ON IMPROVEMENTS OF THE BUTTERFLY THEOREM

ZVONKO ČERIN AND GIAN MARIO GIANELLA

Abstract. This paper explores the locus of butterfly points of a quadrangle $ABCD$ in the plane. These are the common midpoints of three segments formed from intersections of a butterfly line with the lines AB, CD, AD, BC, AC, and BD. The locus is the nine-point-conic of $ABCD$ that goes through the midpoints of the segments AB, CD, AD, BC, AC, and BD. We also consider the problem to determine when two quadrangles share the nine-point-conic. Our proofs use analytic geometry of the rectangular Cartesian coordinates.

1. Introduction

The classical Butterfly Theorem claims that whenever chords AB and CD of a circle γ intersect at the midpoint S of the third chord PQ then S is also the midpoint of the segments XY and UV formed by the intersections X, Y, U, and V of the lines AD, BC, AC, and BD with the line PQ (see Figure 1).

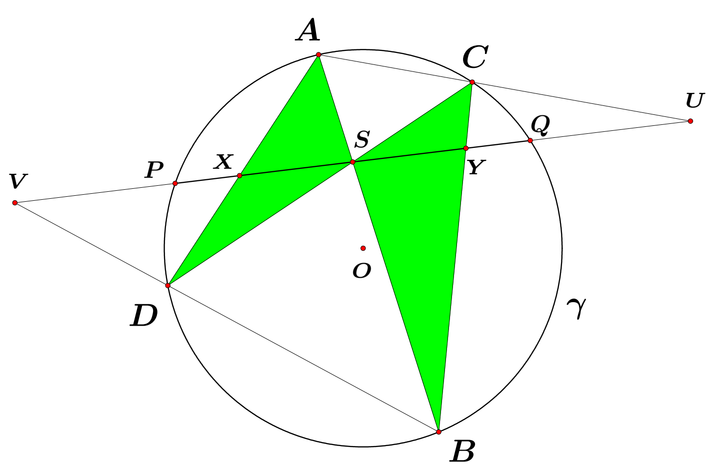

Figure 1. The point S is the body and the triangles ADS and BCS are the wings of the butterfly.

1991 *Mathematics Subject Classification.* Primary 51N20, 51M04, Secondary 14A25, 14Q05.

Key words and phrases. quadrangle, Butterfly Theorem, conic, locus, nine-point-conic, cyclic quadrangle, Kiepert, Jarabek, Feuerbach, center, central point.

In recent years there has been a considerable activity in improving this interesting result. First it was observed in [23] that the line PQ could be replaced by any line ℓ in the plane of the circle and take for the point S the projection of the center O of γ onto ℓ. This was extended by the first author in [2] and [3] where the circle is replaced by any conic γ, the point S is a point on line of symmetry z of γ and the line ℓ is the perpendicular to z at S. The reference [24] contains yet another improvement of this by replacing the line of symmetry z with any line ℓ and taking for the point S the intersection of ℓ with the diameter of the conic γ which is conjugate to the line ℓ.

A further extension is accomplished in the first author's article [4] where he introduced the following technical definition in order to get shorter statements.

A pair $(\ell,\, S)$ consisting of a line ℓ and a point S on it is said to have the *butterfly property* with respect to the quadrangle $ABCD$ provided S is a common midpoint of segments $\ell_a\ell_c$, $\ell_b\ell_d$ and $\ell_e\ell_f$, where ℓ_a, ℓ_b, ℓ_c, ℓ_d, ℓ_e, and ℓ_f are intersections of ℓ with lines AB, BC, CD, DA, AC, and BD. In this situation we shall write $(\ell,\, S) \bowtie ABCD$ or $\ell \underset{S}{\bowtie} ABCD$ and use also the phrase "the line ℓ has the butterfly property with respect to $ABCD$ at the point S". Of course, we consider lines ℓ and quadrangles $ABCD$ for which all six intersections ℓ_a, ℓ_b, ℓ_c, ℓ_d, ℓ_e, and ℓ_f are well-defined points in the (finite) plane.

The main results in [4] show that for most points S in the plane of a conic γ there is a unique line ℓ such that $(\ell,\, S) \bowtie ABCD$ holds for every quadrangle $ABCD$ inscribed to γ.

The article [22] explores for a given cyclic quadrangle $ABCD$ what is the locus of all projections S of the circumcentre O of $ABCD$ on lines ℓ with the property that the relation $(\ell,\, S) \bowtie ABCD$ holds.

This locus is shown to be the equilateral hyperbola that goes through the circumcentre O and the midpoints of segments AB, AC, AD, BC, BD, and CD. It also goes through the intersection of diagonals (AC and BD) and the intersections ($AB \cap CD$ and $AD \cap BC$) of opposite sides.

The goal of this paper is to lift the assumption that $ABCD$ is a cyclic quadrangle from results in [22]. Our approach is through the analytic geometry. Perhaps some or all of our results could be proved synthetically (see the last sentence on page 61 of [22]). However, with this miraculous method in [22] only a very special case of cyclic quadrangles was covered. We hope that one can not impose methods of proofs and discovery in mathematics and that with computers our "heavy calculations" are in fact far easier to follow for an average person than to master projective or affine geometry.

2. Butterfly points of strong quadrangles

In order to avoid repetitions of the phrase "without parallel diagonals or parallel opposite sides" we first introduce a broad class of quadrangles that will be subjects of our investigation.

We shall say that the quadrangle $ABCD$ is *strong* provided the lines AB, AC, and AD intersect in the points E, F, and G with the lines CD, BD, and BC. The triangle EFG is called the *diagonal triangle* of $ABCD$.

Let $ABCD$ be a quadrangle in the plane. A point S from this plane is called a *butterfly point* of $ABCD$ or a β_{ABCD}*-point* if there is a line ℓ through S such that the relation $(\ell,\, S) \bowtie ABCD$ is true.

Let us begin with a technical result which clarifies our definition of the relation $(\ell,\, S) \bowtie ABCD$. It shows that it suffices to require that only midpoints of two among segments $\ell_a\ell_c$, $\ell_b\ell_d$ and $\ell_e\ell_f$ coincide.

Lemma 1. *Let $ABCD$ be a strong quadrangle. Let a line ℓ intersect the lines AB, BC, CD, DA, AC and BD in the points ℓ_a, ℓ_b, ℓ_c, ℓ_d, ℓ_e and ℓ_f. Let M_{ac}, M_{bd} and M_{ef} be midpoints of the segments $\ell_a\ell_c$, $\ell_b\ell_d$ and $\ell_e\ell_f$. If any two of these midpoints coincide then they all coincide.*

Proof (in Cartesian coordinates). We assume that the points A, B, C and D have the rectangular Cartesian coordinates $(0,\, 0)$, $(1,\, 0)$, $(u,\, v)$ and $(U,\, V)$. Let $u_0 = u\,U$, $v_0 = v\,V$, $u_1 = u\,V$, $v_1 = v\,U$, $u_2 = u - U$, $v_2 = v - V$, $w = v_1 - u_1$, $W = v_1 + u_1$. Let $f\,x + g\,y + h = 0$ be the equation of the line ℓ. Note that $f^2 + g^2 \neq 0$. Solving linear equations we easily find coordinates of all points and discover that the distances $|M_{ac}\,M_{bd}|$, $|M_{ac}\,M_{ef}|$ and $|M_{bd}\,M_{ef}|$ are the absolute values of $\frac{M\,K}{2\,\alpha\,\gamma\,\beta\,\delta}$, $\frac{M\,K}{2\,\alpha\,\gamma\,\varepsilon\,\varphi}$ and $\frac{M\,K}{2\,\beta\,\delta\,\varepsilon\,\varphi}$, where $\alpha = f$, $\beta = (u-1)\,f + v\,g$, $\gamma = u_2\,f + v_2\,g$, $\delta = U\,f + V\,g$, $\varepsilon = u\,f + v\,g$, $\varphi = (U-1)\,f + V\,g$, $K = \sqrt{f^2 + g^2}$, $M = f(v_2 - w)(u_0 f^2 + W f\,g + v_0\,g^2) + h(T f^2 + 2\,v_0\,u_2\,fg + v_0\,v_2\,g^2)$ and $T = w - v_1\,U + u\,u_1$. That the lemma holds is now obvious. □

Our first theorem is a version of Theorem 3 in [22] that holds for all strong quadrangles and not only for the ones inscribed to a circle.

Theorem 1. *For any strong quadrangle $ABCD$ the locus of all β_{ABCD}-points is the nine-point-conic $c_9 = c_9(ABCD)$ that goes through the vertices of its diagonal triangle EFG and through the midpoints of segments AB, BC, CD, DA, AC, and BD.*

Proof. In this proof we shall use the same assumption and notation about the points A, B, C and D as we did in the proof of Lemma 1.

Let $P(p,\, q)$ be a β_{ABCD}-point. Let $f\,x + g\,y - f\,p - g\,q = 0$ be the equation of the line ℓ with the property that P is the common midpoint of segments $\ell_a\ell_c$, $\ell_b\ell_d$, and $\ell_e\ell_f$. (Note that the real number f can not be zero because then the lines ℓ and AB would be parallel.) This is true provided the following two conditions K_i hold

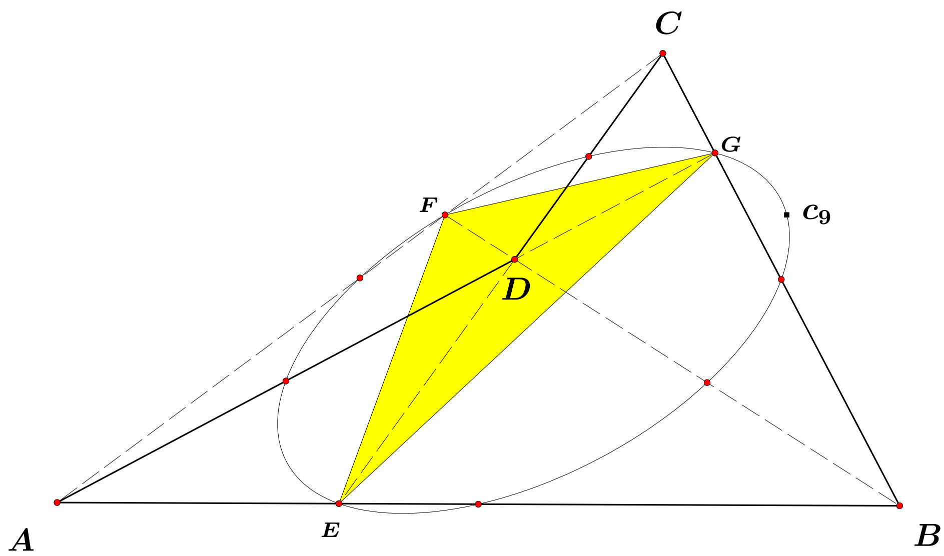

FIGURE 2. The quadrangle $ABCD$ with its nine-point-conic.

$(a_i\, f - 2\, b_i\, g)\, p + (2\, A_i\, f - B_i\, g)\, q + C_i = 0$, with indices $i = 1, 2$ and with coefficients $a_1 = B_1 = -v_2$, $b_1 = 0$, $a_2 = B_2 = V - W$, $b_2 = v_0$, $A_1 = u$, $A_2 = u_0 - U$, $C_1 = w$ and $C_2 = f\, v_1 + g\, v_0$. Since in each equation K_i the variable f appears linearly we can solve it easily and get two quotients Q_i for values of f. Hence, it must be $Q_1 - Q_2 = 0$. The difference on the left hand side has the following polynomial

$$c_9 = c_9(ABCD) = 2\, k_1\, p^2 - 4\, k_2\, p\, q + 2\, k_3\, q^2 - k_4\, p + k_5\, q + k_6$$

as the only factor that could be zero, where $k_1 = v_0\, v_2$, $k_2 = v_0\, u_2$, $k_3 = w + u\, u_1 - U\, v_1$, $k_4 = v_0\,(v_2 + 2\, w)$, $k_5 = v_1^2 - u_1^2 + u_1\,(V + 2\, v) - v_1\,(v + 2\, V)$ and $k_6 = v_0\, w$. We conclude that the locus of all β_{ABCD}-points is a conic whose equation is $c_9 = 0$. It is now easy to check that the midpoints A', B', C', D', E', and F' of segments AB, BC, CD, DA, AC, and BD as well as the vertices E, F, and G of the diagonal triangle lie on this conic. □

Theorem 2. *Let $ABCD$ be a strong quadrangle. The circumcircle O of the triangle ABC is a β_{ABCD}-point if and only if either ABC has a right angle or $ABCD$ is a cyclic quadrangle.*

Proof. This follows immediately from the fact that the value of the polynomial c_9 for $p = \frac{1}{2}$ and $q = \frac{u^2+v^2-u}{2\,v}$ (the coordinates of the circumcircle of the triangle ABC) is the quotient

$$\frac{u\,(u^2 + v^2 - u)\,(1 - u)\,(v(U^2 + V^2 - U) - (u^2 + v^2 - u)\,V)}{2\,v^2}$$

and that $v(p^2 + q^2 - p) - (u^2 + v^2 - u)\, q = 0$ is the equation of the circumcircle of the triangle ABC. □

An easy consequence of Theorem 2 is the following corollary which includes Theorem 3 in [22]. We also describe precisely what are the lines with the butterfly property in this situation.

Corollary 1. *The nine-point conic c_9 of a strong cyclic quadrangle $ABCD$ is an equilateral hyperbola which goes through the center O of its circumcircle.*

For $P = O$ let $\ell = \ell_P$ be the normal of the equilateral hyperbola c_9 in O and for every point $P \in c_9 \setminus \{O\}$ let $\ell = \ell_P$ be the perpendicular at P to the segment OP. Then the line ℓ has the property that P is a common midpoint of segments $\ell_a\ell_c$, $\ell_b\ell_d$, and $\ell_e\ell_f$, where ℓ_a, ℓ_b, ℓ_c, ℓ_d, ℓ_e, and ℓ_f are intersections of ℓ with lines AB, BC, CD, DA, AC, and BD.

Proof. The first part is an easy consequence of Theorem 2 and the following two well-known theorems. A strong quadrangle $ABCD$ is cyclic if and only if the circumcenter O of the triangle ABC is the orthocenter of its diagonal triangle EFG. A conic which goes through the vertices and the orthocenter of a triangle is an equilateral hyperbola.

In order to prove the second part, which describes precisely the position of the line with the butterfly property, we must repeat the proof of Theorem 1 under the assumption that $A = T(0)$, $B = T(u)$, $C = T(v)$, and $D = T(w)$ are points on the unit circle, where $T(x) = \left(\frac{1-x^2}{1+x^2}, \frac{2\,x}{1+x^2}\right)$. Let $s_1 = u + v + w$, $s_2 = v\,w + w\,u + u\,v$ and $s_3 = u\,v\,w$. The equation of c_9 is

$$(s_1 - s_3)p^2 + 2(s_2 - s_3)pq + (s_3 - s_1)q^2 - (s_3 + s_1)p + 2\,q = 0$$

and the parameter u is $\frac{\beta(2(1-v\,w)p+(s_1-s_3)q-2)}{(s_1-s_3)p+2\,u(v+w)q-s_1-s_3}$. Since the perpendicular at P to the line OP is $p\,x + q\,y - p^2 - q^2 = 0$ we conclude that this line will agree with the butterfly line $f\,x + g\,y - f\,p - g\,q = 0$ if and only if P satisfies the above equation of the equilateral hyperbola c_9. When $P = O$, then the normal to c_9 at P and the butterfly line of the same point of course both have the equation $2\,x + (s_1 + s_3)\,y = 0$. □

3. Centers as butterfly points

Our next theorem shows that the center of a conic through the vertices of a triangle ABC will be the butterfly point of a strong quadrangle $ABCD$ if and only if the point D is on this conic. It could therefore be regarded as a converse of Theorem 3 in [4].

Theorem 3. *Let $ABCD$ be a strong quadrangle. The center S of a nondegenerate conic Γ through the vertices of the triangle ABC is a β_{ABCD}-point if and only if D lies on Γ.*

Proof. In this proof we shall use the same assumption about the points A, B, C and D as we did in the proof of Lemma 1 and Theorem 1.

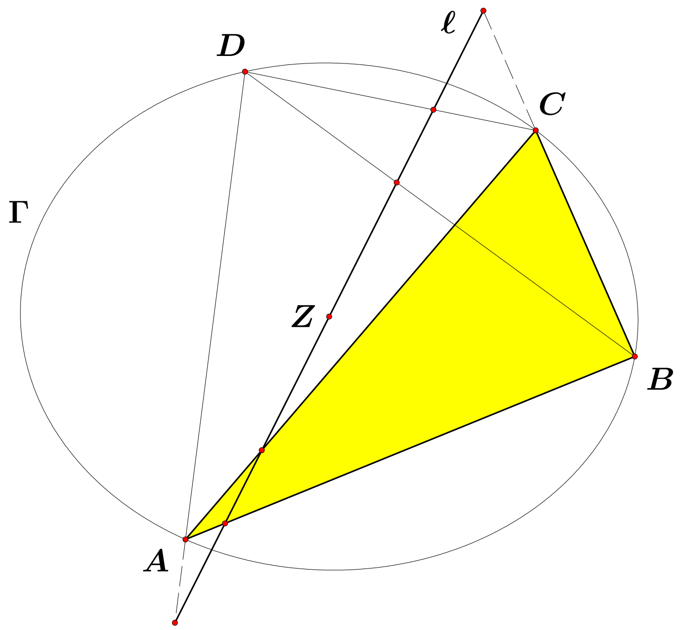

FIGURE 3. The center Z of a conic Γ through the vertices of ABC is a β_{ABCD}-point if and only if D is on Γ.

A conic has the equation

$$a\,x^2 + 2\,b\,xy + c\,y^2 + 2\,d\,x + 2\,e\,y + f = 0.$$

When we substitute coordinates of points A, B and C for x and y and solve these linear equations in d, e and f, we conclude that the equation of our conic Γ that goes through the vertices of ABC is

$$\Gamma(x,\, y) = v\,Q(x,\, y) - Q(u,\, v)\,y = 0,$$

where Q is a function that takes $(x,\, y)$ into $a\,x^2 + 2\,b\,x\,y + c\,y^2 - a\,x$. Let $T = Q(u,\, v)$.

The above conic will have a center (i. e., it will be either an ellipse or a hyperbola) provided $\Delta = a\,c - b^2 \neq 0$. Then the center S is the point $(\frac{a\,c\,v - b\,T}{2\,v\,\Delta},\, \frac{a\,(T - b\,v)}{2\,v\,\Delta})$.

Let S_x and S_y denote the coordinates of S. The equation of a line ℓ through S is $f\,x + g\,y - f\,S_x - g\,S_y = 0$, for some real numbers f and g with $f^2 + g^2 \neq 0$.

We can evaluate $|M_{ac}S|$ and $|M_{bd}S|$ to find that they are absolute values of $\frac{(f\,P_j + g\,Q_j)\,K}{4\,v\,\Delta\,S_j\,T_j}$ for $j = 1,\, 2$ with with $S_1 = \alpha,\ S_2 = \beta,\ T_1 = \gamma,\ T_2 = \delta,\ P_1 = 2\,v\,w\,\Delta + \varrho(a,\, b)T - a\,v\,\varrho(b,\, c),\ Q_1 = a\,v_2(T - b\,v),\ P_2 = 2\,v\,v_1\,\Delta + \sigma(a,\, b)T - a\,v\,\sigma(b,\, c),\ Q_2 = 2\,v\,v_0\,\Delta + \tau(a,\, b)T - a\,v\,\tau(b,\, c)$, $\varrho(a,\, b) = 2\,a\,u_2 + b\,v_2,\ \sigma(a,\, b) = 2\,a\,U(u-1) + b\,(W - V)$ and $\tau(a,\, b) = a\,(W - V) + 2\,b\,v_0$ (for our notation see the proof of Lemma 1).

Let us assume for the moment that $Q_1 \neq 0$. Then the center S is the midpoint M_{ac} if and only if $g = -\frac{f\,P_1}{Q_1}$. Substituted into $f\,P_2 + g\,Q_2$ this value gives $\frac{2\,k\,\Gamma(U,V)}{a\,v_2}$, where $k = (a(u-1) + b\,v)(a\,u + b\,v)$.

When the point D lies on the conic Γ then $\Gamma(U,\,V) = 0$ so that S is also the midpoint M_{bd}. Hence, S is the β_{ABCD}-point by Lemma 1.

Conversely, if S is the β_{ABCD}-point then $k\,\Gamma(U,\,V) = 0$. In other words, either $\Gamma(U,\,V) = 0$ (when the point D lies on the conic Γ), or $a\,u + b\,v = 0$, or $a\,(u-1) + b\,v = 0$. When $a\,u + b\,v = 0$ then $b = -\frac{a\,u}{v}$ so that the center of Γ is the midpoint A_g of the segment BC and the line ℓ agrees with the line BC. In this situation the point ℓ_b is not determined which implies that $a\,u + b\,v = 0$ can not happen. Similarly, when $a\,(u-1) + b\,v = 0$ then $b = \frac{a(1-u)}{v}$ so that the center of Γ is the midpoint B_g of the segment AC and the line ℓ agrees with the line AC and we again conclude that $a\,(u-1) + b\,v = 0$ can not happen.

Note that Q_1 is equal to zero provided either $a = 0$ or $v - V = 0$ or $T - b\,v = 0$. When $T - b\,v = 0$, then $c = \frac{a\,u(1-u) + b\,v(1-2\,u)}{v^2}$ and $S = C_g$, the midpoint of the segment AB. Also, $f\,P_j + g\,Q_j = \frac{f\,k\,F_j}{v}$ where F_1 and F_2 are $v_2 - 2\,w$ and $V + w$. It follows that the center C_g is the β_{ABCD}-point if and only if either $f = 0$ or $F_1 = 0$ and $F_2 = 0$. If $f = 0$, then the line ℓ agrees with the line AB which can not happen for the similar reason which prevents $a\,u + b\,v = 0$ and $a\,(u-1) + b\,v = 0$ to hold. On the other hand, $F_1 = 0$ and $F_2 = 0$ only for $U = 1 - u$ and $V = -v$ when $ABCD$ is a parallelogram which is ruled out by our assumption that $ABCD$ is a strong quadrangle.

It remains to consider the case $a(v - V) = 0$. Of course, there are two subcases $a = 0$ and $V = v$. Since the ordinate of the vertex C is also v we infer that the second subcase is impossible because the lines AB and CD would be parallel.

When $a = 0$, then the center S of Γ is the point $\left(\frac{2\,b\,u + c\,v}{2\,b},\,0\right)$ on the line AB and the conic Γ degenerates into two lines (AB and CS) which we prohibited with our assumptions. □

Remark 1. One can wonder if the statement of Theorem 3 is completely true. Like in the particular case formulated in Theorem 2, there should be an exception, when S is located on a side of the triangle ABC. Then D can be anywhere provided the quadrangle $ABCD$ is still strong.

The following Figure 4 shows that the last claim is wrong.

On this figure $ABCD$ is a strong quadrangle (its diagonal triangle EFG is well-defined), the point S is on the side AB of the triangle ABC and the center of the circumcircle of ABC but it is not the β_{ABCD}-point because for any line through S the point S can not be the midpoint of the segment $\ell_a\ell_c$, where $\ell_a = S$ and ℓ_c are the intersections of the line ℓ with the lines AB and CD.

The above theorem gives the possibility to describe every nondegenerate conic through the vertices of a triangle ABC using the butterfly

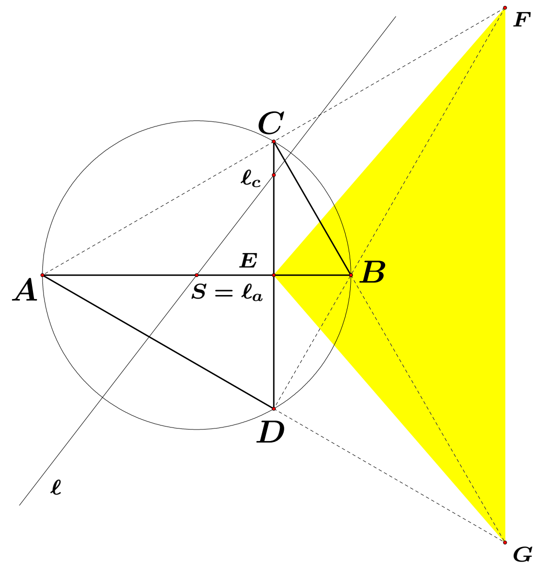

FIGURE 4. The counterexample for the above statement.

property of its center. In particular, we have the following result for the Feuerbach, Jarabek, and Kiepert equilateral hyperbolas of ABC. In the statement we use central points from [15].

Recall that the equilateral hyperbola which goes through the points A, B, C, H (or X_4 – the orthocenter), and I (or X_1 – the incenter) is the Feuerbach hyperbola of the triangle ABC. When the fifth point is G (or X_2 – the centroid) we talk of the Kiepert hyperbola and for O (or X_3 – the circumcenter) as the fifth point we have the Jarabek hyperbola of the triangle ABC.

Corollary 2. *Let $ABCD$ be a strong quadrangle. The central points X_{11}, X_{115}, and X_{125} of the triangle ABC are β_{ABCD}-points if and only if D lies on the Feuerbach, Kiepert, and Jarabek equilateral hyperbola of ABC, respectively.*

Proof. It is well-known that the centers of the three famous named hyperbolas of the triangle are the central points X_{11} (of the Feuerbach hyperbola), X_{115} (of the Kiepert hyperbola), and X_{125} (of the Jarabek hyperbola) (see [16]) so that we can apply Theorem 3 to obtain the desired conclusion. □

Another version of Theorem 3 is the following statement which was formulated for cyclic quadrangles in [22] as Theorem 4.

Theorem 4. *Let $ABCD$ be a strong quadrangle. The locus of centers S of all nondegenerate conics Γ through the vertices of $ABCD$ is the nine-point-conic c_9 of the quadrangle $ABCD$.*

Proof. We shall make the same assumptions about points A, B, C, and D as in the proof of Theorem 3. Replacing x and y with U and V in the polynomial $\Gamma(x, y)$ we can solve for c to obtain that the equation of a nondegenerate conic through the vertices of $ABCD$ is

$$v_0 v_2 x(a x + 2 b y - a) + [(v_1 U - u_1 u - w) a - 2 v_0 u_2 b] y^2 + [(v_1 v - u_1 V - w W) a - 2 v_0 w b] y = 0.$$

Its center S is $\left(\frac{u_1 \alpha(u, v) \beta(V) - v_1 \alpha(U, V) \beta(v)}{2(\varrho a^2 + 2 \sigma a b + \tau b^2)}, \frac{a\left(V^2 \gamma(u, v) - v^2 \gamma(U, V)\right)}{2(\varrho a^2 + 2 \sigma a b + \tau b^2)}\right)$, where $\alpha(u, v) = a(u - 1) + 2 b v$, $\beta(V) = a - b V$, $\varrho = u_1 u - v_1 U + w$, $\sigma = u_1 v - v_1 V$, $\tau = v_0 v_2$ and $\gamma(u, v) = a u(u - 1) + b v(2 u - 1)$.

In order to obtain the locus of these centers we will eliminate the real variable b. This could be done as follows.

Let the coordinates of S be x and y. Let $H = v_2 x - w$, $G = -\varrho$, $F = v_2 - 2 w$, $E = v v_1 - V u_1 - w W$. After the multiplication by the denominator of x and transfer of terms on the left hand side we get the following equations (e_1) and (e_2).

$$(2 x - 1) G a^2 - 2 v_0 H b^2 + [4 v_0 u_2 x + E + 2(U v_1 - v u_1)] a b = 0,$$

$$[E + 2 G y] a^2 - 2 v_0 v_2 y b^2 - v_0 [4 u_2 y - F] a b = 0.$$

We multiply (e_1) by $v_2 y$ and (e_2) by H, make their difference, and solve for b. We get $b = \frac{a(E H + F G y)}{M}$, where $M = K y - v_0 F H$ and $K = u_1^2(V + 3v) + v_1^2(3V + v) - 4 u_0 v_0 (V + v) + v_0(u_1(V + 2v) - v_1(2V + v))$.

Substituting this value back into (e_1) and (e_2) we obtain $\frac{L H \Theta}{M^2} = 0$ and $\frac{a L y \Theta}{M^2} = 0$, where $L = a^2 v_2 w(v - w)(V - w)(v_2 - w)$ and $\Theta = v_0 x (2 H - 4 u_2 y - v_2) + 2(u u_1 - U v_1 + w) y^2 + (2 v u_1 - 2 V v_1 - E) y + v_0 w$.

It is easy to check that $\Theta = 0$ is in fact the equation of the nine-point-conic c_9 of $ABCD$ because the coordinates of the midpoints of segments AB, AC, AD, BC, BD, and CD satisfy it.

On the other hand, if $a = 0$ then the conic degenerates into two lines and if either $v_2 = 0$, $w = 0$, $v - w = 0$, $V - w = 0$, or $v_2 - w = 0$, then the quadrangle $ABCD$ is not strong which happens also when $H = 0$ and $y = 0$ (i. e., when $U = 1 - u$ and $V = -v$). □

4. Quadrangles sharing the nine-point-conics

The last Theorem 5 in the reference [22] considers the question if different quadrangles can share the same nine-point-conic. It shows that for any cyclic quadrangle $ABCD$ with the circumcenter O and for any circle with center at O which intersects the lines AB and CD in points P, R and Q, S the quadrangles $ABCD$ and $PQRS$ have the same nine-point-hyperbola (see Figure 4, i. e., Figures 6 – 8 in [22] without honeycombs).

We shall now prove an analogous result for an arbitrary strong quadrangle $ABCD$. We discover that there is a conic ω with the property that for any of it points there is a simple construction σ that gives a quadrangle $PQRS$ that shares the nine-point-conic with $ABCD$.

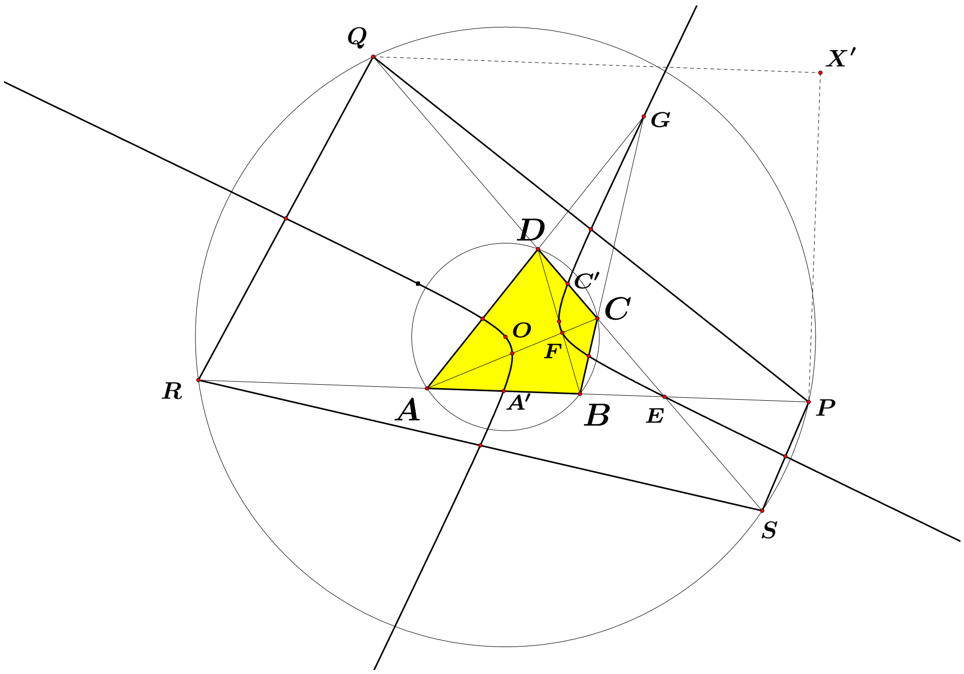

FIGURE 5. Cyclic quadrangles $ABCD$ and $PQRS$ with the concentric circumcircles and the identical nine-point-hyperbolas.

The steps of the construction σ go as follows. Let $ABCD$ be a strong quadrangle. Let A' and C' be midpoints of segments AB and CD. Let X be a point different from A' and C'. Let P be the orthogonal projection of X on the line AB and let Q denote the intersection of the line CD and the parallel through X to the line AB. Let R and S denote the reflections of points P and Q at the points A' and C', respectively. We shall say that $PQRS$ is obtained from X and $ABCD$ by the construction σ and write $PQRS = \sigma(X,\, ABCD)$.

Of course, when $ABCD$ is a cyclic quadrangle with the circumcenter O and k is any circle with center at O which intersects the lines AB and CD in points P, R and Q, S then for the intersection X of the perpendicular at P and the parallel at Q to the line AB we have $PQRS = \sigma(X,\, ABCD)$ so that our construction σ includes the one from [22] as a special case (see Figure 4).

Theorem 5. *Let $ABCD$ be a strong quadrangle. The locus of all points X with the property that the quadrangles $PQRS = \sigma(X,\, ABCD)$ and $ABCD$ share the nine-point-conic is a conic ω. The conics c_9 and ω are of the same type. The lines of symmetry of the conic ω are the perpendicular bisector of the segment AB and the parallel to AB at the midpoint of the segment CD.*

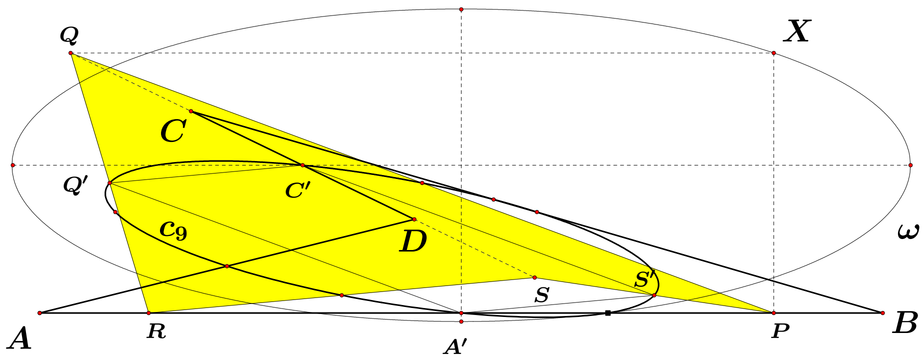

FIGURE 6. The quadrangles $PQRS = \sigma(X,\, ABCD)$ and $ABCD$ share the nine-point-conic if and only if X is on the conic ω.

Proof. We shall retain notation from the proofs of Theorems 3 and 4. Let the coordinates of the point X be s and t. Then the vertices of the quadrangle $PQRS$ have coordinates $P(s,\,0)$, $R(1-s,\,0)$, $Q\left(\frac{w+t\,u_2}{v_2},\,t\right)$, and $S\left(\frac{u\,v-U\,V-t\,u_2}{v_2},\,v+V-t\right)$. It is clear that the nine-point-conics of $ABCD$ and $PQRS$ coincide if and only if the midpoints Q' and S' of the segments QR and SP are on $c_9 = c_9(ABCD)$. Note that $Q'A'S'C'$ is a parallelogram (see Figure 5).

Recall that the equation of the conic c_9 is $\Theta = 0$. If we substitute the coordinates of either Q' or S' for x and y in Θ we shall get $\frac{\Psi}{2\,v_2}$, where

$$\Psi = v_0\,v_2^2\,s\,(s-1) + w(v_2 - w)(t^2 - (v+V)\,t + v_0).$$

We conclude that if the coordinates of the point X satisfy the condition $\Psi = 0$ then the quadrangles $PQRS$ and $ABCD$ will have the same nine-point-conic. Hence, the locus of points X is indeed a conic ω. Since the D-invariants (the expression $a\,c - b^2$ whose sign determines the type of the conic) of c_9 and ω are $4\,v_0\,w(v_2 - w)$ and $v_2^2\,v_0\,w(v_2 - w)$ it follows that c_9 and ω are of the same type. The possibility that $D(\omega) = 0$ and $D(c_9) \neq 0$ (for $V = v$) is ruled out by the assumption that $ABCD$ is a strong quadrangle.

The statement about the lines of symmetry of the conic ω is easily checked by substitution. More precisely, if X is a point on ω then its reflection $(1-s,\,t)$ at the perpendicular bisector of the segment AB and its reflection $(s,\,v+V-t)$ at the parallel to AB through the midpoint of CD are also on ω. □

References

[1] Leon Bankoff, *The Metamorphosis of the Butterfly Problem*, Math. Mag., **60** (1987), 195–210.

[2] Zvonko Čerin, *Poučci o leptirima*, Poučak (Zagreb), **2** (broj 8)(2001), 12–35. (in Croatian).

[3] Zvonko Čerin, *A generalization of the butterfly theorem from circles to conics*, Mathematical Communications (Osijek), **6** (2001), 161–164.

[4] Zvonko Čerin, *Lines with the butterfly property*, Mathematical Communications (Osijek), **8** (2003), No. 1, 35–41.

[5] G. D. Chakerian, M. S. Klamkin, G.T. Sallee, *On the Butterfly Property*, Math. Mag., **47** (1974), 21–23.

[6] Steven R. Conrad, *Another Simple Solution of the Butterfly Problem*, Math. Mag., **46** (1973), 278–280.

[7] H. S. M. Coxeter, *Projective geometry*, Blaisdell, New York, 1964., p. 78.

[8] H. Eves, *A survey of geometry*, Allyn and Bacon, Boston, 1963., p. 171.

[9] G. Geise, *Eine weitere Bemerkung zum Schmetterlingssatz*, Praxis Math., **31** (1989), 367–368.

[10] Larry Hoehn, *A New Proof of the Double Butterfly Theorem*, Math. Mag., **63** (1990), 256–257.

[11] Ross Honsberger, *The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry – Part 1*, College Math. J., **14** (1983), 2–8.

[12] Ross Honsberger, *The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry – Part 2*, College Math. J., **14** (1983), 154–158.

[13] William I. Jacobson, *The Butterfly Problem – Extensions Generalizations*, Math. Mag., **42** (1969), 17–21.

[14] Dixon Jones, *A Double Butterfly Theorem*, Math. Mag., **49** (1976), 86–87.

[15] Clark Kimberling, *Central points and central lines in the plane of a triangle*, Mathematics Magazine, **67** (1994), 163–187.

[16] Clark Kimberling, *Triangle Centers and Central Triangles*, volume 129 of Congressus Numerantium, Utilitas Mathematica Publishing, Winnipeg, 1999.

[17] Murray S. Klamkin, *An extension of the butterfly problem*, Math. Mag., **38** (1965), 206–208.

[18] G. Pickert, *Zum projektiven Beweis des Schmetterlingssatz*, Praxis Math., **30** (1988), 174–175.

[19] H. Schaal, *Bemerkungen zum Schmetterlingssatz*, Praxis Math., **30** (1988), 297–303.

[20] E. Siemon, *Noch eine Bemerkung zum Schmetterlingssatz*, Praxis Math., **31** (1989), 42–43.

[21] J. Sledge, *A generalization of the butterfly theorem*, J. of Undergraduate Math., **5** (1973), 3–4.

[22] A. Sliepčević, *A new generalization of the butterfly theorem*, J. for Geometry and Graphics, **6** (2002), No. 1, 61–68.

[23] Vladimir Volenec, *A generalization of the butterfly theorem*, Mathematical Communications (Osijek), **6** (2001), No. 2, 157–160.

[24] Vladimir Volenec, *The butterfly theorem for conics*, Mathematical Communications (Osijek), **7** (2002), No. 1, 35–38.

Kopernikova 7, 10010 Zagreb, CROATIA, Europe
E-mail address: `cerin@math.hr`

Dipartimento di Matematica, Universita di Torino, Torino, ITALY, Europe
E-mail address: GIANELLA@DM.UNITO.IT