PROPERTIES OF ODD AND EVEN TERMS OF THE FIBONACCI SEQUENCE

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Abstract. We shall improve some results on sums of squares of odd terms of the Fibonacci sequence by Rajesh and Leversha.

The Fibonacci sequence F_n is defined by the recurrence relation

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \text{ for } n \geqslant 3.$$

Let $u_k = F_{2k-1}$, $v_k = F_{2k}$, $U_k = u_k^2$, and $V_k = v_k^2$ denote odd and even terms of the Fibonacci sequence and their squares. The note [3] proves the following relations: $A)$ $u_{k+2} = 3\,u_{k+1} - u_k$, $B)$ $U_{k+1} + 1 = u_{k+2}\,u_k$, $C)$ $U_{k+1} + U_k + 1 = 3\,u_{k+1}\,u_k$, and $D)$ $U_{k+2} + U_k + 2 = 7\,U_{k+1}$.

The purpose of this note is to show that the formulas C) and D) are special cases of the relations:

$$E) \quad \beta_{2j+1} + \sum_{i=0}^{2j+1} U_{k+i} = v_{2j+2}\,u_{k+j}\,u_{k+j+1}, \text{ for } j \geqslant 0 \text{ and } k \geqslant 1,$$

and

$$F) \quad \beta_{2j} + \sum_{i=0}^{2j} U_{k+i} = v_{2j+1}\,U_{k+j}, \text{ for } j \geqslant 0 \text{ and } k \geqslant 1,$$

where the sequence β_k is defined as follows:

$$\beta_k = \begin{cases} 0, & \text{if } k = 0; \\ \beta_{k-1} + V_{j+1}, & \text{if } k = 2j+1,\ j \geqslant 0\ ; \\ \beta_{k-1} + V_j, & \text{if } k = 2j,\ j \geqslant 1\ . \end{cases}$$

Lemma 1. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$G) \quad V_{j+1} + U_{k+2j+1} + v_{2j+1}\,U_{k+j} = v_{2j+2}\,u_{k+j}\,u_{k+j+1}.$$

Proof. In terms of the Fibonacci sequence the relation G) becomes

$$*) \quad F_{2j+2}^2 + F_{2k+4j+1}^2 + F_{4j+2}\,F_{2k+2j-1}^2 - F_{4j+4}\,F_{2k+2j-1}\,F_{2k+2j+1} = 0.$$

Think of the left hand side of *) as the function M of j. The proof will be by induction on j. For $j = 0$ the relation G) agrees with the C)

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above and thus is true. Assume that *) is true for $j = r$. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By the Binet formula F_k is equal to $\frac{\alpha^k - \beta^k}{\alpha - \beta}$ (see [2, section 1.2.8], [4, formula (58)], and [1]). It follows that $M(r+1)$ and $M(r)$ are equal to $\frac{21\sqrt{5}-47}{10}(\mu - a)$ and $\frac{3\sqrt{5}-7}{10}(\mu - a)$, where $\mu = (3\beta + \frac{5}{6})^r$ and $a = \beta^{4r}$. By assumption $M(r) = 0$ so that $\mu = a$. Hence, $M(r+1) = 0$ and the proof is complete. $\square$

Lemma 2. *For every $j \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$H) \quad V_j + U_{k+2j} + v_{2j}\, u_{k+j-1}\, u_{k+j} = v_{2j+1}\, U_{k+j}.$$

Proof. In terms of the Fibonacci sequence the relation H) becomes

$$**) \quad F_{2j}^2 + F_{2k+4j-1}^2 + F_{4j}\, F_{2k+2j-3}\, F_{2k+2j-1} - F_{4j+2}\, F_{2k+2j-1}^2 = 0.$$

Think of the left hand side of **) as the function N of j. The proof will be by induction on j. For $j = 1$ the relation H) follows easily from C) and D) above and thus is true. Assume that **) is true for $j = r$. It follows that $N(r+1)$ and $N(r)$ are equal to $\frac{7-3\sqrt{5}}{10}(a - \mu)$ and $\frac{1}{5}(a - \mu)$, where $\mu = (3\beta + \frac{5}{6})^r$ and $a = \beta^{4r}$. By assumption $N(r) = 0$ so that $\mu = a$. Hence, $N(r+1) = 0$ and the proof is complete. $\square$

Proof of E). The proof is also by induction on j. For $j = 0$ the relation E) is equivalent to the above relation C). Assume that the relations E) and F) are true for $j = r$. Then

$$\beta_{2(r+1)+1} + \sum_{i=0}^{2(r+1)+1} U_{k+i} = \beta_{2(r+1)} + V_{r+2} + U_{k+2(r+1)+1} + \sum_{i=0}^{2(r+1)} U_{k+i} =$$

$$V_{r+2} + U_{k+2(r+1)+1} + v_{2(r+1)+1}\, U_{k+r+1} = v_{2(r+1)+2}\, u_{k+r+1}\, u_{k+r+2},$$

where the last step uses Lemma 1 for $j = r+1$. Hence, E) is true for $j = r+1$ and the proof is completed. $\square$

Proof of F). The proof will be once more by induction on j. For $j = 1$ the relation F) is $2 + U_k + U_{k+1} + U_{k+2} = 8\, U_{k+1}$ that is clearly equivalent to the above relation D). Assume that the relations E) and F) are true for $j = r$. Then

$$\beta_{2(r+1)} + \sum_{i=0}^{2(r+1)} U_{k+i} = \beta_{2(r+1)-1} + V_{r+1} + U_{k+2(r+1)} + \sum_{i=0}^{2r+1} U_{k+i} =$$

$$V_{r+1} + U_{k+2r+2} + v_{2r+2}\, u_{k+r}\, u_{k+r+1} = v_{2(r+1)+1}\, U_{k+r+1},$$

where the last step uses Lemma 2 for $j = r+1$. Hence, F) is true for $j = r+1$ and the proof is completed. $\square$

The formula for the sum of squares of the even terms of the Fibonacci sequence is much simpler.

Theorem 1. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$K) \quad \delta_j + \sum_{i=0}^{j} V_{k+i} = v_{j+1}\, u_{k+1}\, u_{k+j},$$

where the sequence δ_j is defined as follows:

$$\delta_j = \begin{cases} 1, & \text{if } j = 0; \\ \delta_{j-1} + U_j, & \text{if } j \geqslant 1. \end{cases}$$

Proof. The proof will be by induction on j. For $j = 0$ the relation K) is $1 + V_k = u_k\, u_{k+1}$ or $1 + F_{2k}^2 = F_{2k-1}\, F_{2k+1}$ which follows from the famous Cassini identity $F_{n+1}\, F_{n-1} - F_n^2 = (-1)^n$ for $n = 2\,k$ (see [4, formula (29)] or [1]). Assume that K) is true for $j = r$. Then

$$\delta_{r+1} + \sum_{i=0}^{r+1} V_{k+i} = \delta_r + U_{r+1} + V_{k+r+1} + \sum_{i=0}^{r} V_{k+i} =$$

$$U_{r+1} + V_{k+r+1} + v_{r+1}\, u_{k+1}\, u_{k+r} = v_{r+2}\, u_{k+1}\, u_{k+r+1},$$

where the last step uses the relation L) of Lemma 3 below. Hence, K) is true for $j = r + 1$ and the proof is completed. □

Lemma 3. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$L) \quad U_{j+1} + V_{k+j+1} = u_{k+1} \left[v_{j+2}\, u_{k+j+1} - v_{j+1}\, u_{k+j} \right].$$

Proof. In terms of the Fibonacci sequence the relation L) becomes

$$***) \quad F_{2j+1}^2 + F_{2k+2j+2}^2 - F_{2k+1} \left[F_{2j+4}\, F_{2k+2j+1} - F_{2j+2}\, F_{2k+2j-1} \right] = 0.$$

Think of the left hand side of ***) as the function P of j. We shall prove that $P(j) = 0$. Let $\mu = \left(\frac{7+3\sqrt{5}}{2}\right)^{j+1}$ and $\nu = \left(\frac{3+\sqrt{5}}{2}\right)^{k}$. Then $P(j) = -\frac{\sqrt{5}}{50} Q$, where Q is the expression

$$\left(\nu\, \beta^{2\,k} - 1\right) \left[\left(3\sqrt{5} - 5\right) \mu + 2\, \beta^{4\,j} \right] + \left(\sqrt{5} + 2\right) \left(2\,\beta - 1 + \sqrt{5}\right) \beta^{4\,k}.$$

Notice that the second term is zero because $\beta = \frac{1-\sqrt{5}}{2}$ and that

$$\left(\frac{3+\sqrt{5}}{2}\right) \beta^2 = \left(\frac{3+\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right)^2 = \left(\frac{3+\sqrt{5}}{2}\right) \left(\frac{3-\sqrt{5}}{2}\right) = 1,$$

so that $\nu\, \beta^{2\,k} - 1 = 0$. Hence, $P(r)$ is indeed equal to zero and the proof is complete. □

In a similar way it is possible to prove the following theorem.

Theorem 2. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$R) \quad \eta_m + \sum_{i=0}^{m} u_{k+i}\, v_{k+i} = \begin{cases} v_{k+1}, & \text{if } m = 0,\ k = 1; \\ u_{k-1}\, v_{k+1}, & \text{if } m = 0,\ k \geqslant 2; \\ v_{m+1}\, u_{k+j}\, v_{k+j}, & \text{if } m = 2j,\ j \geqslant 1; \\ v_{m+1}\, u_{k+j}\, v_{k+j+1}, & \text{if } m = 2j+1,\ j \geqslant 0, \end{cases}$$

where the sequence η_m is defined as follows:

$$\eta_m = \begin{cases} 2,\ 2,\ 1, & \text{if } m = 0,\ 1,\ 2; \\ \eta_{2j-2} + V_j, & \text{if } m = 2j \text{ and } j \geqslant 2; \\ \eta_{2j+2} + u_{j+1}\, v_{j+1}, & \text{if } m = 2j+1 \text{ and } j \geqslant 1. \end{cases}$$

In the rest of this note I will describe how one can discover these results and check the above proofs with the help of the computer. The presentation is for the software Maple V.

The input `with(combinat):` calls the package that contains the function `fibonacci` that computes the terms of the Fibonacci sequence. We first define functions `fu`, `fv`, `fU`, `fV`, and `fL` that give terms of the sequences u_k, v_k, U_k, V_k, and the left hand sides of E) and F).

```
f:=x->fibonacci(x): fu:=x->f(2*x-1): fv:=x->f(2*x): fU:=x->
fu(x)^2: fV:=x->fv(x)^2: fL:=(b,j,k)->b+sum(fU(k+i),i=0..j):
```

The following procedure tests for values b and k between $b0$ and $b1$ and $k0$ and $k1$ and for a given value j if the quotient $q = \frac{fL(b,j,k)}{fu(k+m)fu(k+n)}$ is an integer and prints out b, k, and q.

```
fS:=proc(b0,b1,j,k0,k1,m,n) local b,k,q;for b from b0 to
b1 do for k from k0 to k1 do q:=fL(b,j,k)/fu(k+m)/fu(k+n):
if denom(q)=1 then print([b,k,q]);fi:od;od;end:
```

The input `fS(1,100,3,17,17,1,2);` is asking to determine among first hundred integers the number β_3 that makes the quotient q of the left hand side of the formula E) for $j = 1$ by the product $u_{k+1}u_{k+2}$ an integer for $k = 17$. The output $[11, 17, 21]$ indicates that $\beta_3 = 11$ and $q = 21$ are good candidates. This is further confirmed when we input `fS(11,11,3,1,100,1,2);` whose output has the same number 21 as the third term in each of the hundred triples.

When we repeat this for values of j between 0 and 10 we discover that numbers β_j and γ_j (the coefficients on the right hand sides of E) and F)) are 1, 1, 10, 11, 75, 84, 148, 589, 1030, 4055, 7080 and 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711. Of course, some experimentation is needed to figure out the correct indices of u's on the right sides. These eleven values are sufficient to discover the rule by which β_j's are built and to recognize that γ_j's are in fact v_j's (with correct indices). All other formulas in this paper are discovered by similar procedures.

It remains to explain how to invoke the help of the computer in the proof of Lemmas 1 – 3. We shall prove only Lemma 1 because the proofs for Lemmas 2 and 3 are analogous. This proof is direct so that it differs from the proof above by induction.

We first define the function that expresses the Binet formula and make the assumption that j and k are positive integers.

```
a:=(1+sqrt(5))/2:b:=-1/a:f:=x->(1/(a-b))*(a^x-b^x):
assume(k,posint):assume(j,posint):
```

The left hand side of *) is evaluated by the the following input.

```
simplify(f(2*j+2)^2+f(4*j+2*k+1)^2+f(4*j+2)*f(2*j+2*k-1)^2
-f(4*j+4)*f(2*j+2*k-1)*f(2*j+2*k+1));
```

The output is the quotient $\frac{32\,M}{\left(1+\sqrt{5}\right)^3\left(5+\sqrt{5}\right)^3}$, where M is the following complicated expression.

$$\begin{gathered}
123\left(7/2+3/2\sqrt{5}\right)^j\sqrt{5}-123\,2^j\left(3+\sqrt{5}\right)^{-j}\left(9+4\sqrt{5}\right)^j\sqrt{5}+\\
5\,2^j\left(7+3\sqrt{5}\right)^{-j}-5\,2^{-j}\left(3+\sqrt{5}\right)^j\left(9+4\sqrt{5}\right)^{-j}+\\
3\sqrt{5}2^j\left(7+3\sqrt{5}\right)^{-j}-3\sqrt{5}2^{-j}\left(3+\sqrt{5}\right)^j\left(9+4\sqrt{5}\right)^{-j}+\\
275\left(7/2+3/2\sqrt{5}\right)^j-275\,2^j\left(3+\sqrt{5}\right)^{-j}\left(9+4\sqrt{5}\right)^j.
\end{gathered}$$

Each line of this expression is zero. Indeed, the first line is

$$123\sqrt{5}\left(\left(7/2+3/2\sqrt{5}\right)^j-2^j\left(3+\sqrt{5}\right)^{-j}\left(9+4\sqrt{5}\right)^j\right).$$

But, $\frac{2(9+4\sqrt{5})}{3+\sqrt{5}}=7/2+3/2\sqrt{5}$ so that the terms in the parenthesis after 123 are identical and the first line is zero. In the same fashion we argue that all other lines are zero so that $M=0$ and the relation *) has been proved.

References

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