REGULAR HEPTAGON'S INTERSECTIONS CIRCLES

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Abstract. This paper describes two interesting circles containing intersections of many lines associated to a regular heptagon. These intersections are vertices of regular heptagons. In the proofs we use the complex numbers and the Maple V software.

1. Introduction

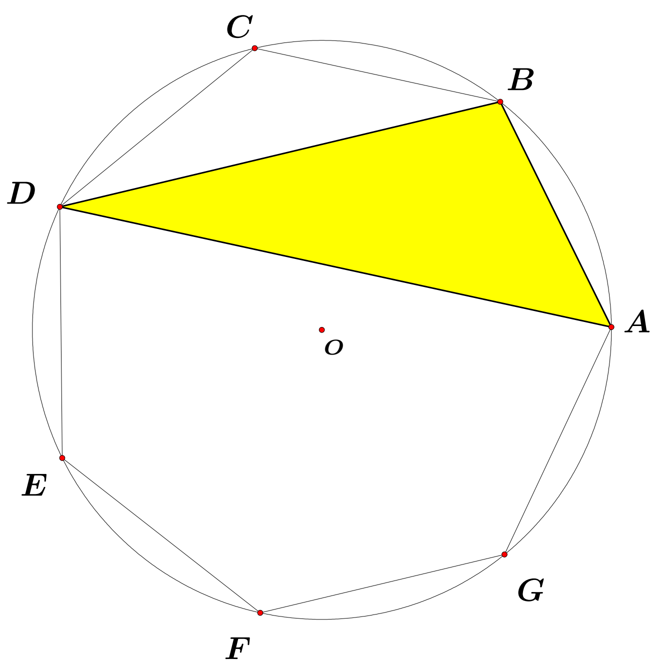

Figure 1: Regular heptagon $ABCDEFG$ and one of its heptagonal triangles ABD.

The regular heptagon (i. e., the planar regular convex polygon with seven vertices) has not been studied extensively like its cousins the

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equilateral triangle, the square, the regular pentagon, and the regular hexagon. Perhaps the reason is because this is the regular polygon with the smallest number of vertices that cannot be constructed only with compass and straightedge. The few sporadic known results on regular heptagons were reviewed by Leon Bankoff and Jack Garfunkel 30 years ago in the reference [1].

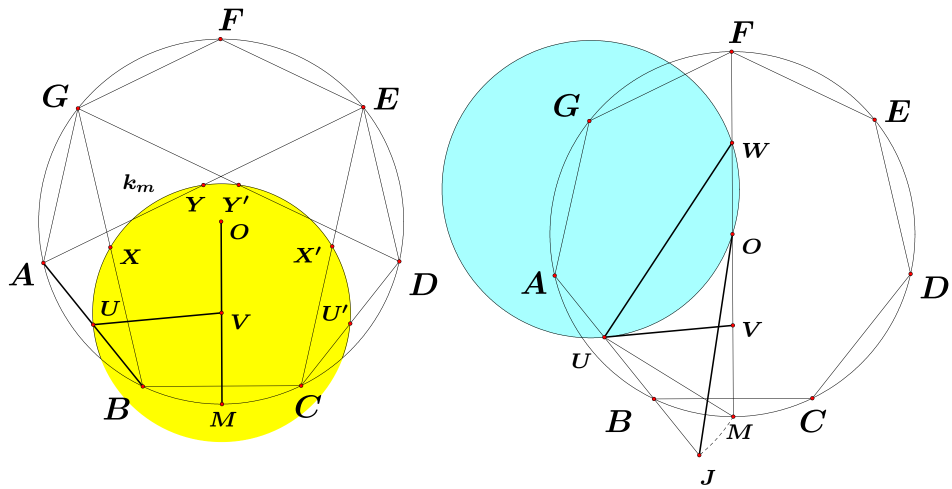

Figure 2: Illustration of two results by Victor Thébault.

They first recall the following result by Victor Thébault:

The distance from the midpoint U of side AB of a regular convex heptagon $ABCDEFG$ inscribed in a circle with center O to the midpoint V of the radius perpendicular to BC and cutting this side, is equal to half the side of a square inscribed in the circle.

In other words, we have $|UV| = \frac{|AO|\sqrt{2}}{2}$. Extending this to diagonals, Hüseyin Demir observed that the circle k_m of radius UV, centered at V, bisects the segments AB, BG, EA, GD, CE and DC in the midpoints U, X, Y, Y', X' and U' (see the left part of Figure 2).

The right part of Figure 2 shows the second result also by Thébault:

If W is the midpoint of OF, M is the point diametrically opposite F and J is the point on UB produced such that $|UJ| = |UM|$, then $|UW| = |UO|\sqrt{2}$, $|OJ| = \frac{|AO|\sqrt{6}}{2}$ and the line UV is tangent to the circle through U, O and W.

The rest of [1] is a study of the heptagonal triangle (for example, the triangle ABD in Figure 1) whose angles are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$ radians. We mention only the following four of their properties from an extensive list (see pages 14, 17 and 19 of [1]):

- *The sum of cotangents of angles is equal to* $\sqrt{7}$.
- *The sum of squares of cotangents of angles is equal to* 5.
- *The triangle formed by joining the feet of the internal angle bisectors of the heptagonal triangle is isosceles.*
- *The two tangents from the orthocenter to the circumcircle of the heptagonal triangle are mutually perpendicular.*

Today we can add new results to the above list with some help from computers. In papers [2], [3] and [4] the author has improved some of the above theorems. We added six more midpoints of segments in Demir observation that also lie on the circle k_m in [2]. Later in [4] we recognized two regular heptagons inscribed in k_m whose vertices are these midpoints. The reference [3] contains the improvement of the second Thébault result above and some new geometric relationships in regular heptagons.

In this paper we show that the intersections of many lines associated to a regular heptagon $ABCDEFG$ lie on its interesting circles determined either by incenters or by the excenters of the triangles DEB and ABG. In other words we discover many regular heptagons related to a given regular heptagon which all have easy construction with compass and straightedge.

Recall that every triangle ABC has the incircle and three excircles which touch the lines BC, CA and AB. Their centers are the incenter I and the excenters I_a, I_b and I_c. The incenter is inside while the excenters are outside the triangle and in the natural order I_a is called the first excenter since it lies on the first angle bisector AI.

In order to simplify our statements we use the following notation. The parallel and the perpendicular to the line ℓ through the point X are $X \parallel \ell$ and $X \perp \ell$.

In our proofs we shall use complex numbers because they provide simple expressions and arguments. There are several excellent books, for example [7], [5], [9], [6], [10], and [8], that give introductions into the method which we utilize. In an appendix we implement this approach in Maple V. The reader can see there how the intersection of two lines is computed. This is in fact the only thing to learn.

A point P in the Gauss plane is identified with a complex number P (its *affix*). The complex conjugate of P is denoted $\bar{P}$. We shall always assume that the complex coordinates of the vertices of the heptagon $ABCDEFG$ are $A = 1,\ B = f^2,\ C = f^4,\ D = f^6,\ E = f^8,\ F = f^{10}$, and $G = f^{12}$, where f is the 14^{th} root of unity. Had we used the 7^{th} roots of unity some important points like the midpoints P and Q of the shorter arcs AG and AB would have complicated affixes. Hence, all these points are on the unit circle k whose center is the origin O.

2. The first circle from incenters

We begin our study with the circle m whose center is the incenter V of the triangle ABG and which goes through the incenter U of the triangle BED.

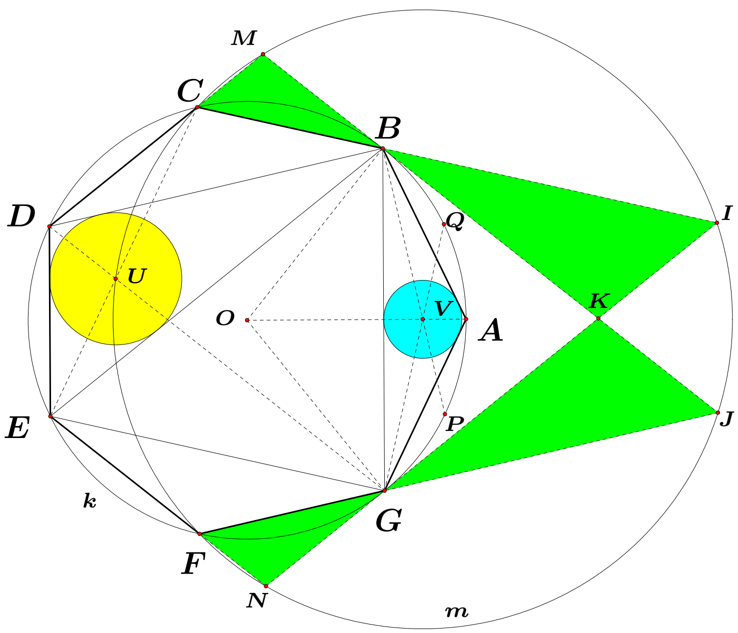

Figure 3: The circle m with the center at the incenter of ABG and the radius $\sqrt{2}$ has interesting properties. (Theorems 1–3).

Theorem 1. *The circle m has the radius $\sqrt{2}$ and it goes through the points C and F.*

Let $K = IN \cap JM$ where the points I, N and J, M are intersections of BC, EF and FG, CD with $G \perp GO$ and $B \perp BO$.

Theorem 2. *The points I, J, M and N are on the circle m and the point V is the midpoint of the segment KO.*

Theorem 3. *The triangles BIK, GJK, BCM, FGN are heptagonal.*

Proof of Theorems 1–3. The points P and Q are f^{13} and f. Note that $|BC| = |CD|$ so that $\sphericalangle BEC = \sphericalangle CED$. It follows that EC is the bisector of the angle E in the triangle BDE. In the same way we see that the line DG is the bisector of the angle D in the triangle BDE and that BP, GQ and AO are the bisectors of the angles B, G and A in the triangle ABG. The incenters U and V are therefore the intersections $CE \cap DG$ and $BP \cap GQ$. Hence, $U = -f^5 + f^4 - 1$ and $V = f^2 - f^3 + f^4 - f^5$. The equation of the circle m with the center V and the radius $\sqrt{2}$ is $(z - V)\overline{(z - V)} = 2$ or

$$z\,\bar{z} + f^2\,(f-1)(f^2+1)(z+\bar{z}) + f^5 - 2\,f^4 + 2\,f^3 - f^2 - 1 = 0.$$

When we substitute the coordinates of the points C, F, and U for z into this equation we obtain an expression that has the polynomial $p_- = f^6 - f^5 + f^4 - f^3 + f^2 - f + 1$ as a factor. Since $f^{14} - 1$ factors as $(f-1)(f+1)\,p_-\,p_+$, with $p_+ = f^6 + f^5 + f^4 + f^3 + f^2 + f + 1$ and $p_+ = 1 + 2\,i\,(1 + 2\,\cos\frac{\pi}{7})\,\sin\frac{2\pi}{7} \neq 0$, we see that $p_- = 0$ so that the points C, F, and U are on the circle m.

In order to find the affix of the point I (the intersection of the line BC with the perpendicular $G \perp GO$ to the line GO in G) notice that BC has the equation $(f^5 - f^3)\,z - (f^4 - f^2)\,\bar{z} + f^5 + f^2 = 0$ while the equation of $G \perp GO$ is $f^2\,z - f^5\,\bar{z} - 2 = 0$. Now we must solve in z and $\bar{z}$ the system formed by these two equations in order to obtain $I = f + f^2 - f^5$. For the points J, M, and N we get similarly $J = \bar{I}$, $M = f^4 - f^3 + f^2 + f - 1$, and $N = \bar{M}$.

Once we know the points I, J, M, and N the rest of the proof is a routine verification. The substitution of their coordinates into the equation of the circle m always contain the factor p_- which is zero. Notice that the lines IN and JM are tangents of the circle k. Finally, solving linear equations we can compute the affix of the intersection $K = 2\,V$ of these tangents. Clearly, the point V is the midpoint of the segment KO. Then we look for conditions (see [5] and the appendix) that the triangles JKG and FNG are directly similar to the heptagonal triangle DEG and that the triangles IKB and CMB are reversely similar to the heptagonal triangle DEG. In all four cases the above factor p_- of $f^{14} - 1$ (which is zero) appears. □

3. Three regular heptagons inscribed in m

In the next two theorems we shall describe three regular heptagons inscribed in the circle m whose easy constructions with compass and straightedge depend on the points I, J, M and N.

Theorem 4. *Let the points H, J', S, U', H', I', S' be intersections of AP, AC, CG, BE, BF, AF, DG with BE, $N \parallel FG$, $K \parallel AG$, $M \parallel CE$, $K \parallel CG$, $K \parallel BF$, $M \parallel CG$, respectively. Then $FUMHIJ'S$ and $NU'CH'I'JS'$ are regular heptagons inscribed in m (see Fig. 4).*

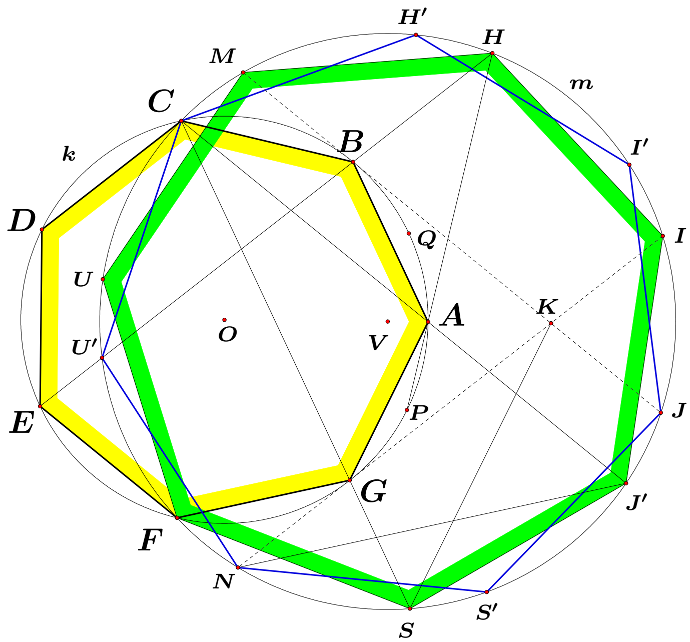

Figure 4: The regular heptagons $FUMHIJ'S$ and $NU'CH'I'JS'$ inscribed in the circle m. (Theorem 4).

The point H lies also on $N \parallel BF$ and U' is the incenter of the triangle DEG.

Theorem 5. *The midpoints B_0, A_0, G_0, F_0, E_0, D_0, and C_0 of the shorter arcs NF, $U'U$, CM, HH', $I'I$, JJ', and SS' are vertices of a regular heptagon whose sides are parallel to the corresponding sides of $BAGFEDC$ (see Fig. 5).*

Proof of Theorems 4 and 5. The equations of the lines AP and BE are

$$(1-f)z + (f^{13}-1)\bar{z} + f - f^{13} = 0$$

and

$$(f^{12}-f^6)z + (f^8-f^2)\bar{z} + f^8 - f^{20} = 0.$$

Their intersection H is $-f^5 + 2\,f^4 - f^3 + 2\,f^2 - f + 1$. Also,

$$J' = -2\,f^5 + 2\,f^4 - 2\,f^3 + f^2 + 1, \quad S = -2\,f^5 + f^4 - 2\,f^3 + 2\,f^2 - f,$$

$$I' = -f^5 + 2\,f^4 - 2\,f^3 + 2\,f^2 + 1, \;\; H' = -f^5 + f^4 + f^2 + f - 1,$$

$U' = -f^3 + f^2 - 1$ and $S' = -f^5 - f^3 + f$. Let us define the number w to be $f - f^2 - f^4$. Then $|S'V|^2 = (S' - V)(\bar{S}' - \bar{V}) = w(1-w)$ is equal to 2. In the same way we verify that $|HV|^2$, $|J'V|^2$, $|SV|^2$, $|U'V|^2$, $|H'V|^2$, and $|I'V|^2$ are also 2 so that the heptagons $FUMHIJ'S$ and

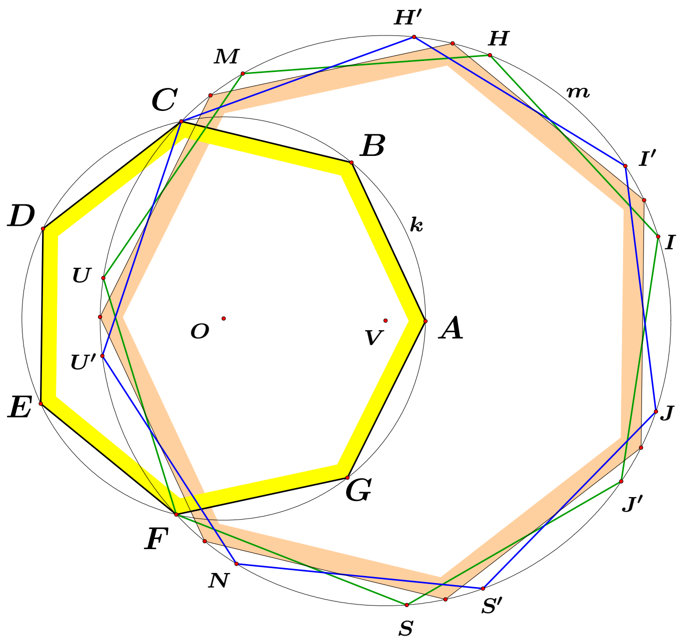

Figure 5: The regular heptagon on midpoints of shorter arcs NF, $U'U$, CM, HH', $I'I$, JJ', and SS' has sides parallel to the corresponding sides of $BAGFEDC$. (Theorem 5).

$NU'CH'I'JS'$ are inscribed in m. That these are regular heptagons follows from the fact that $|FU|^2$, $|UM|^2$, $|MH|^2$, $|HI|^2$, $|IJ'|^2$, $|J'S|^2$, $|SF|^2$, $|NU'|^2$, $|U'C|^2$, $|CH'|^2$, $|H'I'|^2$, $|I'J|^2$, $|JS'|^2$, and $|S'N|^2$ all have the same value $2f^5 - 2f^2 + 4$.

In order to find the midpoint B_0 of the shorter arc FN we use that it has equal distances from the points F and N, that it lies on m, that its distance to the point F is less than $\sqrt{2}$ (the radius of m) and that it is a polynomial of order at most five in f. Hence, $B_0 = -f^5 + f^4 - f^3 + (1 - \sqrt{2})f^2$. Similarly,

$$A_0 = -f^5 + f^4 - f^3 + f^2 - \sqrt{2}, \quad C_0 = -f^5 - (1 - \sqrt{2})f^4 - f^3 + f^2,$$
$$D_0 = (1+\sqrt{2})(1-f)(f^4+f^2+2-\sqrt{2}), \quad E_0 = \overline{D_0}, \quad F_0 = \overline{C_0}, \quad G_0 = \overline{B_0}.$$

It is now easy to check that the regular heptagons $B_0A_0G_0F_0E_0D_0C_0$ and $BAGFEDC$ have parallel corresponding sides. □

4. Four inscribed regular heptagons

In this section we describe four regular heptagons inscribed in the circumcircles of the triangles BIK, GJK, FGN and BCM and show that their centers are vertices of a rectangle.

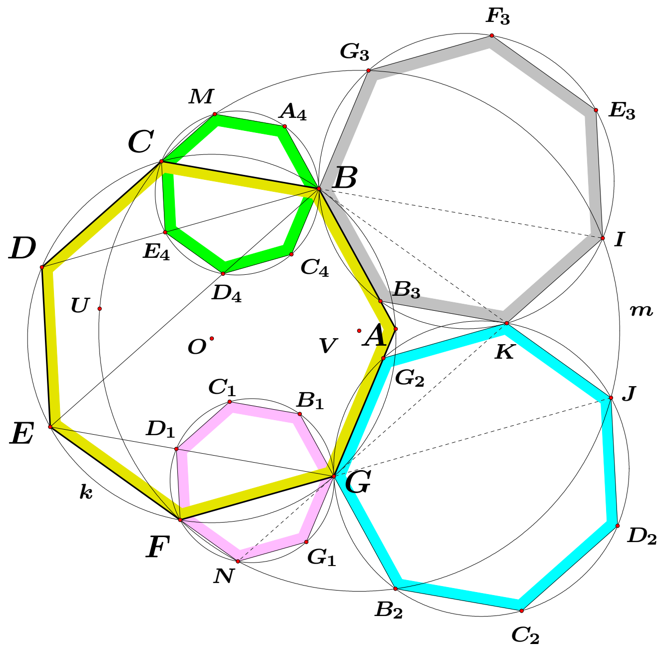

Figure 6: Four regular heptagons inscribed in circumcircles of the triangles FGN, GJK, BIK, BCM. (Theorem 6).

Theorem 6. *Let D_1, C_1, B_1, G_1, G_2, D_2, C_2, B_2, B_3, G_3, F_3, E_3, A_4, E_4, D_4, C_4 be intersections of EG, BF, CG, AG, AG, EG, DG, CG, AB, BF, BE, BD, AB, BD, BE, BF with CF, DG, AF, $F \parallel BC$, $K \parallel FG$, $J \parallel DE$, $J \parallel BF$, $N \parallel BC$, $K \parallel BC$, $I \parallel EF$, $I \parallel AB$, $I \parallel BG$, $C \parallel FG$, CF, CG, AC. Then $NFD_1C_1B_1GG_1$, $GG_2KJD_2C_2B_2$, $IKB_3BG_3F_3E_3$, and $BA_4MCE_4D_4C_4$ are regular heptagons inscribed in the circumcircles of FGN, GJK, BIK, and BCM whose sides are parallel to the corresponding sides of $FEDCBAG$, $AGFEDCB$, $DCBAGFE$, and $BAGFEDC$ (see Fig. 6).*

Proof. The circumcenter O_1 of the triangle FGN is $-f^5 + f^4 - f^3$ and the equation of its circumcircle m_1 is

$$(f^4 + f^2 + 1)z\,\bar{z} - f^4(f^2 + 1)(z + f^8\,\bar{z}) + f^{16} = 0.$$

The points D_1, C_1, B_1, G_1 are $-f^5 + f^4 - f^2$, $-f^5 + f^4 - f^3 + f - 1$, $-f^5 + 2\,f^4 - 2\,f^3 + f^2 - f + 1$, $-f^5 + f^4 - f^3 - f^2 + f$, respectively.

As the expression $f^{2n}(N - O_1) + O_1$, for $n = 1, \ldots, 6$ is G_1, G, B_1, C_1, D_1, and F, we infer that $NFD_1C_1B_1GG_1$ is a regular heptagon inscribed in m_1. That its sides are parallel with the corresponding sides of the heptagon $FEDCBAG$ is now easy to verify. The remaining three circumcircles of the triangles GJK, BIK, and BCM are treated similarly. □

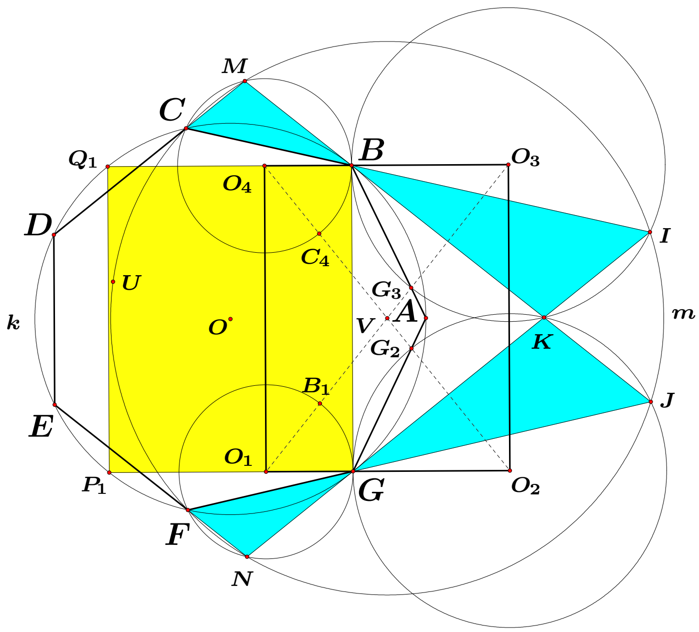

Figure 7: The circumcenters of the triangles FGN, GJK, BIK, BCM are vertices of a rectangle. (Theorem 7).

Theorem 7. *The circumcenters O_1, O_2, O_3, O_4 of the triangles FGN, GJK, BIK, and BCM are vertices of a rectangle – the translation for the vector $\vec{OV}$ of the rectangle P_1GBQ_1 where P_1 and Q_1 are the midpoints of the shorter arcs EF and CD (see Fig. 7).*

Proof. Notice that $P_1 = f^9$, $Q_1 = f^5$, $O_2 = -2\,f^5 + f^4 - f^3 + f^2$, $O_3 = -f^5 + f^4 - f^3 + 2\,f^2$ and $O_4 = f^4 - f^2 + f^2$. The claim follows from $P_1 + V = O_1$, $G + V = O_2$, $B + V = O_3$ and $Q_1 + V = O_4$. □

5. The second circle from excenters

Since the circle m is determined by the incenters of the triangles DEB and ABG, we can ask if the excenters of these triangles give a circle containing intersections of some lines related to the regular heptagon $ABCDEFG$. The answer to this natural question is given in the following theorems.

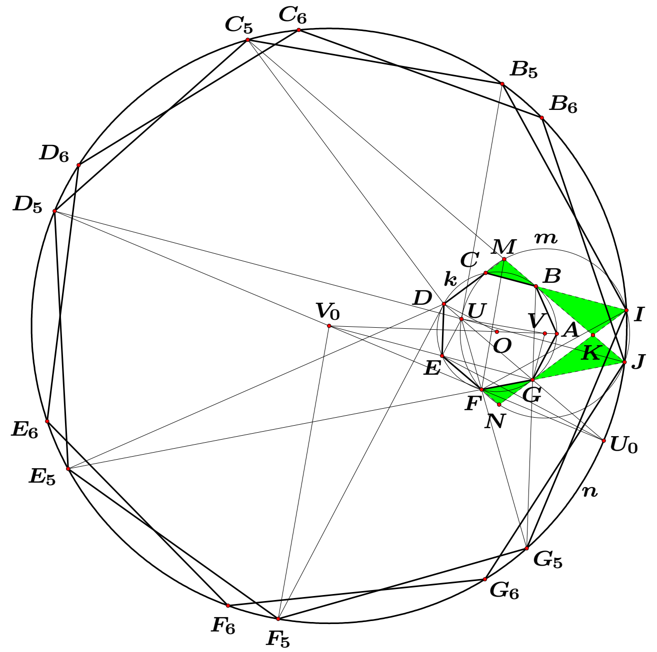

Figure 8: The circle n determined by excenters U_0 and V_0 and two regular heptagons inscribed in it. (Theorems 8 and 9).

Theorem 8. *Let U_0 and V_0 be the first excenters of the triangles DEB and ABG in the regular heptagon $ABCDEFG$ inscribed to the circle k with the center O and the radius R. Then the circle n with the center V_0 and the radius $\frac{R\sqrt{2}\cos\frac{3\pi}{14}}{\sin\frac{\pi}{14}}$ goes through the points U_0, I and J (see Fig. 8).*

Proof. Since the excenter V_0 is the intersection of lines AO and $G \perp GV$ we get $V_0 = f^5 - f^4 + f^3 - f^2 - 2$. Similarly, the excenter U_0 is the intersection of the lines DG and $E \perp EU$ so that $U_0 = -f^5 - f^4 + 1$. The equation of the circle n with the center at the point V_0 through the point U_0 is $z\,\bar{z} - V_0(z + \bar{z}) + 7\,f^5 - 2\,f^4 + 2\,f^3 - 7\,f^2 - 9 = 0$. Its radius is $\sqrt{14 - 10\,f^5 + 4\,f^4 - 4\,f^3 + 10\,f^2}$ which reduces to $\frac{\sqrt{2}\cos\frac{3\pi}{14}}{\sin\frac{\pi}{14}}$. By substitution of coordinates of the points I and J in the above equation we can verify that they lie on the circle n. □

Theorem 9. *Let the points G_5, F_5, E_5, D_5, C_5, B_5, G_6, F_6, E_6, D_6, C_6, B_6 be intersections of BG, $D \perp DO$, FG, JU, JM, $I \perp FI$, $J \perp CJ$, IN, CV_0, JV, $E \perp EO$, BG with FU, $V_0 \parallel FM$, $D \parallel BU$,*

$U_0 \parallel IM$, $D \perp CD$, $U \perp UV$, $E \parallel BP$, $E \perp EF$, $I \parallel AE$, $E \perp CE$, $J \parallel MQ$, $D \perp DO$. Then $IG_5F_5E_5D_5C_5B_5$ and $JG_6F_6E_6D_6C_6B_6$ are regular heptagons inscribed in n (see Fig. 8). The midpoints A_7, G_7, F_7, E_7, D_7, C_7, B_7 of the shorter arcs IJ, G_5G_6, F_5F_6, E_5E_6, D_5D_6, C_5C_6, B_5B_6 are vertices of a regular heptagon whose sides are parallel to the corresponding sides of $AGFEDCB$ (see Fig . 10).

Proof. Solving linear equations we get $G_5 = -2f^5 - 2f^3 + f^2 - 2f + 1$, $F_5 = -3f^4 - 2f^2 - f - 2$, $E_5 = 2f^5 - 2f^4 + f^3 - 4f^2 - 4$, $D_5 = 3f^5 - f^4 + 2f^3 - 2f^2 - 5$, $C_5 = 4f^5 - f^4 + 3f^3 - f^2 + f - 3$, $B_5 = f^5 + 3f^3 + f - 1$, $G_6 = \overline{B_5}$, $F_6 = \overline{C_5}$, $E_6 = \overline{D_5}$, $D_6 = \overline{E_5}$, $C_6 = \overline{F_5}$ and $B_6 = \overline{G_5}$. Since the expressions $f^{2k}(I - V_0) + V_0$ and $f^{2k}(J - V_0) + V_0$ for k from 1 to 6 are B_5, C_5, D_5, E_5, F_5, G_5 and B_6, C_6, D_6, E_6, F_6, G_6 we conclude that $IG_5F_5E_5D_5C_5B_5$ and $JG_6F_6E_6D_6C_6B_6$ are regular heptagons inscribed in n.

Let $\eta = \frac{\sqrt{14}}{7}$. As in the proof of Theorem 5 we find

$$A_7 = (1 - 3\,\eta)\left(f^5 + \frac{1 + 14\,\eta}{11} f^3(f - 1) - f^2 - 2\right),$$

$$B_7 = f^5 - (1 - 2\,\eta)f^4 + (1 + \eta)f^3 - (1 - 5\,\eta)f^2 + \eta\, f - 2(1 - \eta).$$

The condition for the lines AB and A_7B_7 to be parallel (which must be zero) holds because it contains p_- as a factor. Since $A_7B_7C_7D_7E_7F_7G_7$ is obviously a regular heptagon it follows that its sides are parallel with the corresponding sides of $ABCDEFG$. □

Theorem 10. *Let the points C_8, B_8, A_8, G_8, F_8, E_8, D_9, C_9, B_9, A_9, G_9, F_9, E_9 be intersections of AG, BE, EJ, IM, CG, BF, CG, BF, JN, DI, DG, AB, BE with $J \perp JM$, FI, IV_0, $J \parallel CQ$, FU, $E \perp EF$, $J \perp JV$, $D \parallel FO$, $U_0 \parallel BD$, JV_0, CJ, $N \parallel CO$, CV_0. Then $U_0C_8B_8A_8G_8F_8E_8$ and $D_9C_9B_9A_9G_9F_9E_9$ are regular heptagons inscribed in n (see Fig. 9). The midpoints D_{10}, C_{10}, B_{10}, A_{10}, G_{10}, F_{10}, E_{10} of the shorter arcs U_0D_9, C_8C_9, B_8B_9, A_8A_9, G_8G_9, F_8F_9, E_8E_9 are vertices of a regular heptagon whose sides are parallel to the corresponding sides of $DCBAGFE$ (see Fig. 10).*

Proof. From linear equations we get $C_8 = -2f^5 - f^4 - f^3 - f^2 - f - 1$, $B_8 = f^5 - 2f^4 - f^3 - 2f^2 - f - 3$, $A_8 = 3f^5 - 2f^4 + 2f^3 - 3f^2 - f - 4$, $G_8 = 4f^5 - 2f^4 + 4f^3 - 3f^2 + 2f - 5$, $F_8 = 2f^5 + f^4 + 2f^3 + f - 2$, $E_8 = f^3 + 2f^2$, $D_9 = \overline{E_8}$, $C_9 = \overline{F_8}$, $B_9 = \overline{G_8}$, $A_9 = \overline{A_8}$, $G_9 = \overline{B_8}$, $F_9 = \overline{C_8}$ and $E_9 = \overline{U_0}$. Since $f^{2k}(U_0 - V_0) + V_0$ and $f^{2k}(D_9 - V_0) + V_0$ for k from 1 to 6 are E_8, F_8, G_8, A_8, B_8, C_8 and E_9, F_9, G_9, A_9, B_9, C_9 we conclude that $U_0C_8B_8A_8G_8F_8E_8$ and $D_9C_9B_9A_9G_9F_9E_9$ are regular heptagons inscribed in n.

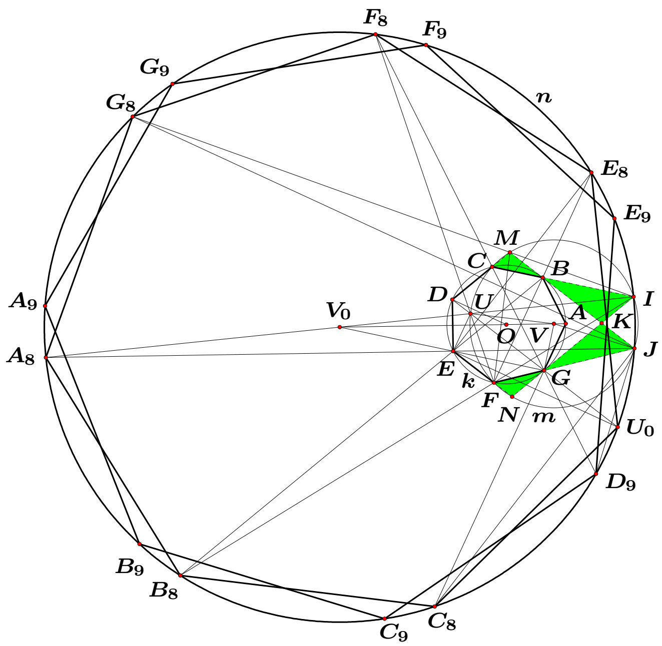

Figure 9: Another two easily constructible regular heptagons inscribed in the circle n. (Theorem 10).

This time we get

$$A_{10} = (1+3\,\eta)\left(f^5 + \frac{1-14\,\eta}{11} f^3(f-1) - f^2 - 2\right),$$

$$B_{10} = f^5 - (1+2\,\eta)f^4 + (1-\eta)f^3 - (1+5\,\eta)f^2 - \eta\, f - 2(1+\eta).$$

The condition for the lines AB and $A_{10}B_{10}$ to be parallel again holds because it contains p_- as a factor. Since $A_{10}B_{10}C_{10}D_{10}E_{10}F_{10}G_{10}$ is obviously a regular heptagon it follows that its sides are parallel with the corresponding sides of $ABCDEFG$. □

Theorem 11. *The points A_7, D_{10}, G_7, C_{10}, F_7, B_{10}, E_7, A_{10}, D_7, G_{10}, C_7, F_{10}, B_7, E_{10} are the vertices of the regular* 14*-gon (see Fig. 10).*

Proof. Since $A_{10} = f(D_7 - V_0) + V_0$, it follows that by rotating D_7 for the angle of $\frac{\pi}{14}$ radians we get A_{10}. This implies the claim of the theorem. Notice that the regular heptagons $A_7B_7C_7D_7E_7F_7G_7$ and $A_{10}B_{10}C_{10}D_{10}E_{10}F_{10}F_{10}$ are symmetric with respect to the perpendicular at O to the line OA. □

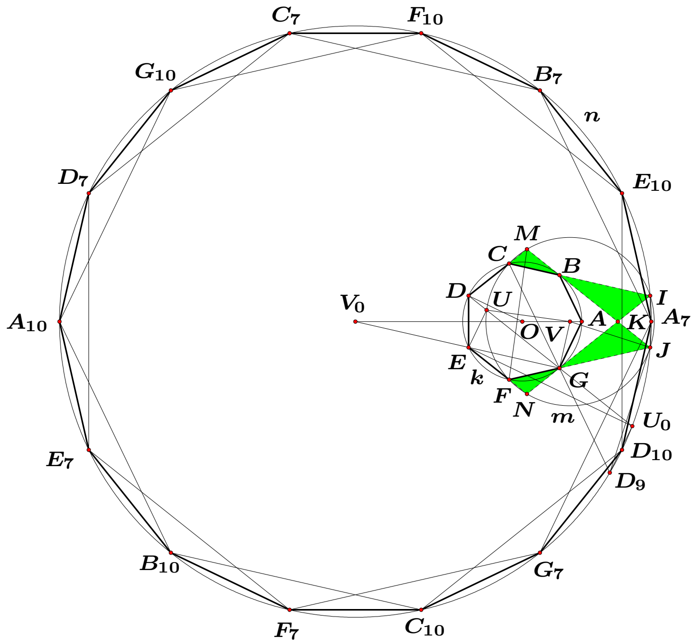

Figure 10: Two regular heptagons on midpoints of shorter arcs inscribed in the circle n and the regular 14-gon from their vertices. (Theorems 8–11).

References

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Appendix

This note is an example of a new approach to geometry offered by computers. In this appendix we will reveal how one can check our results on a computer.

The figures are made in the software The Geometer's Sketchpad that could also be used for approximate verification of statements and in the discovery of new theorems about geometric objects like regular heptagons.

Our mathematically correct proofs where realized on a computer in the software Maple V (version 8). We will describe how to prove Theorems 1–3 in Maple V.

First we give points as ordered pairs $[p,\, q]$ of the complex number p and its conjugate q. The complex number f is the 14^{th} root of unity.

```
hA:=[1,1]:hB:=[f^2,f^12]:hC:=[f^4,f^10]:hD:=[f^6,f^8]:
hF:=[f^8,f^6]:hE:=[f^10,f^4]:hG:=[f^12,f^2]:hO:=[0,0]:
hP:=[f^13,f]:  hQ:=[f,f^13]:
```

Here we use `hA` instead of `A` as a name of the first vertex because with plain letters we run into problems as some letters are reserved in Maple V (for example `D`).

We introduce the shortening `FS` for the simultaneous use of commands `factor` and `simplify` to reduce typing.

```
FS:=x->factor(simplify(x)):
```

The following function computes the square of the distance between two points a and x.

```
di:=(a,x)->FS((a[1]-x[1])*(a[2]-x[2])):
```

Lines are represented as ordered triples $[u,\, v,\, w]$ of coefficients of their equations $u\,z + v\,\bar{z} + w = 0$. The function `li` gives the line through two different points.

```
li:=(a,b)->FS([a[2]-b[2],b[1]-a[1],a[1]*b[2]-a[2]*b[1]]):
```

The function `ins` gives the intersection of two lines. (The names `in` and `int` are reserved!). When its usage results in the error message

```
Error, numeric exception:  division by zero
```

then the lines are parallel (when they do not have an intersection).

```
ins:=(p,q)->FS([(p[2]*q[3]-p[3]*q[2])/(p[1]*q[2]-p[2]*q[1]),
          (p[3]*q[1]-p[1]*q[3])/(p[1]*q[2]-p[2]*q[1])]):
```

This short introduction into the analytic plane geometry via complex numbers concludes with the simple functions for the midpoint of two given points and the parallel and the perpendicular through a given point to a given line.

```
mid:=(a,b)->FS([(a[1]+b[1])/2,(a[2]+b[2])/2]):
par:=(t,p)->FS([p[1],p[2],-t[1]*p[1]-t[2]*p[2]]):
per:=(t,p)->FS([p[1],-p[2],t[2]*p[2]-t[1]*p[1]]):
```

The points U and V are now obtained as follows:

```
hU:=ins(li(hC,hE),li(hD,hG)):hV:=ins(li(hB,hP),li(hG,hQ)):
```

The circle m is the locus of all points whose square of distance to the point V is equal to 2. The following function `hm` associates to a point the difference of the square of its distance from V and 2. A point T will lie on the circle m if and only if the value $hm(T)$ is zero.

```
hm:=x->FS(di(x,hV)-2):
```

We check now the values of hm in the points C, F, and U.

```
hm(hC); hm(hF); hm(hU);
```

The output for the first two inputs is $p_- K$ where K is

$$f^{18} - f^{17} - f^{15} + f^{14} - f^{13} + f^{12} - 2\,f^{11} + \\ f^{10} - f^{9} + 3\,f^{8} + f^{7} - f^{5} + f^{4} - 2\,f - 2$$

while for the third is $\frac{p_- M}{N^2}$ where $N = (f^2 + f + 1)(f^2 - f + 1)$ and

$$M = -2 - 2f - 9f^{19} - 3f^{11} - 4f^{3} - 7f^{5} - 10f^{15} + 11f^{14} - \\ 11f^{17} + 9f^{16} - 3f^{23} - 6f^{21} - 4f^{2} - f^{25} + f^{26} + 9f^{10} - 4f^{9} + \\ 4f^{8} - 5f^{7} - f^{6} + 7f^{18} + 5f^{20} + 4f^{22} + 3f^{24} - 5f^{4} + 13f^{12} - 7f^{13}.$$

Since all of these expressions contain p_- as a factor we infer that they are equal to zero.

The points I, N, J, M, and K are defined as follows.

```
hI:=ins(li(hB,hC),per(hG,li(hG,hO))):
hN:=ins(li(hE,hF),per(hG,li(hG,hO))):
hJ:=ins(li(hF,hG),per(hB,li(hB,hO))):
hM:=ins(li(hC,hD),per(hB,li(hB,hO))):
hK:=ins(li(hI,hN),li(hJ,hM)):
```

We compute the values of hm in the points I, N, J, and M to verify that they lie on the circle m. Next we find the midpoint of the segment KO and show that it is at the distance zero from the point V.

```
hm(hI); hm(hN); hm(hJ); hm(hM); di(hV,mid(hK,kO));
```

For the last claim we will use the following functions that test if two triangles are directly or reversely similar (see [5]).

```
sid:=((a,b,c),(p,q,r))->FS(a[1]*q[1]-a[1]*r[1]-b[1]*p[1]+
                           b[1]*r[1]+c[1]*p[1]-c[1]*q[1]):
sir:=((a,b,c),(p,q,r))->FS(q[2]*a[1]-r[2]*a[1]-p[2]*b[1]+
                           r[2]*b[1]+c[1]*p[2]-q[2]*c[1]):
sid((hD,hE,hG),(hJ,hK,hG)); sid((hD,hE,hG),(hF,hN,hG));
sir((hD,hE,hG),(hI,hK,hB)); sir((hD,hE,hG),(hC,hM,hB));
```

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