ON SUMS OF SQUARES OF ODD AND EVEN TERMS OF THE LUCAS SEQUENCE

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1. Introduction

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \ \text{ for } n \geqslant 3,$$

and

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \ \text{ for } n \geqslant 3.$$

We consider sums of squares of odd and even terms of the Lucas sequence and sums of their products. These sums have nice representations as products of appropriate Fibonacci and Lucas numbers.

Let $u_k = F_{2k-1}$, $s_k = L_{2k-1}$, $v_k = F_{2k}$, $t_k = L_{2k}$, $U_k = u_k^2$, $S_k = s_k^2$, $V_k = v_k^2$ and $T_k = t_k^2$ denote odd and even terms of the Fibonacci and Lucas sequences and their squares. The note [1] includes formulas for the sums $\sum_{i=0}^{j} U_{k+i}$, $\sum_{i=0}^{j} V_{k+i}$ and $\sum_{i=0}^{j} u_{k+i}\, v_{k+i}$ that improved some results from [4]. The purpose of this paper is to establish similar results for the sums $\sum_{i=0}^{j} S_{k+i}$, $\sum_{i=0}^{j} T_{k+i}$ and $\sum_{i=0}^{j} s_{k+i}\, t_{k+i}$.

2. Lucas even squares

The following lemma is needed to accomplish the inductive step in the proof of our first theorem.

Lemma 1. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$S_{m+1} + T_{k+m} + 5\, v_m\, u_k\, u_{k+m} = 5\, v_{m+1}\, u_k\, u_{k+m+1}. \qquad (2.1)$$

Proof. In terms of the Fibonacci and Lucas sequences the relation (2.1) becomes

$$L_{2m+1}^2 + L_{2k+2m}^2 + 5\, F_{2k-1}\, [F_{2k+2m-1}\, F_{2m} - F_{2m+2}\, F_{2k+2m+1}] = 0. \quad (2.2)$$

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When we apply the formula (I_{12}) from [2] which says that L_n^2 is equal to $5\,F_n^2 + 4\,(-1)^n$ the relation (2.2) transforms into

$$F_{2m+1}^2 + F_{2k+2m}^2 + F_{2k-1}\,[F_{2k+2m-1}\,F_{2m} - F_{2m+2}\,F_{2k+2m+1}] = 0. \quad (2.3)$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Note that $\beta = -\frac{1}{\alpha}$. By the Binet formula F_k is equal to $\frac{\alpha^k - \beta^k}{\alpha - \beta}$ (see [2] and [3]). It follows that the left hand side of (2.3) is equal to $\frac{\alpha^3\,M}{(\alpha^2+1)^3}$, where M is the expression

$$\alpha^{4\,m+3} + \alpha^{4\,m+1} + \alpha^{-4\,m-1} + \alpha^{-4\,m-3} + \alpha^{4\,k+4\,m+1} + \alpha^{4\,k+4\,m-1} + \\ \alpha^{-4\,k-4\,m+1} + \alpha^{-4\,k-4\,m-1} + \alpha^{4\,m+4\,k-2} + \alpha^{4\,m} - \alpha^{-4\,m} - \\ \alpha^{-4\,k-4\,m+2} - \alpha^{4\,m+2+4\,k} - \alpha^{4\,m+4} + \alpha^{-4\,m-4} + \alpha^{-4\,k-4\,m-2}.$$

The sixteen terms of M can be considered as four groups of four terms that have similar exponents. The first group is

$$\alpha^{4\,m+3} + \alpha^{4\,m+1} + \alpha^{4\,m} - \alpha^{4\,m+4} = -\alpha^{4\,m}\left(\alpha^2 - \alpha - 1\right)\left(\alpha^2 + 1\right) = 0$$

because α is the root of the equation $x^2 - x - 1 = 0$. One can see similarly that the other three groups are zero. Hence, $M = 0$ and the proof is complete. □

Theorem 1. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$\alpha_m + \sum_{i=0}^{m} T_{k+i} = 5\,v_{m+1}\,u_k\,u_{k+m+1}, \qquad (2.4)$$

where the sequence α_m is defined as follows:

$$\alpha_m = \begin{cases} 1, & \text{if } m = 0; \\ \alpha_{m-1} + S_{m+1}, & \text{if } m \geqslant 1. \end{cases}$$

Proof. The proof is by induction on m. For $m = 0$ the relation (2.4) is $1 + L_{2k}^2 = 5\,F_{2k-1}\,F_{2k+1}$ which is true by formulas (I_{12}) and (I_{13}) in [2]. Assume that the relation (2.4) is true for $m = r$. Then

$$\alpha_{r+1} + \sum_{i=0}^{r+1} T_{k+i} = \alpha_r + S_{r+2} + T_{k+r+1} + \sum_{i=0}^{r} T_{k+i} =$$

$$S_{r+2} + T_{k+r+1} + 5\,v_{r+1}\,u_k\,u_{k+r+1} = 5\,v_{(r+1)+1}\,u_k\,u_{k+(r+1)+1},$$

where the last step uses Lemma 1 for $m = r + 1$. Hence, (2.4) is true for $m = r + 1$ and the proof is completed. □

Remark 1. In a similar way it is possible to prove also: For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:

$$\alpha'_m + \sum_{i=0}^{m} T_{k+i} = 5\, v_{m+1}\, u_{k+1}\, u_{k+m}, \tag{2.5}$$

where the sequence α'_m is defined as follows:

$$\alpha'_m = \begin{cases} 1, & \text{if } m = 0; \\ 1 + \alpha_{m-1}, & \text{if } m \geqslant 1. \end{cases}$$

Indeed, this follows easily from Theorem 1 with the help of the relation:

$$S_{m+1} - 1 = 5\, v_{m+1}\, [u_k\, u_{k+m+1} - u_{k+1}\, u_{k+m}].$$

3. Lucas odd squares

The following lemma is needed to accomplish the inductive step in the proof of our second theorem.

Lemma 2. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$S_{k+m+1} + T_{m+1} + 5\, v_{m+1}\, u_k\, u_{k+m} = 5\, v_{m+2}\, u_k\, u_{k+m+1}. \tag{3.1}$$

Proof. In terms of the Fibonacci and Lucas sequences the left hand side Q of the relation (3.1) is

$$L^2_{2k+2m+1} + L^2_{2m+2} + 5\, F_{2k-1}\, [F_{2k+2m-1}\, F_{2m+2} - F_{2m+4}\, F_{2k+2m+1}].$$

When we apply the formula (I_{12}) from [2] to Q it becomes $5\, Q^*$ with

$$Q^* = F^2_{2m+2} + F^2_{2k+2m+1} + F_{2k-1}\, [F_{2k+2m-1}\, F_{2m+2} - F_{2m+4}\, F_{2k+2m+1}].$$

It follows that Q^* is equal to $\frac{(\alpha^2-\alpha-1)(\alpha^2-4\,\alpha-1)(A^4+\alpha^2)(\alpha^6\, A^4\, B^8+1)}{(\alpha^2+1)^2\alpha^4\, A^4\, B^4}$, where $A = \alpha^k$ and $B = \alpha^m$. Hence, $Q^* = 0$ because α is the root of the equation $x^2 - x - 1 = 0$. □

Theorem 2. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$\beta_m + \sum_{i=0}^{m} S_{k+i} = 5\, v_{m+1}\, u_k\, u_{k+m}, \tag{3.2}$$

where the sequence β_m is defined as follows:

$$\beta_m = \begin{cases} 4, & \text{if } m = 0; \\ \beta_{m-1} + T_m, & \text{if } m \geqslant 1, \end{cases}$$

Proof. The proof is by induction on m. For $m = 0$ the relation (3.2) is $4 + L_{2k-1}^2 = 5\, F_{2k-1}^2$ which is true by the formula (I_{12}) in [2]. Assume that the relation (3.2) is true for $m = r$. Then

$$\beta_{r+1} + \sum_{i=0}^{r+1} S_{k+i} = \beta_r + T_{r+1} + S_{k+r+1} + \sum_{i=0}^{r} S_{k+i} =$$

$$T_{r+1} + S_{k+r+1} + 5\, v_{r+1}\, u_k\, u_{k+r} = 5\, v_{(r+1)+1}\, u_k\, u_{k+(r+1)},$$

where the last step uses Lemma 2. Hence, (3.2) is true also for $m = r + 1$ and the proof is complete. □

Remark 2. The alternative result to Theorem 2 is the following statement: For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:

$$\beta'_m + \sum_{i=0}^{m} S_{k+i} = 5\, v_{m+1}\, u_{k+1}\, u_{k+m-1}, \tag{3.3}$$

where the sequence β'_m is defined as follows:

$$\beta'_m = \begin{cases} 9, & \text{if } m = 0; \\ \beta'_{m-1} + T_m, & \text{if } m \geqslant 1. \end{cases}$$

4. Lucas products

With the following lemma we shall make the inductive step in the proof of the third theorem.

Lemma 3. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$s_{m+2}\, t_{m+2} + s_{k+m+1}\, t_{k+m+1} + 5\, v_{m+1}\, u_k\, v_{k+m} = 5\, v_{m+2}\, u_k\, v_{k+m+1}. \tag{4.1}$$

Proof. In terms of the Fibonacci and Lucas sequences the relation (4.1) is equivalent to $R = 0$, where R is $L_{2k+2m}\, L_{2k+2m+1} + L_{2m+2}\, L_{2m+3} + 5\, F_{2k-1}\, [F_{2k+2m}\, F_{2m+2} - F_{2m+4}\, F_{2k+2m+2}]$. It follows that R is equal to $\frac{(\alpha^2-\alpha-1)(\alpha^2-4\,\alpha-1)(\alpha^2+A^4)(\alpha^2AB^2-1)(\alpha^2AB^2+1)(\alpha^4A^2B^4+1)}{(\alpha^2+1)^2\alpha^5A^4B^4}$, with $A = \alpha^k$ and $B = \alpha^m$. Hence, $R = 0$ since $\alpha^2 - \alpha - 1 = 0$. □

Theorem 3. *For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$\gamma_m + \sum_{i=0}^{m} s_{k+i}\, t_{k+i} = 5\, v_{m+1}\, u_k\, v_{k+m}, \tag{4.2}$$

where the sequence γ_m is defined as follows:

$$\gamma_m = \begin{cases} 2, & \text{if } m = 0; \\ \gamma_{m-1} + s_{m+1}\, t_m, & \text{if } m \geqslant 1. \end{cases}$$

Proof. The proof is by induction on m. For $m = 0$ the relation (4.2) is $2 + L_{2k-1}\, L_{2k} = 5\, F_{2k-1}\, F_{2k}$ which is true. Indeed, if $A = \alpha^k$ and $B = \beta^k$, then the difference $2 + L_{2k-1}\, L_{2k} - 5\, F_{2k-1}\, F_{2k}$ is zero since it reduces to $10\,(AB-1)\,(AB+1)$ and $\alpha\,\beta = -1$.

Assume that the relation (4.2) is true for $m = r$. Then

$$\gamma_{r+1} + \sum_{i=0}^{r+1} s_{k+i}\, t_{k+i} = \gamma_r + s_{r+2}\, t_{r+1} + s_{k+r+1}\, t_{k+r+1} + \sum_{i=0}^{r} s_{k+i}\, t_{k+i} =$$

$$s_{r+2}\, t_{r+1} + s_{k+r+1}\, t_{k+r+1} + 5\, v_{r+1}\, u_k\, v_{k+r} = 5\, v_{(r+1)+1}\, u_k\, v_{k+(r+1)},$$

where the last step uses Lemma 3. Hence, (4.2) is true also for $m = r + 1$. □

Remark 3. The alternative result to Theorem 3 is the following statement: For every $m \geqslant 0$ and $k \geqslant 1$ the following equality holds:

$$\gamma'_m + \sum_{i=0}^{m} s_{k+i}\, t_{k+i} = 5\, v_{m+1}\, v_{k+2}\, u_{k+m-2}, \tag{4.3}$$

where the sequence γ'_m is defined as follows:

$$\gamma'_m = \begin{cases} 77,\ 89,\ 91, & \text{if } m = 0,\ 1,\ 2; \\ \gamma'_{m-1} - s_{m-2}\, t_{m-2}, & \text{if } m \geqslant 3. \end{cases}$$

5. Computer experimental solutions

In the rest of this note we describe how one can discover these results and check the above proofs with the help of the computer. The presentation is for the software Maple V.

The input `with(combinat):` calls the package that contains the function `fibonacci` that computes the terms of the Fibonacci sequence. We first define functions `fF`, `fL`, `fu`, `fv`, `fU`, `fV`, `fs`, `ft`, `fS`, `fT`, and `fT1` that give terms of the sequences F_k, L_k, u_k, v_k, U_k, V_k, s_k, t_k, S_k, T_k and the left hand side of (2.4).

```
fF:=x->fibonacci(x): fL:=x->fF(2*x)/fF(x): fu:=x->fF(2*x-1):
fv:=x->fF(2*x): fU:=x->fu(x)^2: fV:=x->fv(x)^2:
fs:=x->fL(2*x-1):ft:=x->fL(2*x): fS:=x->fs(x)^2:
fT:=x->ft(x)^2: fT1:=(a,m,k)->a+sum(fT(k+i),i=0..m):
```

The following procedure tests for values a and k between $a0$ and $a1$ and $k0$ and $k1$ and for given values m and n if the quotient $q = \frac{fT1(a,m,k)}{fu(k+n)}$ is an integer and prints out a, k, and q (factored into primes).

```
gT1:=proc(a0,a1,m,k0,k1,n) local a,k,q;for a from a0 to
a1 do for k from k0 to k1 do q:=fT1(a,m,k)/fu(k+n):
```

```
if denom(q)=1 then print([a,k,ifactor(q)]);fi:od;od;end:
```

The input `gT1(1,1000,2,17,17,n);` for $n = 0$ and $n = 3$ is asking to determine among first thousand integers the number α_3 that makes the quotient q of the left hand side of the formula (2.4) for $m = 2$ by u_{k+n} an integer for $k = 17$. The outputs [138, 17, $(2)^4(5)(233)(135721)$] and [138, 17, $(2)^4(5)(89)(19801)$] indicate that $\alpha_3 = 138$ is a good candidate. This is "confirmed" when we input `gT1(138,138,2,1,100,0);` and `gT1(138,138,2,1,100,3);` whose outputs are all integers as the third term in each of the hundred triples. All this suggests to modify `gT1` as follows:

```
gT1:=proc(a0,a1,m,k0,k1) local a,k,q;for a from a0 to a1
do for k from k0 to k1 do q:=fT1(a,m,k)/fu(k)/fu(k+3)/5:
if denom(q)=1 then print([a,k,ifactor(q)]);fi:od;od;end:
```

With the new function the command `gT1(138,138,2,1,100);` has for output the same number 8 as the third term in each of the hundred triples.

When we repeat this for values of m between 0 and 4 we discover that numbers α_m and the third terms are 1, 17, 138, 979, 6755 and 1, 3, 8, 21, 55 (i. e., F_2, F_4, F_6, F_8, F_{10}). Of course, some experimentation is needed to figure out the correct indices of u's on the right sides. These five values are sufficient to discover the rule by which α_m's are built. All other formulas in this paper are discovered by similar procedures.

It remains to explain how to invoke the help of the computer in the proof of Lemmas 1 – 3. We shall prove only Lemma 1 because the proofs for Lemmas 2 and 3 are analogous.

We first define the function that expresses the Binet formula and make the assumption that $\alpha > 0$ and that m and k are positive integers.

```
a:=alpha:b:=-1/a:f:=x->(1/(a-b))*(a^x-b^x):
assume(alpha>0):assume(m,posint):assume(k,posint):
```

The numerator of the left hand side of (2.3) is evaluated by the the following input.

```
N:=numer(simplify(f(2*m+1)^2+f(2*k+2*m)^2+
f(2*k-1)*(f(2*k+2*m-1)*f(2*m)-f(2*k+2*m+1)*f(2*m+2))));
```

The output is the expression:

$$\alpha^{6+4\,m} + \alpha^{4\,m+4} + \alpha^{2-4\,m} + \alpha^{-4\,m} + \alpha^{4\,k+4+4\,m} + \alpha^{4\,k+4\,m+2} + \alpha^{-4\,k+4-4\,m} + \alpha^{-4\,k-4\,m+2} + \alpha^{1+4\,k+4\,m} - \alpha^{3-4\,m} - \alpha^{5+4\,k+4\,m} + \alpha^{-1-4\,m} + \alpha^{3+4\,m} - \alpha^{5-4\,k-4\,m} - \alpha^{4\,m+7} + \alpha^{1-4\,k-4\,m}.$$

The first group of four terms with similar exponents consists of the first, the second, the thirteenth and the fifteenth term.

```
fG:=factor(expand((op(1,N)+op(2,N)+
op(13,N)+op(15,N))/alpha^(4*m)));
```

The output $-\alpha^3\,(\alpha^2+1)\,(\alpha^2-\alpha-1)$ is zero because α is the root of the equation $x^2 - x - 1 = 0$.

In the same fashion we argue that the other three groups are zero so that $N = 0$ and the relation (2.3) has been proved.

References

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