CENTRES OF THE GOLDEN RATIO ARCHIMEDEAN TWIN CIRCLES

ZVONKO ČERIN

ABSTRACT. We explore some properties of the geometric configuration when arbelos of the same ratio are constructed on sides of a triangle. The centers of the Archimedean twin circles of these arbelos determine two triangles that are either orthologic or homologic to the base triangle only when the common ratio of arbelos is the number related to the golden ratio. We also consider several triangles associated to the base triangle and build arbelos of the same ratio on their sides and seek when their centers of the Archimedean twin circles give triangles that are either orthologic or homologic to the base triangle. When we construct arbelos on sides of pedal and antipedal triangles of points analogous statements are possible only for points on the Brocard axis and on the Kiepert hyperbola of the base triangle.

1. INTRODUCTION

For points X and Y in the plane and a positive real number s, let Z be the point such that $|XZ| : |ZY| = s$ and let (X, Y, s) be the figure formed by three mutually tangent semicircles O, O_1, and O_2 on segments XY, XZ, and ZY respectively. Let W denote the intersection of O with the perpendicular to XY at the point Z. The figure (X, Y, s) is called the *arbelos* or the *shoemaker's knife*. It has been the subject of intensive research since Greek times when Archimedes noticed the existence of two circles W_1 and W_2 with the same radius such that W_1 touches O, O_1, and ZW while W_2 touches O, O_2, and ZW (see Fig. 2).

The initial idea for this paper is to use centres of the Archimedean twin circles W_1 and W_2 of arbelos on sides of an arbitrary triangle $X_1X_2X_3$ and the notions of orthology and homology for triangles to show the appearance of two significant real numbers $s_1 = \frac{\sqrt{5}+1}{2}$ (the golden ratio) and $s_2 = \frac{\sqrt{5}-1}{2}$.

Recall that triangles $X_1X_2X_3$ and $Y_1Y_2Y_3$ are *orthologic* provided the perpendiculars at vertices of $X_1X_2X_3$ onto sides Y_2Y_3, Y_3Y_1 and Y_1Y_2 of $Y_1Y_2Y_3$ are concurrent. The point of concurrence of these perpendiculars is denoted by $[X_1X_2X_3, Y_1Y_2Y_3]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of $Y_1Y_2Y_3$

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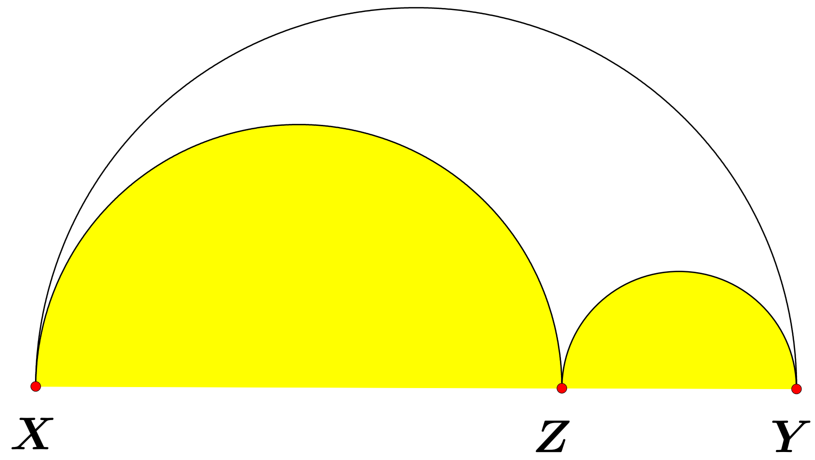

FIGURE 1. The arbelos (X, Y, s), where $s = |XZ|/|ZY|$.

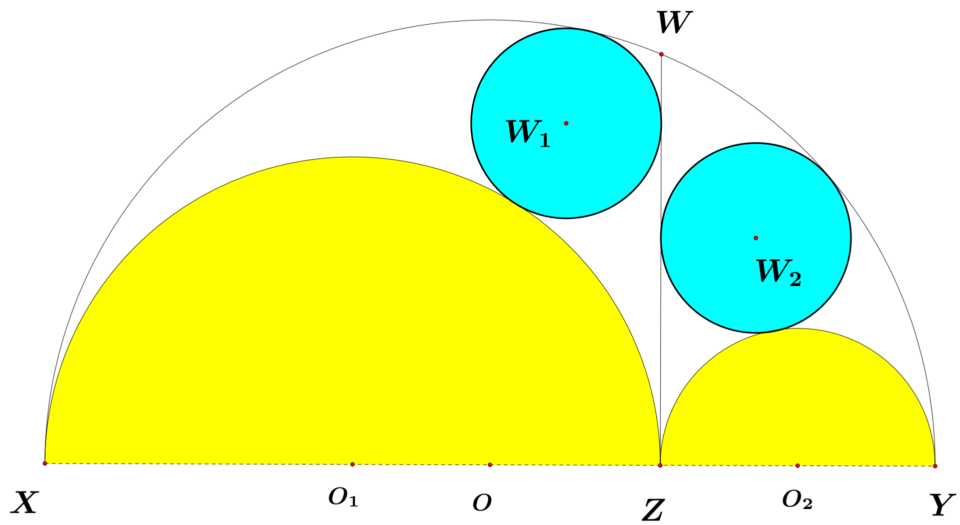

FIGURE 2. The Archimedean circles W_1 and W_2 together.

onto sides X_2X_3, X_3X_1, and X_1X_2 of $X_1X_2X_3$ are concurrent at the point $[Y_1Y_2Y_3, X_1X_2X_3]$.

More precisely, our first goal is to prove (in section 3) the following theorems (see Fig. 3).

Theorem 1. *The triangle $W_{11}W_{12}W_{13}$ on centres $W_1(X_2, X_3, s)$, $W_1(X_3, X_1, s)$, and $W_1(X_1, X_2, s)$ of the Archimedean first twin circle of arbelos on sides of a triangle $X_1X_2X_3$ is orthologic with $X_1X_2X_3$ if and only if s is the golden ratio (i.e., if and only if $s = \frac{\sqrt{5}+1}{2}$).*

Theorem 2. *The triangle $W_{21}W_{22}W_{23}$ on centres $W_2(X_2,\, X_3,\, s)$, $W_2(X_3,\, X_1,\, s)$, and $W_2(X_1,\, X_2,\, s)$ of the Archimedean second twin circles of arbelos on sides of a triangle $X_1X_2X_3$ is orthologic with $X_1X_2X_3$ if and only if $s = \frac{\sqrt{5}-1}{2}$.*

In the sections 4 and 5 we show versions of these results where the relation of orthology is replaced with the relation of homology.

The rest of the paper explores certain triangles $Y_1Y_2Y_3$ associated to the triangle $X_1X_2X_3$ (like its first Brocard triangle, positive and negative Torricelli triangles and positive and negative Napoleon triangles) with the property that building arbelos on their sides lead to analogous conclusions. In the last two sections the triangle $Y_1Y_2Y_3$ is the pedal and the antipedal triangles of carefully chosen points in the plane. In both cases we get a one parameter family of such triangles.

2. Archimedean circles W_1 and W_2

In this section we shall obtain expressions for the coordinates of the points $W_1(X,\, Y,\, s)$ and $W_2(X,\, Y,\, s)$ when the points X and Y are arbitrary points in the plane. Of course, in doing this we reprove the observation by Archimedes that his circles have equal radii.

We use $P(p,\, q)$ or $(p,\, q)$ to denote points by their rectangular coordinates. Let $X(x,\, a)$ and $Y(y,\, b)$. Then $O(\frac{x+y}{2},\, \frac{a+b}{2})$ is the midpoint of the segment XY. Since $\frac{|XZ|}{|ZY|} = s$, the point Z is $\left(\frac{x+s\,y}{s+1},\, \frac{a+s\,b}{s+1}\right)$. Moreover, semicircles O_1 and O_2 have centres at $\left(\frac{(s+2)\,x+s\,y}{2(s+1)},\, \frac{(s+2)\,a+s\,b}{2(s+1)}\right)$ and $\left(\frac{x+(2\,s+1)\,y}{2(s+1)},\, \frac{a+(2\,s+1)\,b}{2(s+1)}\right)$ (the midpoints of segments XZ and ZY).

The intersection $W(p,\, q)$ of the circle O and the perpendicular to XY at Z satisfies the equation

$$p^2 + q^2 - (x+y)\,p - (a+b)\,q + x\,y + a\,b = 0$$

of the circle O and the condition

$$(s+1)\,[(x-y)\,p + (a-b)\,q] - x^2 + (1-s)\,x\,y + s\,y^2 - (a-b)\,(a+b\,s) = 0$$

for the lines XY and ZW to be perpendicular. The solution that makes the triangle XYW negatively oriented is

$$W\left(\frac{x+sy-(a-b)\sqrt{s}}{s+1},\, \frac{a+b\,s+(x-y)\sqrt{s}}{s+1}\right).$$

Our goal now is to prove that besides the circle O_2 there is a unique circle W_1 in the arbelos $(X,\, Y,\, s)$ which touches the line ZW, the circle O_1 from outside, and the circle O from inside.

Let its centre be the point $W_1(p,\, q)$ and the radius a positive real number ϱ. Since W_1 touches ZW, the distance from W_1 to the projection

$$\left(\frac{(s+1)\,B\,(B\,p-A\,q)+A\,(C\,s+D)}{(s+1)\,(A^2+B^2)},\, \frac{(s+1)\,A\,(A\,q-B\,p)+B\,(C\,s+D)}{(s+1)\,(A^2+B^2)}\right)$$

of W_1 onto ZW is equal ϱ, where $A = x - y$, $B = a - b$, $C = y\,A + b\,B$, and $D = x\,A + a\,B$. Hence,

$$(s+1)(A\,p + B\,q) - C\,s - D = \pm\,\varrho\,(s+1)\,\sqrt{A^2+B^2}. \tag{1}$$

Since W_1 touches O_1 from outside, the distance $|W_1O_1|$ is equal to the sum

$$\varrho + \frac{s\,\sqrt{A^2+B^2}}{2\,(s+1)}$$

of their radii. It follows that

$$\left(p - \frac{(s+2)\,x + s\,y}{2(s+1)}\right)^2 + \left(q - \frac{(s+2)\,a + s\,b}{2(s+1)}\right)^2 = \left(\varrho + \frac{s\,\sqrt{A^2+B^2}}{2\,(s+1)}\right)^2. \tag{2}$$

Finally, since W_1 touches O from inside, the distance $|W_1O|$ is equal to the difference $\frac{1}{2}\sqrt{A^2+B^2} - \varrho$ of their radii. This condition leads to the relation

$$4\left(p - \frac{x+y}{2}\right)^2 + 4\left(q - \frac{a+b}{2}\right)^2 = (\sqrt{A^2+B^2} - 2\,\varrho)^2. \tag{3}$$

From (1) we can express ϱ and substitute these values into (2) and (3) and solve for p and q. We obtain

$$p = \frac{(2+3\,s)\,x + s\,(1+2\,s)\,y - 2\,s\,\sqrt{s+1}\,(a-b)}{2\,(s+1)^2},$$

$$q = \frac{(2+3\,s)\,a + s\,(1+2\,s)\,b + 2\,s\,\sqrt{s+1}\,(x-y)}{2\,(s+1)^2},$$

and

$$\varrho = \frac{s\,\sqrt{(x-y)^2+(a-b)^2}}{2\,(s+1)^2}.$$

We can now repeat the above argument to show that besides the circle O_1 there is a unique circle W_2 which touches the line ZW, the circle O_2 from outside, and the circle O from inside. The surprise (see Fig. 2) is that W_2 also has the above number ϱ for radius while its centre W_2 has coordinates

$$\frac{(2+s)\,x + s\,(3+2\,s)\,y - 2\,\sqrt{s\,(s+1)}\,(a-b)}{2\,(s+1)^2},$$

$$\frac{(2+s)\,a + s\,(3+2\,s)\,b + 2\,\sqrt{s\,(s+1)}\,(x-y)}{2\,(s+1)^2}.$$

Recall that the condition for the triangle with vertices $X_1(x_1, a_1)$, $Y_1(y_1, b_1)$, and $Z_1(z_1, c_1)$ to be orthologic to the triangle with vertices $X_2(x_2, a_2)$, $Y_2(y_2, b_2)$, and $Z_2(z_2, c_2)$ is (see [1])

$$\begin{aligned}(y_2 - z_2)\,x_1 + (z_2 - x_2)\,y_1 + (x_2 - y_2)\,z_1 &+ \\ (b_2 - c_2)\,a_1 + (c_2 - a_2)\,b_1 + (a_2 - b_2)\,c_1 &= 0.\end{aligned} \tag{4}$$

3. Proof of Theorems 1 and 2

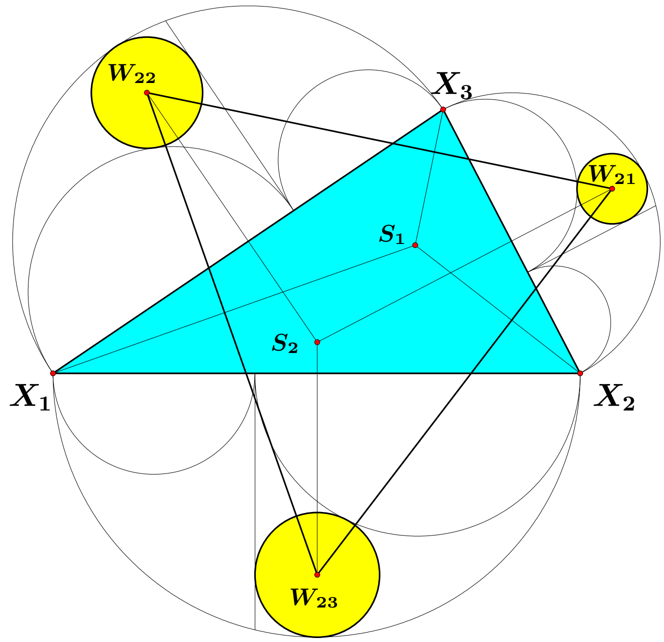

Figure 3. The triangle $W_{21}W_{22}W_{23}$ is orthologic to $X_1X_2X_3$ when $s = \frac{\sqrt{5}-1}{2}$.

Without loss of generality we can assume that $X_i(c_i,\, d_i)$, with $c_i = \cos x_i$ and $d_i = \sin x_i$ (for $i = 1,\, 2,\, 3$). Then $W_{11}\left(\frac{A\,c_2+B\,c_3-2\,C\,(d_2-d_3)}{2\,(s+1)^2},\, \frac{A\,d_2+B\,d_3+2\,C\,(c_2-c_3)}{2\,(s+1)^2}\right)$, where $A = 2+3\,s$, $B = s\,(1+2\,s)$, and $C = s\,\sqrt{s+1}$. The points W_{12} and W_{13} have similar coordinates with $x_3,\, x_1$ and $x_1,\, x_2$ replacing $x_2,\, x_3$ above. The orthology condition (4) for triangles $X_1X_2X_3$ and $W_{11}W_{12}W_{13}$ is

$$\frac{(s^2-s-1)(3-\cos(x_2-x_3)-\cos(x_3-x_1)-\cos(x_1-x_2))}{(s+1)^2} = 0.$$

Clearly, for a non-degenerate triangle $X_1X_2X_3$ and a positive real number s this condition holds if and only if $s = \frac{1+\sqrt{5}}{2}$.

The proof of Theorem 2 is similar. The orthology condition for triangles $X_1X_2X_3$ and $W_{21}W_{22}W_{23}$ is

$$\frac{(s^2+s-1)(3-\cos(x_2-x_3)-\cos(x_3-x_1)-\cos(x_1-x_2))}{(s+1)^2} = 0.$$

4. Homology of triangles on Archimedean centers

In this section we shall see that the same results hold for the relation of homology of triangles. Recall that triangles $X_1X_2X_3$ and $Y_1Y_2Y_3$ are homologic provided the lines X_1Y_1, X_2Y_2, and X_3Y_3 are concurrent. In stead of homologic many authors use the term *perspective*.

Theorem 3. *The triangle $W_{11}W_{12}W_{13}$ on centres $W_1(X_2,\, X_3,\, s)$, $W_1(X_3,\, X_1,\, s)$, and $W_1(X_1,\, X_2,\, s)$ of the Archimedean first twin circle of arbelos on sides of a triangle $X_1X_2X_3$ is homologic with $X_1X_2X_3$ if and only if $s = \frac{\sqrt{5}+1}{2}$.*

5. Proof of Theorem 3

We shall position the triangle $X_1X_2X_3$ in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex X_1 is the origin with coordinates $(0,\, 0)$, the vertex X_2 is on the x-axis and has coordinates $(r\,h,\, 0)$, and the vertex X_3 has coordinates $(f_u\,g\,r/k,\, 2\,f\,g\,r/k)$, where $h = f + g$, $k = f\,g - 1$, $f_v = f^2 + 1$, $f_u = f^2 - 1$, $g_v = g^2 + 1$, $g_u = g^2 - 1$, $f_w = f^4 + 1$, and $g_w = g^4 + 1$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices X_1 and X_2.

Nice features of this placement are that most central points (like the incenter, the centroid, the circumcenter, the orthocenter, the center of the nine-point circle, the symmedian point, etc.) from Table 1 in [3] or [4] have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to the side lengths a, b, and c and back with substitutions $c = r\,h$, $a = r\,f\,g_v/k$, $b = r\,f_v\,g/k$, and

$$f = \frac{(b+c)^2 - a^2}{4\,S}, \quad g = \frac{(a+c)^2 - b^2}{4\,S}, \quad r = \frac{2\,S}{a+b+c},$$

where $S = \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a-b+c)(a+b-c)}$ is the area.

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given trilinears (i. e. with the three real numbers proportional to the distances of the point to the sidelines of the base triangle – see [3]). More precisely, if a point P with coordinates x and y has projections P_a, P_b, and P_c onto the side lines X_2X_3, X_3X_1, and X_1X_2 and $\lambda = |PP_a|/|PP_b|$ and $\mu = |PP_b|/|PP_c|$, then $x = \frac{u}{w}$ and $y = \frac{v}{w}$ with $u = g\,h\,(f_v\,\mu + f_u)\,r$, $v = 2\,f\,g\,h\,r$, and $w = f\,g_v\,\lambda\,\mu + g\,f_v\,\mu + h\,k$.

These formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first trilinear coordinate. For example, we write $X_6[a]$ to indicate that the symmedian point X_6 (i. e., the intersection of symmedians – the reflections of medians in the interior angle bisectors) has trilinears equal to $a : b : c$. Then we use the above formulas with $\lambda = a/b$ and $\mu = b/c$ to get the coordinates $x = \frac{u}{2\,w}$ and $y = \frac{v}{w}$ of X_6 in our coordinate system, where $u = (f\,f_u\,g_u + 2\,g\,f_w)\,g\,h\,r$, $v = f\,g\,h^2\,k\,r$, and $w = f^2\,g_w + f\,g\,f_u\,g_u + g^2\,f_w$.

Let $T = \sqrt{s+1}$. By applying the above formula for $W_1(X,\, Y,\, s)$ we obtain

$$W_{11}\left(\frac{r\,A}{2\,k\,T^4},\, \frac{f\,r\,s\,B}{k\,T^4}\right),\; W_{12}\left(\frac{g\,r\,C}{2\,k\,T^4},\, \frac{g\,r\,D}{k\,T^4}\right),\; W_{13}\left(\frac{h\,r\,s\,(1+2\,s)}{2\,T^4},\, -\frac{h\,r\,s}{T^3}\right),$$

with $A = 2\,g\,f_u\,s^2 + (4\,g\,f_u + 3\,f g_u + 4\,f g\,T)\,s + 2\,h\,k$, $B = g\,(1+2\,s) + g_u T$, $C = (2+3\,s)\,f_u - 4\,f s\,T$ and $D = (2+3\,s)\,f + f_u\,s\,T$. The lines X_1W_{11}, X_2W_{12}, and X_3W_{13} have equations

$$2\,f s\,B\,x - A\,y = 0, \tag{5}$$

$$2\,g\,D\,(x - h\,r) + (2\,h\,k\,s^2 + E\,s + 2\,f g_u)\,y = 0, \tag{6}$$

$$2\,T^2\,F\,x + T\,G\,y - 2\,g\,h\,r\,H = 0, \tag{7}$$

with $H = f_u\,s^2 + (f_u - 3\,fT)\,s - 2\,fT$, $G = 2\,g\,f_u\,s^2 + (g\,f_u - 3\,f g_u)\,s - 2\,f g_u$, $F = h\,k\,s - 2\,f g\,T^3$ and $E = 4\,f g\,T + f_u\,g + 4\,f g_u$.

The intersection of lines X_1W_{11} and X_2W_{12} is at the point

$$\left(\frac{g\,h\,r\,A\,D}{K},\, \frac{2\,f g\,h\,r\,s\,B\,D}{K}\right),$$

with $K = 2\,s\,T^5(g^2 f_w + f g\,f_u\,g_u + f^2 g_w) + f g\,h\,k\,(4\,s^4 + 14\,s^3 + 21\,s^2 + 14\,s + 4)$. This point will be on the line X_3W_{13} (i. e., the lines X_1W_{11}, X_2W_{12}, and X_3W_{13} are concurrent and triangles $X_1X_2X_3$ and $W_{11}W_{12}W_{13}$ are homologic) if and only if $\frac{2\,f g\,h\,r^2\,(1+s-s^2)}{k\,T^4} = 0$. Hence, it follows that this will happen if and only if $s = s_1$ because we are interested only in positive values of s.

In a similar fashion one can also prove the following theorem (see Fig. 4).

Theorem 4. *The triangle $W_{21}W_{22}W_{23}$ on centres $W_2(X_2,\, X_3,\, s)$, $W_2(X_3,\, X_1,\, s)$, and $W_2(X_1,\, X_2,\, s)$ of the Archimedean second twin circle of arbelos on sides of a triangle $X_1X_2X_3$ is homologic with $X_1X_2X_3$ if and only if $s = \frac{\sqrt{5}-1}{2}$.*

It is easy to check that the centre $W_1(X,\, Y,\, s)$ of the first Archimedean twin circle of the arbelos $(X,\, Y,\, s)$ lies on the perpendicular bisector of the segment XY if and only if $s = s_1$. Similarly the centre $W_2(X,\, Y,\, s)$ of the second Archimedean twin circle of the arbelos $(X,\, Y,\, s)$ lies on the perpendicular bisector of the segment XY if and only if $s = s_2$. Moreover, points $W_1(X,\, Y,\, s_1)$ and $W_2(X,\, Y,\, s_2)$ are identical. This implies that one implication in Theorems 2 and 4 follow from Theorems 1 and 3, respectively.

6. Arbelos on various triangles

In this section we shall look for triangles $\Gamma = Y_1Y_2Y_3$ other than $\Delta = X_1X_2X_3$ such that the triangle formed by centres of Archimedean twin circles of arbelos with the ratio $s > 0$ on sides of Γ is orthologic to Δ if and only if either $s = s_1$ or $s = s_2$. Let Φ_j^s and Ψ_j^s denote $W_j(X_2,\, X_3,\, s)W_j(X_3,\, X_1,\, s)W_j(X_1,\, X_2,\, s)$ and $W_j(Y_2,\, Y_3,\, s)W_j(Y_3,\, Y_1,\, s)W_j(Y_1,\, Y_2,\, s)$ for $j \in \{1,\, 2\}$.

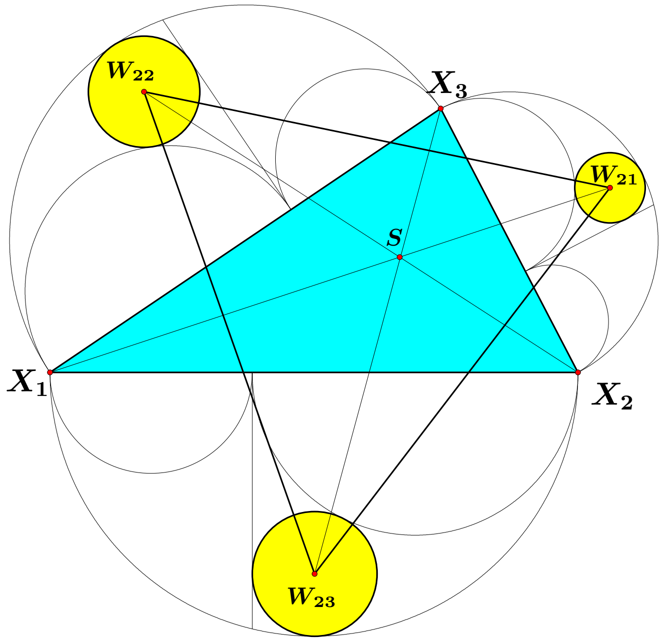

FIGURE 4. The triangle $W_{21}W_{22}W_{23}$ is homologic to $X_1X_2X_3$ when $s = \frac{\sqrt{5}-1}{2}$.

Theorem 5. *If triangles Γ and Δ are homothetic, then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

Proof for $j = 1$. Since the triangles Γ and Δ are homothetic there is a point P and a real number $m \neq -1$ such that the vertices of Γ divide the segments X_1P, X_2P, X_3P in the ratio m. Let us assume that the vertices of Δ are selected as in the proof of Theorem 3 and that P has the coordinates (p, q). The vertices of Γ have the coordinates $\left[\frac{mp}{m+1}, \frac{mq}{m+1}\right]$, $\left[\frac{hr+m}{m+1}, \frac{mq}{m+1}\right]$, and $\left[\frac{f_u g r + k m p}{k(m+1)}, \frac{2fgr+kmq}{k(m+1)}\right]$. The triangles Ψ and Δ are orthologic if and only if

$$\frac{(f^2 g_w + fg\, f_u g_u + g^2 f_w)(1 + s - s^2)}{k^2(m+1)(s+1)^2} = 0.$$

The first parenthesis in the numerator of the left hand side is clearly always positive and this implies that the theorem holds. □

Recall that the first Brocard triangle of Δ has as vertices the projections of its symmedian point K onto the perpendicular bisectors of sides.

Theorem 6. *Let Γ be the first Brocard triangle of a scalene triangle Δ. Then Φ_j^s is homologic with Γ and Ψ_j^s is homologic with Δ. Moreover, Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

Proof for $j = 1$. Let us again assume that the vertices of Δ are selected as in the proof of Theorem 3. The vertices of Γ have the trilinear coordinates $abc : c^3 : b^3$, $c^3 : abc : a^3$, and $b^3 : a^3 : abc$, where a, b, c denote the lengths of sides of Δ.

For the part about the homology the straightforward proof amounts to computing coordinates of all vertices and checking that the lines joining corresponding vertices are concurrent perhaps with the help of a computer because rather complicated expressions appear.

The triangles Ψ and Δ are orthologic if and only if

$$\frac{(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)(1+s-s^2)}{(a^2+b^2+c^2)(s+1)^2} = 0.$$

The first parenthesis in the numerator of the left hand side is equal to

$$\frac{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}{2}$$

so that it is always positive except in the case when $a = b = c$. This implies that the theorem holds. □

In a similar way it is possible to prove also the following theorems. We shall use the Torricelli and Napoleon triangles of a given triangle that have simple constructions related to equilateral triangles built on its sides. The history of these triangles is long and rich with many beautiful results.

Let A_u, B_u, and C_u be vertices of equilateral triangles built on sides Y_2Y_3, Y_3Y_1, and Y_1Y_2 of $Y_1Y_2Y_3$ towards inside. When they are built towards outside then their vertices are denoted A_v, B_v, and C_v. The negative Torricelli triangle is $A_uB_uC_u$ while $A_vB_vC_v$ is the positive Torricelli triangle of $Y_1Y_2Y_3$. The centers A_x, B_x, C_x of the triangles $Y_2Y_3A_u$, $Y_3Y_1B_u$, $Y_1Y_2C_u$ are the vertices of the negative Napoleon (equilateral) triangle while the centers A_y, B_y, C_y of the triangles $Y_2Y_3A_v$, $Y_3Y_1B_v$, $Y_1Y_2C_v$ are the vertices of the positive Napoleon (equilateral) triangle.

Theorem 7. *Let the triangle Δ satisfy $a^2+b^2+c^2 \neq 12\sqrt{3}\,S$. Let Γ be the negative Torricelli triangle of Δ. Then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

Theorem 8. *Let Γ be the positive Torricelli triangle of a scalene triangle Δ. Then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

Theorem 9. *Let Γ be the negative Napoleon triangle of a scalene triangle Δ. Then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

Theorem 10. *Let Γ be the positive Napoleon triangle of a triangle Δ. Then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

The proofs of Theorems 7 – 10 follow the method of the proof of Theorem 6. From the trilinear coordinates we compute the rectangular coordinates of the vertices and then apply the criterion (4) for the orthology relation. The transfer of this criterion into the expression in side lengths and factoring out gives defining

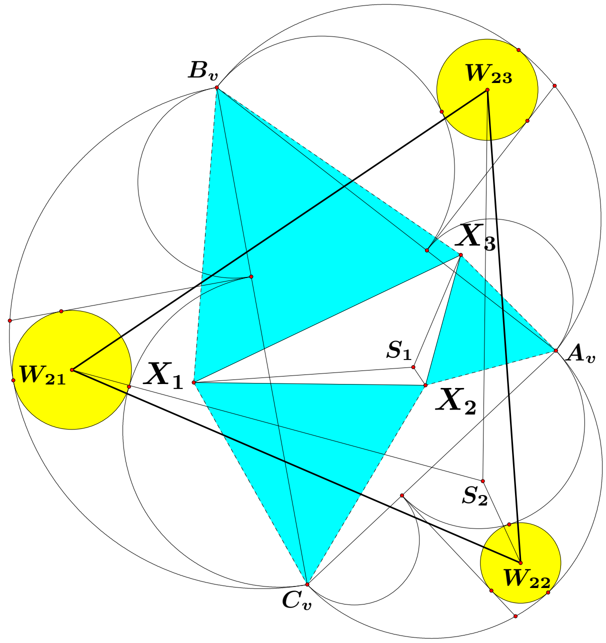

FIGURE 5. The triangle $W_{21}W_{22}W_{23}$ on the positive Torricelli triangle $A_vB_vC_v$ is orthologic to $X_1X_2X_3$ if and only if $s = \frac{\sqrt{5}-1}{2}$.

quadratic equations for either s_1 or s_2 and the factor which is zero only when the exception from the statement holds.

7. Arbelos on pedal triangles

The pedal triangle of a point P with respect to the triangle Δ has as vertices the orthogonal projections of P onto the sidelines of Δ.

Let Q denote the central point of the triangle Δ with the trilinear coordinates $a(b^2+c^2-2a^2) : b(c^2+a^2-2b^2) : c(a^2+b^2-2c^2)$.

Theorem 11. *Let Γ be the pedal triangle of any point $P \neq Q$ on the line joining the circumcenter and the symmedian point of the triangle Δ. Then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

Proof for $j = 1$ and Ψ. Let the point P has the trilinear coordinates $x_0 : y_0 : z_0$. The trilinear coordinates of the orthogonal projection of P onto the sideline X_2X_3 are $0 : 2y_0 + \frac{a^2+b^2-c^2}{ab}\, x_0 : 2z_0 + \frac{c^2+a^2-b^2}{ca}\, x_0$. It follows that Ψ_j^s is orthologic with Δ if and only if

$$2S[bc\, x_0 + ca\, y_0 + ab\, z_0](s^2 - s - 1)\sqrt{s+1} \\ - [bc(b^2-c^2)\, x_0 + ca(c^2-a^2)\, y_0 + ab(a^2-b^2)\, z_0]s(s+1) = 0.$$

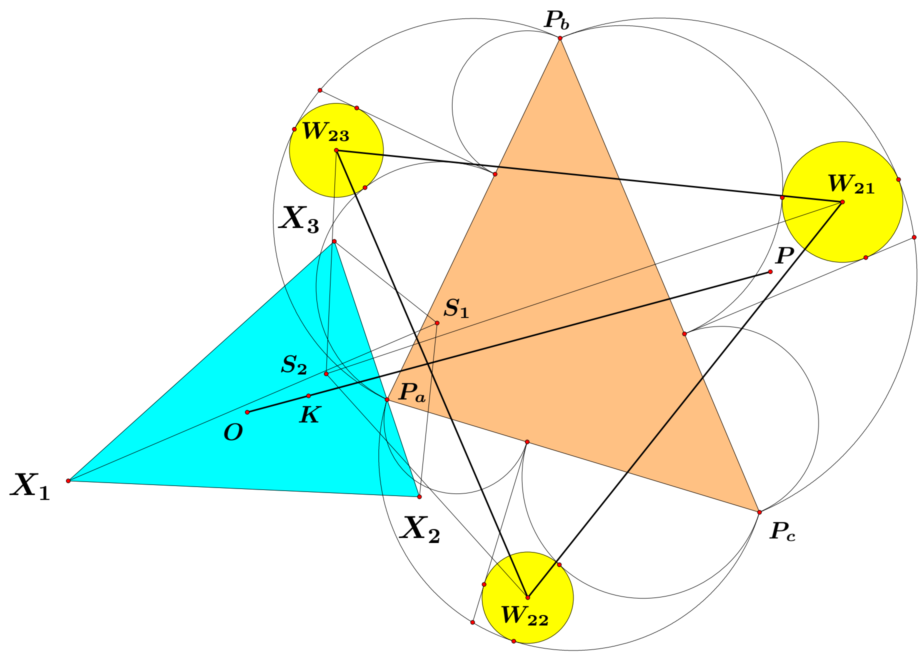

FIGURE 6. The triangle $W_{21}W_{22}W_{23}$ on the pedal triangle $P_aP_bP_c$ of a point from $OK \setminus \{Q\}$ is orthologic to $X_1X_2X_3$ if and only if $s = \frac{\sqrt{5}-1}{2}$.

The second square parenthesis is the equation of the line OK joining the circumcenter O with the symmedian point K while the first is the equation of a line perpendicular to it through the point Q (their intersection). This completes the proof. □

8. Arbelos on antipedal triangles

The antipedal triangle of a point P with respect to the triangle Δ has as vertices the intersections of the perpendiculars in the vertices of Δ to the lines PX_1, PX_2, PX_3.

Let Q^* denote the isogonal conjugate of the central point Q of the triangle Δ. Its trilinear coordinates are $\frac{1}{a(b^2+c^2-2a^2)} : \frac{1}{b(c^2+a^2-2b^2)} : \frac{1}{c(a^2+b^2-2c^2)}$.

Recall that the Kiepert hyperbola of a triangle Δ is a unique (equilateral) hyperbola that goes through its vertices, the orthocenter (the intersection of the altitudes), and the centroid (the intersection of the medians).

Theorem 12. *Let Γ be the antipedal triangle of any point $P \neq X_1$, X_2, X_3, Q^* on the Kiepert hyperbola of the triangle Δ. Then Φ_j^s is orthologic with Γ and/or Ψ_j^s is orthologic with Δ if and only if $s = s_j$.*

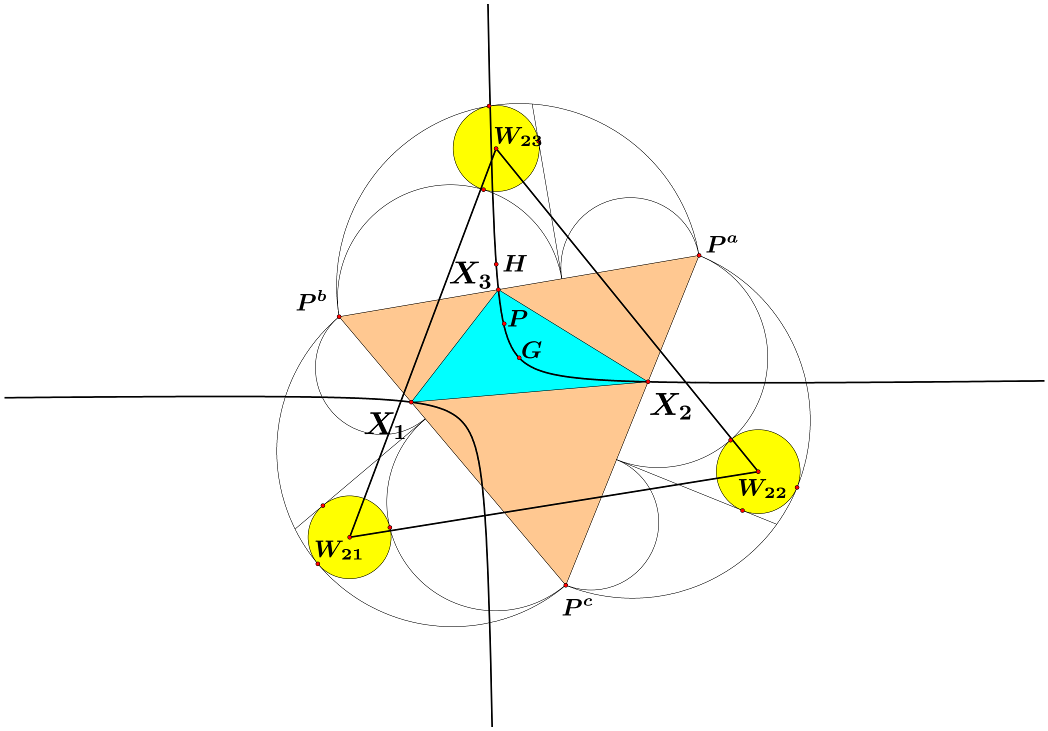

FIGURE 7. The triangle $W_{21}W_{22}W_{23}$ on the antipedal triangle $P^aP^bP^c$ of a point $P \neq Q^*$ on the Kiepert hyperbola of Δ is orthologic to $X_1X_2X_3$ if and only if $s = \frac{\sqrt{5}-1}{2}$.

Proof for $j = 1$ and Ψ. Let the point P has the trilinear coordinates $x_0 : y_0 : z_0$. The trilinear coordinates of the first vertex P^a of the antipedal triangle are

$$-[(c^2+a^2-b^2)\,x_0+2\,c\,a\,z_0][(a^2+b^2-c^2)\,x_0+2\,a\,b\,y_0] : \\ [(c^2+a^2-b^2)\,x_0+2\,c\,a\,z_0][(a^2+b^2-c^2)\,y_0+2\,a\,b\,x_0] : \\ [(a^2+b^2-c^2)\,x_0+2\,a\,b\,z_0][(c^2+a^2-b^2)\,x_0+2\,c\,a\,x_0].$$

It follows that Ψ_j^s is orthologic with Δ if and only if

$$2S[bc\,y_0\,z_0+ca\,z_0\,x_0+ab\,x_0\,y_0](s^2-s-1)\sqrt{s+1} \\ -\,[bc(b^2-c^2)\,y_0\,z_0+ca(c^2-a^2)\,z_0\,x_0+ab(a^2-b^2)\,x_0\,y_0]s(s+1)=0.$$

The second square parenthesis is the equation of the Kiepert hyperbola of Δ while the first is the equation of the Steiner ellipse of Δ. These conics intersect in the vertices of Δ and in the point Q^*. This completes the proof. $\square$

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