Formulas for sums of squares and products of Pell numbers

Nota di Zvonko ČERIN e Gian Mario GIANELLA
presentata dal socio Corrispondente Marius I. STOKA
nell'adunanza del 14 Giugno 2006.

Riassunto. *In questo lavoro si provano diverse formule per somme di quadrati di numeri pari di Pell, somme di quadrati di numeri dispari di Pell e somme di prodotti di numeri pari e dispari di Pell. Queste somme hanno delle rappresentazioni interessanti come prodotti di numeri appropriati di Pell e con coefficienti tratti da certe successioni di interi.*

Abstract. *In this note we prove several formulas for sums of squares of even Pell numbers, sums of squares of odd Pell numbers and sums of products of even and odd Pell numbers. These sums have nice representations as products of appropriate Pell numbers with terms from certain integer sequences.*

1. **Introduction**

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, \qquad P_1 = 2, \qquad P_n = 2\,P_{n-1} + P_{n-2} \ \text{ for } n \geqslant 2,$$

and

$$Q_0 = 2, \qquad Q_1 = 2, \qquad Q_n = 2\,Q_{n-1} + Q_{n-2} \ \text{ for } n \geqslant 2.$$

In the sections 2–4 we consider sums of squares of odd and even terms of the Pell sequence and sums of their products. These sums have nice representations as products of appropriate Pell numbers.

The numbers Q_k make the integer sequence $A002203$ from [11] while the numbers $\frac{1}{2}\,P_k$ make $A000129$. In this paper we shall also need the sequence $A001109$ that we shorten to a_k with $k \geqslant 0$. We shall see later that $a_k = \frac{1}{4}\,P_{2k}$.

These formulas for sums have been discovered with the help of a PC computer and all algebraic identities needed for the verification of our theorems

1991 *Mathematics Subject Classification.* Primary 11B39, 11Y55, 05A19.
Key words and phrases. Pell numbers, Pell-Lucas numbers, integer sequences, sum of squares, sums of products.

can be easily checked in either Derive, Mathematica or Maple V. Running times of all these calculations are in the range of a few seconds.

Similar results for Fibonacci, Lucas and Pell-Lucas numbers have recently been discovered in papers [1], [2], [3], [4] and [5]. They improved some results in [8].

2. **Pell even squares**

The following lemma is needed to accomplish the inductive step in the proof of our first theorem.

Lemma 1. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$4\,a_k\left[a_{m+2}\,P_{2k+2m+2} - a_{m+1}\,P_{2k+2m}\right] = P_{2k+2m+2}^2 - P_{2m+2}^2. \tag{2.1}$$

Proof. Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$. Note that $\alpha+\beta = 2$ and $\alpha\cdot\beta = -1$ so that the numbers α and β are solutions of the equation $x^2 - 2\,x - 1 = 0$. Since $P_j = \frac{2(\alpha^j - \beta^j)}{\alpha-\beta}$ and

$$a_j = \frac{7\alpha+\beta}{16}\,\alpha^{2j-2} + \frac{\alpha+7\beta}{16}\,\beta^{2j-2}$$

for every $j \geqslant -1$, the difference of the left hand side and the right hand side of the relation (2.1) (after the substitutions $\alpha^m = A$, $\beta^m = B$, $\alpha^k = U$, $\beta^k = V$) and the replacement of α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get the rational function whose numerator is

$$128\left(17-12\sqrt{2}\right)\left(1-(U\,V)^2\right)\left(B^2-\left(17+12\sqrt{2}\right)A^2\right)^2.$$

Since $U\,V = \pm 1$ we conclude that the above difference is zero and the proof is complete. □

Theorem 1. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$\sum_{i=0}^{m} P_{2k+2i}^2 = \sum_{i=0}^{m} P_{2i}^2 + 4\cdot a_{m+1}\cdot a_k\cdot P_{2k+2m}. \tag{2.2}$$

Proof. When $m = 0$ the above formula is $P_{2k}^2 = 4\,a_k\,P_{2k}$. It holds as the consequence of the relation $P_{2k} = 4\,a_k$ that we prove easily as follows.

Since $P_{2k} = \frac{2(\alpha^{2\,k} - \beta^{2\,k})}{\alpha-\beta}$ and $4\,a_k = \frac{7\,\alpha+\beta}{4}\,\alpha^{2\,k-2} + \frac{\alpha+7\,\beta}{4}\,\beta^{2\,k-2}$, we get

$$P_{2k} - 4\,a_k = \frac{\left(\alpha^2+6\,\beta\,\alpha+\beta^2\right)\left(A\beta - B\alpha\right)\left(B\alpha + A\beta\right)}{4\,\left(\alpha-\beta\right)\alpha^2\beta^2} = 0,$$

where $A = \alpha^k$, $B = \beta^k$ and $\alpha^2 + 6\,\beta\,\alpha + \beta^2 = 0$ for $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Assume that the relation (2.2) is true for $m = r$. Then

$$\sum_{i=0}^{r+1} P_{2k+2i}^2 = \sum_{i=0}^{r} P_{2k+2i}^2 + P_{2k+2r+2}^2 =$$
$$\sum_{i=0}^{r} P_{2i}^2 + 4\,a_{r+1}\,a_k\,P_{2k+2r} + P_{2k+2r+2}^2 = \sum_{i=0}^{r+1} P_{2i}^2 + 4\,a_{r+1}\,a_k\,P_{2k+2r}$$
$$+ P_{2k+2r+2}^2 - P_{2r+2}^2 = \sum_{i=0}^{r+1} P_{2i}^2 + 4\,a_{r+2}\,a_k\,P_{2k+2r+2},$$

where the last step uses Lemma 1 for $m = r$. Hence, (2.2) is true for $m = r + 1$ and the proof is completed by induction. □

The formula (2.2) shows that in order to sum up m squares of consecutive even indexed Pell numbers it suffices to know the initial such sum. Hence, we can view this as the appropriate translation property for sums of consecutive squares. All other formulas in this note have analogous properties.

The next remark lists some other variants of the formula (2.2). It uses the integer sequence b_k whose first ten terms are 1, 1, 5, 29, 169, 985, 5741, 33461, 195025 and 1136689 and which is not in the Sloane depository [11]. The explicit formula for its terms is

$$b_k = \frac{(3 + 2\sqrt{2})^k\,(2 - \sqrt{2})}{4} + \frac{(3 - 2\sqrt{2})^k\,(2 + \sqrt{2})}{4}$$

or

$$b_k = \frac{(4 - \alpha + \beta)(3 + \alpha - \beta)^k}{8} + \frac{(4 + \alpha - \beta)(3 - \alpha + \beta)^k}{8}$$

for any integer k. We shall see later that $b_k = \frac{1}{2}\,P_{2k-1}$.

Remark 1. The following are four additional versions of Theorem 1:

For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities holds:

$$\sum_{i=0}^{m} P_{2k+2i}^2 + 4\left(\sum_{i=0}^{m+1} b_i^2\right) = 2 \cdot a_{m+1} \cdot b_{k+1} \cdot P_{2k+2m-1}, \qquad (2.5)$$

$$\sum_{i=0}^{m} P_{2k+2i}^2 = 16\left(\sum_{i=0}^{m} a_{i-1}^2\right) + 4 \cdot a_{m+1} \cdot a_{k+1} \cdot P_{2k+2m-2}, \qquad (2.6)$$

$$\sum_{i=0}^{m} P_{2k+2i}^2 + 4\left(\sum_{i=0}^{m+1} b_i^2\right) = 4 + 2 \cdot a_{m+1} \cdot b_k \cdot P_{2k+2m+1}, \tag{2.7}$$

$$\sum_{i=0}^{m} P_{2k+2i}^2 = \left(\sum_{i=0}^{m+1} P_{2i}^2\right) + 4 \cdot a_{m+1} \cdot a_{k-1} \cdot P_{2k+2m+2}. \tag{2.8}$$

3. **Pell odd squares**

The exceptional case (when $m = 0$) of our second theorem is the following lemma.

Lemma 2. *For every $k \geqslant 0$ the following identity holds:*

$$P_{2k+1}^2 = 4 + 4\, a_{k+1}\, P_{2k}. \tag{3.1}$$

Proof. By the Binet formula $P_k = \frac{2(\alpha^k - \beta^k)}{\alpha - \beta}$ so that we have

$$P_{2k+1}^2 - 4\, a_{k+1}\, P_{2k} - 4 = P_{2k+1}^2 - P_{2k+2}\, P_{2k} - 4 =$$
$$\left(\frac{2(\alpha^{2k+1} - \beta^{2k+1})}{\alpha - \beta}\right)^2 - \left(\frac{2(\alpha^{2k+2} - \beta^{2k+2})}{\alpha - \beta}\right)\left(\frac{2(\alpha^{2k} - \beta^{2k})}{\alpha - \beta}\right) - 4$$
$$= 4(A\,B - 1) = 0,$$

where $A = \alpha^{2k}$ and $B = \beta^{2k}$. □

The initial step in an inductive proof of our second theorem uses the following lemma.

Lemma 3. *For every $k \geqslant 0$ the following identity holds:*

$$P_{2k+1}^2 + P_{2k+3}^2 = 8 + 24\, a_{k+1}\, P_{2k+2}. \tag{3.2}$$

Proof. By the Binet formula $P_k = \frac{2(\alpha^k - \beta^k)}{\alpha - \beta}$ so that we have

$$P_{2k+1}^2 + P_{2k+3}^2 - 24\, a_{k+1}\, P_{2k+2} - 8 = P_{2k+1}^2 + P_{2k+3}^2 - 6\, P_{2k+2}^2 - 8 =$$
$$\left(\frac{2(\alpha^{2k+1} - \beta^{2k+1})}{\alpha - \beta}\right)^2 + \left(\frac{2(\alpha^{2k+3} - \beta^{2k+3})}{\alpha - \beta}\right)^2 - 6\left(\frac{2(\alpha^{2k+2} - \beta^{2k+2})}{\alpha - \beta}\right)^2$$
$$- 8 = 8(A\,B - 1) = 0,$$

where again $A = \alpha^{2k}$ and $B = \beta^{2k}$ and we substitute $1 + \sqrt{2}$ and $1 - \sqrt{2}$ for α and β (and their small concrete powers). □

The following lemma is needed to accomplish the inductive step in the proof of our second theorem.

Lemma 4. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$4\, a_{k+1} \left[a_{m+2}\, P_{2k+2m+2} - a_{m+1}\, P_{2k+2m}\right] = P^2_{2k+2m+3} - P^2_{2m+1}. \qquad (3.3)$$

Proof. The difference of the left hand side and the right hand side of the relation (3.3) (after the substitutions $\alpha^m = A,\ \beta^m = B,\ \alpha^k = U,\ \beta^k = V$) and the replacement of α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get the rational function whose numerator is

$$128\,\left(-3+2\sqrt{2}\right)\left((U\,V)^2-1\right)\left((3+2\sqrt{2})\,A^2+B^2\right)^2.$$

Since $U\,V = \pm 1$ we conclude that the above difference is zero and the proof is complete. □

Theorem 2. *For every $m \geqslant 1$ and $k \geqslant 0$ the following equality holds:*

$$\sum_{i=0}^{m} P^2_{2k+2i+1} = \left(\sum_{i=0}^{m-1} P^2_{2i+1}\right) + 4\left(1 + a_{m+1}\cdot a_{k+1}\cdot P_{2k+2m}\right). \qquad (3.4)$$

Proof. The proof is by induction on m. For $m = 1$ the relation (3.4) is true by the relation (3.2) in Lemma 3. Assume that the relation (3.4) is true for $m = r$.

$$\sum_{i=0}^{r+1} P^2_{2k+2i+1} = \sum_{i=0}^{r} P^2_{2k+2i+1} + P^2_{2k+2r+3} =$$

$$\sum_{i=0}^{r-1} P^2_{2i+1} + 4 + 4\, a_{r+1}\, a_{k+1}\, P_{2k+2r} + P^2_{2k+2r+3} = 4 + 4\, a_{r+1}\, a_{k+1}\, P_{2k+2r}$$

$$+ \sum_{i=0}^{r} P^2_{2i+1} + P^2_{2k+2r+3} - P^2_{2r+1} = \sum_{i=0}^{r} P^2_{2i+1} + 4 + 4\, a_{r+2}\, a_{k+1}\, P_{2k+2r+2},$$

where the last step uses Lemma 4. Hence, (3.4) is true also for $m = r+1$ and the proof by induction is complete. □

Other versions of Theorem 2 are listed in the following statement:

Theorem 3. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equalities holds:*

$$\left(\sum_{i=0}^{m} P^2_{2k+2i+1}\right) + \left(\sum_{i=0}^{m} P^2_{2i}\right) = a_{m+1}\cdot P_{2k+1}\cdot P_{2k+2m+1}, \qquad (3.5)$$

$$\sum_{i=0}^{m} P_{2k+2i+1}^2 = \left(\sum_{i=0}^{m} P_{2i+1}^2\right) + 4 \cdot a_{m+1} \cdot a_k \cdot P_{2k+2m+2}, \tag{3.6}$$

$$\left(\sum_{i=0}^{m} P_{2k+2i+1}^2\right) + \left(\sum_{i=0}^{m+1} P_{2i}^2\right) = 2 \cdot a_{m+1} \cdot b_k \cdot P_{2k+2m+3}, \tag{3.7}$$

$$\sum_{i=0}^{m} P_{2k+2i+1}^2 = \left(\sum_{i=0}^{m+1} P_{2i+1}^2\right) + 4\left(a_{m+1} \cdot a_{k-1} \cdot P_{2k+2m+4} - 1\right). \tag{3.8}$$

Proof. We shall only outline the key steps in an inductive proof of the formula (3.6) leaving the details to the reader because they are analogous to the proof of Theorem 2.

The initial step is the equality $P_{2k+1}^2 = 4 + P_{2k}\, P_{2k+2}$ which holds for every $k \geqslant 0$. On the other hand, the inductive step is realized with the following equality:

$$4\, a_k\, [a_{m+2}\, P_{2k+2m+4} - a_{m+1}\, P_{2k+2m+2}] = P_{2k+2m+3}^2 - P_{2m+3}^2,$$

which holds for every $k \geqslant 0$ and every $m \geqslant 0$. □

4. **Sums of products of Pell numbers**

For the first two steps in a proof by induction of our next theorem we require the following lemma.

Lemma 5. *For every $k \geqslant 0$ the following equalities hold:*

$$P_{2k+1} = 2\, b_{k+1}. \tag{4.1}$$

$$8 + P_{2k}\, P_{2k+1} + P_{2k+2}\, P_{2k+3} = 12\, P_{2k+2}\, b_{k+1}. \tag{4.2}$$

Proof of (4.1). By the Binet formula we have

$$P_{2k+1} - 2\, b_{k+1} = \frac{2(\alpha^{2\,k-1} - \beta^{2\,k-1})}{\alpha - \beta} - \frac{(4 - \alpha + \beta)(3 + \alpha - \beta)^{k+1}}{4} - \frac{(4 + \alpha - \beta)(3 - \alpha + \beta)^{k+1}}{4} = \frac{\left(2 + \sqrt{2}\right)\left(A^2 - B\right)}{2} = 0,$$

where $A = (1 - \sqrt{2})^k$, $B = (3 - 2\sqrt{2})^k$ and $(1 - \sqrt{2})^2 = 3 - 2\sqrt{2}$. □

Proof of (4.2). When we apply the Binet formula to the terms in the difference of the left hand side and the right hand side of (4.2) and substitute $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ we get

$$(3\sqrt{2}-5)A^2\,C + 3(7-5\sqrt{2})(A^2-B)A^2 + (7+5\sqrt{2})(46\sqrt{2}-65+3\,D-3\,C^2),$$

where we have $D=(17+12\sqrt{2})^k$, $C=(3+2\sqrt{2})^k$, $B=(3-2\sqrt{2})^k$ and $A=(1-\sqrt{2})^k$. Since $A^2=B$, $A^2\,C=1$ and $D=C^2$, the above expression is equal to

$$(3\sqrt{2}-5)+(7+5\sqrt{2})(46\sqrt{2}-65)=0.$$

□

With the following lemma we shall make the inductive step in the proof of the fourth theorem.

Lemma 6. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$2\,b_{k+1}\left[a_{m+2}\,P_{2k+2m+2} - a_{m+1}\,P_{2k+2m}\right] = P_{2k+2m+2}\,P_{2k+2m+3} + P_{2m+1}\,P_{2m+2}. \tag{4.3}$$

Proof. Let R denote the difference of the left hand side and the right hand side of the above relation. We need to show that $R=0$.

Using the Binet formula and replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get

$$R = \frac{1}{16}\left(10-7\sqrt{2}\right)\Bigg(\left(3264+2308\sqrt{2}\right)a^2\,z - \left(16-12\sqrt{2}\right)b^4d^4 + \left(17-12\sqrt{2}\right)d^4\,y - \left(3264+2308\sqrt{2}\right)x\,z + \left(3+2\sqrt{2}\right)d^4 - c^4 y - \left(3+2\sqrt{2}\right)c^4\Bigg),$$

with $a=(3+2\sqrt{2})^k$, $b=(\sqrt{2}-1)^k$, $c=(1-\sqrt{2})^m$, $d=(\sqrt{2}-1)^m$, $x=(17+12\sqrt{2})^k$, $y=(17-12\sqrt{2})^k$ and $z=(17+12\sqrt{2})^m$. Since $a^2=x$, $y=b^4$ and $d=-c$, it follows that $R=0$. □

Theorem 4. *For every $m \geqslant 1$ and $k \geqslant 0$ the following equality holds:*

$$\sum_{i=0}^{m} P_{2k+2i}\cdot P_{2k+2i+1} = 2\cdot a_{m+1}\cdot b_{k+1}\cdot P_{2k+2m} - \sum_{i=0}^{m-1} P_{2i+1}\cdot P_{2i+2} \tag{4.4}$$

Proof. The proof is by induction on m. For $m = 1$ the relation (4.4) is true by (4.2).

Assume that the relation (4.4) is true for $m = r$. Then

$$\sum_{i=0}^{r+1} P_{2k+2i}\, P_{2k+2i+1} + \sum_{i=0}^{r} P_{2i+1}\, P_{2i+2} = \sum_{i=0}^{r} P_{2k+2i}\, P_{2k+2i+1} +$$
$$\sum_{i=0}^{r-1} P_{2i+1}\, P_{2i+2} + P_{2r+1}\, P_{2r+2} + P_{2k+2r+2}\, P_{2k+2r+3} = P_{2r+1}\, P_{2r+2}$$
$$+ 2\, a_{r+1}\, b_{k+1}\, P_{2k+2r} + P_{2k+2r+2}\, P_{2k+2r+3} = 2\, a_{r+2}\, b_{k+1}\, P_{2k+2r+2},$$

where the last step uses Lemma 6. Hence, (4.4) is true also for $m = r + 1$. □

Other versions of Theorem 4 are listed in the following statement:

Theorem 5. *For every $j \geqslant 0$ and $k \geqslant 0$ the sum $\sum_{i=0}^{m} P_{2k+2i} \cdot P_{2k+2i+1}$ is equal to the following expressions:*

$$\left(\sum_{i=0}^{m} P_{2i} \cdot P_{2i+1} \right) + 4 \cdot a_{m+1} \cdot a_k \cdot P_{2k+2m+1}, \tag{4.5}$$

$$2 \cdot a_{m+1} \cdot b_k \cdot P_{2k+2m+2} - \left(\sum_{i=0}^{m} P_{2i+1} \cdot P_{2i+2} \right), \tag{4.6}$$

$$\left(\sum_{i=0}^{m+1} P_{2i} \cdot P_{2i+1} \right) + 4 \cdot a_{m+1} \cdot a_{k-1} \cdot P_{2k+2m+3}, \tag{4.7}$$

$$2\,(a_{m+1} \cdot b_{k-1} \cdot P_{2k+2m+4} + 4) - \left(\sum_{i=0}^{m+1} P_{2i+1} \cdot P_{2i+2} \right), \tag{4.8}$$

Proof. We shall only outline the key steps in an inductive proof of the part (4.6) leaving the details and other parts to the reader because they are analogous to the proof of Theorem 4.

The initial step is the equality $2\, b_k\, P_{2k+2} - 8 = P_{2k}\, P_{2k+1}$ which holds for every $k \geqslant 0$. On the other hand, the inductive step is realized with the following equality:

$$2\, b_k \left[a_{m+2}\, P_{2k+2m+4} - a_{m+1}\, P_{2k+2m+2}\right] =$$
$$P_{2k+2m+2}\, P_{2k+2m+3} + P_{2m+3}\, P_{2m+4},$$

which holds for every $k \geqslant 0$ and every $m \geqslant 0$. □

References

[1] Z. Čerin, *Properties of odd and even terms of the Fibonacci sequence*, Demonstratio Mathematica, **39**, No. 1, 2006, 55–60.

[2] Z. Čerin, *On sums of squares of odd and even terms of the Lucas sequence*, Proccedings of the 11th Fibonacci Conference, (to appear).

[3] Z. Čerin, *Some alternating sums of Lucas numbers*, Central European Journal of Mathematics **3**, No. 1, 2005, 1 – 13.

[4] Z. Čerin, *Alternating Sums of Fibonacci Products*, Atti del Seminario Matematico e Fisico dell'Università di Modena e Reggio Emilia, (to appear).

[5] Z. Čerin and G. M. Gianella, *On sums of squares of Pell-Lucas numbers*, INTEGERS: Electronic Journal of Combinatorial Number Theory, (to appear).

[6] V. E. Hoggatt, Jr., *Fibonacci and Lucas numbers*, The Fibonacci Association, Santa Clara, 1969.

[7] R. Knott, *Fibonacci numbers and the Golden Section*, `http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`.

[8] V. Rajesh and G. Leversha, *Some properties of odd terms of the Fibonacci sequence*, Mathematical Gazette, **88** 2004, 85–86.

[9] R. Ram, *Pell Numbers Formulae*, `http://users.tellurian.net/hsejar/maths/`.

[10] T. Reix, *Properties of Pell numbers modulo prime Fermat numbers*, (preprint).

[11] N. Sloane, *On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

[12] S. Vajda, *Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications*, Halsted Press, Chichester 1989.