CONFIGURATIONS ON CENTERS OF BANKOFF CIRCLES

ZVONKO ČERIN

Abstract. We study configurations built from centers of Bankoff circles of arbelos erected on sides of a given triangle or on sides of various related triangles.

1. Introduction

For points X and Y in the plane and a positive real number λ, let Z be the point on the segment XY such that $|XZ| : |ZY| = \lambda$ and let $\zeta = \zeta(X, Y, \lambda)$ be the figure formed by three mutually tangent semicircles σ, σ_1, and σ_2 on the same side of segments XY, XZ, and ZY respectively. Let S, S_1, S_2 be centers of σ, σ_1, σ_2. Let W denote the intersection of σ with the perpendicular to XY at the point Z. The figure ζ is called the *arbelos* or the *shoemaker's knife* (see Fig. 1).

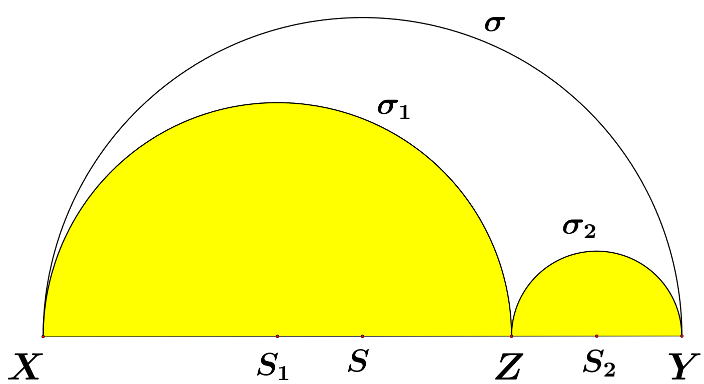

Figure 1. The arbelos $\zeta = \zeta(X, Y, \lambda)$, where $\lambda = \frac{|XZ|}{|ZY|}$.

It has been the subject of studies since Greek times when Archimedes proved the existence of the circles $\omega_1 = \omega_1(\zeta)$ and $\omega_2 = \omega_2(\zeta)$ of equal radius such that ω_1 touches σ, σ_1, and ZW while ω_2 touches σ, σ_2, and ZW (see Fig. 2).

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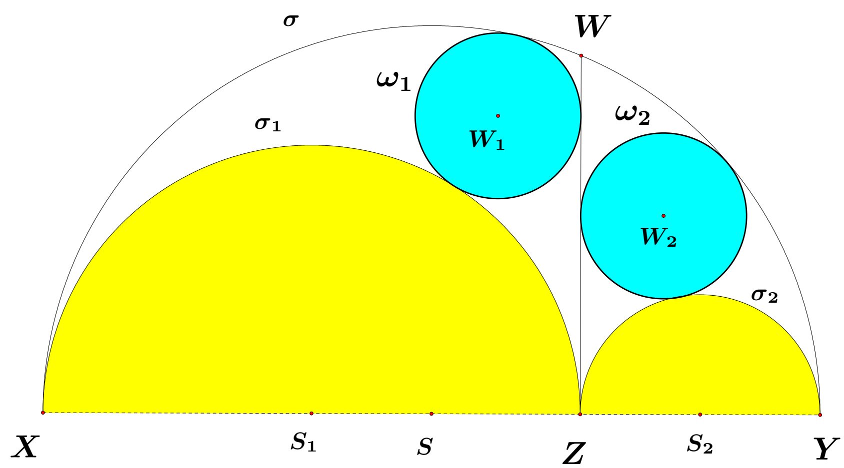

FIGURE 2. The Archimedean circles ω_1 and ω_2 together.

In [1] Bankoff discovered that the circumcircle $\omega_3 = \omega_3(\zeta)$ of the triangle MNZ has the same radius as the Archimedean twin circles ω_1 and ω_2, where $M = \sigma_3 \cap \sigma_1$ and $N = \sigma_3 \cap \sigma_2$ and σ_3 is the circle that touches σ from inside and σ_1 and σ_2 from outside (see Fig. 3).

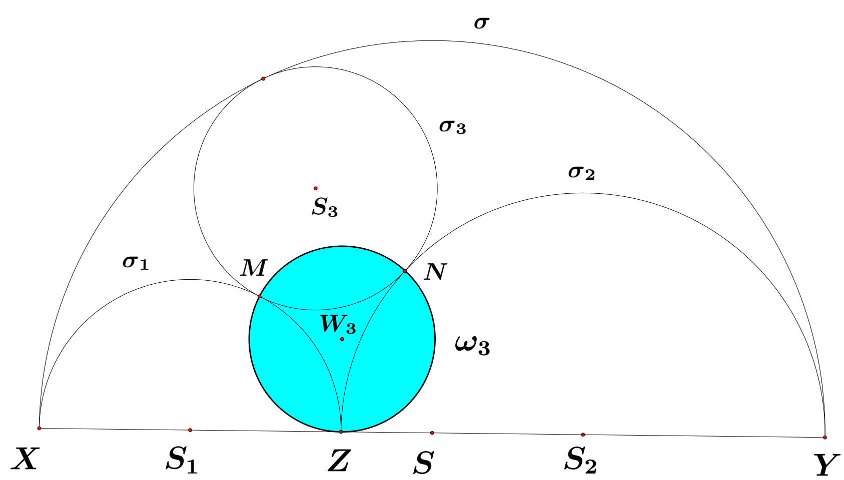

FIGURE 3. The Bankoff circle ω_3 and the circle σ_3.

The purpose of this paper is to study triangles on centers of the Bankoff circles of arbelos either on sides of an arbitrary triangle ABC or on sides of some of its associated triangles.

More precisely, our first goal is to prove the following theorem (see Fig. 4). All other results in this paper are similar.

Let τ denote the base triangle ABC. Then τ_b is a short notation for its first Brocard triangle $A_bB_bC_b$. Its vertices are the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides (see [6]). Let $\zeta_a = \zeta_a(\tau) = \zeta(B, C, \lambda)$, $\zeta_b = \zeta_b(\tau) = \zeta(C, A, \lambda)$

and $\zeta_c = \zeta_c(\tau) = \zeta(A,\, B,\, \lambda)$. Let W^a, W^b, W^c denote centers of $\omega_3(\zeta_a)$, $\omega_3(\zeta_b)$, $\omega_3(\zeta_c)$, respectively.

Recall that triangles τ and $\theta = XYZ$ are *homologic* and we write $\tau \langle\rangle\, \theta$ provided lines AX, BY, and CZ are concurrent. The point Q in which they concur is their homology *center* and the line ℓ containing intersections of pairs of lines $(BC,\, YZ)$, $(CA,\, ZX)$, and $(AB,\, XY)$ is their homology *axis*. In this situation we also use the notation $\tau\, \langle Q;\, \ell\rangle\, \theta$ where ℓ and/or Q can be omitted. For Q and ℓ sometimes we use $\langle \tau,\, \theta\rangle$ and $\langle\langle \tau,\, \theta\rangle\rangle$. In stead of homologic, homology center, and homology axis many authors use *perspective*, *perspector*, and *perspectrix*.

Theorem 1. *For every $\lambda \geq 0$ the triangle $W^aW^bW^c$ on the centers of Bankoff circles of arbelos on sides of a triangle τ is homologic to its Brocard triangle τ_b. The centre of this homology lies on the Brocard circle of τ (i.e., on the circumcircle of τ_b). The triangles τ, τ_b, and $W^aW^bW^c$ have the same centroid.*

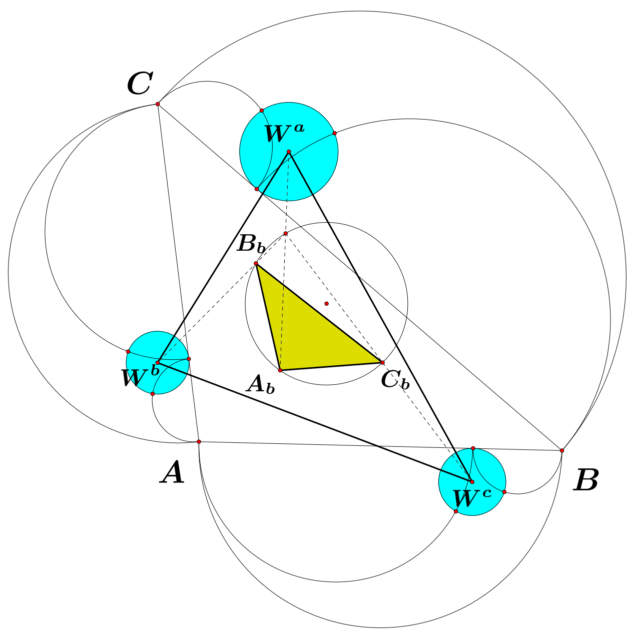

FIGURE 4. The triangle $W^aW^bW^c$ and the first Brocard triangle $A_bB_bC_b$ are homologic.

2. Bankoff circle ω_3

Let $X(x,\, a)$ and $Y(y,\, b)$. Then $S(\frac{x+y}{2},\, \frac{a+b}{2})$ is the midpoint of the segment XY. Since $\frac{|XZ|}{|ZY|} = \lambda$, the point Z is $\left(\frac{x+\lambda\, y}{\lambda+1},\, \frac{a+\lambda\, b}{\lambda+1}\right)$. Moreover, semicircles σ_1 and σ_2 have centres at $\left(\frac{(\lambda+2)\, x+\lambda\, y}{2(\lambda+1)},\, \frac{(\lambda+2)\, a+\lambda\, b}{2(\lambda+1)}\right)$ and $\left(\frac{x+(2\,\lambda+1)\, y}{2(\lambda+1)},\, \frac{a+(2\,\lambda+1)\, b}{2(\lambda+1)}\right)$ (the midpoints of segments XZ and ZY).

Our goal now is to find the center and the radius of the unique circle ω_3 in the arbelos ζ which touches the semicircles σ_1 and σ_2 from outside and the semicircle σ from inside.

Let its centre be the point $S_3(p, q)$ and the radius a positive real number ϱ. Let $A = x - y$ and $B = a - b$.

Since σ_3 touches σ from inside, the distance $|S_3 S|$ is equal to the difference $\frac{1}{2}\sqrt{A^2+B^2} - \varrho$ of their radii. This condition leads to the relation

$$\left(p - \frac{x+y}{2}\right)^2 + \left(q - \frac{a+b}{2}\right)^2 = \left(\frac{\sqrt{A^2+B^2}}{2} - \varrho\right)^2. \tag{1}$$

Since σ_3 touches σ_1 and σ_2 from outside, the distances $|S_3 S_1|$ and $|S_3 S_2|$ are equal to the sums $\varrho + \frac{\lambda\sqrt{A^2+B^2}}{2\,(\lambda+1)}$ and $\varrho + \frac{\sqrt{A^2+B^2}}{2\,(\lambda+1)}$ of their radii. It follows that

$$\left(p - \frac{(\lambda+2)\,x + \lambda\,y}{2\,(\lambda+1)}\right)^2 + \left(q - \frac{(\lambda+2)\,a + \lambda\,b}{2\,(\lambda+1)}\right)^2 = \left(\varrho + \frac{\lambda\sqrt{A^2+B^2}}{2\,(\lambda+1)}\right)^2, \tag{2}$$

$$\left(p - \frac{x + (2\,\lambda+1)\,y}{2\,(\lambda+1)}\right)^2 + \left(q - \frac{a + (2\,\lambda+1)\,b}{2\,(\lambda+1)}\right)^2 = \left(\varrho + \frac{\sqrt{A^2+B^2}}{2\,(\lambda+1)}\right)^2. \tag{3}$$

These equations have two solutions in p, q, and ϱ. We shall use the solution $\varrho = \frac{\lambda\sqrt{A^2+B^2}}{2\,(\lambda^2+\lambda+1)}$ and

$$S_3\left(\frac{(\lambda+2)\,x + \lambda\,(2\,\lambda+1)\,y - 2\,\lambda\,(a-b)}{2\,(\lambda^2+\lambda+1)},\ \frac{(\lambda+2)\,a + \lambda\,(2\,\lambda+1)\,b + 2\,\lambda\,(x-y)}{2\,(\lambda^2+\lambda+1)}\right).$$

It is now easy to find the coordinates of the points M and N because we know that they divide the segments S_1S_3 and S_2S_3 in the ratio of the radii of the circles σ_1, σ_3 and σ_2, σ_3. Hence,

$$M\left(\frac{(\lambda+2)\,x + \lambda\,(\lambda+1)\,y - \lambda\,(a-b)}{\lambda^2+2\,\lambda+2},\ \frac{(\lambda+2)\,a + \lambda\,(\lambda+1)\,b + \lambda\,(x-y)}{\lambda^2+2\,\lambda+2}\right),$$

$$N\left(\frac{(\lambda+1)\,x + \lambda\,(2\,\lambda+1)\,y - \lambda\,(a-b)}{2\,\lambda^2+2\,\lambda+1},\ \frac{(\lambda+1)\,a + \lambda\,(2\,\lambda+1)\,b + \lambda\,(x-y)}{2\,\lambda^2+2\,\lambda+1}\right).$$

The circumcircle ω_3 of the triangle MNZ has the radius $\frac{\lambda\sqrt{A^2+B^2}}{2\,(\lambda+1)^2}$ (of the Archimedean twin circles) and its center is at the point

$$W_3\left(\frac{x+\lambda\,y}{\lambda+1} - \frac{\lambda\,(a-b)}{2\,(\lambda+1)^2},\ \frac{a+\lambda\,b}{\lambda+1} + \frac{\lambda\,(x-y)}{2\,(\lambda+1)^2}\right).$$

Remark 1. It is possible to prove the above by inversion (see [9, p. 224]). But, since in this paper we use analytic approach, we need functions that describe coordinates of the center of the Bankoff's circle. Behind the scene we work in rectangular coordinates with complicated expressions and then transfer the results into trilinear coordinates where the expressions are often much simpler.

3. Proof of Theorem 1

In our proofs we shall use trilinear coordinates. Recall that the *actual trilinear coordinates* of a point P with respect to the triangle ABC are signed distances f, g, and h of P from the lines BC, CA, and AB. We shall regard P as lying on the positive side of BC if P lies on the same side of BC as A. Similarly, we shall regard P as lying on the positive side of CA if it lies on the same side of CA as B, and similarly with regard to the side AB. Ordered triples $x : y : z$ of real numbers proportional to (f, g, h) (that is such that $x = \mu f$, $y = \mu g$, and $z = \mu h$, for some real number μ different from zero) are called *trilinear coordinates* of P. The advantage of their use is that a high degree of symmetry is present so that it usually suffices to describe part of the information and the rest is self evident. For example, when we write $X_1(1)$ or $I(1)$ or simply say I is 1 this indicates that the incenter has trilinear coordinates $1 : 1 : 1$. We gave only the first coordinate while the other two are cyclic permutations of the first. Similarly, $X_2(\frac{1}{a})$ or $G(\frac{1}{a})$ say that the centroid has has trilinears $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, where a, b, c are lengths of sides of ABC. We use X_n to denote the n-th central point of the triangle ABC (see [8]). The expressions in terms of sides a, b, and c can be shortened using the following notation.

$$d_a = b - c,\;\; d_b = c - a,\;\; d_c = a - b,\;\; z_a = b + c,\;\; z_b = c + a,\;\; z_c = a + b,$$

$$t = a + b + c, \quad t_a = b + c - a, \quad t_b = c + a - b, \quad t_c = a + b - c,$$

$$m = abc, \quad m_a = bc, \quad m_b = ca, \quad m_c = ab, \quad T = \tfrac{1}{4}\sqrt{t t_a t_b t_c},$$

For an integer n, let $t_n = a^n + b^n + c^n$ and $d_{na} = b^n - c^n$ and similarly for other cases.

In order to achieve even greater economy in our presentation, we shall describe coordinates or equations of only one object from triples of related objects and use cyclic permutations φ and ψ to obtain the rest. For example, the first vertex A_b of the first Brocard triangle τ_b of τ has trilinears $abc : c^3 : b^3$. Then the trilinears of B_b and C_b need not be described because they are easily figured out and memorized by relations $B_b = \varphi(A_b)$ and $C_b = \psi(A_b)$. One must remember always that transformations φ and ψ are not only permutations of letters but also of positions, i. e., if P has trilinear coordinates $f_1(a, b, c) : f_2(a, b, c) : f_3(a, b, c)$, then the associated points $Q = \varphi(P)$ and $R = \psi(P)$ have $f_3(b, c, a) : f_1(b, c, a) : f_2(b, c, a)$ and $f_2(c, a, b) : f_3(c, a, b) : f_1(c, a, b)$ for trilinear coordinates. Note that $\psi = \varphi^{-1}$. Therefore, the trilinears of B_b and C_b are $c^3 : abc : a^3$ and $b^3 : a^3 : abc$.

The trilinear coordinates of W^a are

$$-2\,\lambda\, a : \frac{(t_{2\,c} + 8\,T)\,\lambda + 8\,T}{b} : \frac{(t_{2\,b} + 8\,T)\,\lambda + 8\,T\,\lambda^2}{c}.$$

The equation of the line $W^a A_b$ is

$$(8\,c^2\,T\,\lambda^2 - d_{2a}(t_2 + 8\,T)\,\lambda - 8\,b^2\,T)\,x - a\,b\,\lambda\,(8\,T\,\lambda + t_2 + 8\,T)\,y + a\,c\,((t_2 + 8\,T)\,\lambda + 8\,T)\,z = 0.$$

It is now easy to check that the determinant from the coefficients of the lines $W^a A_b$, $W^b B_b$, and $W^c C_b$ is zero. Hence, $W^a W^b W^c \langle\rangle \tau_b$.

The first trilinear coordinate of $\langle W^a W^b W^c,\, \tau_b\rangle$ is

$$a\,(64\,b^2\,T^2\,\lambda^4 - 8\,T\,(a^2 - 2\,b^2)(t_2 + 8\,T)\,\lambda^3 + U\,\lambda^2 - 8\,T\,(a^2 - 2\,c^2)(t_2 + 8\,T)\,\lambda + 64\,c^2\,T^2),$$

where $U = 16\,t_2\,t_{2a}\,T + 7\,a^6 - 21\,z_{2a}\,a^4 + (17\,z_{4a} + 2\,m_a^2)\,a^2 + (b^2 - 3\,c^2)(c^2 - 3\,b^2)\,z_{2a}$. By simple substitution we can check that this point lies on the Brocard circle of τ whose equation is $m \sum x^2 - \sum a^3\,y\,z = 0$. Of course, the same could be proved by eliminating the parameter λ. Finally, the verification of the statement about the centroids is easily accomplished in rectangular coordinates.

4. The dual of Theorem 1

It is interesting that in Theorem 1 we can interchange triangles τ and τ_b. Let W_b^a denote the center of the Bankoff circle for the arbelos $\zeta(B_b,\, C_b,\, \lambda)$. The points W_b^b and W_b^c are defined similarly.

Theorem 2. *For every $\lambda \geq 0$ the triangle $W_b^a W_b^b W_b^c$ is homologic with the triangle τ. The locus of the homology center $\langle W_b^a W_b^b W_b^c,\, \tau\rangle$ is a part of a quartic that goes through the vertices of τ and its Tarry point X_{98}. The triangles τ, τ_b, and $W_b^a W_b^b W_b^c$ have the same centroid.*

Proof. The trilinear coordinates of W_b^a are

$$\frac{8\,T\,(b^2\lambda^2 + c^2) + \lambda\,U}{a} : \frac{8\,T\,(a^2\lambda^2 + b^2) + \lambda\,V}{b} : \frac{8\,T\,(c^2\lambda^2 + a^2) + \lambda\,W}{c},$$

with $U = 8\,T\,z_{2a} + 2\,a^4 - z_{2a}\,a^2 + d_{2a}^2$, $V = 8\,T\,z_{2a} - a^4 + a^2\,c^2 - b^2\,d_{2a}$, $W = 8\,T\,z_{2b} - a^4 + a^2\,b^2 + c^2\,d_{2a}$. The equation of the line joining A with W_b^a is

$$b\,[8\,T\,(c^2\lambda^2 + a^2) + \lambda\,W]\,y - c\,[8\,T\,(a^2\lambda^2 + b^2) + \lambda\,V]\,z = 0.$$

It is now easy to check that the determinant from the coefficients of the lines $W_b^a A$, $W_b^b B$, and $W_b^c C$ is zero. Hence, $W_b^a W_b^b W_b^c \langle\rangle \tau$.

The first trilinear coordinate of $D = \langle W_b^a W_b^b W_b^c,\, \tau\rangle$ is

$$\frac{1}{a\,[8\,T\,(b^2\,\lambda^2 + c^2) + \lambda\,(z_{2a}(8\,T + a^2) - z_{4a})]}.$$

By simple substitution we can check that this point lies on the quartic Γ whose equation is

$$\sum b\,c\,y\,z\,[2\,a^2\,x^2\,P + b\,c\,y\,z\,Q] = 0,$$

with $P = 8\,P_1\,P_2\,T + P_3$ and $Q = Q_1\,Q_2 - 16\,T\,a^2\,(a^4 - m_{2a})\,P_2$, where $P_1 = z_{4a}\,a^2 - m_{2a}\,z_{2a}$, $P_2 = a^4 - z_{2a}\,a^2 + z_{4a} - m_{2a}$,

$$P_3 = (z_{4a} + 3\,m_{2a})\,a^8 - z_{2a}^3\,a^6 + m_{2a}\,(2\,z_{4a} - m_{2a})\,a^4 - \\ m_{2a}\,z_{2a}\,(2\,z_{4a} - 3\,m_{2a})\,a^2 + m_{2a}\,((z_{4a} - m_{2a})^2 + m_{4a}),$$

$$Q_1 = a^6 - z_{2a}\,a^4 - c^2\,(b^2 - 2\,c^2)\,a^2 - m_{2a}\,d_{2a},$$

and

$$Q_2 = a^6 - z_{2a}\,a^4 - b^2\,(c^2 - 2\,b^2)\,a^2 + m_{2a}\,d_{2a}.$$

Of course, the same could be proved by eliminating the parameters λ and μ from the equations $D_x = \mu\,x$, $D_y = \mu\,y$, $D_z = \mu\,z$, where D_x, D_y, D_z are trilinear coordinates of the homology center D. It is easy to check that the quartic Γ goes through the vertices of τ and through its Tarry point X_{98} whose first trilinear coordinate is $\frac{1}{a\,(z_{2a}\,a^2 - z_{4a})}$. Finally, the verification of the statement about the centroids is easily accomplished in rectangular coordinates. □

Theorem 3. *For every $\lambda \geq 0$ the triangles $W^aW^bW^c$ and $W_b^aW_b^bW_b^c$ are homologic.*

Proof. Since we know the trilinear coordinates for the vertices of both triangles $W^aW^bW^c$ and $W_b^aW_b^bW_b^c$ it is easy to write down the equations joining their corresponding vertices and verify that they concur again by calculating the determinant from the coefficients of these linear equations. □

In this way, it is possible also to show the following homology relations: $W_a^aW_a^bW_a^c \,\langle\rangle\, \tau_b$, $W_b^aW_b^bW_b^c \,\langle\rangle\, \tau_a$, $W_g^aW_g^bW_g^c \,\langle\rangle\, \tau_b$, $W_b^aW_b^bW_b^c \,\langle\rangle\, \tau_g$, $W_x^aW_x^bW_x^c \,\langle\rangle\, \tau_y$, $W_y^aW_y^bW_y^c \,\langle\rangle\, \tau_x$, where τ_a, τ_g, τ_x, and τ_y denote the anticomplementary, the complementary, and the two Napoleon triangles of τ. The loci of their homology centers are quartics that go through few points related to τ.

Another group of interesting homology relations appears when we consider two triangles and we build an arbelos with the same parameter λ on each side of both.

Let $\tau_0 = \tau$. Let τ_u and τ_v be Torricelli triangles of τ (whose vertices are apexes of equilateral triangles erected on sidelines towards either all inwards or all outwards). For the following pairs: $(0,\,u)$, $(0,\,v)$, $(0,\,x)$, $(0,\,y)$, $(a,\,b)$, $(a,\,u)$, $(a,\,v)$, $(a,\,x)$, $(a,\,y)$, $(b,\,g)$, $(b,\,u)$, $(b,\,v)$, $(b,\,x)$, $(b,\,y)$, $(g,\,u)$, $(g,\,v)$, $(g,\,x)$, $(g,\,y)$, $(u,\,v)$, $(u,\,x)$, $(u,\,y)$, $(v,\,x)$, $(v,\,y)$, and $(x,\,y)$, the relation $W_p^aW_p^bW_p^c \,\langle\rangle\, W_q^aW_q^bW_q^c$ holds, where p is the first and q is the second member of the pair. The loci of the centers of these homologies are curves of order six. Moreover, if τ_m and τ_n are homothetic triangles, then $W_m^aW_m^bW_m^c \,\langle\rangle\, W_n^aW_n^bW_n^c$. The homology center agrees with the center of the homothety.

5. Arbelos on Apollonian segments

Let U and V be the points on sideline BC met by the interior and exterior bisectors of angle A. The circle having diameter UV is the A-Apollonian circle and its center the midpoint A_π of UV is the A-Apollonian point. The B- and C- Apollonian circles and points are similarly constructed. Each circle passes through one vertex and both isodynamic points X_{15} and X_{16}. The Apollonian points A_π, B_π, C_π are collinear and we regard them as vertices of degenerate triangle τ_π. The trilinear coordinates of A_π are $0 : b : -c$.

It is known [6, p. 118] that the midpoints A_c, B_c, C_c of the chords of the circumcircle containing the vertex and the symmedian point K are vertices of the second Brocard triangle τ_c of τ. The trilinear coordinates of A_c are $\frac{t_{2a}}{a} : b : c$.

Theorem 4. *For every $\lambda \geq 0$ the second Brocard triangle τ_c of τ is homologic to the triangle $W_\pi^a W_\pi^b W_\pi^c$. The locus of the homology center $\langle W_\pi^a W_\pi^b W_\pi^c,\, \tau_c\rangle$ for a scalene triangle τ is a part of a parabola.*

Proof. The trilinear coordinates of W_π^a are $k_1 : k_2 : k_3$ with

$$k_1 = a\,(8\,T\,(d_{2b}\,\lambda^2 - d_{2c}) + (8\,T\,(z_{2a} - 2\,a^2) + a^2\,z_{2a} - z_{4a})\,\lambda),$$
$$k_2 = \lambda\,b\,(8\,T\,d_{2b}\,(\lambda+1) - b^2\,z_{2b} + z_{4b}),$$
$$k_3 = c\,((8\,T\,d_{2c} + c^2\,z_{2c} - z_{4c})\,\lambda + 8\,T\,d_{2c}).$$

The equation of the line joining A_c with W_π^a is

$$a\,b\,c\,E\,x + c\,F\,y + b\,G\,z = 0,$$

where $E = 8\,T(d_{2b}\,\lambda^2 + d_{2c}) - \lambda\,E_1$, $E_1 = d_{2a}\,(t_2 + 8\,T)$, $F = 8\,T(a^2\,d_{2b}\,\lambda^2 - d_{2c}\,z_{2a}) - \lambda\,F_1$, $F_1 = F_2 + 8\,T F_3$, $F_2 = a^6 - (2\,z_{2a} + c^2)a^4 + 2\,z_{4a}a^2 - b^2\,d_{2a}\,z_{2a}$, $F_3 = a^4 - b^2\,t_{2a}$, $G = 8\,T\,(d_{2b}\,z_{2a}\,\lambda^2 + a^2\,d_{2c}) - \lambda\,G_1$, $G_1 = G_2 + 8\,T\,G_3$, $G_2 = a^6 - (2\,z_{2a} + b^2)\,a^4 + 2\,z_{4a}\,a^2 + c^2\,d_{2a}\,z_{2a}$, and $G_3 = a^4 + c^2\,t_{2a}$. It is now simple to check that the determinant from the coefficients of the lines joining the corresponding vertices of the triangles $W_\pi^a W_\pi^b W_\pi^c$ and τ_c is zero. Hence, $W_\pi^a W_\pi^b W_\pi^c \,\langle\rangle\, \tau_c$.

The first trilinear coordinate of $D = \langle W_\pi^a W_\pi^b W_\pi^c,\, \tau_c\rangle$ is

$$a\,[8\,T\,(b^2\,\lambda^2 + c^2) + \lambda\,(z_{2a}(8\,T - a^2) + z_{4a})].$$

By simple substitution we can check that this point lies on the parabola Γ whose equation is

$$\sum_{cyclic} [\,b\,c\,P\,x^2 + 2\,a^2\,Q\,y\,z\,] = 0,$$

with $P = Q_2\,Q_3 + 16\,a^2\,(a^4 - m_a^2)\,T\,P_2$ and $Q = P_3 - 8\,T\,P_1\,P_2$, where P_1, P_2, P_3, Q_2, and Q_3 have been defined in the proof of Theorem 2. Of course, the same could be proved by eliminating the parameters λ and μ from the equations $D_x = \mu\,x$, $D_y = \mu\,y$, $D_z = \mu\,z$, where D_x, D_y, D_z are trilinear coordinates of the homology center D. It is easy to check that the focus of the parabola Γ is at the central point with the first trilinear coordinate $a(16\,T\,z_{2a} + 3\,(a^2\,z_{2a} - z_{4a}))$ and that its directrix has the equation

$$\sum_{cyclic} b\,c\,[16\,(a^4 - m_a^2)\,T + 5\,a^2\,P_2]\,x = 0.$$

□

Theorem 5. *For every real number $\lambda \geq 0$, the triangles $W_\pi^a W_\pi^b W_\pi^c$ and τ_π are homologic. For a scalene triangle τ, the locus of the homology center $\langle W_\pi^a W_\pi^b W_\pi^c,\, \tau_\pi \rangle$ is a part of a parabola.*

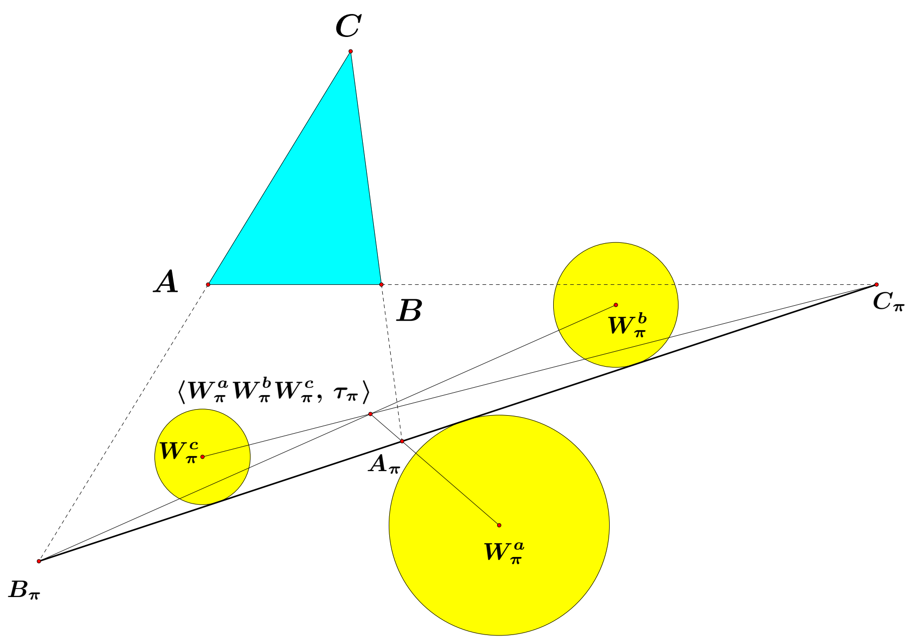

FIGURE 5. The triangle $W_\pi^a W_\pi^b W_\pi^c$ and the triangle $A_\pi B_\pi C_\pi$ on Apollonian points are homologic.

Proof. The equation of the line joining A_π with W_π^a is

$$b\,c\,U\,x + c\,a\,V\,y + a\,b\,V\,z = 0,$$

with U and V equal to $8\,(\lambda+1)(\lambda\,d_{2b} + d_{2c})\,T + \lambda\,(2\,a^4 - z_{2a}\,a^2 + d_{2a}^2)$ and $8\,(\lambda+1)(\lambda\,d_{2b} + d_{2c})\,T + \lambda\,(z_{2a}\,a^2 - z_{4a})$, respectively. It is now easy to check that the determinant from the coefficients of the lines joining the corresponding vertices of the triangles $W_\pi^a W_\pi^b W_\pi^c$ and τ_π is zero. Hence, $W_\pi^a W_\pi^b W_\pi^c \,\langle\rangle\, \tau_\pi$.

The first trilinear coordinate of $D = \langle W_\pi^a W_\pi^b W_\pi^c,\, \tau_\pi \rangle$ is

$$a\,[8\,T\,(d_{2b}\,\lambda^2 - d_{2c}) + \lambda\,(8\,T\,(z_{2a} - 2\,a^2) + a^2\,z_{2a} - z_{4a})].$$

By direct substitution we can check that this point lies on the parabola Γ whose equation is

$$\sum_{cyclic} [\, b\, c\, (t_{2b}\, t_{2c} - 16\, a^2\, T)\, x^2 + 2\, a^2\, (2\, m_{2a} - a^2\, t_{2a} - 8\, z_{2a}\, T)\, y\, z\,] = 0.$$

It is easy to check that the focus of the parabola Γ is at the central point with the first trilinear coordinate $a\,(3\,(z_{4a} - a^2\, z_{2a}) + 16\, T\,(z_{2a} - 2\, a^2))$ and that its directrix has the equation

$$\sum_{cyclic} b\, c\, [8\, T\, (\lambda + 1)(\lambda\, d_{2b} + d_{2c}) + \lambda\, (2\, a^4 - z_{2a}\, a^2 + d_{2a}^2)]\, x = 0.$$

□

In a similar way one can prove the following theorem for the triangle τ_ε whose vertices are points of intersection of external angle bisectors with the corresponding sidelines. These points are collinear so that τ_ε is also a degenerate triangle.

Theorem 6. *The triangles τ_ε and $W_\varepsilon^a W_\varepsilon^b W_\varepsilon^c$ are homologic for every real number $\lambda \geq 0$. For a scalene triangle τ, the locus of the homology center $\langle W_\varepsilon^a W_\varepsilon^b W_\varepsilon^c,\, \tau_\varepsilon \rangle$ is a part of a parabola.*

Remark 2. In the above results we have always build arbelos outwards. Of course, it is possible to build them inwards with similar conclusions. For example, in Theorem 1 in stead of the points W^a, W^b, W^c we can take the centers of the Bankoff circles of arbelos $\zeta(C,\, B,\, \frac{1}{\lambda})$, $\zeta(A,\, C,\, \frac{1}{\lambda})$, $\zeta(B,\, A,\, \frac{1}{\lambda})$. With these extended concept of building arbelos on sides of triangles our statements about loci are true without the words "part of". Another possibility is to allow negative values for the parameter λ and drop out arbelos altogether by considering vertices P, Q, R of three similar triangles BCP, CAQ, ABR build on sides of a triangle. In this form our results are closely related to the following 19th century results:

(1) The centers of similitude of each pair of these triangles are the vertices of the Brocard's second triangle.

(The center of similitude of two similar figures is the unique point such that a suitable rotation and a dilatation with that point as center transforms one figure into the other.)

(2) Three homologous lines through the vertices of the Brocard's first triangle meet on the Brocard's circle.

(3) If a triangle is formed by three homologous lines its symmedians pass through the vertices of the Brocard's second triangle and meet at a point on the Brocard's circle.

(see [4, pp. 189–204] and [7, pp. 302–312]).

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