Acc. Sc. Torino – Atti Sc. Fis.
140 (2006), xx-xx.

TEORIA DEI NUMERI

On sums of Pell numbers

Nota di ZVONKO ČERIN[1] e GIAN MARIO GIANELLA[2]
presentata dal Socio corrispondente Marius J. STOKA
nell'adunanza del 14 Novembre 2006.

Riassunto. *In questa nota che è un miglioramento della nostra nota* [6]*, si sono ottenute formule esplicite per somme di numeri di Pell, numeri di Pell pari e numeri di Pell dispari, per somme di quadrati di numeri di Pell, numeri di Pell pari e numeri di Pell dispari e somme di prodotti di numeri di Pell. Si sono pure determinate le forme alternate di queste somme. Esse hanno tutte interessanti rappresentazioni mediante numeri di Pell e di Pell-Lucas.*

Abstract. *In this improvement of our paper* [6] *we give explicit formulas for sums of Pell numbers, even Pell numbers and odd Pell numbers, for sums of squares of Pell numbers, even Pell numbers and odd Pell numbers and for sums of products of even and odd Pell numbers. The alternating forms of these sums are also considered. They all have nice representations in terms of Pell and Pell-Lucas numbers.*

1. **Introduction**

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, \qquad P_1 = 2, \qquad P_n = 2\,P_{n-1} + P_{n-2} \ \text{ for } n \geqslant 2,$$

and

$$Q_0 = 2, \qquad Q_1 = 2, \qquad Q_n = 2\,Q_{n-1} + Q_{n-2} \ \text{ for } n \geqslant 2.$$

The numbers Q_k make the integer sequence $A002203$ from [10] while the numbers $\frac{1}{2}\,P_k$ make $A000129$.

In this paper we prove twentyfour formulas for sums of a finite number of consecutive terms of various integer sequences related to the Pell numbers. More precisely, for any integers $k \geq 0$ and $m \geq 0$ we consider the sum

[1]Kopernikova 7, 10010 Zagreb, Croatia. E-mail: `cerin@math.hr`

[2]Dipartimento di Matematica, Università di Torino, via C. Alberto 10, 10123 Torino, Italy. E-mail: `gianella@dm.unito.it`

AMS 2000 *Math. Subject Classification:* Primary 11B39, 11Y55, 05A19.

Key words and phrases. Pell numbers, Pell-Lucas numbers, integer sequences, sum of squares, sums of products.

$\sum_{i=0}^{m} P_{k+i}$ and show that it can be evaluated as $\frac{1}{2}\,Q_{k+m} + P_{k+m} - \frac{1}{2}\,Q_k$. In other words, we discover the formula for the sum of $m+1$ consecutive members of the Pell sequence. Then we accomplish the same thing for even and for odd Pell numbers, for their squares and for six kinds of products of adjacent Pell numbers. Finally, we treat also the alternating sums for those sequences. For example, we prove that the alternating sum $\sum_{i=0}^{m} (-1)^i\, P_{k+i}$ is equal to $(-1)^m\, \frac{1}{2}\, Q_{k+m} - \frac{1}{2}\, Q_k + P_k$.

These formulas for sums have been discovered with the help of a PC computer and all algebraic identities needed for the verification of our theorems can be easily checked in either Derive, Mathematica or Maple V. Running times of all these calculations are in the range of a few seconds.

Similar results for Fibonacci, Lucas and Pell-Lucas numbers have recently been discovered in papers [1], [2], [3], [4] and [5]. They improved some results in [7].

2. **Statements of results**

The best method to sum finitely many consecutive integers

$$k + (k+1) + (k+2) + \cdots + (k+n)$$

is to group the initial number k and use the formula for the sum of the first n natural numbers. Hence, the above sum is

$$k(n+1) + \frac{n(n+1)}{2} = \frac{(n+1)(2k+n)}{2}.$$

In this paper our first goal is to show that the analogous explicit formula exists for the sum of finitely many consecutive members of the Pell integer sequence. In other words, we first want to find the formula for the sum $\sum_{i=0}^{m} P_{k+i}$ when $m \geq 0$ and $k \geq 0$ are integers.

We use the symbols $\mathbb{N}_* = \{0,\, 1,\, 2,\, 3,\, 4,\, \dots\}$ for all natural numbers (with zero included) and $\mathbb{N}_{**}$ for the product $\mathbb{N}_* \times \mathbb{N}_*$.

Theorem 1. *For $(m,\, k) \in \mathbb{N}_{**}$ the following equality holds:*

$$\sum_{i=0}^{m} P_{k+i} = \frac{1}{2}\, Q_{k+m} + P_{k+m} - \frac{1}{2}\, Q_k = \frac{1}{2}\, (Q_{k+m+1} - Q_k). \tag{2.1}$$

Next we sum up consecutive even and odd Pell numbers.

Theorem 2. *For $(m,\, k) \in \mathbb{N}_{**}$ the following equalities holds:*

$$\sum_{i=0}^{m} P_{2k+2i} = \frac{1}{2}\left[Q_{2k+2m} + P_{2k+2m} - Q_{2k} + P_{2k}\right] = \\ = \frac{1}{2}\left[Q_{2k+2m+2} - P_{2k+2m+2} + Q_{2k} - P_{2k}\right]. \quad (2.2)$$

$$\sum_{i=0}^{m} P_{2k+2i+1} = \frac{1}{2}\left[2\,Q_{2k+2m} + 3\,P_{2k+2m} - P_{2k}\right]. \quad (2.3)$$

The alternating sums of consecutive Pell numbers, even Pell numbers and odd Pell numbers are covered in the following theorem.

Theorem 3. *For $(m,\,k) \in \mathbb{N}_{**}$ the following equalities holds:*

$$\sum_{i=0}^{m} (-1)^i\, P_{k+i} = (-1)^m\, \frac{1}{2}\, Q_{k+m} - \frac{1}{2}\, Q_k + P_k. \quad (2.4)$$

$$\sum_{i=0}^{m} (-1)^i\, P_{2k+2i} = (-1)^m \left[\frac{1}{4}\, Q_{2k+2m} + \frac{1}{2}\, P_{2k+2m}\right] - \frac{1}{4}\, Q_{2k} + \frac{1}{2}\, P_{2k}. \quad (2.5)$$

$$\sum_{i=0}^{m} (-1)^i\, P_{2k+2i+1} = (-1)^m \left[\frac{3}{4}\, Q_{2k+2m} + P_{2k+2m}\right] + \frac{1}{4}\, Q_{2k}. \quad (2.6)$$

The sums of squares of consecutive Pell numbers, even Pell numbers and odd Pell numbers are treated in the following theorem.

Theorem 4. *For $(m,\,k) \in \mathbb{N}_{**}$ the following equalities holds:*

$$\sum_{i=0}^{m} P_{k+i}^2 = \frac{1}{4}\, Q_{2k+2m} + \frac{1}{2}\, P_{2k+2m} + \frac{1}{4}\, Q_k - \frac{1}{2} P_k - \frac{(-1)^{k+m} + (-1)^k}{2}. \quad (2.7)$$

$$\sum_{i=0}^{m} P_{2k+2i}^2 = \frac{1}{4}\, Q_{4k+4m} + \frac{3}{8}\, P_{4k+4m} + \frac{1}{4}\, Q_{4k} - \frac{3}{8} P_{4k} - m - 1. \quad (2.8)$$

$$\sum_{i=0}^{m} P_{2k+2i+1}^2 = \frac{3}{2}\, Q_{4k+4m} + \frac{17}{8}\, P_{4k+4m} - \frac{1}{8} P_{4k} + m + 1. \quad (2.9)$$

Of course, the alternating sums of squares of consecutive Pell numbers, even Pell numbers and odd Pell numbers are next.

Theorem 5. *For $(m,\, k) \in \mathbb{N}_{**}$ the following equalities holds:*

$$\sum_{i=0}^{m} (-1)^i \, P_{k+i}^2 = \frac{(-1)^m}{4} \, [Q_{2k+2m} + P_{2k+2m}] + \frac{1}{4} \, [Q_{2k} - P_{2k}] - (-1)^k \, (m+1). \tag{2.10}$$

$$\sum_{i=0}^{m} (-1)^i \, P_{2k+2i}^2 = \frac{(-1)^m}{12} \, [3\, Q_{4k+4m} + 4\, P_{4k+4m} - 6] + \frac{1}{12} \, [3\, Q_{4k} - 4\, P_{4k} - 6] \,. \tag{2.11}$$

$$\sum_{i=0}^{m} (-1)^i \, P_{2k+2i+1}^2 = \frac{(-1)^m}{12} \, [17\, Q_{4k+4m} + 24\, P_{4k+4m} + 6] + \frac{1}{12} \, Q_{4k} + \frac{1}{2}. \tag{2.12}$$

Now we shall consider six kinds of sums of products of Pell numbers.

Theorem 6. *For $(m,\, k) \in \mathbb{N}_{**}$ the following equalities holds:*

$$\sum_{i=0}^{m} P_{k+i} \, P_{k+i+1} = \frac{3}{4} \, Q_{2k+2m} + P_{2k+2m} - \frac{1}{4} \, Q_{2k} - \frac{(-1)^k + (-1)^{k+m}}{2}. \tag{2.13}$$

$$\sum_{i=0}^{m} P_{k+2i} P_{k+2i+1} = \frac{5}{8} \, Q_{2k+4m} + \frac{7}{8} \, P_{2k+4m} - \frac{1}{8} \, Q_{2k} + \frac{1}{8} \, P_{2k} - (-1)^k (m+1). \tag{2.14}$$

$$\sum_{i=0}^{m} P_{2k+2i} P_{2k+2i+1} = \frac{5}{8} \, Q_{4k+4m} + \frac{7}{8} \, P_{4k+4m} - \frac{1}{8} \, Q_{4k} + \frac{1}{8} \, P_{4k} - m - 1. \tag{2.15}$$

$$\sum_{i=0}^{m} P_{2k+2i+1} \, P_{2k+2i+2} = \frac{29}{8} \, Q_{4k+4m} + \frac{41}{8} \, P_{4k+4m} - \frac{1}{8} \, Q_{4k} - \frac{1}{8} \, P_{4k} + m + 1. \tag{2.16}$$

$$\sum_{i=0}^{m} P_{2k+4i}\, P_{2k+4i+2} =$$
$$\frac{35}{24}\, Q_{4k+8m} + \frac{33}{16}\, P_{4k+8m} + \frac{1}{24}\, Q_{4k} - \frac{1}{16}\, P_{4k} - 3(m+1). \tag{2.17}$$
$$\sum_{i=0}^{m} P_{2k+4i+1}\, P_{2k+4i+3} =$$
$$\frac{17}{2}\, Q_{4k+8m} + \frac{577}{48}\, P_{4k+8m} - \frac{1}{48}\, P_{4k} + 3(m+1). \tag{2.18}$$

The last six formulas cover the alternating forms of the relations from the previous theorem.

Theorem 7. *For* $(m,\, k) \in \mathbb{N}_{**}$ *the following equalities holds:*

$$\sum_{i=0}^{m} (-1)^i\, P_{k+i}\, P_{k+i+1} =$$
$$(-1)^m \left[\frac{1}{2}\, Q_{2k+2m} + \frac{3}{4}\, P_{2k+2m}\right] + \frac{1}{4}\, P_{2k} - (-1)^k (m+1). \tag{2.19}$$
$$\sum_{i=0}^{m} (-1)^i\, P_{k+2i}\, P_{k+2i+1} =$$
$$(-1)^m \left[\frac{7}{12} Q_{2k+4m} + \frac{5}{6} P_{2k+4m}\right] - \frac{1}{12} Q_{2k} + \frac{1}{6} P_{2k} - \frac{(-1)^k (1 + (-1)^m)}{2}. \tag{2.20}$$
$$\sum_{i=0}^{m} (-1)^i\, P_{2k+2i}\, P_{2k+2i+1} =$$
$$(-1)^m \left[\frac{7}{12}\, Q_{4k+4m} + \frac{5}{6}\, P_{4k+4m}\right] - \frac{1}{12}\, Q_{4k} + \frac{1}{6}\, P_{4k} - \frac{1 + (-1)^m}{2}. \tag{2.21}$$
$$\sum_{i=0}^{m} (-1)^i\, P_{2k+2i+1}\, P_{2k+2i+2} =$$
$$(-1)^m \left[\frac{41}{12}\, Q_{4k+4m} + \frac{29}{6}\, P_{4k+4m}\right] + \frac{1}{12}\, Q_{4k} + \frac{1}{6}\, P_{4k} + \frac{1 + (-1)^m}{2}. \tag{2.22}$$
$$\sum_{i=0}^{m} (-1)^i\, P_{2k+4i}\, P_{2k+4i+2} =$$
$$(-1)^m \left[\frac{99}{68}\, Q_{4k+8m} + \frac{35}{17}\, P_{4k+8m}\right] + \frac{3}{68}\, Q_{4k} - \frac{1}{17}\, P_{4k} - 3\, \frac{1 + (-1)^m}{2}. \tag{2.23}$$

$$\sum_{i=0}^{m}(-1)^i\,P_{2k+4i+1}\,P_{2k+4i+3} = (-1)^m\left[\frac{577}{68}\,Q_{4k+8m} + 12\,P_{4k+8m}\right] + \frac{1}{68}\,Q_{4k} + 3\frac{1+(-1)^m}{2}. \qquad (2.24)$$

3. **Comments on the proofs**

All twentyfour formulas have been discovered with the help from computers. Of course, it is possible that they could also be found without machines using in a clever way the formula for the sum of finitely many terms of a geometric series.

Once we have guessed the formula its proof by induction is usually quite easy provided we know some other relations among numbers P_k and Q_k. These relations are often proved also by induction or by algebraic manipulation.

We shall present only few proofs because they are all very similar so that the reader should not have difficulties to supply their own proofs.

4. **Proof of Theorem 1**

The formula (2.1) clearly holds when $m = 0$. If we assume that it holds for $m = r$, then we have

$$\sum_{i=0}^{r+1} P_{k+i} = \sum_{i=0}^{r} P_{k+i} + P_{k+r+1} = \frac{1}{2}\,Q_{k+r} + P_{k+r} - \frac{1}{2}\,Q_k + P_{k+r+1} = \frac{1}{2}\,Q_{k+r+1} + P_{k+r+1} - \frac{1}{2}\,Q_k,$$

where the last step uses the formula $Q_{n+1} - Q_n = 2\,P_n$ for $n = k + r$. Hence, (2.1) is true also for $m = r + 1$ and the proof by induction is complete.

The following direct proof is even simpler. Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Note that $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$ and $\alpha \cdot \beta = -1$ so that the numbers α and β are solutions of the equation $x^2 - 2\,x - 1 = 0$. Since $P_j = \frac{2(\alpha^j-\beta^j)}{\alpha-\beta} = \frac{\alpha^j-\beta^j}{\sqrt{2}}$ and $Q_j = \alpha^j + \beta^j$ we have

$$\sum_{i=0}^{m} P_{k+i} = \sum_{i=0}^{m} \frac{\alpha^{k+i} - \beta^{k+i}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\alpha^k\,\frac{\alpha^{m+1}-1}{\alpha-1} - \beta^k\,\frac{\beta^{m+1}-1}{\beta-1}\right] = \frac{1}{\sqrt{2}}\left[\frac{\alpha^{k+m+1}-\alpha^k}{\sqrt{2}} - \frac{\beta^{k+m+1}-\beta^k}{-\sqrt{2}}\right]$$

$$= \frac{1}{2}\left[\alpha^{k+m+1} + \beta^{k+m+1} - \left(\alpha^k + \beta^k\right)\right] = \frac{1}{2}\left(Q_{k+m+1} - Q_k\right).$$

5. Proof of (2.2)

The formula (2.2) clearly holds when $m = 0$. If we assume that it holds for $m = r$, then we have

$$\sum_{i=0}^{r+1} P_{2k+2i} = \sum_{i=0}^{r} P_{2k+2i} + P_{2k+2r+2} =$$
$$= \frac{1}{2}\left[Q_{2k+2r} + P_{2k+2r} - Q_{2k} + P_{2k}\right] + P_{2k+2r+2} =$$
$$= \frac{1}{2}\left[Q_{2k+2(r+1)} + P_{2k+2(r+1)} - Q_{2k} + P_{2k}\right],$$

where the last step uses the formula $Q_{n+1} = P_{n+1} + P_n$. Hence, (2.2) is true also for $m = r + 1$ and the proof by induction is complete.

The following direct proof is also simple. We have

$$\sum_{i=0}^{m} P_{2k+2i} = \sum_{i=0}^{m} \frac{\alpha^{2k+2i} - \beta^{2k+2i}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\alpha^{2k}\frac{\alpha^{2m+2}-1}{\alpha^2-1} - \beta^{2k}\frac{\beta^{2m+2}-1}{\beta^2-1}\right]$$
$$= \frac{1}{\sqrt{2}}\left[\frac{\alpha^{2k+2m+2} - \alpha^{2k}}{2+2\sqrt{2}} - \frac{\beta^{2k+2m+2} - \beta^{2k}}{2-2\sqrt{2}}\right] =$$
$$\frac{\alpha^{2k+2m+2} + \beta^{2k+2m+2}}{2} - \frac{\alpha^{2k+2m+2} - \beta^{2k+2m+2}}{2\sqrt{2}} - \frac{\alpha^{2k} - \beta^{2k}}{2\sqrt{2}} + \frac{\alpha^{2k} + \beta^{2k}}{2}$$
$$= \frac{1}{2}\left[Q_{2k+2m+2} - P_{2k+2m+2} - P_{2k} + Q_{2k}\right].$$

6. Proof of (2.4)

$$\sum_{i=0}^{m} (-1)^i P_{k+i} = \sum_{i=0}^{m} (-1)^i \frac{\alpha^{k+i} - \beta^{k+i}}{\sqrt{2}} =$$
$$\frac{1}{\sqrt{2}}\left[\alpha^k \frac{(-\alpha)^{m+1}-1}{-\alpha-1} - \beta^k \frac{(-\beta)^{m+1}-1}{-\beta-1}\right] =$$
$$\frac{1}{\sqrt{2}}\left[\frac{\alpha^k - (-1)^{m+1}\alpha^{k+m+1}}{\alpha+1} - \frac{\beta^k - (-1)^{m+1}\beta^{k+m+1}}{\beta+1}\right] =$$
$$\frac{\alpha^k - \beta^k}{\sqrt{2}} - \frac{\alpha^k + \beta^k}{2} + (-1)^m \left[\frac{\alpha^{k+m+1} - \beta^{k+m+1}}{\sqrt{2}} - \frac{\alpha^{k+m+1} + \beta^{k+m+1}}{2}\right]$$
$$= P_k - \frac{1}{2} Q_k + (-1)^m \left(P_{k+m+1} - \frac{1}{2} Q_{k+m+1}\right) = P_k - \frac{1}{2} Q_k + (-1)^m \frac{1}{2} Q_{k+m}.$$

7. **Proof of (2.7)**

$$\sum_{i=0}^{m} P_{k+i}^2 = \sum_{i=0}^{m} \left(\frac{\alpha^{k+i} - \beta^{k+i}}{\sqrt{2}}\right)^2 =$$

$$\frac{\alpha^{2k}}{2}\left(\frac{\alpha^{2m+2}-1}{\alpha^2-1}\right) + \frac{\beta^{2k}}{2}\left(\frac{\beta^{2m+2}-1}{\beta^2-1}\right) - (\alpha\,\beta)^k \sum_{i=0}^{m} (\alpha\,\beta)^i =$$

$$\frac{\alpha^{2k+2m+2} - \alpha^{2k}}{4+4\sqrt{2}} + \frac{\beta^{2k+2m+2} - \beta^{2k}}{4-4\sqrt{2}} - \frac{(-1)^{k+m} + (-1)^k}{2} =$$

$$\frac{1}{2}\,P_{2k+2m+2} - \frac{1}{4}\,Q_{2k+2m+2} + \frac{1}{4}\,Q_{2k} - \frac{1}{2}\,P_{2k} - \frac{(-1)^{k+m} + (-1)^k}{2} =$$

$$\frac{1}{4}\,Q_{2k+2m} + \frac{1}{2}\,P_{2k+2m} + \frac{1}{4}\,Q_k - \frac{1}{2}P_k - \frac{(-1)^{k+m} + (-1)^k}{2},$$

where we used the fact that $\alpha\,\beta = -1$ and the formula $P_{n+2} - P_n = \frac{Q_{n+2}+Q_n}{2}$ for $n = 2k + 2m$.

8. **Proof of (2.10)**

$$\sum_{i=0}^{m} (-1)^i\, P_{k+i}^2 = \sum_{i=0}^{m} (-1)^i \left(\frac{\alpha^{k+i} - \beta^{k+i}}{\sqrt{2}}\right)^2 =$$

$$\sum_{i=0}^{m} (-1)^i\, \frac{\alpha^{2k+2i} - 2(\alpha\,\beta)^{k+i} + \beta^{2k+2i}}{2} =$$

$$\frac{\alpha^{2k}}{2} \sum_{i=0}^{m} (-\alpha^2)^i - (\alpha\,\beta)^k \sum_{i=0}^{k} (-\alpha\,\beta)^i + \frac{\beta^{2k}}{2} \sum_{i=0}^{m} (-\beta^2)^i =$$

$$\frac{\alpha^{2k}}{2}\left(\frac{(-\alpha^2)^{m+1}-1}{-\alpha^2-1}\right) + \frac{\beta^{2k}}{2}\left(\frac{(-\beta^2)^{m+1}-1}{-\beta^2-1}\right) - (-1)^k\,(m+1) =$$

$$\frac{\alpha^{2k} + (-1)^m\,\alpha^{2k+2m+2}}{8+4\sqrt{2}} + \frac{\beta^{2k} + (-1)^m\,\beta^{2k+2m+2}}{8-4\sqrt{2}} - (-1)^k(m+1) =$$

$$\frac{1}{4}\,[Q_{2k} - P_{2k}] - (-1)^k(m+1) + \frac{(-1)^m}{4}\,[Q_{2k+2m+2} - P_{2k+2m+2}] =$$

$$\frac{(-1)^m}{4}\,[Q_{2k+2m} + P_{2k+2m}] + \frac{1}{4}\,[Q_{2k} - P_{2k}] - (-1)^k\,(m+1),$$

where we used the formula $Q_{n+2} - P_{n+2} = Q_n + P_n$ for $n = 2k + 2m$ and the fact that $\alpha\,\beta = -1$.

9. Proof of (2.13)

$$\sum_{i=0}^{m} P_{k+i}\, P_{k+i+1} = \sum_{i=0}^{m} \left(\frac{\alpha^{k+i} - \beta^{k+i}}{\sqrt{2}}\right) \left(\frac{\alpha^{k+i+1} - \beta^{k+i+1}}{\sqrt{2}}\right) =$$

$$\frac{1}{2} \sum_{i=0}^{m} \alpha^{2k+2i+1} - \frac{\alpha+\beta}{2} \sum_{i=0}^{m} (\alpha\,\beta)^{k+i} + \frac{1}{2} \sum_{i=0}^{m} \beta^{2k+2i+1} =$$

$$\frac{\alpha^{2k+1}}{2} \left(\frac{(\alpha^2)^{m+1} - 1}{\alpha^2 - 1}\right) - \frac{(-1)^{k+m} + (-1)^k}{2} + \frac{\beta^{2k+1}}{2} \left(\frac{(\beta^2)^{m+1} - 1}{\beta^2 - 1}\right) =$$

$$\frac{\alpha^{2k+2m+3} - \alpha^{2k+1}}{4 + 4\sqrt{2}} + \frac{\beta^{2k+2m+3} - \beta^{2k+1}}{4 - 4\sqrt{2}} - \frac{(-1)^{k+m} + (-1)^k}{2} =$$

$$\frac{1}{2} P_{2k+2m+3} - \frac{1}{4} Q_{2k+2m+3} - \frac{1}{2} P_{2k+1} + \frac{1}{4} Q_{2k+1} - \frac{(-1)^{k+m} + (-1)^k}{2}$$

$$= \frac{3}{4} Q_{2k+2m} + P_{2k+2m} - \frac{1}{4} Q_{2k} - \frac{(-1)^k + (-1)^{k+m}}{2}.$$

References

[1] Z. Čerin, *Properties of odd and even terms of the Fibonacci sequence*, Demonstratio Mathematica, **39**, No. 1, 2006, 55–60.

[2] Z. Čerin, *On sums of squares of odd and even terms of the Lucas sequence*, Proccedings of the 11th Fibonacci Conference, (to appear).

[3] Z. Čerin, *Some alternating sums of Lucas numbers*, Central European Journal of Mathematics **3**, No. 1, 2005, 1 – 13.

[4] Z. Čerin, *Alternating Sums of Fibonacci Products*, Atti del Seminario Matematico e Fisico dell'Università di Modena e Reggio Emilia, (to appear).

[5] Z. Čerin and G. M. Gianella, *On sums of squares of Pell-Lucas numbers*, INTEGERS: Electronic Journal of Combinatorial Number Theory, 6 (2006), #A15.

[6] Z. Čerin and G. M. Gianella, *Formulas for sums of squares and products of Pell numbers*, Accad. Sc. Torino - Atti Sci. Fis. 140 (2006), (to appear).

[7] V. Rajesh and G. Leversha, *Some properties of odd terms of the Fibonacci sequence*, Mathematical Gazette, **88** 2004, 85–86.

[8] R. Ram, *Pell Numbers Formulae*, `http://users.tellurian.net/hsejar/maths/`.

[9] T. Reix, *Properties of Pell numbers modulo prime Fermat numbers*, (preprint).

[10] N. Sloane, *On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/∼njas/sequences/`.