ALTERNATING SUMS OF FIBONACCI PRODUCTS

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Abstract. We consider alternating sums of squares of odd and even terms of the Fibonacci sequence and alternating sums of their products. These alternating sums are related to the products of appropriate Fibonacci and Lucas numbers and to the integer sequences $A049685$ and $A004178$.

1. Introduction

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \ \text{ for } n \geqslant 3,$$

and

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \ \text{ for } n \geqslant 3.$$

The note [1] derived formulas for the sums $\sum_{i=0}^{j} F_{2k+2i-1}^2$, $\sum_{i=0}^{j} F_{2k+2i}^2$ and $\sum_{i=0}^{j} F_{2k+2i-1}\, F_{2k+2i}$ of squares of odd and even and products of consecutive Fibonacci numbers that improved some results from [6]. The aim of this work is to give similar results for the alternating sums $\sum_{i=0}^{j} (-1)^i\, F_{2k+2i-1}^2$, $\sum_{i=0}^{j} (-1)^i\, F_{2k+2i}$ and $\sum_{i=0}^{j} (-1)^i\, F_{2k+2i-1}\, F_{2k+2i}$. For Lucas numbers this was done in [2] and [3].

Therefore we shall prove the following three theorems covering each of these alternating sums. Their proofs are by mathematical induction.

In each theorem we must treat odd and even summations separately. The results are similar in form and the method of proof. They could be viewed as examples of situations where the following integer sequences $A049685$ and $A004178$ in [7] appear.

Let the sequence a_k be defined by the recurrence relation $a_0 = 1$, $a_1 = 6$, and $a_k = 7\, a_{k-1} - a_{k-2}$, for $k \geqslant 2$. Then $a_k = \frac{L_{4k+2}}{3}$.

For $k \geqslant 0$, let $b_k = \sum_{i=0}^{k} a_i$. Since $a_k = b_k - b_{k-1}$ we see that the terms b_k satisfy the recurrence relations $b_0 = 1$, $b_1 = 7$, $b_2 = 48$, and $b_k = 8\, b_{k-1} - 8\, b_{k-2} + b_{k-3}$ for $k \geqslant 3$. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We

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easily find that $b_k = u(\alpha,\, \beta)^k\, v(\alpha,\, \beta) + u(\beta,\, \alpha)^k\, v(\beta,\, \alpha)$ for every $k \geqslant 0$, where $u(\alpha,\, \beta) = 5\,\alpha + 2\,\beta$ and $v(\alpha,\, \beta) = \frac{11\,\alpha + 4\,\beta}{15}$. Also, $b_{k-1} = \frac{F_{4k}}{3}$.

In order to state our theorems we shall also need the following integer sequences which are not listed in [7].

Let c_k denote the sequence whose k-th term for $k \geqslant 0$ is given by $c_0 = 1$ and $c_k = 1 + 7\,(\sum_{i=0}^{k-1} b_k)$ for $k \geqslant 1$. The first seven terms of this sequence are 1, 8, 57, 393, 2696, 18481, 126673. In a standard way one can prove that $c_k = \frac{21+7\sqrt{5}}{30}\left(\frac{7+3\sqrt{5}}{2}\right)^k + \frac{21-7\sqrt{5}}{30}\left(\frac{7-3\sqrt{5}}{2}\right)^k - \frac{2}{5}$.

Let d_k denote the sequence whose k-th term for $k \geqslant 0$ is given by $d_0 = 2$ and $d_k = 2 + 11\,(\sum_{i=0}^{k-1} b_k)$ for $k \geqslant 1$. The first seven terms of this sequence are 2, 13, 90, 618, 4237, 29042, 199058. In a standard way one can prove that $d_k = \frac{33+11\sqrt{5}}{30}\left(\frac{7+3\sqrt{5}}{2}\right)^k + \frac{33-11\sqrt{5}}{30}\left(\frac{7-3\sqrt{5}}{2}\right)^k - \frac{1}{5}$.

Here are our three theorems on alternating sums.

Theorem 1. *(a) For every $n \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$D)\quad F_{4\,n} + \sum_{i=0}^{2\,n-1} (-1)^i\, F_{2k+2i}^2 = -\frac{F_{4n}}{3}\, F_{2k+2n+1}\, L_{2k+2n-3};$$

(b) For every $n \geqslant 0$ and $k \geqslant 1$ the following equality holds:

$$E)\quad F_{2\,n+1}^2 + \sum_{i=0}^{2\,n} (-1)^i\, F_{2k+2i}^2 = \frac{L_{4n+2}}{3}\, F_{2k+2n+1}\, F_{2k+2n-1}.$$

Theorem 2. *(a) For every $n \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$K)\quad \frac{F_{4\,n}}{3} + \sum_{i=0}^{2\,n-1} (-1)^i\, F_{2k+2i-1}^2 = -\frac{F_{4n}}{3}\, F_{2k+2n-1}\, L_{2k+2n-3};$$

(b) For every $n \geqslant 0$ and $k \geqslant 1$ the following equality holds:

$$L)\quad c_n + \sum_{i=0}^{2\,n} (-1)^i\, F_{2k+2i-1}^2 = \frac{L_{4n+2}}{3}\, F_{2k+2n+1}\, F_{2k+2n-3}.$$

Theorem 3. *(a) For every $n \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$P)\quad \frac{F_{4\,n}}{3} + \sum_{i=0}^{2\,n-1} (-1)^i\, F_{2k+2i-1}\, F_{2k+2i} = -\frac{F_{4n}}{3}\, F_{2k+2n-2}\, L_{2k+2n-1};$$

(b) For $n = 0$ and every $k \geqslant 2$ and also for every $n \geqslant 1$ and $k \geqslant 1$ the following equality holds:

$$Q) \quad d_n + \sum_{i=0}^{2\,n} (-1)^i \, F_{2k+2i-1} \, F_{2k+2i} = \frac{L_{4n+2}}{3} \, F_{2k+2n+2} \, F_{2k+2n-3}.$$

2. Lemmas for Theorem 1

For the initial step in an inductive proof of the part a) of our first theorem we shall use the following lemma.

Lemma 1. *For every $k \geqslant 1$ the following equality holds:*

$$A) \quad 3 + F_{2\,k}^2 - F_{2\,k+2}^2 + F_{2\,k+3} \, L_{2\,k-1} = 0.$$

Proof. First note that

$$F_{2k+2}^2 - F_{2k}^2 = (F_{2k+2} - F_{2k})(F_{2k+2} + F_{2k}) = F_{2k+1} \, L_{2k+1} = F_{4k+2}$$

so that it suffices to prove that

$$F_{2k+3} \, L_{2k-1} = F_{4k+2} - 3.$$

Now using the Binet formulas $F_k = \frac{\alpha^k - \beta^2}{\alpha - \beta}$, $L_k = \alpha^k + \beta^k$, (see [4] and [5]) we have

$$F_{2k+3} \, L_{2k-1} = \left(\frac{\alpha^{2k+3} - \beta^{2k+3}}{\alpha - \beta} \right) \left(\alpha^{2k-1} + \beta^{2k-1} \right) =$$

$$\frac{\alpha^{4k+2} - \beta^{4k+2} + \alpha^{2k+3} \, \beta^{2k-1} - \alpha^{2k-1} \, \beta^{2k+3}}{\alpha - \beta} =$$

$$F_{4k+2} + (\alpha \, \beta)^{2k-1} \left(\frac{\alpha^4 - \beta^4}{\alpha - \beta} \right) = F_{4k+2} + (-1)^{2k-1} \, F_4 = F_{4k+2} - 3.$$

□

The above proof was supplied by the referee. The proofs of the other lemmas are modelled on this example. They all use basic Fibonacci identities such as can be found in [4] or [8]. The author's original proofs used the symbolic computation software Maple V. The following was the proof of Lemma 1 which explains our method. All other lemmas in this paper can be easily proved in this way.

Proof. By the Binet formula F_k is equal to $\frac{\alpha^k - \beta^k}{\alpha - \beta}$ and L_k is equal to $\alpha^k + \beta^k$ (see [4] and [5]). It follows that the left hand side of A) is equal to $W = -\frac{M}{\alpha^3 \, (\alpha^2 + 1)^2}$, where M denotes the expression

$$\alpha^{10} + \alpha^8 - 3\,\alpha^7 - 6\,\alpha^5 - 3\,\alpha^3 - \alpha^2 - 1 + \alpha^{4\,k+9} - \alpha^{4\,k+8} - \alpha^{4\,k+6} - \alpha^{4\,k+5} - \alpha^{-4\,k+5} + \alpha^{-4\,k+4} + \alpha^{-4\,k+2} + \alpha^{-4\,k+1}.$$

The fifteen terms of M can be considered as one group of seven terms and two groups of four terms that have similar exponents. The first group is $\alpha^{10} + \alpha^8 - 3\,\alpha^7 - 6\,\alpha^5 - 3\,\alpha^3 - \alpha^2 - 1$. It is equal to zero because it factors as $(\alpha^2 - \alpha - 1)\,(\alpha^4 + \alpha^3 + \alpha^2 - \alpha + 1)\,(\alpha^2 + 1)^2$ and α is the root of the equation $x^2 - x - 1 = 0$. One can prove similarly that the other two groups are zero. Hence, $M = 0$ and the proof is complete.

We shall now describe input for the proof of Lemma 1 in Maple V.

We first use `fF` and `fL` as short names for functions of Fibonacci and Lucas numbers by Binet formula. We make the assumption that k and r are positive integers and that α is positive.

```
fF:=k->(alpha^k-beta^k)/(alpha-beta): fL:=k->alpha^k+beta^k:
assume(k,posint):  assume(r,posint):  assume(alpha>0):
```

The following will give us the left hand side W of the relation A).

```
sumA:=3+fF(2*k)^2-fF(2*k+2)^2+fF(2*k+3)*fL(2*k-1):
fW:=simplify(factor(subs(beta=-1/alpha,sumA)));
```

The part M is extracted as follows. In some situations M is not the first part but second or third.

```
fM:=collect(op(1,fW),alpha,distributed,factor):
```

The first group is factored out with the following input. Here it can happen that these seven terms are not the initial seven so that we pick up these terms by counting parts in an output of `fM`.

```
simplify(factor(add(op(i,fM),i in [1,2,3,4,5,6,7])));
```

□

For the induction step in the proof of the part a) of Theorem 1 we shall use the following lemma.

Lemma 2. *For every $r \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$B) \quad F_{2\,k+4\,r}^2 + b_r\,(3 + F_{2\,k+2\,r+3}\,L_{2\,k+2\,r-1}) \\ - F_{2\,k+4\,r+2}^2 - b_{r-1}\,(3 + F_{2\,k+2\,r+1}\,L_{2\,k+2\,r-3}) = 0.$$

Proof. First note that, by the formulas (I_8) and (I_7) in [4],

$$F_{2k+4r+2}^2 - F_{2k+4r}^2 = (F_{2k+4r+2} - F_{2k+4r})(F_{2k+4r+2} + F_{2k+4r}) \\ = F_{2k+4r+1}\,L_{2k+4r+1} = F_{4k+8r+2}.$$

Also, using the Binet formulas we have

$$\begin{aligned} F_{2n+3}\,L_{2n-1} &= \left(\frac{\alpha^{2n+3}-\beta^{2n+3}}{\alpha-\beta}\right)\left(\alpha^{2n-1}-\beta^{2n-1}\right) \\ &= \frac{\alpha^{4n+2}-\alpha^{2n-1}\,\beta^{2n+3}+\alpha^{2n+3}\,\beta^{2n-1}+\beta^{4n+2}}{\alpha-\beta} \\ &= F_{4n+2}+(\alpha\,\beta)^{2n-1}\left(\frac{\alpha^4-\beta^4}{\alpha-\beta}\right) = F_{4n+2}-3, \end{aligned}$$

since $\alpha\,\beta=-1$ and $\frac{\alpha^4-\beta^4}{\alpha-\beta}=F(4)=3$. For $n=k+r$ and $n=k+r-1$ we get

$$3+F_{2k+2r+3}\,L_{2k+2r-1}=F_{4k+4r+2}$$

and

$$3+F_{2k+2r+1}\,L_{2k+2r-3}=F_{4k+4r-2}.$$

Hence, if we use $b_r=\frac{F_{4r+4}}{3}$, it suffices to prove

$$F_{4r+4}\,F_{4k+4r+2}-F_{4r}\,F_{4k+4r-2}=3\,F_{4k+8r+2}.$$

This is the special case of the formula

$$F_{m+4}\,F_{m+n+2}-F_m\,F_{m+n-2}=3\,F_{2m+n+2},$$

that is proved using Binet formulas as follows.

$$\begin{aligned} &F_{m+4}\,F_{m+n+2}-F_m\,F_{m+n-2}= \\ &\frac{(\alpha^{m+4}-\beta^{m+4})\,(\alpha^{m+n+2}-\beta^{m+n+2})-(\alpha^m-\beta^m)\,(\alpha^{m+n-2}-\beta^{m+n-2})}{(\alpha-\beta)^2} \\ &= \frac{1}{(\alpha-\beta)^2}\Big[\alpha^{2m+n+6}+\beta^{2m+n+6}-(\alpha\,\beta)^{m+2}\,\beta^2\,\alpha^n-(\alpha\,\beta)^{m+2}\,\beta^n\,\alpha^2 \\ &\quad +(\alpha\,\beta)^{m-2}\,\beta^n\,\alpha^2+(\alpha\,\beta)^{m-2}\,\beta^2\,\alpha^n-\alpha^{2m+n-2}+\beta^{2m+n-2}\Big] \\ &= \frac{\alpha^4\,\alpha^{2m+n+2}+\beta^4\,\beta^{2m+n+2}-\alpha^{-4}\,\alpha^{2m+n+2}-\beta^{-4}\,\beta^{2m+n+2}}{(\alpha-\beta)^2} \\ &= \frac{\alpha^4-\alpha^{-4}}{(\alpha-\beta)^2}\,\alpha^{2m+n+2}+\frac{\beta^4-\beta^{-4}}{(\alpha-\beta)^2}\,\beta^{2m+n+2} \\ &= 3\,\frac{\alpha^{2m+n+2}-\beta^{2m+n+2}}{\alpha-\beta}=3\,F_{2\,m+n+2}. \end{aligned}$$

□

For the induction step in the proof of the part b) of our first theorem we shall use the following lemma.

Lemma 3. *For every $r \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$C) \quad 3\,[F_{2\,r+3}^2 - F_{2\,r+1}^2 + F_{2\,k+4\,r+4}^2 - F_{2\,k+4\,r+2}^2] - \\ F_{2\,k+2\,r+1}\,[L_{4r+6}\,F_{2\,k+2\,r+3} - L_{4\,r+2}\,F_{2\,k+2\,r-1}] = 0.$$

Proof. Representing differences of squares as products in the same way as in the proofs of Lemmas 1 and 2 and applying the formula (13) in [8, p. 176] we conclude that the first square bracket is $F_{4r+4} + F_{4k+8r+6}$. The application of the formula (10a) in [8, p. 176] to each product in the second square bracket implies that it is equal to $F_{2k+6r+9} - F_{2k+6r+1}$. Hence, it suffices to prove that

$$F_{2k+2r+1}(F_{2k+6r+9} - F_{2k+6r+1}) = 3\,(F_{4r+4} + F_{4k+8r+6}).$$

Then we again use the Binet formula on the left hand side and reduce terms containing powers of both α and β to infer that

$$F_{2k+2r+1}(F_{2k+6r+9} - F_{2k+6r+1}) = \\ \frac{1}{5}\,[L_{4k+8r+10} + L_{4r+8} - L_{4k+8r+2} - L_{4r}]\,.$$

But, by the formula (17b) in [8, p. 177]

$$L_{4k+8r+10} - L_{4k+8r+2} = 5\,F_4\,F_{4k+8r+6}$$

and

$$L_{4r+8} - L_{4r} = 5\,F_4\,F_{4r+4},$$

so that the left hand side of the above alleged equality is indeed equal to the right hand side because $F_4 = 3$. □

3. Proof of the first theorem

Proof of Theorem 1. a). The proof is by induction on n. For $n = 1$ the relation D) is $3 + F_{2k}^2 - F_{2k+2}^2 = -F_{2k+3}\,L_{2k-1}$ (i. e., the relation A)) which is true by Lemma 1. Assume that the relation D) is true for $n = r$. Then

$$3\,b_r + \sum_{i=0}^{2(r+1)-1} (-1)^i\,F_{2k+2i}^2 =$$

$$3\,b_r + \sum_{i=0}^{2\,r-1} (-1)^i\,F_{2k+2i}^2 + F_{2k+4\,r}^2 - F_{2k+4\,r+2}^2 =$$

$$3\,b_r - 3\,b_{r-1} + F_{2k+4\,r}^2 - F_{2k+4\,r+2}^2 - b_{r-1}\,F_{2k+2r+1}\,L_{2k+2r-3} =$$

$$-b_r\,F_{2k+2(r+1)+1}\,L_{2k+2(r+1)-3},$$

where the last step uses Lemma 2. Hence, D) is true for $n = r + 1$.

b). The proof is also by induction on n. For $n = 0$ the relation E) is $1 + F_{2k}^2 = F_{2k+1}\, F_{2k-1}$ which is the special case (for $n = 2\,k$) of the formula (29) in [8].

Assume that the relation E) is true for $n = r$. Then

$$F_{2(r+1)+1}^2 + \sum_{i=0}^{2(r+1)} (-1)^i\, F_{2k+2i}^2 =$$

$$F_{2(r+1)+1}^2 + \sum_{i=0}^{2\,r} (-1)^i\, F_{2k+2i}^2 - F_{2k+4\,r+2}^2 + F_{2k+4\,r+4}^2 =$$

$$a_r\, F_{2k+2r+1}\, F_{2k+2r-1} + F_{2r+3}^2 - F_{2r+1}^2 - F_{2k+4\,r+2}^2 + F_{2k+4\,r+4}^2 =$$
$$a_{r+1}\, F_{2k+2(r+1)+1}\, F_{2k+2(r+1)-1},$$

where the last step uses Lemma 3. Hence, E) is true for $n = r + 1$. □

4. Lemmas for Theorem 2

For the initial step in an inductive proof of the part a) of our second theorem we shall use the following lemma.

Lemma 4. *For every $k \geqslant 1$ the following equality holds:*

$$F) \quad 1 + F_{2\,k-1}^2 - F_{2\,k+1}^2 + F_{2\,k+1}\, L_{2\,k-1} = 0.$$

Proof. First note that, by the formulas (1) and (6) in [8, p. 176],

$$F_{2k+1}^2 - F_{2k-1}^2 = (F_{2k+1} - F_{2k-1})F_{2k+1} + F_{2k-1}) = F_{2k}\, L_{2k},$$

so that it suffices to prove that

$$F_{2k+1}\, L_{2k-1} - F_{2k}\, L_{2k} = -1.$$

But, this is the special case of the formula (19b) in [8, p. 177] for $n = 2k$, $h = 1$ and $k = -1$ since $F_1 = 1$ and $L_{-1} = -1$. □

For the induction step in the proof of the part a) of our second theorem we shall use the following lemma.

Lemma 5. *For every $r \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$G) \quad b_r(1 + F_{2\,k+2\,r+1}\, L_{2\,k+2\,r-1}) -$$
$$b_{r-1}(1 + F_{2\,k+2\,r-1}\, L_{2\,k+2\,r-3}) + F_{2\,k+4\,r-1}^2 - F_{2\,k+4\,r+1}^2 = 0.$$

Proof. First we apply the formula (19b) in [8, p. 177] for $n = 2k + 2r$, $h = 1$ and $k = -1$ to get

$$F_{2k+2r+1}\, L_{2k+2r-1} - F_{2k+2r}\, L_{2k+2r} = (-1)^{2k+2r}\, F_1\, L_{-1} = -1.$$

Hence,

$$1 + F_{2\,k+2\,r+1}\, L_{2\,k+2\,r-1} = F_{2k+2r}\, L_{2k+2r} = F_{4k+4r},$$

by the formula (13) in [8, p. 177]. In a similar way we obtain

$$1 + F_{2\,k+2\,r-1}\,L_{2\,k+2\,r-3} = F_{2k+2r}\,L_{2k+2r-4} - 3.$$

On the other hand, from the formulas (1), (6) and (13) in [8], we have

$$F^2_{2k+4r+1} - F^2_{2k+4r+1} =$$
$$(F_{2k+4r+1} - F_{2k+4r-1})(F_{2k+4r+1} + F_{2k+4r-1}) = F_{2k+4r}\,L_{2k+4r} = F_{4k+8r}.$$

Since $b_{r-1} = \frac{F_{4r}}{3}$ and $b_r = \frac{F_{4r+4}}{3}$ the equality G) is equivalent with

$$F_{2k+2r}\,[F_{4r+4}\,L_{2k+2r} - F_{4r}\,L_{2k+2r-4}] = 3\,[F_{4k+8r} - F_{4r}]\,.$$

Appling the formula (10a) on each product in the square bracket on the left hand side we get further simplification

$$F_{2k+2r}\,[F_{2k+6r+4} - F_{2k+6r-4}] = 3\,[F_{4k+8r} - F_{4r}]\,.$$

Then we again use the Binet formula on the left hand side and reduce terms containing powers of both α and β to infer that

$$F_{2k+2r}(F_{2k+6r+4} - F_{2k+6r-4}) =$$
$$\frac{1}{5}\,[L_{4k+8r+4} + L_{4r-4} - L_{4k+8r-4} - L_{4r+4}]\,.$$

But, by the formula (17b) in [8, p. 177]

$$L_{4k+8r+4} - L_{4k+8r-4} = 5\,F_4\,F_{4k+8r}$$

and

$$L_{4r+4} - L_{4r-4} = 5\,F_4\,F_{4r},$$

so that the left hand side of the last alleged equality is indeed equal to its right hand side because $F_4 = 3$. □

For the induction step in the proof of the part b) of our second theorem we shall use the following lemma.

Lemma 6. *For every $r \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$H) \quad 21\,b_r + 3\,F^2_{2\,k+4\,r+3} - 3\,F^2_{2\,k+4\,r+1} -$$
$$L_{4\,r+6}\,F_{2\,k+2\,r-1}\,F_{2\,k+2\,r+3} + L_{4\,r+2}\,F_{2\,k+2\,r-3}\,F_{2\,k+2\,r+1} = 0.$$

Proof. From the formulas (1), (6) and (13) in [8], we have

$$3\,\left[F^2_{2\,k+4\,r+3} - F^2_{2\,k+4\,r+1}\right] = 3\,(F_{2k+4r+3} - F_{2k+4r+1})\cdot$$
$$(F_{2k+4r+3} + F_{2k+4r+1}) = 3\,F_{2k+4r+2}\,L_{2k+4r+2} = 3\,F_{4k+8r+4}.$$

Since $b_r = \frac{F_{4r+4}}{3}$ the relation H) is equivalent with

$$3\,F_{4k+8r+4} + 7\,F_{4r+4} = \\ L_{4r+6}\,F_{2k+2r-1}\,F_{2k+2r+3} - L_{4r+2}\,F_{2k+2r-3}\,F_{2k+2r+1}.$$

Then we again use the Binet formula on the right hand side and reduce terms containing powers of both α and β to conclude that

$$L_{4r+6}\,F_{2k+2r-1}\,F_{2k+2r+3} - L_{4r+2}\,F_{2k+2r-3}\,F_{2k+2r+1} = \\ \frac{1}{5}\left[L_{4k+8r+8} + L_{4r+10} + L_{4r+2} - L_{4k+8r} - L_{4r+6} - L_{4r-2}\right].$$

But, by the formula (17b) in [8, p. 177]

$$L_{4k+8r+8} - L_{4k+8r} = 5\,F_4\,F_{4k+8r+4},$$

$$L_{4r+10} - L_{4r-2} = 5\,F_6\,F_{4r+4},$$

and

$$L_{4r+2} - L_{4r+6} = -5\,F_2\,F_{4r+4},$$

so that the right hand side of the last alleged equality is indeed equal to its left hand side because $F_2 = 1$ and $F_6 = 8$. □

5. Proof of the second theorem

Proof of Theorem 2. a). The proof is by induction on n. For $n = 0$ the relation K) is $1 + F_{2k-1}^2 - F_{2k+1}^2 = -F_{2k+1}\,L_{2k-1}$ (i. e., the relation F)) which is true by Lemma 4. Assume that the relation K) is true for $n = r$. Then

$$\frac{F_{4(r+1)}}{3} + \sum_{i=0}^{2(r+1)-1} (-1)^i\,F_{2k+2i-1}^2 =$$

$$\frac{F_{4r+4}}{3} + \sum_{i=0}^{2\,r-1} (-1)^i\,F_{2k+2i-1}^2 + F_{2k+4\,r-1}^2 - F_{2k+4\,r+1}^2 =$$

$$\frac{F_{4r+4}}{3} - \frac{F_{4r}}{3} + F_{2k+4\,r-1}^2 - F_{2k+4\,r+1}^2 - \frac{F_{4r}}{3}\,F_{2k+2r-1}\,L_{2k+2r-3} =$$

$$-\frac{F_{4(r+1)}}{3}\,F_{2k+2(r+1)-1}\,L_{2k+2(r+1)-3},$$

where the last step uses Lemma 5. Hence, K) is true for $n = r + 1$.

b). The proof is also by induction on n. For $n = 0$ the relation L) is $1 + F_{2k-1}^2 = F_{2k+1}\,F_{2k-3}$ that we prove as follows.

For $m = 2\,k + 1$ and $n = 2\,k - 3$ the formula (17b) in [8] says that $L_{4\,k-2} - (-1)^{2\,k+1}\,L_4 = 5\,F_{2\,k-3}\,F_{2\,k+1}$ and since $L_4 = 7$ we get $L_{4\,k-2} + 7$ $= 5\,F_{2\,k-3}\,F_{2\,k+1}$. On the other hand, the formula (23) in [8] implies the relation $L_{4\,k-2} + 2 = 5\,F_{2\,k-1}^2$ so that $1 + F_{2k-1}^2 = F_{2k+1}\,F_{2k-3}$.

Assume that the relation L) is true for $n = r$. Then

$$c_{r+1} + \sum_{i=0}^{2(r+1)} (-1)^i \, F_{2k+2i-1}^2 =$$

$$c_{r+1} + \sum_{i=0}^{2\,r} (-1)^i \, F_{2k+2i-1} - F_{2k+4\,r+1}^2 + F_{2k+4\,r+3}^2 =$$

$$a_r \, F_{2k+2r-3} \, F_{2k+2r+1} + c_{r+1} - c_r - F_{2k+4\,r+1}^2 + F_{2k+4\,r+3}^2 =$$

$$a_{r+1} \, F_{2k+2r-1} \, F_{2k+2r+3},$$

where the last step uses Lemma 6. Hence, L) is true for $n = r + 1$ and the proof is completed. □

6. Lemmas for Theorem 3

For the initial step in an inductive proof of the part a) of our third theorem we shall use the following lemma.

Lemma 7. *For every $k \geqslant 1$ the following equality holds:*

$$M) \quad 1 + F_{2\,k} \, (F_{2\,k-1} + L_{2\,k+1}) - F_{2\,k+1} \, F_{2\,k+2} = 0.$$

Proof. First we look into the bracket and use the formula (6) in [8] to write $L_{2k+1} = F_{2\,k} + F_{2k+2}$ and then note that $F_{2k-1} + F_{2k} = F_{2k+1}$ so that after the multiplication and factoring out terms that contain F_{2k+1} we get that the left hand side is equal to

$$1 - F_{2k+1}(F_{2\,k+2} - F_{2\,k}) + F_{2\,k} \, F_{2\,k+2}$$

and thus to

$$1 - F_{2k+1}^2 + F_{2\,k} \, F_{2\,k+2},$$

since $F_{2\,k+2} - F_{2\,k} = F_{2k+1}$. This is indeed zero by the famous Cassini formula (i. e., the formula (29) in [8]). □

For the induction step in the proof of the part a) of our third theorem we shall use the following lemma.

Lemma 8. *For every $r \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$N) \quad b_r \, (1 + F_{2\,k+2\,r} \, L_{2\,k+2\,r+1}) - b_{r-1} \, (1 + F_{2\,k+2\,r-2} \, L_{2\,k+2\,r-1}) - F_{2\,k+4\,r+1} \, F_{2\,k+4\,r+2} + F_{2\,k+4\,r-1} \, F_{2\,k+4\,r} = 0.$$

Proof. The formula (15b) in [8, p. 177] implies that

$$1 + F_{2\,k+2\,r} \, L_{2\,k+2\,r+1} = F_{4k+4r+1}$$

and

$$1 + F_{2\,k+2\,r-2} \, L_{2\,k+2\,r-1} = F_{4k+4r-3}.$$

On the other hand, since $F_{2k+4r+2} = F_{2k+4r+1} + F_{2k+4r}$ and

$$F_{2k+4r} = F_{2k+4r+1} - F_{2k+4r-1}$$

we see that

$$F_{2\,k+4\,r+1}\,F_{2\,k+4\,r+2} - F_{2\,k+4\,r-1}\,F_{2\,k+4\,r} = F^2_{2k+4r+1} + F^2_{2k+4r}.$$

This sum of squares is $F_{4k+8r+1}$ by the formula (11) in [8]. Since $b_{r-1} = \frac{F_{4r}}{3}$ and $b_r = \frac{F_{4r+4}}{3}$, it follows that the equality N) is equivalent with

$$F_{4r+4}\,F_{4k+4r+1} - F_{4r}\,F_{4k+4r-3} = 3\,F_{4k+8r+1}.$$

We now apply the formula (17b) trice to get

$$5\,F_{4r+4}\,F_{4k+4r+1} = L_{4k+8r+5} - L_{4k-3},$$

$$5\,F_{4r}\,F_{4k+4r-3} = L_{4k+8r-3} - L_{4k-3},$$

and

$$5\,F_4\,F_{4k+8r+1} = L_{4k+8r+5} - L_{4k+8r-3}.$$

This obviously completes the proof because $F_4 = 3$. □

For the induction step in the proof of the part b) of our third theorem we shall use the following lemma.

Lemma 9. *For every $r \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$O)\quad 33\,b_r + 3\,F_{2\,k+4\,r+3}\,F_{2\,k+4\,r+4} - 3\,F_{2\,k+4\,r+1}\,F_{2\,k+4\,r+2} + L_{4\,r+2}\,F_{2\,k+2\,r-3}\,F_{2\,k+2\,r+2} - L_{4\,r+6}\,F_{2\,k+2\,r-1}\,F_{2\,k+2\,r+4} = 0.$$

Proof. Let us multiply the left hand side of O) by number 5 and use the formula (17b) from [8, p. 177] four times to get

$$5\,F_{2\,k+4\,r+3}\,F_{2\,k+4\,r+4} = L_{4k+8r+7} + 1,$$

$$5\,F_{2\,k+4\,r+1}\,F_{2\,k+4\,r+2} = L_{4k+8r+3} + 1,$$

$$5\,F_{2\,k+2\,r-3}\,F_{2\,k+2\,r+2} = L_{4k+4r-1} + 11,$$

and

$$5\,F_{2\,k+2\,r-1}\,F_{2\,k+2\,r+4} = L_{4k+4r+3} + 11.$$

Since $b_r = \frac{F_{4r+4}}{3}$, the five times the left hand side of O) is equal to

$$55\,F_{4r+4} - 11(L_{4r+6} - L_{4r+2}) + 3(L_{4k+8r+7} - L_{4k+8r+3}) - (L_{4r+6}\,L_{4k+4r+3} - L_{4r+2}\,L_{4k+4r-1}).$$

Once again the formula (17b) in [8] implies $L_{4r+6} - L_{4r+2} = 5\,F_{4r+4}$ so that the second term is the opposite of the first term. The third term, by the same formula, is equal to $15\,F_{4k+8r+5}$. The first product of Lucas numbers in the last term is, by the formula (17a) in [8], equal to

$$L_{4r+6}\,L_{4k+4r+3} = L_{4k+8r+9} + L_{4k-3},$$

while the second is

$$L_{4r+2}\, L_{4k+4r-1} = L_{4k+8r+1} + L_{4k-3}.$$

Another application of the formula (17b) on the difference of these products shows that the last term is the opposite of the third term so that the relation O) holds. □

7. Proof of the third theorem

Proof of Theorem 3. a). The proof is by induction on n. For $n = 1$ the relation P) is $1 + F_{2k-1}\, F_{2k} - F_{2k+1}\, F_{2k+2} = -F_{2k}\, L_{2k+1}$ (i. e., the relation M)) which we already proved in Lemma 7. Assume that the relation P) is true for $n = r$. Then

$$b_r + \sum_{i=0}^{2(r+1)-1} (-1)^i\, F_{2k+2i-1}\, F_{2k+2i} =$$

$$b_r + \sum_{i=0}^{2\,r-1} (-1)^i F_{2k+2i-1} F_{2k+2i} + F_{2k+4\,r-1} F_{2k+4\,r} - F_{2k+4r+1} F_{2k+4r+2} =$$

$$-b_{r-1} F_{2k+2r-2} L_{2k+2r-1} + b_r - b_{r-1} + F_{2k+4r-1} F_{2k+4r} - F_{2k+4r+1} F_{2k+4r+2}$$

$$= -b_r\, F_{2k+2(r+1)-2}\, L_{2k+2(r+1)-1},$$

where the last step uses Lemma 8. Hence, P) is true for $n = r + 1$ and the proof is completed.

b). The proof is also by induction on n. For $n = 0$ and $k \geqslant 2$ the relation Q) is

$$2 + F_{2\,k-1}\, F_{2\,k} = F_{2\,k-3}\, F_{2\,k+2}.$$

This identity follows immediately from the formula (20a) in [8] for $n = 2\,k$, $h = -3$ and $k = 2$.

Assume that the relation Q) is true for $n = r$ with $r \geqslant 1$. Let $U = 2k + 4r$. Then

$$d_{r+1} + \sum_{i=0}^{2(r+1)} (-1)^i\, F_{2k+2i-1}\, F_{2k+2i} =$$

$$d_{r+1} + \sum_{i=0}^{2\,r} (-1)^i\, F_{2k+2i-1}\, F_{2k+2i} - F_{U+1}\, F_{U+2} + F_{U+3}\, F_{U+4} =$$

$$a_r\, F_{2k+2r-3}\, F_{2k+2r+2} + d_{r+1} - d_r - F_{U+1}\, F_{U+2} + F_{U+3}\, F_{U+4} =$$

$$a_{r+1}\, F_{2k+2(r+1)-3}\, F_{2k+2(r+1)+2},$$

where we get the last line from the previous using Lemma 9. Hence, Q) is true for $n = r + 1$. □

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