_{m+2} + (Q_{m+2} - Q_{m+1})P_{m+4}.$$

Theorem 10. *The triangles Δ_m and Φ_m have equal Brocard angles.*

Proof. It is well-known that the cotangent of the Brocard angle of the triangle with vertices $A(x,\,a)$, $B(y,\,b)$ and $C(z,\,c)$ is equal to

$$\frac{2\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}}{(y-z)^2 + (z-x)^2 + (x-y)^2 + (b-c)^2 + (c-a)^2 + (a-b)^2}.$$

Hence, by direct substitution of coordinates and simplification we discover that the triangles Δ_m and Φ_m both have cotangents of its Brocard angles equal to $\frac{8}{6+(-1)^m(82\,P_{2m}+59\,Q_{2m})}$. $\square$

In a similar way one can show the following result.

Theorem 11. *The cotangent of the Brocard angle of the triangle Ψ_{m+1} is equal to $\frac{8}{6+(-1)^m\,(298\,P_{2m}+211\,Q_{2m})}$.*

Theorem 12. *For all positive integers m the lines D_mX_{m+1}, E_mY_{m+1} and F_mZ_{m+1} are parallel to the line $y = 2\,x$ so that the triangles Φ_m and Γ_{m+1} are homologic. Their homology center is the point at infinity and their homology axis is the line $y = x$. They are never orthologic neither paralogic.*

Proof. The lines D_mX_{m+1}, E_mY_{m+1} and F_mZ_{m+1} have equations

$$2x - y + A + B = 0, \qquad 2x - y + \theta_{A-B}^{A+B} = 0, \quad \text{and} \quad 2x - y + \theta_{2(A-B)}^{3(A+B)} = 0.$$

It follows that they are parallel to the line $y = 2x$.

Since

$$E_mF_m \cap Y_{m+1}Z_{m+1} = E_mF_m \cap V_{m+1}W_{m+1},$$
$$F_mD_m \cap Z_{m+1}X_{m+1} = F_mD_m \cap W_{m+1}U_{m+1},$$

and

$$D_mE_m \cap X_{m+1}Y_{m+1} = D_mE_m \cap U_{m+1}V_{m+1},$$

we conclude that the homology axis of the triangles Φ_m and Γ_{m+1} is the line $y = x$.

The above conditions for the triangles Φ_m and Γ_{m+1} to be orthologic and paralogic are both equal to $24\,(-1)^{m+1} = 0$ which is not true for any value m. $\square$

Theorem 13. *For all positive integers m the triangle Δ_m is orthologic to both triangles Φ_{m+1} and Ψ_{m+1}.*

Proof. The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}$ for triangles Δ_m and Φ_{m+1} is equal to $2(\alpha-1)(\beta-1)(\alpha^2\,\beta+\alpha\,\beta^2+2)\,\alpha^m\,\beta^m$ and therefore to zero because $\alpha\,\beta=-1$ and $\alpha+\beta=2$. For the triangles Δ_m and Ψ_{m+1} we get that this expression is equal to

$$2(\alpha-1)(\beta-1)(2\alpha\,\beta+\alpha+\beta)\,\alpha^m\,\beta^m$$

so that the same conclusion holds. □

In fact, it is possible to prove the following better results:

- For natural numbers m and n the triangles Δ_m and Φ_n are orthologic if and only if $n=m+1$.
- For natural numbers m and n the triangles Δ_m and Ψ_n are orthologic if and only if $n=m+1$.

On the other hand, for the triangles Γ_m, we can analogously prove the following results.

- For all natural numbers m and n the triangle Γ_m is not orthologic neither with the triangle Φ_n nor with the triangle Ψ_n.

Theorem 14. *For all positive integers m the triangle Δ_m is paralogic to the triangle Ψ_m. Moreover,*

$$2\,|\langle\Delta_m,\,\Psi_m\rangle\,\langle\Psi_m,\,\Delta_m\rangle|^2=P_m^2\,P_{m+1}.$$

Proof. The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix}$ for triangles Δ_m and Ψ_m is equal to $2(\alpha-1)(\beta-1)(2-\alpha-\beta)\,\alpha^m\,\beta^m$ and therefore to zero because $\alpha+\beta=2$.

With the notation from the proof of Theorem 2 we get that the points $\langle\Delta_m,\,\Psi_m\rangle$ and $\langle\Psi_m,\,\Delta_m\rangle$ have

$$\left(\frac{-\theta^{(A+B)(3A^2-14AB+3B^2)}_{2(A-B)(A^2-3AB+B^2)}}{4AB},\ \frac{\theta^{(B-A)(7A^2-22AB+7B^2)}_{(A+B)(A-5B)(B-5A)}}{4\sqrt{2}AB}\right)$$

and

$$\left(\frac{\theta^{(B-A)(B-3A)(3B-A)}_{-2(A+B)(A^2-6AB+B^2)}}{4\sqrt{2}AB},\ \frac{\theta^{(B-A)(B-3A)(3B-A)}_{-2(A+B)(A^2-6AB+B^2)}}{4\sqrt{2}AB}\right)$$

as coordinates. The square of their distance is

$$\frac{(A-B)^4\,\theta_{2(A^2-B^2)}^{3A^2+2AB+3B^2}}{32}$$

which is equal to $\frac{P_m^2\,P_{m+1}}{2}$. $\square$

Of course, again it is possible to prove the following better result:

- For natural numbers m and n the triangles Δ_m and Ψ_n are paralogic if and only if $n = m$.

On the other hand, we can also prove the following results.

- For all natural numbers m and n the triangle Δ_m is not paralogic with the triangle Φ_n.
- For all natural numbers m and n the triangle Γ_m is not paralogic neither with the triangle Φ_n nor with the triangle Ψ_n.

In closing, let us observe that the following are also pairs of homologic triangles: (Δ_m, Φ_m), (Δ_m, Ψ_m), (Γ_m, Φ_m) and (Γ_m, Ψ_m). The reason in these cases is quite simple – the corresponding vertices have identical either first or second coordinates.

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