# PROBLEMS IN PLANE AND SOLID GEOMETRY
## v.1 Plane Geometry

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ABSTRACT. This book has no equal. The priceless treasures of elementary geometry are nowhere else exposed in so complete and at the same time transparent form. The short solutions take barely $1.5 - 2$ times more space than the formulations, while still remaining *complete*, with no gaps whatsoever, although many of the problems are quite difficult. Only this enabled the author to squeeze about 2000 problems on plane geometry in the book of volume of ca 600 pages thus embracing practically all the known problems and theorems of elementary geometry.

The book contains non-standard geometric problems of a level higher than that of the problems usually offered at high school. The collection consists of two parts. It is based on three Russian editions of Prasolov's books on plane geometry.

The text is considerably modified for the English edition. Many new problems are added and detailed structuring in accordance with the methods of solution is adopted.

The book is addressed to high school students, teachers of mathematics, mathematical clubs, and college students.

# Contents

## Editor's preface

The enormous number of problems and theorems of elementary geometry was considered too wide to grasp in full even in the last century. Even nowadays the stream of new problems is still wide. (The majority of these problems, however, are either well-forgotten old ones or those recently pirated from a neighbouring country.)

Any attempt to collect an encyclopedia of all the problems seems to be doomed to failure for many reasons.

First of all, this is an impossible task because of the huge number of the problems, an enormity too vast to grasp. Second, even if this might have been possible, the book would be terribly overloaded, and therefore of no interest to anybody.

However, in the book *Problems in plane geometry* followed by *Problems in solid geometry* this task is successfully perfomed.

In the process of writing the book the author used the books and magazines published in the last century as well as modern ones. The reader can judge the completeness of the book by, for instance, the fact that *American Mathematical Monthly* yearly[1] publishes, as "new", 1–2 problems already published in the Russian editions of this book.

The book turned out to be of interest to a vast audience: about 400 000 copies of the first edition of each of the Parts (Parts 1 and 2 — Plane and Part 3 — Solid) were sold; the second edition, published 5 years later, had an even larger circulation, the total over 1 000 000 copies. The 3rd edition of *Problems in Plane Geometry* was issued in 1996 and the latest one in 2001.

The readers' interest is partly occasioned by a well-thought classification system.

The collection consists of three parts.

Part 1 covers classical subjects of plane geometry. It contains nearly 1000 problems with complete solutions and over 100 problems to be solved on one's own. Still more will be added for the English version of the book.

Part 2 includes more recent topics, geometric transformations and problems more suitable for contests and for use in mathematical clubs. The problems cover cuttings, colorings, the pigeonhole (or Dirichlet's) principle, induction, and so on.

Part 3 is devoted to solid geometry.

A rather detailed table of contents serves as a guide in the sea of geometric problems. It helps the experts to easily find what they need while the uninitiated can quickly learn what exactly is that they are interested in in geometry. Splitting the book into small sections (5 to 10 problems in each) made the book of interest to the readers of various levels.

FOR THE ENGLISH VERSION of the book about 150 new problems are already added and several hundred more of elementary and intermideate level problems will be added to make the number of more elementary problems sufficient to use the book in the ordinary school: the Russian editions are best suited for coaching for a mathematical Olympiad than for a regular class work: the level of difficulty increases rather fast.

Problems in each section are ordered difficulty-wise. The first problems of the sections are simple; they are a match for many. Here are some examples:

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[1]Here are a few samples: v. 96, n. 5, 1989, p. 429–431 (here the main idea of the solution is the right illustration — precisely the picture from the back cover of the 1st Russian edition of *Problems in Solid Geometry*, Fig. to Problem 13.22); v. 96, n. 6, p. 527, Probl. E3192 corresponds to Problems 5.31 and 18.20 of *Problems in Plane Geometry* — with their two absolutely different solutions, the one to Problem 5.31, unknown to AMM, is even more interesting.

**Plane 1.1**. The bases of a trapezoid are $a$ and $b$. Find the length of the segment that the diagonals of the trapezoid intersept on the trapezoid's midline.

**Plane 1.52**. Let $AA_1$ and $BB_1$ be the altitudes of $\triangle ABC$. Prove that $\triangle A_1B_1C$ is similar to $\triangle ABC$. What is the similarity coefficient?

**Plane 2.1**. A line segment connects vertex $A$ of an acute $\triangle ABC$ with the center $O$ of the circumscribed circle. The altitude $AH$ is dropped from $A$. Prove that $\angle BAH = \angle OAC$.

**Plane 6.1**. Prove that if the center of the circle inscribed in a quadrilateral coincides with the intersection point of the quadrilateral's diagonals, then the quadrilateral is a rhombus.

**Solid 1**. Arrange 6 match sticks to get 4 equilateral triangles with side length equal to the length of a stick.

**Solid 1.1**. Consider the cube $ABCDA_1B_1C_1D_1$ with side length $a$. Find the angle and the distance between the lines $A_1B$ and $AC_1$.

**Solid 6.1**. Is it true that in every tetrahedron the heights meet at one point?

The above problems are not difficult. The last problems in the sections are a challenge for the specialists in geometry. It is important that the passage from simple problems to complicated ones is not too long; there are no boring and dull long sequences of simple similar problems. (In the Russian edition these sequences are, perhaps, too short, so more problems are added.)

The final problems of the sections are usually borrowed from scientific journals. Here are some examples:

**Plane 10.20**. Prove that $l_a + l_b + m_c \leq \sqrt{3}p$, where $l_a$, $l_b$ are the lengths of the bisectors of the angles $\angle A$ and $\angle B$ of the triangle $\triangle ABC$, $m_c$ is the length of the median of the side $AB$, and $p$ is the semiperimeter.

**Plane 19.55**. Let $O$ be the center of the circle inscribed in $\triangle ABC$, $K$ the Lemoine's point, $P$ and $Q$ Brocard's points. Prove that $P$ and $Q$ belong to the circle with diameter $KO$ and that $OP = OQ$.

**Plane 22.29**. The numbers $\alpha_1, \ldots, \alpha_n$, whose sum is equal to $(n-2)\pi$, satisfy inequalities $0 < \alpha_i < 2\pi$. Prove that there exists an $n$-gon $A_1 \ldots A_n$ with the angles $\alpha_1, \ldots, \alpha_n$ at the vertices $A_1, \ldots, A_n$, respectively.

**Plane 24.12**. Prove that for any $n$ there exists a circle on which there lie precisely $n$ points with integer coordinates.

**Solid 4.48**. Consider several arcs of great circles on a sphere with the sum of their angle measures $< \pi$. Prove that there exists a plane that passes through the center of the sphere but does not intersect any of these arcs.

**Solid 14.22**. Prove that if the centers of the escribed spheres of a tetrahedron belong to the circumscribed sphere, then the tetrahedron's faces are equal.

**Solid 15.34**. In space, consider 4 points not in one plane. How many various parallelipipeds with vertices in these points are there?

### From the Author's preface

The book underwent extensive revision. The solutions to many of the problems were rewritten and about 600 new problems were added, particularly those concerning the geometry of the triangle. I was greatly influenced in the process by the second edition of the book by I. F. Sharygin *Problems on Geometry. Plane geometry*, Nauka, Moscow,1986 and a wonderful and undeservedly forgotten book by D. Efremov *New Geometry of the Triangle*, Matezis, Odessa, 1902.

The present book can be used not only as a source of optional problems for students but also as a self-guide for those who wish (or have no other choice but) to study geometry

independently. Detailed headings are provided for the reader's convenience. Problems in the two parts of Plane are spread over 29 Chapters, each Chapter comprising 6 to 14 sections. The classification is based on the methods used to solve geometric problems. The purpose of the division is basically to help the reader find his/her bearings in this large array of problems. Otherwise the huge number of problems might be somewhat depressingly overwhelming.

Advice and comments given by Academician A. V. Pogorelov, and Professors A. M. Abramov, A. Yu. Vaintrob, N. B. Vasiliev, N. P. Dolbilin, and S. Yu. Orevkov were a great help to me in preparing the first Soviet edition. I wish to express my sincere gratitude to all of them.

To save space, sections with background only contain the material directly pertinent to the respective chapter. It is collected just to remind the reader of notations. Therefore, the basic elements of a triangle are only defined in chapter 5, while in chapter 1 we assume that their definition is known. For the reader's convenience, cross references in this translation are facilitated by a very detailed index.

# Chapter 1. SIMILAR TRIANGLES

## Background

1) Triangle $ABC$ is said to be *similar* to triangle $A_1B_1C_1$ (we write $\triangle ABC \sim \triangle A_1B_1C_1$) if and only if one of the following equivalent conditions is satisfied:

   a) $AB : BC : CA = A_1B_1 : B_1C_1 : C_1A_1$;

   b) $AB : BC = A_1B_1 : B_1C_1$ and $\angle ABC = \angle A_1B_1C_1$;

   c) $\angle ABC = \angle A_1B_1C_1$ and $\angle BAC = \angle B_1A_1C_1$.

2) *Triangles $AB_1C_1$ and $AB_2C_2$ cut off from an angle with vertex $A$ by parallel lines are similar and $AB_1 : AB_2 = AC_1 : AC_2$ (here points $B_1$ and $B_2$ lie on one leg of the angle and $C_1$ and $C_2$ on the other leg).*

3) A *midline* of a triangle is the line connecting the midpoints of two of the triangle's sides. *The midline is parallel to the third side and its length is equal to a half length of the third side.*

The *midline of a trapezoid* is the line connecting the midpoints of the trapezoid's sides. *This line is parallel to the bases of the trapezoid and its length is equal to the halfsum of their lengths.*

4) *The ratio of the areas of similar triangles is equal to the square of the similarity coefficient*, i.e., to the squared ratio of the lengths of respective sides. This follows, for example, from the formula $S_{ABC} = \frac{1}{2}AB \cdot AC \sin \angle A$.

5) Polygons $A_1A_2 \ldots A_n$ and $B_1B_2 \ldots B_n$ are called *similar* if $A_1A_2 : A_2A_3 : \cdots : A_nA_1 = B_1B_2 : B_2B_3 : \cdots : B_nB_1$ and the angles at the vertices $A_1$, ..., $A_n$ are equal to the angles at the vertices $B_1$, ..., $B_n$, respectively.

*The ratio of the respective diagonals of similar polygons is equal to the similarity coefficient. For the circumscribed similar polygons, the ratio of the radii of the inscribed circles is also equal to the similarity coefficient.*

## Introductory problems

**1**. Consider heights $AA_1$ and $BB_1$ in acute triangle $ABC$. Prove that $A_1C \cdot BC = B_1C \cdot AC$.

**2**. Consider height $CH$ in right triangle $ABC$ with right angle $\angle C$. Prove that $AC^2 = AB \cdot AH$ and $CH^2 = AH \cdot BH$.

**3**. Prove that the medians of a triangle meet at one point and this point divides each median in the ratio of $2 : 1$ counting from the vertex.

**4**. On side $BC$ of $\triangle ABC$ point $A_1$ is taken so that $BA_1 : A_1C = 2 : 1$. What is the ratio in which median $CC_1$ divides segment $AA_1$?

**5**. Square $PQRS$ is inscribed into $\triangle ABC$ so that vertices $P$ and $Q$ lie on sides $AB$ and $AC$ and vertices $R$ and $S$ lie on $BC$. Express the length of the square's side through $a$ and $h_a$.

## §1. Line segments intercepted by parallel lines

**1.1**. Let the lengths of bases $AD$ and $BC$ of trapezoid $ABCD$ be $a$ and $b$ $(a > b)$.

a) Find the length of the segment that the diagonals intercept on the midline.

b) Find the length of segment $MN$ whose endpoints divide $AB$ and $CD$ in the ratio of $AM : MB = DN : NC = p : q$.

**1.2**. Prove that the midpoints of the sides of an arbitrary quadrilateral are vertices of a parallelogram. For what quadrilaterals this parallelogram is a rectangle, a rhombus, a square?

**1.3**. Points $A_1$ and $B_1$ divide sides $BC$ and $AC$ of $\triangle ABC$ in the ratios $BA_1 : A_1C = 1 : p$ and $AB_1 : B_1C = 1 : q$, respectively. In what ratio is $AA_1$ divided by $BB_1$?

**1.4**. Straight lines $AA_1$ and $BB_1$ pass through point $P$ of median $CC_1$ in $\triangle ABC$ ($A_1$ and $B_1$ lie on sides $BC$ and $CA$, respectively). Prove that $A_1B_1 \parallel AB$.

**1.5**. The straight line which connects the intersection point $P$ of the diagonals in quadrilateral $ABCD$ with the intersection point $Q$ of the lines $AB$ and $CD$ bisects side $AD$. Prove that it also bisects $BC$.

**1.6**. A point $P$ is taken on side $AD$ of parallelogram $ABCD$ so that $AP : AD = 1 : n$; let $Q$ be the intersection point of $AC$ and $BP$. Prove that $AQ : AC = 1 : (n + 1)$.

**1.7**. The vertices of parallelogram $A_1B_1C_1D_1$ lie on the sides of parallelogram $ABCD$ (point $A_1$ lies on $AB$, $B_1$ on $BC$, etc.). Prove that the centers of the two parallelograms coincide.

**1.8**. Point $K$ lies on diagonal $BD$ of parallelogram $ABCD$. Straight line $AK$ intersects lines $BC$ and $CD$ at points $L$ and $M$, respectively. Prove that $AK^2 = LK \cdot KM$.

**1.9**. One of the diagonals of a quadrilateral inscribed in a circle is a diameter of the circle. Prove that (the lengths of) the projections of the opposite sides of the quadrilateral on the other diagonal are equal.

**1.10**. Point $E$ on base $AD$ of trapezoid $ABCD$ is such that $AE = BC$. Segments $CA$ and $CE$ intersect diagonal $BD$ at $O$ and $P$, respectively. Prove that if $BO = PD$, then $AD^2 = BC^2 + AD \cdot BC$.

**1.11**. On a circle centered at $O$, points $A$ and $B$ single out an arc of $60°$. Point $M$ belongs to this arc. Prove that the straight line passing through the midpoints of $MA$ and $OB$ is perpendicular to that passing through the midpoints of $MB$ and $OA$.

**1.12**. a) Points $A$, $B$, and $C$ lie on one straight line; points $A_1$, $B_1$, and $C_1$ lie on another straight line. Prove that if $AB_1 \parallel BA_1$ and $AC_1 \parallel CA_1$, then $BC_1 \parallel CB_1$.

b) Points $A$, $B$, and $C$ lie on one straight line and $A_1$, $B_1$, and $C_1$ are such that $AB_1 \parallel BA_1$, $AC_1 \parallel CA_1$, and $BC_1 \parallel CB_1$. Prove that $A_1$, $B_1$ and $C_1$ lie on one line.

**1.13**. In $\triangle ABC$ bisectors $AA_1$ and $BB_1$ are drawn. Prove that the distance from any point $M$ of $A_1B_1$ to line $AB$ is equal to the sum of distances from $M$ to $AC$ and $BC$.

**1.14**. Let $M$ and $N$ be the midpoints of sides $AD$ and $BC$ in rectangle $ABCD$. Point $P$ lies on the extension of $DC$ beyond $D$; point $Q$ is the intersection point of $PM$ and $AC$. Prove that $\angle QNM = \angle MNP$.

**1.15**. Points $K$ and $L$ are taken on the extensions of the bases $AD$ and $BC$ of trapezoid $ABCD$ beyond $A$ and $C$, respectively. Line segment $KL$ intersects sides $AB$ and $CD$ at $M$ and $N$, respectively; $KL$ intersects diagonals $AC$ and $BD$ at $O$ and $P$, respectively. Prove that if $KM = NL$, then $KO = PL$.

**1.16**. Points $P$, $Q$, $R$, and $S$ on sides $AB$, $BC$, $CD$ and $DA$, respectively, of convex quadrilateral $ABCD$ are such that $BP : AB = CR : CD = \alpha$ and $AS : AD = BQ : BC = \beta$. Prove that $PR$ and $QS$ are divided by their intersection point in the ratios $\beta : (1 - \beta)$ and $\alpha : (1 - \alpha)$, respectively.

## §2. The ratio of sides of similar triangles

**1.17**. a) In $\triangle ABC$ bisector $BD$ of the external or internal angle $\angle B$ is drawn. Prove that $AD : DC = AB : BC$.

b) Prove that the center $O$ of the circle inscribed in $\triangle ABC$ divides the bisector $AA_1$ in the ratio of $AO : OA_1 = (b + c) : a$, where $a$, $b$ and $c$ are the lengths of the triangle's sides.

**1.18**. The lengths of two sides of a triangle are equal to $a$ while the length of the third side is equal to $b$. Calculate the radius of the circumscribed circle.

**1.19**. A straight line passing through vertex $A$ of square $ABCD$ intersects side $CD$ at $E$ and line $BC$ at $F$. Prove that $\frac{1}{AE^2} + \frac{1}{AF^2} = \frac{1}{AB^2}$.

**1.20**. Given points $B_2$ and $C_2$ on heights $BB_1$ and $CC_1$ of $\triangle ABC$ such that $AB_2C = AC_2B = 90°$, prove that $AB_2 = AC_2$.

**1.21**. A circle is inscribed in trapezoid $ABCD$ ($BC \parallel AD$). The circle is tangent to sides $AB$ and $CD$ at $K$ and $L$, respectively, and to bases $AD$ and $BC$ at $M$ and $N$, respectively.

a) Let $Q$ be the intersection point of $BM$ and $AN$. Prove that $KQ \parallel AD$.

b) Prove that $AK \cdot KB = CL \cdot LD$.

**1.22**. Perpendiculars $AM$ and $AN$ are dropped to sides $BC$ and $CD$ of parallelogram $ABCD$ (or to their extensions). Prove that $\triangle MAN \sim \triangle ABC$.

**1.23**. Straight line $l$ intersects sides $AB$ and $AD$ of parallelogram $ABCD$ at $E$ and $F$, respectively. Let $G$ be the intersection point of $l$ with diagonal $AC$. Prove that $\frac{AB}{AE} + \frac{AD}{AF} = \frac{AC}{AG}$.

**1.24**. Let $AC$ be the longer of the diagonals in parallelogram $ABCD$. Perpendiculars $CE$ and $CF$ are dropped from $C$ to the extensions of sides $AB$ and $AD$, respectively. Prove that $AB \cdot AE + AD \cdot AF = AC^2$.

**1.25**. Angles $\alpha$ and $\beta$ of $\triangle ABC$ are related as $3\alpha + 2\beta = 180°$. Prove that $a^2 + bc = c^2$.

**1.26**. The endpoints of segments $AB$ and $CD$ are gliding along the sides of a given angle, so that straight lines $AB$ and $CD$ are moving parallelly (i.e., each line moves parallelly to itself) and segments $AB$ and $CD$ intersect at a point, $M$. Prove that the value of $\frac{AM \cdot BM}{CM \cdot DM}$ is a constant.

**1.27**. Through an arbitrary point $P$ on side $AC$ of $\triangle ABC$ straight lines are drawn parallelly to the triangle's medians $AK$ and $CL$. The lines intersect $BC$ and $AB$ at $E$ and $F$, respectively. Prove that $AK$ and $CL$ divide $EF$ into three equal parts.

**1.28**. Point $P$ lies on the bisector of an angle with vertex $C$. A line passing through $P$ intercepts segments of lengths $a$ and $b$ on the angle's legs. Prove that the value of $\frac{1}{a} + \frac{1}{b}$ does not depend on the choice of the line.

**1.29**. A semicircle is constructed outwards on side $BC$ of an equilateral triangle $ABC$ as on the diameter. Given points $K$ and $L$ that divide the semicircle into three equal arcs, prove that lines $AK$ and $AL$ divide $BC$ into three equal parts.

**1.30**. Point $O$ is the center of the circle inscribed in $\triangle ABC$. On sides $AC$ and $BC$ points $M$ and $K$, respectively, are selected so that $BK \cdot AB = BO^2$ and $AM \cdot AB = AO^2$. Prove that $M$, $O$ and $K$ lie on one straight line.

**1.31**. Equally oriented similar triangles $AMN$, $NBM$ and $MNC$ are constructed on segment $MN$ (Fig. 1).

Prove that $\triangle ABC$ is similar to all these triangles and the center of its curcumscribed circle is equidistant from $M$ and $N$.

**1.32**. Line segment $BE$ divides $\triangle ABC$ into two similar triangles, their similarity ratio being equal to $\sqrt{3}$.

Find the angles of $\triangle ABC$.
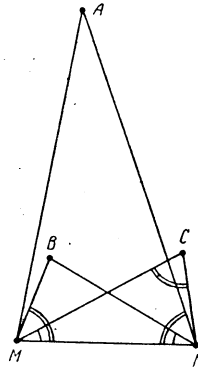
Figure 1 (1.31)

## §3. The ratio of the areas of similar triangles

**1.33**. A point $E$ is taken on side $AC$ of $\triangle ABC$. Through $E$ pass straight lines $DE$ and $EF$ parallel to sides $BC$ and $AB$, respectively; $D$ and $E$ are points on $AB$ and $BC$, respectively. Prove that $S_{BDEF} = 2\sqrt{S_{ADE} \cdot S_{EFG}}$.

**1.34**. Points $M$ and $N$ are taken on sides $AB$ and $CD$, respectively, of trapezoid $ABCD$ so that segment $MN$ is parallel to the bases and divides the area of the trapezoid in halves. Find the length of $MN$ if $BC = a$ and $AD = b$.

**1.35**. Let $Q$ be a point inside $\triangle ABC$. Three straight lines are pass through $Q$ parallelly to the sides of the triangle. The lines divide the triangle into six parts, three of which are triangles of areas $S_1$, $S_2$ and $S_3$. Prove that the area of $\triangle ABC$ is equal to $\left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$.

**1.36**. Prove that the area of a triangle whose sides are equal to the medians of a triangle of area $S$ is equal to $\frac{3}{4}S$.

**1.37**. a) Prove that the area of the quadrilateral formed by the midpoints of the sides of convex quadrilateral $ABCD$ is half that of $ABCD$.

b) Prove that if the diagonals of a convex quadrilateral are equal, then its area is the product of the lengths of the segments which connect the midpoints of its opposite sides.

**1.38**. Point $O$ lying inside a convex quadrilateral of area $S$ is reflected symmetrically through the midpoints of its sides. Find the area of the quadrilateral with its vertices in the images of $O$ under the reflections.

## §4. Auxiliary equal triangles

**1.39**. In right triangle $ABC$ with right angle $\angle C$, points $D$ and $E$ divide leg $BC$ of into three equal parts. Prove that if $BC = 3AC$, then $\angle AEC + \angle ADC + \angle ABC = 90°$.

**1.40**. Let $K$ be the midpoint of side $AB$ of square $ABCD$ and let point $L$ divide diagonal $AC$ in the ratio of $AL : LC = 3 : 1$. Prove that $\angle KLD$ is a right angle.

**1.41**. In square $ABCD$ straight lines $l_1$ and $l_2$ pass through vertex $A$. The lines intersect the square's sides. Perpendiculars $BB_1$, $BB_2$, $DD_1$, and $DD_2$ are dropped to these lines. Prove that segments $B_1B_2$ and $D_1D_2$ are equal and perpendicular to each other.

**1.42**. Consider an isosceles right triangle $ABC$ with $CD = CE$ and points $D$ and $E$ on sides $CA$ and $CB$, respectively. Extensions of perpendiculars dropped from $D$ and $C$ to $AE$ intersect the hypotenuse $AB$ at $K$ and $L$. Prove that $KL = LB$.

**1.43**. Consider an inscribed quadrilateral $ABCD$. The lengths of sides $AB$, $BC$, $CD$, and $DA$ are $a$, $b$, $c$, and $d$, respectively. Rectangles are constructed outwards on the sides of

the quadrilateral; the sizes of the rectangles are $a \times c$, $b \times d$, $c \times a$ and $d \times b$, respectively. Prove that the centers of the rectangles are vertices of a rectangle.

**1.44**. Hexagon $ABCDEF$ is inscribed in a circle of radius $R$ centered at $O$; let $AB = CD = EF = R$. Prove that the intersection points, other than $O$, of the pairs of circles circumscribed about $\triangle BOC$, $\triangle DOE$ and $\triangle FOA$ are the vertices of an equilateral triangle with side $R$.

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**1.45**. Equilateral triangles $BCK$ and $DCL$ are constructed outwards on sides $BC$ and $CD$ of parallelogram $ABCD$. Prove that $AKL$ is an equilateral triangle.

**1.46**. Squares are constructed outwards on the sides of a parallelogram. Prove that their centers form a square.

**1.47**. Isosceles triangles with angles $2\alpha$, $2\beta$ and $2\gamma$ at vertices $A'$, $B'$ and $C'$ are constructed outwards on the sides of triangle $ABC$; let $\alpha + \beta + \gamma = 180°$. Prove that the angles of $\triangle A'B'C'$ are equal to $\alpha$, $\beta$ and $\gamma$.

**1.48**. On the sides of $\triangle ABC$ as on bases, isosceles similar triangles $AB_1C$ and $AC_1B$ are constructed outwards and an isosceles triangle $BA_1C$ is constructed inwards. Prove that $AB_1A_1C_1$ is a parallelogram.

**1.49**. a) On sides $AB$ and $AC$ of $\triangle ABC$ equilateral triangles $ABC_1$ and $AB_1C$ are constructed outwards; let $\angle C_1 = \angle B_1 = 90°$, $\angle ABC_1 = \angle ACB_1 = \varphi$; let $M$ be the midpoint of $BC$. Prove that $MB_1 = MC_1$ and $\angle B_1MC_1 = 2\varphi$.

b) Equilateral triangles are constructed outwards on the sides of $\triangle ABC$. Prove that the centers of the triangles constructed form an equilateral triangle whose center coincides with the intersection point of the medians of $\triangle ABC$.

**1.50**. Isosceles triangles $AC_1B$ and $AB_1C$ with an angle $\varphi$ at the vertex are constructed outwards on the unequal sides $AB$ and $AC$ of a scalene triangle $\triangle ABC$.

a) Let $M$ be a point on median $AA_1$ (or on its extension), let $M$ be equidistant from $B_1$ and $C_1$. Prove that $\angle B_1MC_1 = \varphi$.

b) Let $O$ be a point of the midperpendicular to segment $BC$, let $O$ be equidistant from $B_1$ and $C_1$. Prove that $\angle B_1OC = 180° - \varphi$.

**1.51**. Similar rhombuses are constructed outwards on the sides of a convex rectangle $ABCD$, so that their acute angles (equal to $\alpha$) are adjacent to vertices $A$ and $C$. Prove that the segments which connect the centers of opposite rhombuses are equal and the angle between them is equal to $\alpha$.

## §5. The triangle determined by the bases of the heights

**1.52**. Let $AA_1$ and $BB_1$ be heights of $\triangle ABC$. Prove that $\triangle A_1B_1C \sim \triangle ABC$. What is the similarity coefficient?

**1.53**. Height $CH$ is dropped from vertex $C$ of acute triangle $ABC$ and perpendiculars $HM$ and $HN$ are dropped to sides $BC$ and $AC$, respectively. Prove that $\triangle MNC \sim \triangle ABC$.

**1.54**. In $\triangle ABC$ heights $BB_1$ and $CC_1$ are drawn.

a) Prove that the tangent at $A$ to the circumscribed circle is parallel to $B_1C_1$.

b) Prove that $B_1C_1 \perp OA$, where $O$ is the center of the circumscribed circle.

**1.55**. Points $A_1$, $B_1$ and $C_1$ are taken on the sides of an acute triangle $ABC$ so that segments $AA_1$, $BB_1$ and $CC_1$ meet at $H$. Prove that $AH \cdot A_1H = BH \cdot B_1H = CH \cdot C_1H$ if and only if $H$ is the intersection point of the heights of $\triangle ABC$.

**1.56**. a) Prove that heights $AA_1$, $BB_1$ and $CC_1$ of acute triangle $ABC$ bisect the angles of $\triangle A_1B_1C_1$.

b) Points $C_1$, $A_1$ and $B_1$ are taken on sides $AB$, $BC$ and $CA$, respectively, of acute triangle $ABC$. Prove that if $\angle B_1A_1C = \angle BA_1C_1$, $\angle A_1B_1C = \angle AB_1C_1$ and $\angle A_1C_1B = \angle AC_1B_1$, then points $A_1$, $B_1$ and $C_1$ are the bases of the heights of $\triangle ABC$.

**1.57**. Heights $AA_1$, $BB_1$ and $CC_1$ are drawn in acute triangle $ABC$. Prove that the point symmetric to $A_1$ through $AC$ lies on $B_1C_1$.

**1.58**. In acute triangle $ABC$, heights $AA_1$, $BB_1$ and $CC_1$ are drawn. Prove that if $A_1B_1 \parallel AB$ and $B_1C_1 \parallel BC$, then $A_1C_1 \parallel AC$.

**1.59**. Let $p$ be the semiperimeter of acute triangle $ABC$ and $q$ the semiperimeter of the triangle formed by the bases of the heights of $\triangle ABC$. Prove that $p : q = R : r$, where $R$ and $r$ are the radii of the circumscribed and the inscribed circles, respectively, of $\triangle ABC$.

## §6. Similar figures

**1.60**. A circle of radius $r$ is inscribed in a triangle. The straight lines tangent to the circle and parallel to the sides of the triangle are drawn; the lines cut three small triangles off the triangle. Let $r_1$, $r_2$ and $r_3$ be the radii of the circles inscribed in the small triangles. Prove that $r_1 + r_2 + r_3 = r$.

**1.61**. Given $\triangle ABC$, draw two straight lines $x$ and $y$ such that the sum of lengths of the segments $MX_M$ and $MY_M$ drawn parallel to $x$ and $y$ from a point $M$ on $AC$ to their intersections with sides $AB$ and $BC$ is equal to 1 for any $M$.

**1.62**. In an isosceles triangle $ABC$ perpendicular $HE$ is dropped from the midpoint of base $BC$ to side $AC$. Let $O$ be the midpoint of $HE$. Prove that lines $AO$ and $BE$ are perpendicular to each other.

**1.63**. Prove that projections of the base of a triangle's height to the sides between which it lies and on the other two heights lie on the same straight line.

**1.64**. Point $B$ lies on segment $AC$; semicircles $S_1$, $S_2$, and $S_3$ are constructed on one side of $AC$, as on diameter. Let $D$ be a point on $S_3$ such that $BD \perp AC$. A common tangent line to $S_1$ and $S_2$ touches these semicircles at $F$ and $E$, respectively.

a) Prove that $EF$ is parallel to the tangent to $S_3$ passing through $D$.

b) Prove that $BFDE$ is a rectangle.

**1.65**. Perpendiculars $MQ$ and $MP$ are dropped from an arbitrary point $M$ of the circle circumscribed about rectangle $ABCD$ to the rectangle's two opposite sides; the perpendiculars $MR$ and $MT$ are dropped to the extensions of the other two sides. Prove that lines $PR \perp QT$ and the intersection point of $PR$ and $QT$ belongs to a diagonal of $ABCD$.

**1.66**. Two circles enclose non-intersecting areas. Common tangent lines to the two circles, one external and one internal, are drawn. Consider two straight lines each of which passes through the tangent points on one of the circles. Prove that the intersection point of the lines lies on the straight line that connects the centers of the circles.

## Problems for independent study

**1.67**. The (length of the) base of an isosceles triangle is a quarter of its perimeter. From an arbitrary point on the base straight lines are drawn parallel to the sides of the triangle. How many times is the perimeter of the triangle greater than that of the parallelogram?

**1.68**. The diagonals of a trapezoid are mutually perpendicular. The intersection point divides the diagonals into segments. Prove that the product of the lengths of the trapezoid's bases is equal to the sum of the products of the lengths of the segments of one diagonal and those of another diagonal.

**1.69**. A straight line is drawn through the center of a unit square. Calculate the sum of the squared distances between the four vertices of the square and the line.

**1.70**. Points $A_1$, $B_1$ and $C_1$ are symmetric to the center of the circumscribed circle of $\triangle ABC$ through the triangle's sides. Prove that $\triangle ABC = \triangle A_1 B_1 C_1$.

**1.71**. Prove that if $\angle BAC = 2\angle ABC$, then $BC^2 = (AC + AB)AC$.

**1.72**. Consider points $A$, $B$, $C$ and $D$ on a line $l$. Through $A$, $B$ and through $C$, $D$ parallel straight lines are drawn. Prove that the diagonals of the parallelograms thus formed (or their extensions) intersect $l$ at two points that do not depend on parallel lines but depend on points $A$, $B$, $C$, $D$ only.

**1.73**. In $\triangle ABC$ bisector $AD$ and midline $A_1 C_1$ are drawn. They intersect at $K$. Prove that $2A_1 K = |b - c|$.

**1.74**. Points $M$ and $N$ are taken on sides $AD$ and $CD$ of parallelogram $ABCD$ such that $MN \parallel AC$. Prove that $S_{ABM} = S_{CBN}$.

**1.75**. On diagonal $AC$ of parallelogram $ABCD$ points $P$ and $Q$ are taken so that $AP = CQ$. Let $M$ be such that $PM \parallel AD$ and $QM \parallel AB$. Prove that $M$ lies on diagonal $BD$.

**1.76**. Consider a trapezoid with bases $AD$ and $BC$. Extensions of the sides of $ABCD$ meet at point $O$. Segment $EF$ is parallel to the bases and passes through the intersection point of the diagonals. The endpoints of $EF$ lie on $AB$ and $CD$. Prove that $AE : CF = AO : CO$.

**1.77**. Three straight lines parallel to the sides of the given triangle cut three triangles off it leaving an equilateral hexagon. Find the length of the side of the hexagon if the lengths of the triangle's sides are $a$, $b$ and $c$.

**1.78**. Three straight lines parallel to the sides of a triangle meet at one point, the sides of the triangle cutting off the line segments of length $x$ each. Find $x$ if the lengths of the triangle's sides are $a$, $b$ and $c$.

**1.79**. Point $P$ lies inside $\triangle ABC$ and $\angle ABP = \angle ACP$. On straight lines $AB$ and $AC$, points $C_1$ and $B_1$ are taken so that $BC_1 : CB_1 = CP : BP$. Prove that one of the diagonals of the parallelogram whose two sides lie on lines $BP$ and $CP$ and two other sides (or their extensions) pass through $B_1$ and $C_1$ is parallel to $BC$.

## Solutions

**1.1**. a) Let $P$ and $Q$ be the midpoints of $AB$ and $CD$; let $K$ and $L$ be the intersection points of $PQ$ with the diagonals $AC$ and $BD$, respectively. Then $PL = \frac{a}{2}$ and $PK = \frac{1}{2}b$ and so $KL = PL - PK = \frac{1}{2}(a - b)$.

b) Take point $F$ on $AD$ such that $BF \parallel CD$. Let $E$ be the intersection point of $MN$ with $BF$. Then

$$MN = ME + EN =$$

$$\frac{q \cdot AF}{p + q} + b = \frac{q(a - b) + (p + q)b}{p + q} = \frac{qa + pb}{p + q}.$$

**1.2**. Consider quadrilateral $ABCD$. Let $K$, $L$, $M$ and $N$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$, respectively. Then $KL = MN = \frac{1}{2}AC$ and $KL \parallel MN$, that is $KLMN$ is a parallelogram. It becomes clear now that $KLMN$ is a *rectangle* if the diagonals $AC$ and $BD$ are perpendicular, a *rhombus* if $AC = BD$, and a *square* if $AC$ and $BD$ are of equal length and perpendicular to each other.

**1.3**. Denote the intersection point of $AA_1$ with $BB_1$ by $O$. In $\triangle B_1 BC$ draw segment $A_1 A_2$ so that $A_1 A_2 \parallel BB_1$. Then $\frac{B_1 C}{B_1 A_2} = 1 + p$ and so $AO : OA_1 = AB_1 : B_1 A_2 = B_1 C : qB_1 A_2 = (1 + p) : q$.

**1.4**. Let $A_2$ be the midpoint of $A_1B$. Then $CA_1 : A_1A_2 = CP : PC_1$ and $A_1A_2 : A_1B = 1 : 2$. So $CA_1 : A_1B = CP : 2PC_1$. Similarly, $CB_1 : B_1A = CP : 2PC_1 = CA_1 : A_1B$.

**1.5**. Point $P$ lies on the median $QM$ of $\triangle AQD$ (or on its extension). It is easy to verify that the solution of Problem 1.4 remains correct also for the case when $P$ lies on the extension of the median. Consequently, $BC \parallel AD$.

**1.6**. We have $AQ : QC = AP : BC = 1 : n$ because $\triangle AQP \sim \triangle CQB$. So $AC = AQ + QC = (n+1)AQ$.

**1.7**. The center of $A_1B_1C_1D_1$ being the midpoint of $B_1D_1$ belongs to the line segment which connects the midpoints of $AB$ and $CD$. Similarly, it belongs to the segment which connects the midpoints of $BC$ and $AD$. The intersection point of the segments is the center of $ABCD$.

**1.8**. Clearly, $AK : KM = BK : KD = LK : AK$, that is $AK^2 = LK \cdot KM$.

**1.9**. Let $AC$ be the diameter of the circle circumscribed about $ABCD$. Drop perpendiculars $AA_1$ and $CC_1$ to $BD$ (Fig. 2).
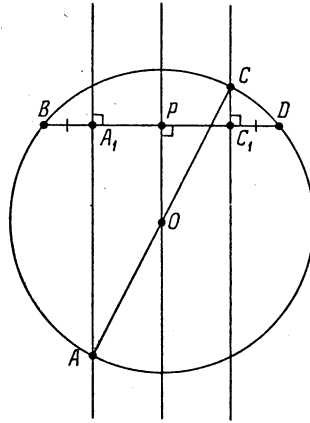


Figure 2 (Sol. 1.9)

We must prove that $BA_1 = DC_1$. Drop perpendicular $OP$ from the center $O$ of the circumscribed circle to $BD$. Clearly, $P$ is the midpoint of $BD$. Lines $AA_1$, $OP$ and $CC_1$ are parallel to each other and $AO = OC$. So $A_1P = PC_1$ and, since $P$ is the midpoint of $BD$, it follows that $BA_1 = DC_1$.

**1.10**. We see that $BO : OD = DP : PB = k$, because $BO = PD$. Let $BC = 1$. Then $AD = k$ and $ED = \frac{1}{k}$. So $k = AD = AE + ED = 1 + \frac{1}{k}$, that is $k^2 = 1 + k$. Finally, observe that $k^2 = AD^2$ and $1 + k = BC^2 + BC \cdot AD$.

**1.11**. Let $C$, $D$, $E$ and $F$ be the midpoints of sides $AO$, $OB$, $BM$ and $MA$, respectively, of quadrilateral $AOMB$. Since $AB = MO = R$, where $R$ is the radius of the given circle, $CDEF$ is a rhombus by Problem 1.2. Hence, $CE \perp DF$.

**1.12**. a) If the lines containing the given points are parallel, then the assertion of the problem is obviously true. We assume that the lines meet at $O$. Then $OA : OB = OB_1 : OA_1$ and $OC : OA = OA_1 : OC_1$. Hence, $OC : OB = OB_1 : OC_1$ and so $BC_1 \parallel CB_1$ (the ratios of the segment should be assumed to be *oriented*).

b) Let $AB_1$ and $CA_1$ meet at $D$, let $CB_1$ and $AC_1$ meet at $E$. Then $CA_1 : A_1D = CB : BA = EC_1 : C_1A$. Since $\triangle CB_1D \sim \triangle EB_1A$, points $A_1$, $B_1$ and $C_1$ lie on the same line.

**1.13**. A point that lies on the bisector of an angle is equidistant from the angle's legs. Let $a$ be the distance from point $A_1$ to lines $AC$ and $AB$, let $b$ be the distance from point $B_1$ to lines $AB$ and $BC$. Further, let $A_1M : B_1M = p : q$, where $p + q = 1$. Then the distances

from point $M$ to lines $AC$ and $BC$ are equal to $qa$ and $pb$, respectively. On the other hand, by Problem 1.1 b) the distance from point $M$ to line $AB$ is equal to $qa + pb$.

**1.14**. Let the line that passes through the center $O$ of the given rectangle parallel to $BC$ intersect line segment $QN$ at point $K$ (Fig. 3).



Figure 3 (Sol. 1.14)

Since $MO \parallel PC$, it follows that $QM : MP = QO : OC$ and, since $KO \parallel BC$, it follows that $QO : OC = QK : KN$. Therefore, $QM : MP = QK : KN$, i.e., $KM \parallel NP$. Hence, $\angle MNP = \angle KMO = \angle QNM$.

**1.15**. Let us draw through point $M$ line $EF$ so that $EF \parallel CD$ (points $E$ and $F$ lie on lines $BC$ and $AD$). Then $PL : PK = BL : KD$ and $OK : OL = KA : CL = KA : KF = BL : EL$. Since $KD = EL$, we have $PL : PK = OK : OL$ and, therefore, $PL = OK$.

**1.16**. Consider parallelogram $ABCD_1$. We may assume that points $D$ and $D_1$ do not coincide (otherwise the statement of the problem is obvious). On sides $AD_1$ and $CD_1$ take points $S_1$ and $R_1$, respectively, so that $SS_1 \parallel DD_1$ and $RR_1 \parallel DD_1$. Let segments $PR_1$ and $QS_1$ meet at $N$; let $N_1$ and $N_2$ be the intersection points of the line that passes through $N$ parallel to $DD_1$ with segments $PR$ and $QS$, respectively.

Then $\overrightarrow{N_1N} = \beta\overrightarrow{RR_1} = \alpha\beta\overrightarrow{DD_1}$ and $\overrightarrow{N_2N} = \alpha\overrightarrow{SS_1} = \alpha\beta\overrightarrow{DD_1}$. Hence, segments $PR$ and $QS$ meet at $N_1 = N_2$. Clearly, $PN_1 : PR = PN : PR_1 = \beta$ and $QN_2 : QS = \alpha$.

REMARK. If $\alpha = \beta$, there is a simpler solution. Since $BP : BA = BQ : BC = \alpha$, it follows that $PQ \parallel AC$ and $PQ : AC = \alpha$. Similarly, $RS \parallel AC$ and $RS : AC = 1 - \alpha$. Therefore, segments $PR$ and $QS$ are divided by their intersection point in the ratio of $\alpha : (1 - \alpha)$.

**1.17**. a) From vertices $A$ and $C$ drop perpendiculars $AK$ and $CL$ to line $BD$. Since $\angle CBL = \angle ABK$ and $\angle CDL = \angle KDA$, we see that $\triangle BLC \sim \triangle BKA$ and $\triangle CLD \sim \triangle AKD$. Therefore, $AD : DC = AK : CL = AB : BC$.

b) Taking into account that $BA_1 : A_1C = BA : AC$ and $BA_1 + A_1C = BC$ we get $BA_1 = \frac{ac}{b+c}$. Since $BO$ is the bisector of triangle $ABA_1$, it follows that $AO : OA_1 = AB : BA_1 = (b + c) : a$.

**1.18**. Let $O$ be the center of the circumscribed circle of isosceles triangle $ABC$, let $B_1$ be the midpoint of base $AC$ and $A_1$ the midpoint of the lateral side $BC$. Since $\triangle BOA_1 \sim \triangle BCB_1$, it follows that $BO : BA_1 = BC : BB_1$ and, therefore, $R = BO = \frac{a^2}{\sqrt{4a^2 - b^2}}$.

**1.19**. If $\angle EAD = \varphi$, then $AE = \frac{AD}{\cos \varphi} = \frac{AB}{\cos \varphi}$ and $AF = \frac{AB}{\sin \varphi}$. Therefore,

$$\frac{1}{AE^2} + \frac{1}{AF^2} = \frac{\cos^2 \varphi + \sin^2 \varphi}{AB^2} = \frac{1}{AB^2}.$$

**1.20**. It is easy to verify that $AB_2^2 = AB_1 \cdot AC = AC_1 \cdot AB = AC_2^2$.

**1.21**. a) Since $BQ : QM = BN : AM = BK : AK$, we have: $KQ \parallel AM$.

b) Let $O$ be the center of the inscribed circle. Since $\angle CBA + \angle BAD = 180°$, it follows that $\angle ABO + \angle BAO = 90°$. Therefore, $\triangle AKO \sim \triangle OKB$, i.e., $AK : KO = OK : KB$. Consequently, $AK \cdot KB = KO^2 = R^2$, where $R$ is the radius of the inscribed circle. Similarly, $CL \cdot LD = R^2$.

**1.22**. If angle $\angle ABC$ is obtuse (resp. acute), then angle $\angle MAN$ is also obtuse (resp. acute). Moreover, the legs of these angles are mutually perpendicular. Therefore, $\angle ABC = \angle MAN$. Right triangles $ABM$ and $ADN$ have equal angles $\angle ABM = \angle ADN$, therefore, $AM : AN = AB : AD = AB : CB$, i.e., $\triangle ABC \sim \triangle MAN$.

**1.23**. On diagonal $AC$, take points $D'$ and $B'$ such that $BB' \parallel l$ and $DD' \parallel l$. Then $AB : AE = AB' : AG$ and $AD : AF = AD' : AG$. Since the sides of triangles $ABB'$ and $CDD'$ are pairwise parallel and $AB = CD$, these triangles are equal and $AB' = CD'$. Therefore,

$$\frac{AB}{AE} + \frac{AD}{AF} = \frac{AB'}{AG} + \frac{AD'}{AG} = \frac{CD' + AD'}{AG} = \frac{AC}{AG}.$$

**1.24**. Let us drop from vertex $B$ perpendicular $BG$ to $AC$ (Fig. 4).
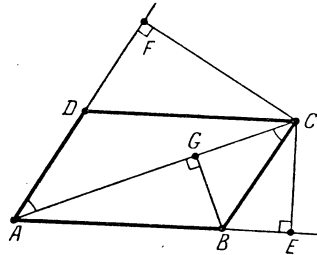


Figure 4 (Sol. 1.24)

Since triangles $ABG$ and $ACE$ are similar, $AC \cdot AG = AE \cdot AB$. Lines $AF$ and $CB$ are parallel, consequently, $\angle GCB = \angle CAF$. We also infer that right triangles $CBG$ and $ACF$ are similar and, therefore, $AC \cdot CG = AF \cdot BC$. Summing the equalities obtained we get

$$AC \cdot (AG + CG) = AE \cdot AB + AF \cdot BC.$$

Since $AG + CG = AC$, we get the equality desired.

**1.25**. Since $\alpha + \beta = 90° - \frac{1}{2}\alpha$, it follows that $\gamma = 180° - \alpha - \beta = 90° + \frac{1}{2}\alpha$. Therefore, it is possible to find point $D$ on side $AB$ so that $\angle ACD = 90° - \frac{1}{2}\alpha$, i.e., $AC = AD$. Then $\triangle ABC \sim \triangle CBD$ and, therefore, $BC : BD = AB : CB$, i.e., $a^2 = c(c - b)$.

**1.26**. As segments $AB$ and $CD$ move, triangle $AMC$ is being replaced by another triangle similar to the initial one. Therefore, the quantity $\frac{AM}{CM}$ remains a constant. Analogously, $\frac{BM}{DM}$ remains a constant.

**1.27**. Let medians meet at $O$; denote the intersection points of median $AK$ with lines $FP$ and $FE$ by $Q$ and $M$, respectively; denote the intersection points of median $CL$ with lines $EP$ and $FE$ by $R$ and $N$, respectively (Fig. 5).

Clearly, $FM : FE = FQ : FP = LO : LC = 1 : 3$, i.e., $FM = \frac{1}{3}FE$. Similarly, $EN = \frac{1}{3}FE$.
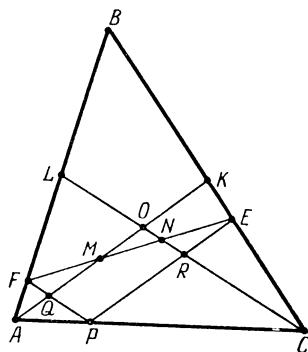
Figure 5 (Sol. 1.27)

**1.28**. Let $A$ and $B$ be the intersection points of the given line with the angle's legs. On segments $AC$ and $BC$, take points $K$ and $L$, respectively, so that $PK \parallel BC$ and $PL \parallel AC$. Since $\triangle AKP \sim \triangle PLB$, it follows that $AK : KP = PL : LB$ and, therefore, $(a-p)(b-p) = p^2$, where $p = PK = PL$. Hence, $\frac{1}{a} + \frac{1}{b} = \frac{1}{p}$.

**1.29**. Denote the midpoint of side $BC$ by $O$ and the intersection points of $AK$ and $AL$ with side $BC$ by $P$ and $Q$, respectively. We may assume that $BP < BQ$. Triangle $LCO$ is an equilateral one and $LC \parallel AB$. Therefore, $\triangle ABQ \sim \triangle LCQ$, i.e., $BQ : QC = AB : LC = 2 : 1$. Hence, $BC = BQ + QC = 3QC$. Similarly, $BC = 3BP$.

**1.30**. Since $BK : BO = BO : AB$ and $\angle KBO = \angle ABO$, it follows that $\triangle KOB \sim \triangle OAB$. Hence, $\angle KOB = \angle OAB$. Similarly, $\angle AOM = \angle ABO$. Therefore,

$$\angle KOM = \angle KOB + \angle BOA + \angle AOM = \angle OAB + \angle BOA + \angle ABO = 180°,$$

i.e., points $K$, $O$ and $M$ lie on one line.

**1.31**. Since $\angle AMN = \angle MNC$ and $\angle BMN = \angle MNA$, we see that $\angle AMB = \angle ANC$. Moreover, $AM : AN = NB : NM = BM : CN$. Hence, $\triangle AMB \sim \triangle ANC$ and, therefore, $\angle MAB = \angle NAC$. Consequently, $\angle BAC = \angle MAN$. For the other angles the proof is similar.

Let points $B_1$ and $C_1$ be symmetric to $B$ and $C$, respectively, through the midperpendicular to segment $MN$. Since $AM : NB = MN : BM = MC : NC$, it follows that $MA \cdot MC_1 = AM \cdot NC = NB \cdot MC = MB_1 \cdot MC$. Therefore, point $A$ lies on the circle circumscribed about trapezoid $BB_1CC_1$.

**1.32**. Since $\angle AEB + \angle BEC = 180°$, angles $\angle AEB$ and $\angle BEC$ cannot be different angles of similar triangles $ABE$ and $BEC$, i.e., the angles are equal and $BE$ is a perpendicular.

Two cases are possible: either $\angle ABE = \angle CBE$ or $\angle ABE = \angle BCE$. The first case should be discarded because in this case $\triangle ABE = \triangle CBE$.

In the second case we have $\angle ABC = \angle ABE + \angle CBE = \angle ABE + \angle BAE = 90°$. In right triangle $ABC$ the ratio of the legs' lengths is equal to $1 : \sqrt{3}$; hence, the angles of triangle $ABC$ are equal to $90°$, $60°$, $30°$.

**1.33**. We have $\frac{S_{BDEF}}{2S_{ADE}} = \frac{S_{BDE}}{S_{ADE}} = \frac{DB}{AD} = \frac{EF}{AD} = \sqrt{\frac{S_{EFC}}{S_{ADE}}}$. Hence,

$$S_{BDEF} = 2\sqrt{S_{ADE} \cdot S_{EFC}}.$$

**1.34**. Let $MN = x$; let $E$ be the intersection point of lines $AB$ and $CD$. Triangles $EBC$, $EMN$ and $EAD$ are similar, hence, $S_{EBC} : S_{EMN} : S_{EAD} = a^2 : x^2 : b^2$. Since $S_{EMN} - S_{EBC} = S_{MBCN} = S_{MADN} = S_{EAD} - S_{EMN}$, it follows that $x^2 - a^2 = b^2 - x^2$, i.e., $x^2 = \frac{1}{2}(a^2 + b^2)$.

**1.35**. Through point $Q$ inside triangle $ABC$ draw lines $DE$, $FG$ and $HI$ parallel to $BC$, $CA$ and $AB$, respectively, so that points $F$ and $H$ would lie on side $BC$, points $E$ and $I$ on side $AC$, points $D$ and $G$ on side $AB$ (Fig. 6).
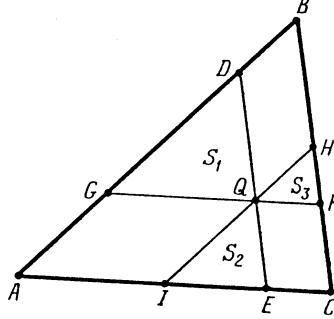


Figure 6 (Sol. 1.35)

Set $S = S_{ABC}$, $S_1 = S_{GDQ}$, $S_2 = S_{IEQ}$, $S_3 = S_{HFQ}$. Then

$$\sqrt{\frac{S_1}{S}} + \sqrt{\frac{S_2}{S}} + \sqrt{\frac{S_3}{S}} = \frac{GQ}{AC} + \frac{IE}{AC} + \frac{FQ}{AC} = \frac{AI + IE + EC}{AC} = 1,$$

i.e., $S = (\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$.

**1.36**. Let $M$ be the intersection point of the medians of triangle $ABC$; let point $A_1$ be symmetric to $M$ through the midpoint of segment $BC$. The ratio of the lengths of sides of triangle $CMA_1$ to the lengths of the corresponding medians of triangle $ABC$ is to $2 : 3$. Therefore, the area to be found is equal to $\frac{9}{4}S_{CMA_1}$. Clearly, $S_{CMA_1} = \frac{1}{3}S$ (cf. the solution of Problem 4.1).

**1.37**. Let $E$, $F$, $G$ and $H$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$, respectively.

a) Clearly, $S_{AEH} + S_{CFG} = \frac{1}{4}S_{ABD} + \frac{1}{4}S_{CBD} = \frac{1}{4}S_{ABCD}$. Analogously, $S_{BEF} + S_{DGH} = \frac{1}{4}S_{ABCD}$; hence, $S_{EFGH} = S_{ABCD} - \frac{1}{4}S_{ABCD} - \frac{1}{4}S_{ABCD} = \frac{1}{2}S_{ABCD}$.

b) Since $AC = BD$, it follows that $EFGH$ is a rhombus (Problem 1.2). By heading a) we have $S_{ABCD} = 2S_{EFGH} = EG \cdot FH$.

**1.38**. Let $E$, $F$, $G$ and $H$ be the midpoints of sides of quadrilateral $ABCD$; let points $E_1$, $F_1$, $G_1$ and $H_1$ be symmetric to point $O$ through these points, respectively. Since $EF$ is the midline of triangle $E_1OF_1$, we see that $S_{E_1OF_1} = 4S_{EOF}$. Similarly, $S_{F_1OG_1} = 4S_{FOG}$, $S_{G_1OH_1} = 4S_{GOH}$, $S_{H_1OE_1} = 4S_{HOE}$. Hence, $S_{E_1F_1G_1H_1} = 4S_{EFGH}$. By Problem 1.37 a) $S_{ABCD} = 2S_{EFGH}$. Hence, $S_{E_1F_1G_1H_1} = 2S_{ABCD} = 2S$.

**1.39. First solution**. Let us consider square $BCMN$ and divide its side $MN$ by points $P$ and $Q$ into three equal parts (Fig. 7).

Then $\triangle ABC = \triangle PDQ$ and $\triangle ACD = \triangle PMA$. Hence, triangle $\triangle PAD$ is an isosceles right triangle and $\angle ABC + \angle ADC = \angle PDQ + \angle ADC = 45°$.

**Second solution**. Since $DE = 1$, $EA = \sqrt{2}$, $EB = 2$, $AD = \sqrt{5}$ and $BA = \sqrt{10}$, it follows that $DE : AE = EA : EB = AD : BA$ and $\triangle DEA \sim \triangle AEB$. Therefore, $\angle ABC = \angle EAD$. Moreover, $\angle AEC = \angle CAE = 45°$. Hence,

$$\angle ABC + \angle ADC + \angle AEC = (\angle EAD + \angle CAE) + \angle ADC$$

$$= \angle CAD + \angle ADC = 90°.$$
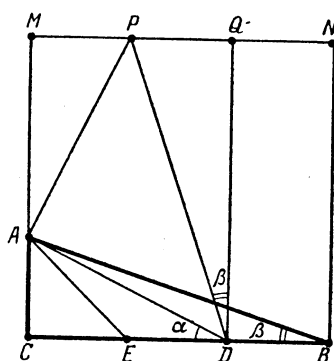
Figure 7 (Sol. 1.39)

**1.40**. From point $L$ drop perpendiculars $LM$ and $LN$ on $AB$ and $AD$, respectively. Then $KM = MB = ND$ and $KL = LB = DL$ and, therefore, right triangles $KML$ and $DNL$ are equal. Hence, $\angle DLK = \angle NLM = 90°$.

**1.41**. Since $D_1A = B_1B, AD_2 = BB_2$ and $\angle D_1AD_2 = \angle B_1BB_2$, it follows that $\triangle D_1AD_2 = \triangle B_1BB_2$. Sides $AD_1$ and $BB_1$ (and also $AD_2$ and $BB_2$) of these triangles are perpendicular and, therefore, $B_1B_2 \perp D_1D_2$.

**1.42**. On the extension of segment $AC$ beyond point $C$ take point $M$ so that $CM = CE$ (Fig. 8).
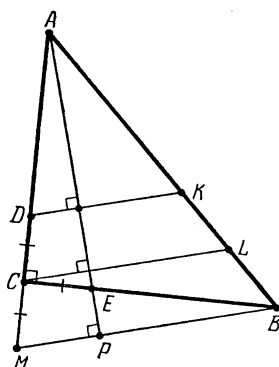


Figure 8 (Sol. 1.42)

Then under the rotation with center $C$ through an angle of $90°$ triangle $ACE$ turns into triangle $BCM$. Therefore, line $MB$ is perpendicular to line $AE$; hence, it is parallel to line $CL$. Since $MC = CE = DC$ and lines $DK$, $CL$ and $MB$ are parallel, $KL = LB$.

**1.43**. Let rectangles $ABC_1D_1$ and $A_2BCD_2$ be constructed on sides $AB$ and $BC$; let $P$, $Q$, $R$ and $S$ be the centers of rectangles constructed on sides $AB$, $BC$, $CD$ and $DA$, respectively. Since $\angle ABC + \angle ADC = 180°$, it follows that $\triangle ADC = \triangle A_2BC_1$ and, therefore, $\triangle RDS = \triangle PBQ$ and $RS = PQ$. Similarly, $QR = PS$. Therefore, $PQRS$ is a parallelogram such that one of triangles $RDS$ and $PBQ$ is constructed on its sides outwards and on the other side inwards; a similar statement holds for triangles $QCR$ and $SAP$ as well. Therefore, $\angle PQR + \angle RSP = \angle BQC + \angle DSA = 180°$ because $\angle PQB = \angle RSD$ and $\angle RQC = \angle PSA$. It follows that $PQRS$ is a rectangle.

**1.44**. Let $K$, $L$ and $M$ be the intersection points of the circumscribed circles of triangles $FOA$ and $BOC$, $BOC$ and $DOE$, $DOE$ and $FOA$, respectively; $2\alpha$, $2\beta$ and $2\gamma$ the angles at the vertices of isosceles triangles $BOC$, $DOE$ and $FOA$, respectively (Fig. 9).
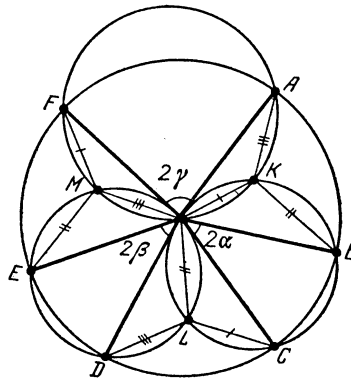
Figure 9 (Sol. 1.44)

Point $K$ lies on arc $\smile OB$ of the circumscribed circle of the isosceles triangle $BOC$ and, therefore, $\angle OKB = 90° + \alpha$. Similarly, $\angle OKA = 90° + \gamma$. Since $\alpha + \beta + \gamma = 90°$, it follows that $\angle AKB = 90° + \beta$. Inside equilateral triangle $AOB$ there exists a unique point $K$ that serves as the vertex of the angles that subtend its sides and are equal to the given angles.

Similar arguments for a point $L$ inside triangle $COD$ show that $\triangle OKB = \triangle CLO$.

Now, let us prove that $\triangle KOL = \triangle OKB$. Indeed, $\angle COL = \angle KBO$; hence, $\angle KOB + \angle COL = 180° - \angle OKB = 90° - \alpha$ and, therefore, $\angle KOL = 2\alpha + (90° - \alpha) = 90° + \alpha = \angle OKB$. It follows that $KL = OB = R$. Similarly, $LM = MK = R$.

**1.45**. Let $\angle A = \alpha$. It is easy to verify that both angles $\angle KCL$ and $\angle ADL$ are equal to $240° - \alpha$ (or $120° + \alpha$). Since $KC = BC = AD$ and $CL = DL$, it follows that $\triangle KCL = \triangle ADL$ and, therefore, $KL = AL$. Similarly, $KL = AK$.

**1.46**. Let $P$, $Q$ and $R$ be the centers of the squares constructed on sides $DA$, $AB$ and $BC$, respectively, in parallelogram $ABCD$ with an acute angle of $\alpha$ at vertex $A$. It is easy to verify that $\angle PAQ = 90° + \alpha = \angle RBQ$; hence, $\triangle PAQ = \triangle RBQ$. Sides $AQ$ and $BQ$ of these triangles are perpendicular, hence, $PQ \perp QR$.

**1.47**. First, observe that the sum of the angles at vertices $A$, $B$ and $C$ of hexagon $AB'CA'BC'$ is equal to $360°$ because by the hypothesis the sum of its angles at the other vertices is equal to $360°$. On side $AC'$, construct outwards triangle $\triangle AC'P$ equal to triangle $\triangle BC'A'$ (Fig. 10).
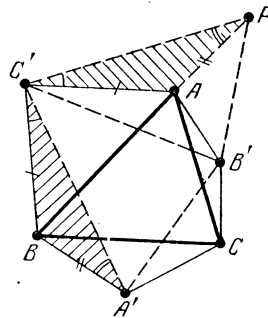


Figure 10 (Sol. 1.47)

Then $\triangle AB'P = \triangle CB'A'$ because $AB' = CB'$, $AP = CA'$ and

$$\angle PAB' = 360° - \angle PAC' - \angle C'AB' = 360° - \angle A'BC' - \angle C'AB' = \angle A'CB'.$$

Hence, $\triangle C'B'A' = \triangle C'B'P$ and, therefore, $2\angle A'B'C' = \angle PB'A' = \angle AB'C$ because $\angle PB'A = \angle A'B'C$.

**1.48**. Since $BA : BC = BC_1 : BA_1$ and $\angle ABC = \angle C_1BA_1$, it follows that $\triangle ABC \sim \triangle C_1BA_1$. Similarly, $\triangle ABC \sim \triangle B_1A_1C$. Since $BA_1 = A_1C$, it follows that $\triangle C_1BA_1 = \triangle B_1A_1C$. Therefore, $AC_1 = C_1B = B_1A_1$ and $AB_1 = B_1C = C_1A_1$. It is also clear that quadrilateral $AB_1A_1C_1$ is a convex one.

**1.49**. a) Let $P$ and $Q$ be the midpoints of sides $AB$ and $AC$. Then $MP = \frac{1}{2}AC = QB_1$, $MQ = \frac{1}{2}AB = PC_1$ and $\angle C_1PM = \angle C_1PB + \angle BPM = \angle B_1QC + \angle CQM = \angle B_1QM$. Hence, $\triangle MQB_1 = \triangle C_1PM$ and, therefore, $MC_1 = MB_1$. Moreover,

$$\angle PMC_1 + \angle QMB_1 = \angle QB_1M + \angle QMB_1 = 180° - \angle MQB_1$$

and

$$\angle MQB_1 = \angle A + \angle CQB_1 = \angle A + (180° - 2\varphi).$$

Therefore, $\angle B_1MC_1 = \angle PMQ + 2\varphi - \angle A = 2\varphi$. (The case when $\angle C_1PB + \angle BPM > 180°$ is analogously treated.)

b) On sides $AB$ and $AC$, take points $B'$ and $C'$, respectively, such that $AB' : AB = AC' : AC = 2 : 3$. The midpoint $M$ of segment $B'C'$ coincides with the intersection point of the medians of triangle $ABC$. On sides $AB'$ and $AC'$, construct outwards right triangles $AB'C_1$ and $AB_1C'$ with angle $\varphi = 60°$ as in heading a). Then $B_1$ and $C_1$ are the centers of right triangles constructed on sides $AB$ and $AC$; on the other hand, by heading a), $MB_1 = MC_1$ and $\angle B_1MC_1 = 120°$.

REMARK. Statements of headings a) and b) remain true for triangles constructed inwards, as well.

**1.50**. a) Let $B'$ be the intersection point of line $AC$ and the perpendicular to line $AB_1$ erected from point $B_1$; define point $C'$ similarly. Since $AB' : AC' = AC_1 : AB_1 = AB : AC$, it follows that $B'C' \parallel BC$. If $N$ is the midpoint of segment $B'C'$, then, as follows from Problem 1.49, $NC_1 = NB_1$ (i.e., $N = M$) and $\angle B_1NC_1 = 2\angle AB'B_1 = 180° - 2\angle CAB_1 = \varphi$.

b) On side $BC$ construct outwards isosceles triangle $BA_1C$ with angle $360° - 2\varphi$ at vertex $A_1$ (if $\varphi < 90°$ construct inwards a triangle with angle $2\varphi$). Since the sum of the angles at the vertices of the three constructed isosceles triangles is equal to $360°$, it follows that the angles of triangle $A_1B_1C_1$ are equal to $180° - \varphi$, $\frac{1}{2}\varphi$ and $\frac{1}{2}\varphi$ (cf. Problem 1.47). In particular, this triangle is an isosceles one, hence, $A_1 = O$.

**1.51**. Let $O_1$, $O_2$, $O_3$ and $O_4$ be the centers of rhombuses constructed on sides $AB$, $BC$, $CA$ and $DA$, respectively; let $M$ be the midpoint of diagonal $AC$. Then $MO_1 = MO_2$ and $\angle O_1MO_2 = \alpha$ (cf. Problem 1.49). Similarly, $MO_3 = MO_4$ and $\angle O_3MO_4 = \alpha$. Therefore, under the rotation through an angle of $\alpha$ about point $M$ triangle $\triangle O_1MO_3$ turns into $\triangle O_2MO_4$.

**1.52**. Since $A_1C = AC|\cos C|$, $B_1C = BC|\cos C|$ and angle $\angle C$ is the common angle of triangles $ABC$ and $A_1B_1C$, these triangles are similar; the similarity coefficient is equal to $|\cos C|$.

**1.53**. Since points $M$ and $N$ lie on the circle with diameter $CH$, it follows that $\angle CMN = \angle CHN$ and since $AC \perp HN$, we see that $\angle CHN = \angle A$. Similarly, $\angle CNM = \angle B$.

**1.54**. a) Let $l$ be the tangent to the circumscribed circle at point $A$. Then $\angle(l, AB) = \angle(AC, CB) = \angle(C_1B_1, AC_1)$ and, therefore, $l \parallel B_1C_1$.

b) Since $OA \perp l$ and $l \parallel B_1C_1$, it follows that $OA \perp B_1C_1$.

**1.55**. If $AA_1$, $BB_1$ and $CC_1$ are heights, then right triangles $AA_1C$ and $BB_1C$ have equal angles at vertex $C$ and, therefore, are similar. It follows that $\triangle A_1BH \sim \triangle B_1AH$, consequently, $AH \cdot A_1H = BH \cdot B_1H$. Similarly, $BH \cdot B_1H = CH \cdot C_1H$.

If $AH \cdot A_1H = BH \cdot B_1H = CH \cdot C_1H$, then $\triangle A_1BH \sim \triangle B_1AH$; hence, $\angle BA_1H = \angle AB_1H = \varphi$. Thus, $\angle CA_1H = \angle CB_1H = 180° - \varphi$.

Similarly, $\angle AC_1H = \angle CA_1H = 180° - \varphi$ and $\angle AC_1H = \angle AB_1H = \varphi$. Hence, $\varphi = 90°$, i.e., $AA_1$, $BB_1$ and $CC_1$ are heights.

**1.56**. a) By Problem 1.52 $\angle C_1A_1B = \angle CA_1B_1 = \angle A$. Since $AA_1 \perp BC$, it follows that $\angle C_1A_1A = \angle B_1A_1A$. The proof of the fact that rays $B_1B$ and $C_1C$ are the bisectors of angles $A_1B_1C_1$ and $A_1C_1B_1$ is similar.

b) Lines $AB$, $BC$ and $CA$ are the bisectors of the outer angles of triangle $A_1B_1C_1$, hence, $A_1A$ is the bisector of angle $\angle B_1A_1C_1$ and, therefore, $AA_1 \perp BC$. For lines $BB_1$ and $CC_1$ the proof is similar.

**1.57**. From the result of Problem 1.56 a) it follows that the symmetry through line $AC$ sends line $B_1A_1$ into line $B_1C_1$.

**1.58**. By Problem 1.52 $\angle B_1A_1C = \angle BAC$. Since $A_1B_1 \parallel AB$, it follows that $\angle B_1A_1C = \angle ABC$. Hence, $\angle BAC = \angle ABC$. Similarly, since $B_1C_1 \parallel BC$, it follows that $\angle ABC = \angle BCA$. Therefore, triangle $ABC$ is an equilateral one and $A_1C_1 \parallel AC$.

**1.59**. Let $O$ be the center of the circumscribed circle of triangle $ABC$. Since $OA \perp B_1C_1$ (cf. Problem 1.54 b), it follows that $S_{AOC_1} + S_{AOB_1} = \frac{1}{2}(R \cdot B_1C_1)$. Similar arguments for vertices $B$ and $C$ show that $S_{ABC} = qR$. On the other hand, $S_{ABC} = pr$.

**1.60**. The perimeter of the triangle cut off by the line parallel to side $BC$ is equal to the sum of distances from point $A$ to the tangent points of the inscribed circle with sides $AB$ and $AC$; therefore, the sum of perimeters of small triangles is equal to the perimeter of triangle $ABC$, i.e., $P_1 + P_2 + P_3 = P$. The similarity of triangles implies that $\frac{r_i}{r} = \frac{P_i}{P}$. Summing these equalities for all the $i$ we get the statement desired.

**1.61**. Let $M = A$. Then $X_A = A$; hence, $AY_A = 1$. Similarly, $CX_C = 1$. Let us prove that $y = AY_A$ and $x = CX_C$ are the desired lines. On side $BC$, take point $D$ so that $AB \parallel MD$, see Fig. 11. Let $E$ be the intersection point of lines $CX_C$ and $MD$. Then, $X_MM + Y_MM = X_CE + Y_MM$. Since $\triangle ABC \sim \triangle MDC$, it follows that $CE = Y_MM$. Therefore, $CE = Y_MM$. Hence, $X_MM + Y_MM = X_CE + CE = X_CC = 1$.
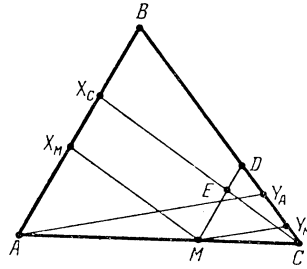


Figure 11 (Sol. 1.61)

**1.62**. Let $D$ be the midpoint of segment $BH$. Since $\triangle BHA \sim \triangle HEA$, it follows that $AD : AO = AB : AH$ and $\angle DAH = \angle OAE$. Hence, $\angle DAO = \angle BAH$ and, therefore, $\triangle DAO \sim \triangle BAH$ and $\angle DOA = \angle BAH = 90°$.

**1.63**. Let $AA_1$, $BB_1$ and $CC_1$ be heights of triangle $ABC$. Let us drop from point $B_1$ perpendiculars $B_1K$ and $B_1N$ to sides $AB$ and $BC$, respectively, and perpendiculars $B_1L$ and $B_1M$ to heights $AA_1$ and $CC_1$, respectively. Since $KB_1 : C_1C = AB_1 : AC = LB_1 : A_1C$, it follows that $\triangle KLB_1 \sim \triangle C_1A_1C$ and, therefore, $KL \parallel C_1A_1$. Similarly, $MN \parallel C_1A_1$. Moreover, $KN \parallel C_1A_1$ (cf. Problem 1.53). It follows that points $K$, $L$, $M$ and $N$ lie on one line.

**1.64**. a) Let $O$ be the midpoint of $AC$, let $O_1$ be the midpoint of $AB$ and $O_2$ the midpoint of $BC$. Assume that $AB \leq BC$. Through point $O_1$ draw line $O_1K$ parallel to $EF$ (point $K$ lies on segment $EO_2$). Let us prove that right triangles $DBO$ and $O_1KO_2$ are equal. Indeed,

$O_1O_2 = DO = \frac{1}{2}AC$ and $BO = KO_2 = \frac{1}{2}(BC - AB)$. Since triangles $DBO$ and $O_1KO_2$ are equal, we see that $\angle BOD = \angle O_1O_2E$, i.e., line $DO$ is parallel to $EO_2$ and the tangent drawn through point $D$ is parallel to line $EF$.

b) Since the angles between the diameter $AC$ and the tangents to the circles at points $F$, $D$, $E$ are equal, it follows that $\angle FAB = \angle DAC = \angle EBC$ and $\angle FBA + \angle DCA = \angle ECB$, i.e., $F$ lies on line segment $AD$ and $E$ lies on line segment $DC$. Moreover, $\angle AFB = \angle BEC = \angle ADC = 90°$ and, therefore, $FDEB$ is a rectangle.

**1.65**. Let $MQ$ and $MP$ be perpendiculars dropped on sides $AD$ and $BC$, let $MR$ and $MT$ be perpendiculars dropped on the extensions of sides $AB$ and $CD$ (Fig. 12). Denote by $M_1$ and $P_1$ the other intersection points of lines $RT$ and $QP$ with the circle.
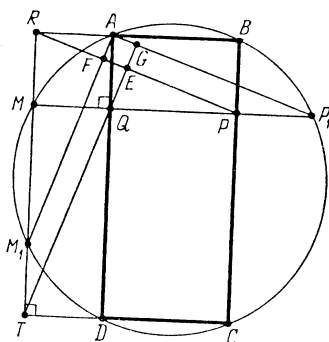


Figure 12 (Sol. 1.65)

Since $TM_1 = RM = AQ$ and $TM_1 \parallel AQ$, it follows that $AM_1 \parallel TQ$. Similarly, $AP_1 \parallel RP$. Since $\angle M_1AP_1 = 90°$, it follows that $RP \perp TQ$.

Denote the intersection points of lines $TQ$ and $RP$, $M_1A$ and $RP$, $P_1A$ and $TQ$ by $E$, $F$, $G$, respectively. To prove that point $E$ lies on line $AC$, it suffices to prove that rectangles $AFEG$ and $AM_1CP_1$ are similar. Since $\angle ARF = \angle AM_1R = \angle M_1TG = \angle M_1CT$, we may denote the values of these angles by the same letter $\alpha$. We have: $AF = RA\sin\alpha = M_1A\sin^2\alpha$ and $AG = M_1T\sin\alpha = M_1C\sin^2\alpha$. Therefore, rectangles $AFEG$ and $AM_1CP_1$ are similar.

**1.66**. Denote the centers of the circles by $O_1$ and $O_2$. The outer tangent is tangent to the first circle at point $K$ and to the other circle at point $L$; the inner tangent is tangent to the first circle at point $M$ and to the other circle at point $N$ (Fig. 13).
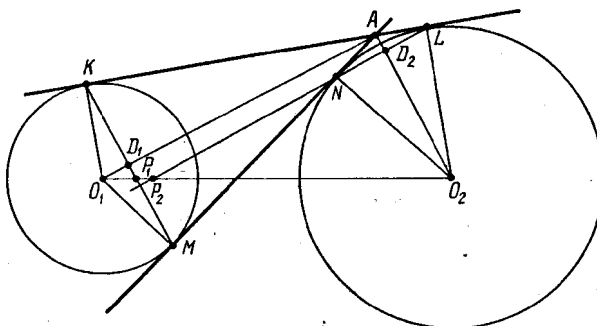


Figure 13 (Sol. 1.66)

Let lines $KM$ and $LN$ intersect line $O_1O_2$ at points $P_1$ and $P_2$, respectively. We have to prove that $P_1 = P_2$. Let us consider points $A$, $D_1$, $D_2$ — the intersection points of $KL$ with $MN$, $KM$ with $O_1A$, and $LN$ with $O_2A$, respectively. Since $\angle O_1AM + \angle NAO_2 = 90°$,

right triangles $O_1MA$ and $ANO_2$ are similar; we also see that $AO_2 \parallel KM$ and $AO_1 \parallel LN$. Since these lines are parallel, $AD_1 : D_1O_1 = O_2P_1 : P_1O_1$ and $D_2O_2 : AD_2 = O_2P_2 : P_2O_1$. The similarity of quadrilaterals $AKO_1M$ and $O_2NAL$ yields $AD_1 : D_1O_1 = D_2O_2 : AD_2$. Therefore, $O_2P_1 : P_1O_1 = O_2P_2 : P_2O_1$, i.e., $P_1 = P_2$.

# CHAPTER 2. INSCRIBED ANGLES

## Background

1. Angle $\angle ABC$ whose vertex lies on a circle and legs intersect this circle is called *inscribed* in the circle. Let $O$ be the center of the circle. Then

$$\angle ABC = \begin{cases} \frac{1}{2}\angle AOC & \text{if points } B \text{ and } O \text{ lie on one side of } AC \\ 180° - \frac{1}{2}\angle AOC & \text{otherwise.} \end{cases}$$

The most important and most often used corollary of this fact is that *equal chords subtend angles that either are equal or the sum of the angles is equal to* 180°.

2. The value of the angle between chord $AB$ and the tangent to the circle that passes through point $A$ is equal to half the angle value of arc $\smile AB$.

3. The angle values of arcs confined between parallel chords are equal.

4. As we have already said, if two angles subtend the same chord, either they are equal or the sum of their values is 180°. In order not to consider various variants of the positions of points on the circle let us introduce the notion of an *oriented angle* between lines. The *value of the oriented angle between lines $AB$ and $CD$* (notation: $\angle(AB, CD)$) is the value of the angle by which we have to rotate line $AB$ counterclockwise in order for it to become parallel to line $CD$. The angles that differ by $n \cdot 180°$ are considered equal.

Notice that, generally, the oriented angle between lines $CD$ and $AB$ is not equal to the oriented angle between lines $AB$ and $CD$ (the sum of $\angle(AB, CD)$ and $\angle(CD, AB)$ is equal to 180° which, according to our convention, is the same as 0°).

It is easy to verify the following properties of the oriented angles:

a) $\angle(AB, BC) = -\angle(BC, AB)$;

b) $\angle(AB, CD) + \angle(CD, EF) = \angle(AB, EF)$;

c) *points $A$, $B$, $C$, $D$ not on one line lie on one circle if and only if* $\angle(AB, BC) = \angle(AD, DC)$. (To prove this property we have to consider two cases: points $B$ and $D$ lie on one side of $AC$; points $B$ and $D$ lie on different sides of $AC$.)

## Introductory problems

**1.** a) From point $A$ lying outside a circle rays $AB$ and $AC$ come out and intersect the circle. Prove that the value of angle $\angle BAC$ is equal to half the difference of the angle measures of the arcs of the circle confined inside this angle.

b) The vertex of angle $\angle BAC$ lies inside a circle. Prove that the value of angle $\angle BAC$ is equal to half the sum of angle measures of the arcs of the circle confined inside angle $\angle BAC$ and inside the angle symmetric to it through vertex $A$.

**2.** From point $P$ inside acute angle $\angle BAC$ perpendiculars $PC_1$ and $PB_1$ are dropped on lines $AB$ and $AC$. Prove that $\angle C_1AP = \angle C_1B_1P$.

**3.** Prove that all the angles formed by the sides and diagonals of a regular $n$-gon are integer multiples of $\frac{180°}{n}$.

**4.** The center of an inscribed circle of triangle $ABC$ is symmetric through side $AB$ to the center of the circumscribed circle. Find the angles of triangle $ABC$.

**5.** The bisector of the exterior angle at vertex $C$ of triangle $ABC$ intersects the circumscribed circle at point $D$. Prove that $AD = BD$.

## §1. Angles that subtend equal arcs

**2.1.** Vertex $A$ of an acute triangle $ABC$ is connected by a segment with the center $O$ of the circumscribed circle. From vertex $A$ height $AH$ is drawn. Prove that $\angle BAH = \angle OAC$.

**2.2.** Two circles intersect at points $M$ and $K$. Lines $AB$ and $CD$ are drawn through $M$ and $K$, respectively; they intersect the first circle at points $A$ and $C$, the second circle at points $B$ and $D$, respectively. Prove that $AC \parallel BD$.

**2.3.** From an arbitrary point $M$ inside a given angle with vertex $A$ perpendiculars $MP$ and $MQ$ are dropped to the sides of the angle. From point $A$ perpendicular $AK$ is dropped on segment $PQ$. Prove that $\angle PAK = \angle MAQ$.

**2.4.**  a) The continuation of the bisector of angle $\angle B$ of triangle $ABC$ intersects the circumscribed circle at point $M$; $O$ is the center of the inscribed circle, $O_b$ is the center of the escribed circle tangent to $AC$. Prove that points $A$, $C$, $O$ and $O_b$ lie on a circle centered at $M$.

b) Point $O$ inside triangle $ABC$ is such that lines $AO$, $BO$ and $CO$ pass through the centers of the circumscribed circles of triangles $BCO$, $ACO$ and $ABO$, respectively. Prove that $O$ is the center of the inscribed circle of triangle $ABC$.

**2.5.** Vertices $A$ and $B$ of right triangle $ABC$ with right angle $\angle C$ slide along the sides of a right angle with vertex $P$. Prove that in doing so point $C$ moves along a line segment.

**2.6.** Diagonal $AC$ of square $ABCD$ coincides with the hypothenuse of right triangle $ACK$, so that points $B$ and $K$ lie on one side of line $AC$. Prove that

$$BK = \frac{|AK - CK|}{\sqrt{2}} \quad \text{and} \quad DK = \frac{AK + CK}{\sqrt{2}}.$$

**2.7.** In triangle $ABC$ medians $AA_1$ and $BB_1$ are drawn. Prove that if $\angle CAA_1 = \angle CBB_1$, then $AC = BC$.

**2.8.** Each angle of triangle $ABC$ is smaller than $120°$. Prove that inside $\triangle ABC$ there exists a point that serves as the vertex for three angles each of value $120°$ and subtending the side of the triangle different from the sides subtended by the other angles.

**2.9.** A circle is divided into equal arcs by $n$ diameters. Prove that the bases of the perpendiculars dropped from an arbitrary point $M$ inside the circle to these diameters are vertices of a regular $n$-gon.

**2.10.** Points $A$, $B$, $M$ and $N$ on a circle are given. From point $M$ chords $MA_1$ and $MB_1$ perpendicular to lines $NB$ and $NA$, respectively, are drawn. Prove that $AA_1 \parallel BB_1$.

**2.11.** Polygon $ABCDEF$ is an inscribed one; $AB \parallel DE$ and $BC \parallel EF$. Prove that $CD \parallel AF$.

**2.12.** Polygon $A_1 A_2 \dots A_{2n}$ as an inscribed one. We know that all the pairs of its opposite sides except one are parallel. Prove that for any odd $n$ the remaining pair of sides is also parallel and for any even $n$ the lengths of the exceptional sides are equal.

**2.13.** Consider triangle $ABC$. Prove that there exist two families of equilateral triangles whose sides (or extensions of the sides) pass through points $A$, $B$ and $C$. Prove also that the centers of triangles from these families lie on two concentric circles.

## §2. The value of an angle between two chords

The following fact helps to solve problems from this section. Let $A$, $B$, $C$, $D$ be points on a circle situated in the order indicated. Then

$$\angle(AC, BD) = \frac{\smile AB + \smile CD}{2} \quad \text{and} \quad \angle(AB, CD) = \frac{|\smile AD - \smile CB|}{2}.$$

To prove this, we have to draw a chord parallel to another chord through the endpoint of one of the chords.

**2.14.** Points $A$, $B$, $C$, $D$ in the indicated order are given on a circle. Let $M$ be the midpoint of arc $\smile AB$. Denote the intersection points of chords $MC$ and $MD$ with chord $AB$ by $E$ and $K$. Prove that $KECD$ is an inscribed quadrilateral.

**2.15.** Concider an equilateral triangle. A circle with the radius equal to the triangle's height rolls along a side of the triangle. Prove that the angle measure of the arc cut off the circle by the sides of the triangle is always equal to $60°$.

**2.16.** The diagonals of an isosceles trapezoid $ABCD$ with lateral side $AB$ intersect at point $P$. Prove that the center $O$ of the inscribed circle lies on the inscribed circle of triangle $APB$.

**2.17.** Points $A$, $B$, $C$, $D$ in the indicated order are given on a circle; points $A_1$, $B_1$, $C_1$ and $D_1$ are the midpoints of arcs $\smile AB$, $\smile BC$, $\smile CD$ and $\smile DA$, respectively. Prove that $A_1C_1 \perp B_1D_1$.

**2.18.** Point $P$ inside triangle $ABC$ is taken so that $\angle BPC = \angle A + 60°$, $\angle APC = \angle B + 60°$ and $\angle APB = \angle C + 60°$. Lines $AP$, $BP$ and $CP$ intersect the circumscribed circle of triangle $ABC$ at points $A'$, $B'$ and $C'$, respectively. Prove that triangle $A'B'C'$ is an equilateral one.

**2.19.** Points $A$, $C_1$, $B$, $A_1$, $C$, $B_1$ in the indicated order are taken on a circle.

a) Prove that if lines $AA_1$, $BB_1$ and $CC_1$ are the bisectors of the angles of triangle $ABC$, then they are the heights of triangle $A_1B_1C_1$.

b) Prove that if lines $AA_1$, $BB_1$ and $CC_1$ are the heights of triangle $ABC$, then they are the bisectors of the angles of triangle $A_1B_1C_1$.

**2.20.** Triangles $T_1$ and $T_2$ are inscribed in a circle so that the vertices of triangle $T_2$ are the midpoints of the arcs into which the circle is divided by the vertices of triangle $T_1$. Prove that in the hexagon which is the intersection of triangles $T_1$ and $T_2$ the diagonals that connect the opposite vertices are parallel to the sides of triangle $T_1$ and meet at one point.

## §3. The angle between a tangent and a chord

**2.21.** Two circles intersect in points $P$ and $Q$. Through point $A$ on the first circle lines $AP$ and $AQ$ are drawn. The lines intersect the second circle in points $B$ and $C$. Prove that the tangent at $A$ to the first circle is parallel to line $BC$.

**2.22.** Circles $S_1$ and $S_2$ intersect at points $A$ and $P$. Tangent $AB$ to circle $S_1$ is drawn through point $A$, and line $CD$ parallel to $AB$ is drawn through point $P$ (points $B$ and $C$ lie on $S_2$, point $D$ on $S_1$). Prove that $ABCD$ is a parallelogram.

**2.23.** The tangent at point $A$ to the inscribed circle of triangle $ABC$ intersects line $BC$ at point $E$; let $AD$ be the bisector of triangle $ABC$. Prove that $AE = ED$.

**2.24.** Circles $S_1$ and $S_2$ intersect at point $A$. Through point $A$ a line that intersects $S_1$ at point $B$ and $S_2$ at point $C$ is drawn. Through points $C$ and $B$ tangents to the circles are drawn; the tangents intersect at point $D$. Prove that angle $\angle BDC$ does not depend on the choice of the line that passes through $A$.

**2.25.** Two circles intersect at points $A$ and $B$. Through point $A$ tangents $AM$ and $AN$, where $M$ and $N$ are points of the respective circles, are drawn. Prove that:

a) $\angle ABN + \angle MAN = 180°$;

b) $\frac{BM}{BN} = \left(\frac{AM}{AN}\right)^2$.

**2.26.** Inside square $ABCD$ a point $P$ is taken so that triangle $ABP$ is an equilateral one. Prove that $\angle PCD = 15°$.

**2.27.** Two circles are internally tangent at point $M$. Let $AB$ be the chord of the greater circle which is tangent to the smaller circle at point $T$. Prove that $MT$ is the bisector of angle $AMB$.

**2.28.** Through point $M$ inside circle $S$ chord $AB$ is drawn; perpendiculars $MP$ and $MQ$ are dropped from point $M$ to the tangents that pass through points $A$ and $B$ respectively. Prove that the value of $\frac{1}{PM} + \frac{1}{QM}$ does not depend on the choice of the chord that passes through point $M$.

**2.29.** Circle $S_1$ is tangent to sides of angle $ABC$ at points $A$ and $C$. Circle $S_2$ is tangent to line $AC$ at point $C$ and passes through point $B$, circle $S_2$ intersects circle $S_1$ at point $M$. Prove that line $AM$ divides segment $BC$ in halves.

**2.30.** Circle $S$ is tangent to circles $S_1$ and $S_2$ at points $A_1$ and $A_2$; let $B$ be a point of circle $S$, let $K_1$ and $K_2$ be the other intersection points of lines $A_1B$ and $A_2B$ with circles $S_1$ and $S_2$, respectively. Prove that if line $K_1K_2$ is tangent to circle $S_1$, then it is also tangent to circle $S_2$.

## §4. Relations between the values of an angle and the lengths of the arc and chord associated with the angle

**2.31.** Isosceles trapezoids $ABCD$ and $A_1B_1C_1D_1$ with parallel respective sides are inscribed in a circle. Prove that $AC = A_1C_1$.

**2.32.** From point $M$ that moves along a circle perpendiculars $MP$ and $MQ$ are dropped on diameters $AB$ and $CD$, respectively. Prove that the length of segment $PQ$ does not depend on the position of point $M$.

**2.33.** In triangle $ABC$, angle $\angle B$ is equal to $60°$; bisectors $AD$ and $CE$ intersect at point $O$. Prove that $OD = OE$.

**2.34.** In triangle $ABC$ the angles at vertices $B$ and $C$ are equal to $40°$; let $BD$ be the bisector of angle $B$. Prove that $BD + DA = BC$.

**2.35.** On chord $AB$ of circle $S$ centered at $O$ a point $C$ is taken. The circumscribed circle of triangle $AOC$ intersects circle $S$ at point $D$. Prove that $BC = CD$.

**2.36.** Vertices $A$ and $B$ of an equilateral triangle $ABC$ lie on circle $S$, vertex $C$ lies inside this circle. Point $D$ lies on circle $S$ and $BD = AB$. Line $CD$ intersects $S$ at point $E$. Prove that the length of segment $EC$ is equal to the radius of circle $S$.

**2.37.** Along a fixed circle another circle whose radius is half that of the fixed one rolls on the inside without gliding. What is the trajectory of a fixed point $K$ of the rolling circle?

## §5. Four points on one circle

**2.38.** From an arbitrary point $M$ on leg $BC$ of right triangle $ABC$ perpendicular $MN$ is dropped on hypothenuse $AP$. Prove that $\angle MAN = \angle MCN$.

**2.39.** The diagonals of trapezoid $ABCD$ with bases $AD$ and $BC$ intersect at point $O$; points $B'$ and $C'$ are symmetric through the bisector of angle $\angle BOC$ to vertices $B$ and $C$, respectively. Prove that $\angle C'AC = \angle B'DB$.

**2.40.** The extensions of sides $AB$ and $CD$ of the inscribed quadrilateral $ABCD$ meet at point $P$; the extensions of sides $BC$ and $AD$ meet at point $Q$. Prove that the intersection points of the bisectors of angles $\angle AQB$ and $\angle BPC$ with the sides of the quadrilateral are vertices of a rhombus.

**2.41.** The inscribed circle of triangle $ABC$ is tangent to sides $AB$ and $AC$ at points $M$ and $N$, respectively. Let $P$ be the intersection point of line $MN$ with the bisector (or its extension) of angle $\angle B$. Prove that:

a) $\angle BPC = 90°$;

b) $S_{ABP} : S_{ABC} = 1 : 2$.

**2.42.** Inside quadrilateral $ABCD$ a point $M$ is taken so that $ABMD$ is a parallelogram. Prove that if $\angle CBM = \angle CDM$, then $\angle ACD = \angle BCM$.

**2.43.** Lines $AP$, $BP$ and $CP$ intersect the circumscribed circle of triangle $ABC$ at points $A_1$, $B_1$ and $C_1$, respectively. On lines $BC$, $CA$ and $AB$ points $A_2$, $B_2$ and $C_2$, respectively, are taken so that $\angle(PA_2, BC) = \angle(PB_2, CA) = \angle(PC_2, AB)$. Prove that $\triangle A_2 B_2 C_2 \sim \triangle A_1 B_1 C_1$.

**2.44.** About an equilateral triangle $APQ$ a rectangular $ABCD$ is circumscribed so that points $P$ and $Q$ lie on sides $BC$ and $CD$, respectively; $P'$ and $Q'$ are the midpoints of sides $AP$ and $AQ$, respectively. Prove that triangles $BQ'C$ and $CP'D$ are equilateral ones.

**2.45.** Prove that if for inscribed quadrilateral $ABCD$ the equality $CD = AD + BC$ holds, then the intersection point of the bisectors of angles $\angle A$ and $\angle B$ lies on side $CD$.

**2.46.** Diagonals $AC$ and $CE$ of a regular hexagon $ABCDEF$ are divided by points $M$ and $N$, respectively, so that $AM : AC = CN : CE = \lambda$. Find $\lambda$ if it is known that points $B$, $M$ and $N$ lie on a line.

**2.47.** The corresponding sides of triangles $ABC$ and $A_1 B_1 C_1$ are parallel and sides $AB$ and $A_1 B_1$ lie on one line. Prove that the line that connects the intersection points of the circumscribed circles of triangles $A_1 BC$ and $AB_1 C$ contains point $C_1$.

**2.48.** In triangle $ABC$ heights $AA_1$, $BB_1$ and $CC_1$ are drawn. Line $KL$ is parallel to $CC_1$; points $K$ and $L$ lie on lines $BC$ and $B_1 C_1$, respectively. Prove that the center of the circumscribed circle of triangle $A_1 KL$ lies on line $AC$.

**2.49.** Through the intersection point $O$ of the bisectors of triangle $ABC$ line $MN$ is drawn perpendicularly to $CO$ so that $M$ and $N$ lie on sides $AC$ and $BC$, respectively. Lines $AO$ and $BO$ intersect the circumscribed circle of triangle $ABC$ at points $A'$ and $B'$, respectively. Prove that the intersection point of lines $A'N$ and $B'M$ lies on the circumscribed circle.

## §6. The inscribed angle and similar triangles

**2.50.** Points $A$, $B$, $C$ and $D$ on a circle are given. Lines $AB$ and $CD$ intersect at point $M$. Prove that

$$\frac{AC \cdot AD}{AM} = \frac{BC \cdot BD}{BM}.$$

**2.51.** Points $A$, $B$ and $C$ on a circle are given; the distance $BC$ is greater than the distance from point $B$ to line $l$ tangent to the circle at point $A$. Line $AC$ intersects the line drawn through point $B$ parallelly to $l$ at point $D$. Prove that $AB^2 = AC \cdot AD$.

**2.52.** Line $l$ is tangent to the circle of diameter $AB$ at point $C$; points $M$ and $N$ are the projections of points $A$ and $B$ on line $l$, respectively, and $D$ is the projection of point $C$ on $AB$. Prove that $CD^2 = AM \cdot BN$.

**2.53.** In triangle $ABC$, height $AH$ is drawn and from vertices $B$ and $C$ perpendiculars $BB_1$ and $CC_1$ are dropped on the line that passes through point $A$. Prove that $\triangle ABC \sim \triangle HB_1 C_1$.

**2.54.** On arc $\smile BC$ of the circle circumscribed about equilateral triangle $ABC$, point $P$ is taken. Segments $AP$ and $BC$ intersect at point $Q$. Prove that

$$\frac{1}{PQ} = \frac{1}{PB} + \frac{1}{PC}.$$

**2.55.** On sides $BC$ and $CD$ of square $ABCD$ points $E$ and $F$ are taken so that $\angle EAF = 45°$. Segments $AE$ and $AF$ intersect diagonal $BD$ at points $P$ and $Q$, respectively. Prove that $\frac{S_{AEF}}{S_{APQ}} = 2$.

**2.56.** A line that passes through vertex $C$ of equilateral triangle $ABC$ intersects base $AB$ at point $M$ and the circumscribed circle at point $N$. Prove that

$$CM \cdot CN = AC^2 \quad \text{and} \quad \frac{CM}{CN} = \frac{AM \cdot BM}{AN \cdot BN}.$$

**2.57.** Consider parallelogram $ABCD$ with an acute angle at vertex $A$. On rays $AB$ and $CB$ points $H$ and $K$, respectively, are marked so that $CH = BC$ and $AK = AB$. Prove that:

a) $DH = DK$;

b) $\triangle DKH \sim \triangle ABK$.

**2.58.** a) The legs of an angle with vertex $C$ are tangent to a circle at points $A$ and $B$. From point $P$ on the circle perpendiculars $PA_1$, $PB_1$ and $PC_1$ are dropped on lines $BC$, $CA$ and $AB$, respectively. Prove that $PC_1^2 = PA_1 \cdot PB_1$.

b) From point $O$ of the inscribed circle of triangle $ABC$ perpendiculars $OA'$, $OB'$, $OC'$ are dropped on the sides of triangle $ABC$ opposite to vertices $A$, $B$ and $C$, respectively, and perpendiculars $OA''$, $OB''$, $OC'''$ are dropped to the sides of the triangle with vertices at the tangent points. Prove that

$$OA' \cdot OB' \cdot OC' = OA'' \cdot OB'' \cdot OC''.$$

**2.59.** Pentagon $ABCDE$ is inscribed in a circle. Distances from point $E$ to lines $AB$, $BC$ and $CD$ are equal to $a$, $b$ and $c$, respectively. Find the distance from point $E$ to line $AD$.

**2.60.** In triangle $ABC$, heights $AA_1$, $BB_1$ and $CC_1$ are drawn; $B_2$ and $C_2$ are the midpoints of heights $BB_1$ and $CC_1$, respectively. Prove that $\triangle A_1 B_2 C_2 \sim \triangle ABC$.

**2.61.** On heights of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ that divide them in the ratio $2 : 1$ counting from the vertex are taken. Prove that $\triangle A_1 B_1 C_1 \sim \triangle ABC$.

**2.62.** Circle $S_1$ with diameter $AB$ intersects circle $S_2$ centered at $A$ at points $C$ and $D$. Through point $B$ a line is drawn; it intersects $S_2$ at point $M$ that lies inside $S_1$ and it intersects $S_1$ at point $N$. Prove that $MN^2 = CN \cdot ND$.

**2.63.** Through the midpoint $C$ of an arbitrary chord $AB$ on a circle chords $KL$ and $MN$ are drawn so that points $K$ and $M$ lie on one side of $AB$. Segments $KN$ and $ML$ intersect $AB$ at points $Q$ and $P$, respectively. Prove that $PC = QC$.

**2.64.** a) A circle that passes through point $C$ intersects sides $BC$ and $AC$ of triangle $ABC$ at points $A_1$ and $B_1$, respectively, and it intersects the circumscribed circle of triangle $ABC$ at point $M$. Prove that $\triangle AB_1 M \sim \triangle BA_1 M$.

b) On rays $AC$ and $BC$ segments $AA_1$ and $BB_1$ equal to the semiperimeter of triangle $ABC$ are drawn. Let $M$ be a point on the circumscribed circle such that $CM \parallel A_1 B_1$. Prove that $\angle CMO = 90°$, where $O$ is the center of the inscribed circle.

## §7. The bisector divides an arc in halves

**2.65.** In triangle $ABC$, sides $AC$ and $BC$ are not equal. Prove that the bisector of angle $\angle C$ divides the angle between the median and the height drawn from this vertex in halves if and only if $\angle C = 90°$.

**2.66.** It is known that in a triangle the median, the bisector and the height drawn from vertex $C$ divide the angle $\angle C$ into four equal parts. Find the angles of this triangle.

**2.67.** Prove that in triangle $ABC$ bisector $AE$ lies between median $AM$ and height $AH$.

**2.68.** Given triangle $ABC$; on its side $AB$ point $P$ is chosen; lines $PM$ and $PN$ parallel to $AC$ and $BC$, respectively, are drawn through $P$ so that points $M$ and $N$ lie on sides $BC$ and $AC$, respectively; let $Q$ be the intersection point of the circumscribed circles of triangles $APN$ and $BPM$. Prove that all lines $PQ$ pass through a fixed point.

**2.69.** The continuation of bisector $AD$ of acute triangle $ABC$ inersects the circumscribed circle at point $E$. Perpendiculars $DP$ and $DQ$ are dropped on sides $AB$ and $AC$ from point $D$. Prove that $S_{ABC} = S_{APEQ}$.

## §8. An inscribed quadrilateral with perpendicular diagonals

In this section $ABCD$ is an inscribed quadrilateral whose diagonals intersect at a right angle. We will also adopt the following notations: $O$ is the center of the circumscribed circle of quadrilateral $ABCD$ and $P$ is the intersection point of its diagonals.

**2.70.** Prove that the broken line $AOC$ divides $ABCD$ into two parts whose areas are equal.

**2.71.** The radius of the circumscribed circle of quadrilateral $ABCD$ is equal to $R$.

a) Find $AP^2 + BP^2 + CP^2 + DP^2$.

b) Find the sum of squared lengths of the sides of $ABCD$.

**2.72.** Find the sum of squared lengths of the diagonals of $ABCD$ if the length of segment $OP$ and the radius of the circumscribed circle $R$ are known.

**2.73.** From vertices $A$ and $B$ perpendiculars to $CD$ that intersect lines $BD$ and $AC$ at points $K$ and $L$, respectively, are drawn. Prove that $AKLB$ is a rhombus.

**2.74.** Prove that the area of quadrilateral $ABCD$ is equal to $\frac{1}{2}(AB \cdot CD + BC \cdot AD)$.

**2.75.** Prove that the distance from point $O$ to side $AB$ is equal to half the length of side $CD$.

**2.76.** Prove that the line drawn through point $P$ perpendicularly to $BC$ divides side $AD$ in halves.

**2.77.** Prove that the midpoints of the sides of quadrilateral $ABCD$ and the projections of point $P$ on the sides lie on one circle.

**2.78.** a) Through vertices $A$, $B$, $C$ and $D$ tangents to the circumscribed circle are drawn. Prove that the quadrilateral formed by them is an inscribed one.

b) Quadrilateral $KLMN$ is simultaneously inscribed and circumscribed; $A$ and $B$ are the tangent points of the inscribed circle with sides $KL$ and $LM$, respectively. Prove that $AK \cdot BM = r^2$, where $r$ is the radius of the inscribed circle.

## §9. Three circumscribed circles intersect at one point

**2.79.** On sides of triangle $ABC$ triangles $ABC'$, $AB'C$ and $A'BC$ are constructed outwards so that the sum of the angles at vertices $A'$, $B'$ and $C'$ is a multiple of $180°$. Prove that the circumscribed circles of the constructed triangles intersect at one point.

**2.80.** a) On sides (or their extensions) $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ distinct from the vertices of the triangle are taken (one point on one side). Prove that the circumscribed circles of triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$ intersect at one point.

b) Points $A_1$, $B_1$ and $C_1$ move along lines $BC$, $CA$ and $AB$, respectively, so that all triangles $A_1B_1C_1$ are similar and equally oriented. Prove that the intersection point of the circumscribed circles of triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$ remains fixed in the process.

**2.81.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ are taken. Prove that if triangles $A_1B_1C_1$ and $ABC$ are similar and have opposite orientations, then circumscribed circles of triangles $AB_1C_1$, $ABC_1$ and $A_1B_1C$ pass through the center of the circumscribed circle of triangle $ABC$.

**2.82.** Points $A'$, $B'$ and $C'$ are symmetric to a point $P$ relative sides $BC$, $CA$ and $AB$, respectively, of triangle $ABC$.

a) The circumscribed circles of triangles $AB'C'$, $A'BC'$, $A'B'C$ and $ABC$ have a common point;

b) the circumscribed circles of triangles $A'BC$, $AB'C$, $ABC'$ and $A'B'C'$ have a common point $Q$;

c) Let $I$, $J$, $K$ and $O$ be the centers of the circumscribed circles of triangles $A'BC$, $AB'C$, $ABC'$ and $A'B'C'$, respectively. Prove that $QI : OI = QJ : OJ = QK : OK$.

## §10. Michel's point

**2.83.** Four lines form four triangles. Prove that

a) The circumscribed circles of these triangles have a common point. (*Michel's point.*)

b) The centers of the circumscribed circles of these triangles lie on one circle that passes through Michel's point.

**2.84.** A line intersects sides (or their extensions) $AB$, $BC$ and $CA$ of triangle $ABC$ at points $C_1$, $B_1$ and $A_1$, respectively; let $O$, $O_a$, $O_b$ and $O_c$ be the centers of the circumscribed circles of triangles $ABC$, $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$, respectively; let $H$, $H_a$, $H_b$ and $H_c$ be the respective orthocenters of these triangles. Prove that

a) $\triangle O_aO_bO_c \sim \triangle ABC$.

b) the midperpendiculars to segments $OH$, $O_aH_a$, $O_bH_b$ and $O_cH_c$ meet at one point.

**2.85.** Quadrilateral $ABCD$ is an inscribed one. Prove that Michel's point of lines that contain its sides lies on the segment that connects the intersection points of the extensions of the sides.

**2.86.** Points $A$, $B$, $C$ and $D$ lie on a circle centered at $O$. Lines $AB$ and $CD$ intersect at point $E$ and the circumscribed circles of triangles $AEC$ and $BED$ ilntersect at points $E$ and $P$. Prove that

a) points $A$, $D$, $P$ and $O$ lie on one circle;

b) $\angle EPO = 90°$.

**2.87.** Given four lines prove that the projections of Michel's point to these lines lie on one line.

See also Problem 19.45.

## §11. Miscellaneous problems

**2.88.** In triangle $ABC$ height $AH$ is drawn; let $O$ be the center of the circumscribed circle. Prove that $\angle OAH = |\angle B - \angle C|$.

**2.89.** Let $H$ be the intersection point of the heights of triangle $ABC$; let $AA'$ be a diameter of its circumscribed circle. Prove that segment $A'H$ divides side $BC$ in halves.

**2.90.** Through vertices $A$ and $B$ of triangle $ABC$ two parallel lines are drawn and lines $m$ and $n$ are symmetric to them through the bisectors of the corresponding angles. Prove that the intersection point of lines $m$ and $n$ lies on the circumscribed circle of triangle $ABC$.

**2.91.** a) Lines tangent to circle $S$ at points $B$ and $C$ are drawn from point $A$. Prove that the center of the inscribed circle of triangle $ABC$ and the center of its escribed circle tangent to side $BC$ lie on circle $S$.

b) Prove that the circle that passes through vertices $B$ and $C$ of any triangle $ABC$ and the center $O$ of its inscribed circle intercepts on lines $AB$ and $AC$ chords of equal length.

**2.92.** On sides $AC$ and $BC$ of triangle $ABC$ squares $ACA_1A_2$ and $BCB_1B_2$ are constructed outwards. Prove that lines $A_1B$, $A_2B_2$ and $AB_1$ meet at one point.

**2.93.** Circles $S_1$ and $S_2$ intersect at points $A$ and $B$ so that the tangents to $S_1$ at these points are radii of $S_2$. On the inner arc of $S_1$ a point $C$ is taken; straight lines connect it with points $A$ and $B$. Prove that the second intersection points of these lines with $S_2$ are the endpoints of a diameter.

**2.94.** From the center $O$ of a circle the perpendicular $OA$ is dropped to line $l$. On $l$, points $B$ and $C$ are taken so that $AB = AC$. Through points $B$ and $C$ two sections are drawn one of which intersects the circle at points $P$ and $Q$ and the other one at points $M$ and $N$. Lines $PM$ and $QN$ intersect line $l$ at points $R$ and $S$, respectively. Prove that $AR = AS$.

## Problems for independent study

**2.95.** In triangle $ABC$ heights $AA_1$ and $BB_1$ are drawn; let $M$ be the midpoint of side $AB$. Prove that $MA_1 = MB_1$.

**2.96.** In convex quadrilateral $ABCD$ angles $\angle A$ and $\angle C$ are right ones. Prove that $AC = BD \sin ABC$.

**2.97.** Diagonals $AD$, $BE$ and $CF$ of an inscribed hexagon $ABCDEF$ meet at one point. Prove that $AB \cdot CD \cdot EF = BC \cdot DE \cdot AF$.

**2.98.** In a convex quadrilateral $AB = BC = CD$, let $M$ be the intersection point of diagonals, $K$ is the intersection point of bisectors of angles $\angle A$ and $\angle D$. Prove that points $A$, $M$, $K$ and $D$ lie on one circle.

**2.99.** Circles centered at $O_1$ and $O_2$ intersect at points $A$ and $B$. Line $O_1A$ intersects the circle centered at $O_2$ at point $N$. Prove that points $O_1$, $O_2$, $B$ and $N$ lie on one circle.

**2.100.** Circles $S_1$ and $S_2$ intersect at points $A$ and $B$. Line $MN$ is tangent to circle $S_1$ at point $M$ and to $S_2$ at point $N$. Let $A$ be the intersection point of the circles, which is more distant from line $MN$. Prove that $\angle O_1AO_2 = 2\angle MAN$.

**2.101.** Given quadrilateral $ABCD$ inscribed in a circle and such that $AB = BC$, prove that $S_{ABCD} = \frac{1}{2}(DA + CD) \cdot h_b$, where $h_b$ is the height of triangle $ABD$ dropped from vertex $B$.

**2.102.** Quadrilateral $ABCD$ is an inscribed one and $AC$ is the bisector of angle $\angle DAB$. Prove that $AC \cdot BD = AD \cdot DC + AB \cdot BC$.

**2.103.** In right triangle $ABC$, bisector $CM$ and height $CH$ are drawn from the vertex of the right angle $\angle C$. Let $HD$ and $HE$ be bisectors of triangles $AHC$ and $CHB$. Prove that points $C$, $D$, $H$, $E$ and $M$ lie on one circle.

**2.104.** Two circles pass through the vertex of an angle and a point on its bisector. Prove that the segments cut by them on the sides of the angle are equal.

**2.105.** Triangle $BHC$, where $H$ is the orthocenter of triangle $ABC$ is complemented to the parallelogram $BHCD$. Prove that $\angle BAD = \angle CAH$.

**2.106.** Outside equilateral triangle $ABC$ but inside angle $\angle BAC$, point $M$ is taken so that $\angle CMA = 30°$ and $\angle BMA = \alpha$. What is the value of angle $\angle ABM$?

**2.107.** Prove that if the inscribed quadrilateral with perpendicular diagonals is also a circumscribed one, then it is symmetric with respect to one of its diagonals.

## Solutions

**2.1.** Let us draw diameter $AD$. Then $\angle CDA = \angle CBA$; hence, $\angle BAH = \angle DAC$ because $\angle BHA = \angle ACD = 90°$.

**2.2.** Let us make use of the properties of oriented angles:

$$\angle(AC, CK) = \angle(AM, MK) = \angle(BM, MK) = \angle(BD, DK) = \angle(BD, CK),$$

i.e., $AC \parallel BD$.

**2.3.** Points $P$ and $Q$ lie on the circle with diameter $AM$. Therefore, $\angle QMA = \angle QPA$ as angles that intersect the same arc. Triangles $PAK$ and $MAQ$ are right ones, therefore, $\angle PAK = \angle MAQ$.

**2.4.** a) Since

$$\angle AOM = \angle BAO + \angle ABO = \frac{\angle A + \angle B}{2}$$

and

$$\angle OAM = \angle OAC + \angle CAM = \frac{\angle A}{2} + \angle CBM = \frac{\angle A + \angle B}{2},$$

we have $MA = MO$. Similarly, $MC = MO$.

Since triangle $OAO_b$ is a right one and $\angle AOM = \angle MAO = \varphi$, it follows that $\angle MAO_b = \angle MO_bA = 90° - \varphi$ and, therefore, $MA = MO_b$. Similarly, $MC = MO_b$.

b) Let $P$ be the center of the circumscribed circle of triangle $ACO$. Then

$$\angle COP = \frac{180° - \angle CPO}{2} = 90° - \angle OAC.$$

Hence, $\angle BOC = 90° + \angle OAC$. Similarly, $\angle BOC = 90° + \angle OAB$ and, therefore, $\angle OAB = \angle OAC$. We similarly establish that point $O$ lies on the bisectors of angles $\angle B$ and $\angle C$.

**2.5.** Points $P$ and $C$ lie on the circle with diameter $AB$, and, therefore, $\angle APC = \angle ABC$, i.e., the value of angle $\angle APC$ is a constant.

REMARK. A similar statement is true for any triangle $ABC$ whose vertices are moving along the legs of angle $\angle MPN$ equal to $180° - \angle C$.

**2.6.** Points $B$, $D$ and $K$ lie on the circle with diameter $AC$. Let, for definiteness sake, $\angle KCA = \varphi \le 45°$. Then

$$BK = AC \sin(45° - \varphi) = \frac{AC(\cos \varphi - \sin \varphi)}{\sqrt{2}}$$

and

$$DK = AC \sin(45° + \varphi) = \frac{AC(\cos \varphi + \sin \varphi)}{\sqrt{2}}.$$

Clearly, $AC \cos \varphi = CK$ and $AC \sin \varphi = AK$.

**2.7.** Since $\angle B_1AA_1 = \angle A_1BB_1$, it follows that points $A$, $B$, $A_1$ and $B_1$ lie on one circle. Parallel lines $AB$ and $A_1B_1$ intercept on it equal chords $AB_1$ and $BA_1$. Hence, $AC = BC$.

**2.8.** On side $BC$ of triangle $ABC$ construct outwards an equilateral triangle $A_1BC$. Let $P$ be the intersection point of line $AA_1$ with the circumscribed circle of triangle $A_1BC$. Then point $P$ lies inside triangle $ABC$ and

$$\angle APC = 180° - \angle A_1PC = 180° - \angle A_1BC = 120°.$$

Similarly, $\angle APB = 120°$.

**2.9.** The bases of perpendiculars dropped from point $M$ on the diameters lie on the circle $S$ with diameter $OM$ (where $O$ is the center of the initial circle). The intersection points of the given diameters with circle $S$ distinct from $O$ divide the circle into $n$ arcs. Since the angles $\frac{180°}{n}$ intersect all the circles that do not contain point $O$, the angle measure of each of these arcs is equal to $\frac{360°}{n}$. Therefore, the angle measure of the arc on which point $O$ lies is equal to $360° - (n-1) \cdot \frac{360°}{n} = \frac{360°}{n}$. Thus, the bases of the perpendiculars divide the circle $S$ into $n$ equal arcs.

**2.10.** Clearly,

$$\angle(AA_1, BB_1) = \angle(AA_1, AB_1) + \angle(AB_1, BB_1) = \angle(MA_1, MB_1) + \angle(AN, BN).$$

Since $MA_1 \perp BN$ and $MB_1 \perp AN$, it follows that

$$\angle(MA_1, MB_1) = \angle(BN, AN) = -\angle(AN, BN).$$

Therefore, $\angle(AA_1, BB_1) = 0°$, i.e., $AA_1 \parallel BB_1$.

**2.11.** Since $AB \parallel DE$, it follows that $\angle ACE = \angle BFD$ and since $BC \parallel EF$, it follows that $\angle CAE = \angle BDF$. Triangles $ACE$ and $BDF$ have two pairs of equal angles and, therefore, their third angles are also equal. The equality of these angles implies the equality of arcs $\smile AC$ and $\smile DF$, i.e., chords $CD$ and $AF$ are parallel.

**2.12.** Let us carry out the proof by induction on $n$. For the quadrilateral the statement is obvious; for the hexagon it had been proved in the preceding problem. Assume that the statement is proved for the $2(n-1)$-gon; let us prove the statement for the $2n$-gon. Let $A_1 \ldots A_{2n}$ be a $2n$-gon in which $A_1A_2 \parallel A_{n+1}A_{n+2}$, ..., $A_{n-1}A_n \parallel A_{2n-1}A_{2n}$. Let us consider $2(n-1)$-gon $A_1A_2 \ldots A_{n-1}A_{n+1} \ldots A_{2n-1}$. By the inductive hypothesis for $n$ odd we have $A_{n-1}A_{n+1} = A_{2n-1}A_1$, and for $n$ even we have $A_{n-1}A_{n+1} \parallel A_{2n-1}A_1$.

Let us consider triangles $A_{n-1}A_nA_{n+1}$ and $A_{2n-1}A_{2n}A_1$. Let $n$ be even. Then vectors $\{A_{n-1}A_n\}$ and $\{A_{2n-1}A_{2n}\}$, as well as $\{A_{n-1}A_{n+1}\}$ and $\{A_{2n-1}A_1\}$ are parallel and directed towards each other; hence, $\angle A_nA_{n-1}A_{n+1} = \angle A_1A_{2n-1}A_{2n}$ and $A_nA_{n+1} = A_{2n}A_1$ as chords that cut equal arcs, as required.

Let $n$ be odd. Then $A_{n-1}A_{n+1} = A_{2n-1}A_1$, i.e., $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$. In hexagon $A_{n-1}A_nA_{n+1}A_{2n-1}A_{2n}A_1$ we have $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$ and $A_{n-1}A_n \parallel A_{2n-1}A_{2n}$; hence, thanks to the preceding problem $A_nA_{n+1} \parallel A_{2n}A_1$, as required.

**2.13.** Let lines $FG$, $GE$ and $EF$ pass through points $A$, $B$ and $C$, respectively, so that triangle $EFG$ is an equilateral one, i.e.,

$$\angle(GE, EF) = \angle(EF, FG) = \angle FG, GE = \pm 60°.$$

Then

$$\angle(BE, EC) = \angle(CF, FA) = \angle(AG, GB) = \pm 60°.$$

Selecting one of the signs we get three circles $S_E$, $S_F$ and $S_G$ on which points $E$, $F$ and $G$ should lie. Any point $E$ of circle $S_E$ uniquely determines triangle $EFG$.

Let $O$ be the center of triangle $EFG$; let $P$, $R$ and $Q$ be the intersection points of lines $OE$, $OF$ and $OG$ with the corresponding circles $S_E$, $S_F$ and $S_G$. Let us prove that $P$, $Q$ and $R$ are the centers of equilateral triangles constructed on sides of triangle $ABC$ (outwards for one family and inwards for the other one), and point $O$ lies on the circumscribed circle of triangle $PQR$.

Clearly,

$$\angle(CB, BP) = \angle(CE, EP) = \angle(EF, EO) = \mp 30°$$

and

$$\angle(BP, CP) = \angle(BE, EC) = \angle(GE, EF) = \pm 60°.$$

Hence,

$$\angle(CB, CP) = \angle(CB, BP) + \angle(BP, CP) = \pm 30°.$$

Therefore, $P$ is the center of an equilateral triangle with side $AB$.

For points $Q$ and $R$ the proof is similar. Triangle $PQR$ is an equilateral one and its center coincides with the intersection point of medians of triangle $ABC$ (cf. Problem 1.49 b)). As is not difficult to verify,

$$\angle(PR, RQ) = \mp 60° = \angle(OE, OG) = \angle(OP, OQ),$$

i.e., point $O$ lies on the circumscribed circle of triangle $PQR$.

**2.14.** Clearly,

$$2(\angle KEC + \angle KDC) = (\smile MB + \smile AC) + (\smile MB + \smile BC) = 360°,$$

since $\smile MB = \smile AM$.

**2.15.** Denote the angle measure of the arc intercepted on the circle by the sides of triangle $ABC$ by $\alpha$. Denote the angle measure of the arc intercepted by the extensions of the sides of the triangle on the circle by $\alpha'$. Then $\frac{1}{2}(\alpha + \alpha') = \angle BAC = 60°$. But $\alpha = \alpha'$ because these arcs are symmetric through the line that passes through the center of the circle parallel to side $BC$. Hence, $\alpha = \alpha' = 60°$.

**2.16.** Since $\angle APB = \frac{1}{2}(\smile AB + \smile CD) = \angle AOB$, point $O$ lies on the circumscribed circle of triangle $APB$.

**2.17.** Let $O$ be the point where lines $A_1C_1$ and $B_1D_1$ meet; let $\alpha$, $\beta$, $\gamma$ and $\delta$ be angle measures of arcs $AB$, $BC$, $CD$ and $DA$. Then

$$\angle A_1OB_1 = \frac{\smile A_1B + \smile BB_1 + \smile C_1D + \smile DD_1}{2} = \frac{\alpha + \beta + \gamma + \delta}{4} = 90°.$$

**2.18.** By summing up the equalities we get

$$\smile C'A + \smile CA' = 2(180° - \angle APC) = 240° - 2\angle B \quad \text{and} \quad \smile AB' + \smile BA' = 240° - 2\angle C.$$

Then by subtracting from their sum the equality $\smile BA' + \smile CA' = 2\angle A$ we get

$$\smile C'B' = \smile C'A + \smile AB' = 480° - 2(\angle A + \angle B + \angle C) = 120°.$$

Similarly, $\smile B'A' = \smile C'A' = 120°$.

**2.19.** a) Let us prove, for example, that $AA_1 \perp C_1B_1$. Let $M$ be the intersection point of these segments. Then

$$\angle AMB_1 = \frac{\smile AB_1 + \smile A_1B + \smile BC_1}{2} = \angle ABB_1 + \angle A_1AB + \angle BCC_1 =$$

$$\frac{\angle B + \angle A + \angle C}{2} = 90°.$$

b) Let $M_1$ and $M_2$ be the intersection points of segments $AA_1$ with $BC$ and $BB_1$ with $AC$. Right triangles $AM_1C$ and $BM_2C$ have a common angle $\angle C$; hence, $\angle B_1BC = \angle A_1AC$. Consequently, $\smile B_1C = \smile A_1C$ and $\angle B_1C_1C = \angle A_1C_1C$, i.e., $CC_1$ is the bisector of angle $\angle A_1C_1B_1$.

**2.20.** Denote the vertices of triangle $T_1$ by $A$, $B$ and $C$; denote the midpoints of arcs $\smile BC$, $\smile CA$, $\smile AB$ by $A_1$, $B_1$, $C_1$, respectively. Then $T_2 = A_1B_1C_1$. Lines $AA_1$, $BB_1$, $CC_1$ are the bisectors of triangle $T_1$; hence, they meet at one point, $O$. Let lines $AB$ and $C_1B_1$ intersect at point $K$. It suffices to verify that $KO \parallel AC$. In triangle $AB_1O$, line $B_1C_1$ is a bisector and height, hence, this triangle is an isosceles one. Therefore, triangle $AKO$ is also an isosceles one. Lines $KO$ and $AC$ are parallel, since $\angle KOA = \angle KAO = \angle OAC$.

**2.21.** Let $l$ be tangent to the first circle at point $A$. Then $\angle(l, AP) = \angle(AQ, PQ) = \angle(BC, PB)$, hence, $l \parallel BC$.

**2.22.** Since

$$\angle(AB, AD) = \angle(AP, PD) = \angle(AB, BC),$$

we have $BC \parallel AD$.

**2.23.** Let, for definiteness, point $E$ lie on ray $BC$. Then $\angle ABC = \angle EAC$ and

$$\angle ADE = \angle ABC + \angle BAD = \angle EAC + \angle CAD = \angle DAE.$$

**2.24.** Let $P$ be the other intersection point of the circles. Then $\angle(AB, DB) = \angle(PA, PB)$ and $\angle(DC, AC) = \angle(PC, PA)$. By summing these equalities we get

$$\angle(DC, DB) = \angle(PC, PB) = \angle(PC, CA) + \angle(BA, PB).$$

The latter two angles subtend constant arcs.

**2.25.** a) Since $\angle MAB = \angle BNA$, the sum of angles $\angle ABN$ and $\angle MAN$ is equal to the sum of the angles of triangle $ABN$.

b) Since $\angle BAM = \angle BNA$ and $\angle BAN = \angle BMA$, it follows that $\triangle AMB \sim \triangle NAB$ and, therefore, $AM : NA = MB : AB$ and $AM : NA = AB : NB$. By multiplying these equalities we get the desired statement.

**2.26.** Point $P$ lies on the circle of radius $BC$ with center $B$ and line $DC$ is tangent to this circle at point $C$. Hence, $\angle PCD = \frac{1}{2}\angle PBC = 15°$.

**2.27.** Let $A_1$ and $B_1$ be intersection points of lines $MA$ and $MB$, respectively, with the smaller circle. Since $M$ is the center of homothety of the circles, $A_1B_1 \parallel AB$. Hence, $\angle A_1MT = \angle A_1TA = \angle B_1A_1T = \angle B_1MT$.

**2.28.** Let $\varphi$ be the angle between chord $AB$ and the tangent that passes through one of the chord's endpoints. Then $AB = 2R\sin\varphi$, where $R$ is the radius of circle $S$. Moreover, $PM = AM\sin\varphi$ and $QM = BM\sin\varphi$. Hence,

$$\frac{1}{PM} + \frac{1}{QM} = \left(\frac{AM + BM}{\sin\varphi}\right) AM \cdot BM = \frac{2R}{AM \cdot BM}.$$

The value $AM \cdot BM$ does not depend on the choice of chord $AB$.

**2.29.** Let line $AM$ intersect circle $S_2$ at point $D$. Then $\angle MDC = \angle MCA = \angle MAB$; hence, $CD \parallel AB$. Further, $\angle CAM = \angle MCB = \angle MDB$; hence, $AC \parallel BD$. Therefore, $ABCD$ is a parallelogram and its diagonal $AD$ divides diagonal $BC$ in halves.

**2.30.** Let us draw line $l_1$ tangent to $S_1$ at point $A_1$. Line $K_1K_2$ is tangent to $S_1$ if and only if $\angle(K_1K_2, K_1A_1) = \angle(K_1A_1, l_1)$. It is also clear that

$$\angle(K_1A_1, l_1) = \angle(A_1B, l_1) = \angle(A_2B, A_1A_2).$$

Similarly, line $K_1K_2$ is tangent to $S_2$ if and only if $\angle(K_1K_2, K_2A_2) = \angle(A_1B, A_1A_2)$. It remains to observe that if $\angle(K_1K_2, K_1A_1) = \angle(A_2B, A_1A_2)$, then

$$\angle(K_1K_2, K_2A_2) = \angle(K_1K_2, A_2B) =$$
$$\angle(K_1K_2, A_1B) + \angle(A_1B, A_1A_2) + \angle(A_1A_2, A_2B) = \angle(A_1B, A_1A_2).$$

**2.31.** Equal angles $ABC$ and $A_1B_1C_1$ intersect chords $AC$ and $A_1C_1$, hence, $AC = A_1C_1$.

**2.32.** Let us denote the center of the circle by $O$. Points $P$ and $Q$ lie on the circle with diameter $OM$, i.e., points $O$, $P$, $Q$ and $M$ lie on a circle of radius $\frac{1}{2}R$. Moreover, either $\angle POQ = \angle AOD$ or $\angle POQ = \angle BOD = 180° - \angle AOD$, i.e., the length of chord $PQ$ is a constant.

**2.33.** Since $\angle AOC = 90° + \frac{1}{2}\angle B$ (cf. Problem 5.3), it follows that

$$\angle EBD + \angle EOD = 90° + \frac{3}{2}\angle B = 180°$$

and, therefore, quadrilateral $BEOD$ is an inscribed one. Equal angles $\angle EBO$ and $\angle OBD$ subtend chords $EO$ and $OD$, hence, $EO = OD$.

**2.34.** On the extension of segment $BD$ beyond point $D$ take a point $Q$ such that $\angle ACQ = 40°$. Let $P$ be the intersection point of lines $AB$ and $QC$. Then $\angle BPC = 60°$ and $D$ is the intersection point of the bisectors of angles of triangle $BCP$. By Problem 2.33 $AD = DQ$. Moreover, $\angle BQC = \angle BCQ = 80°$. Therefore, $BC = BD + DQ = BD + DA$.

**2.35.** It suffices to verify that the exterior angle $ACD$ of triangle $BCD$ is twice greater than the angle at vertex $B$. Clearly, $\angle ACD = \angle AOD = 2\angle ABD$.

**2.36.** Let $O$ be the center of circle $S$. Point $B$ is the center of the circumscribed circle of triangle $ACD$, hence, $\angle CDA = \frac{1}{2}\angle ABC = 30°$ and, therefore, $\angle EOA = 2\angle EDA = 60°$, i.e., triangle $EOA$ is an equilateral one. Moreover, $\angle AEC = \angle AED = \angle AOB = 2\angle AOC$; hence, point $E$ is the center of the circumscribed circle of triangle $AOC$. Therefore, $EC = EO$.

**2.37.** Let us consider two positions of the moving circle: at the first moment, when point $K$ just gets to the fixed circle (the tangent point of the circles at this moment will be denoted by $K_1$) and at some other (second) moment.

Let $O$ be the center of the fixed circle, $O_1$ and $O_2$ be the positions of the center of the moving circle at the first and the second moments, respectively, $K_2$ be the position of point $K$ at the second moment. Let $A$ be the tangent point of the circles at the second moment. Since the moving circle rolls without gliding, the length of arc $\smile K_1A$ is equal to the length of arc $\smile K_2A$. Since the radius of the moving circle is one half of the radius of the fixed circle, $\angle K_2O_2A = 2\angle K_1OA$. Point $O$ lies on the moving circle, hence, $\angle K_2OA = \frac{1}{2}\angle K_2O_2A = \angle K_1OA$, i.e., points $K_2$, $K_1$ and $O$ lie on one line.

The trajectory of point $K$ is the diameter of the fixed circle.

**2.38.** Points $N$ and $C$ lie on the circle with diameter $AM$. Angles $\angle MAN$ and $\angle MCN$ subtend the same arc and therefore, are equal.

**2.39.** The symmetry through the bisector of angle $\angle BOC$ sends lines $AC$ and $DB$ into each other and, therefore, we have to prove that $\angle C'AB' = \angle B'DC'$. Since $BO = B'O$, $CO = C'O$ and $AO : DO = CO : BO$, it follows that $AO \cdot B'O = DO \cdot C'O$, i.e., the quadrilateral $AC'B'D$ is an inscribed one and $\angle C'AB' = \angle B'DC'$.

**2.40.** Denote the intersection points and angles as indicated on Fig. 14.
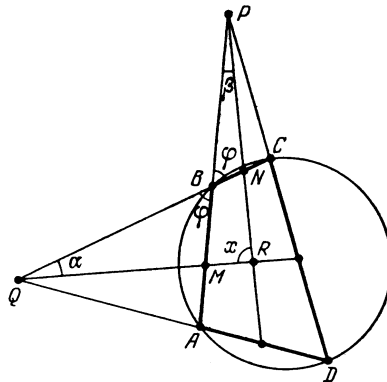


Figure 14 (Sol. 2.40)

It suffices to verify that $x = 90°$. The angles of quadrilateral $BMRN$ are equal to $180° - \varphi$, $\alpha + \varphi$, $\beta + \varphi$ and $x$, hence, the equality $x = 90°$ is equivalent to the equality $(2\alpha + \varphi) + (2\beta + \varphi) = 180°$. It remains to notice that $2\alpha + \varphi = \angle BAD$ and $2\beta + \varphi = \angle BCD$.

**2.41.** a) It suffices to prove that if $P_1$ is the point on the bisector (or its extension) of angle $\angle B$ that serves as the vertex of an angle of $90°$ that subtends segment $BC$, then $P_1$ lies on line $MN$. Points $P_1$ and $N$ lie on the circle with diameter $CO$, where $O$ is the intersection point of bisectors, hence,

$$\angle(P_1N, NC) = \angle(P_1O, OC) = \frac{1}{2}(180° - \angle A) = \angle(MN, NC).$$

b) Since $\angle BPC = 90°$, it follows that $BP = BC \cdot \cos \frac{\angle B}{2}$; hence,

$$S_{ABP} : S_{ABC} = \left( BP \cdot \sin \frac{\angle B}{2} \right) : (BC \sin B) = 1 : 2.$$

**2.42.** Take point $N$ so that $BN \parallel MC$ and $NC \parallel BM$. Then $NA \parallel CD$, $\angle NCB = \angle CBM = \angle CDM = \angle NAB$, i.e., points $A$, $B$, $N$ and $C$ lie on one circle. Hence, $\angle ACD = \angle NAC = \angle NBC = \angle BCM$.

**2.43.** Points $A_2$, $B_2$, $C$ and $P$ lie on one circle, hence,

$$\angle(A_2B_2, B_2P) = \angle(A_2C, CP) = \angle(BC, CP).$$

Similarly, $\angle(B_2P, B_2C_2) = \angle(AP, AB)$. Therefore,

$$\angle(A_2B_2, B_2C_2) = \angle(BC, CP) + \angle(AP, AB) = \angle(B_1B, B_1C_1) + \angle(A_1B_1, B_1B)$$
$$= \angle(A_1B_1, B_1C_1).$$

We similarly verify that all the other angles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ are either equal or their sum is equal to 180°; therefore, these triangles are similar (cf. Problem 5.42).

**2.44.** Points $Q'$ and $C$ lie on the circle with diameter $PQ$, hence, $\angle Q'CQ = \angle Q'PQ = 30°$. Therefore, $\angle BCQ' = 60°$. Similarly, $\angle CBQ' = 60°$ and, therefore, triangle $BQ'C$ is equilateral one. By similar reasons triangle $CP'D$ is an equilateral one.

**2.45.** Let $\angle BAD = 2\alpha$ and $\angle CBA = 2\beta$; for definiteness we will assume that $\alpha \geq \beta$. On side $CD$ take point $E$ so that $DE = DA$. Then $CE = CD - AD = CB$. The angle at vertex $C$ of an isosceles triangle $BCE$ is equal to $180° - 2\alpha$; hence, $\angle CBE = \alpha$. Similarly, $\angle DAE = \beta$. The bisector of angle $B$ intersects $CD$ at a point $F$. Since $\angle FBA = \beta = \angle AED$, quadrilateral $ABFE$ is an inscribed one and, therefore, $\angle FAE = \angle FBE = \alpha - \beta$. It follows that $\angle FAD = \beta + (\alpha - \beta) = \alpha$, i.e., $AF$ is the bisector of angle $\angle A$.

**2.46.** Since $ED = CB$, $EN = CM$ and $\angle DEC = \angle BCA = 30°$ (Fig. 15), it follows that $\triangle EDN = \triangle CBM$. Let $\angle MBC = \angle NDE = \alpha$, $\angle BMC = \angle END = \beta$.
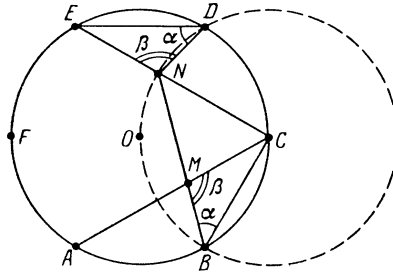


Figure 15 (Sol. 2.46)

It is clear that $\angle DNC = 180° - \beta$. Considering triangle $BNC$ we get $\angle BNC = 90° - \alpha$. Since $\alpha + \beta = 180° - 30° = 150°$, it follows that

$$\angle DNB = \angle DNC + \angle CNB = (180° - \beta) + (90° - \alpha) = 270° - (\alpha + \beta) = 120°.$$

Therefore, points $B$, $O$, $N$ and $D$, where $O$ is the center of the hexagon, lie on one circle. Moreover, $CO = CB = CD$, i.e., $C$ is the center of this circle, hence, $\lambda = CN : CE = CB : CA = 1 : \sqrt{3}$.

**2.47.** Let $D$ be the other intersection point of the circumscribed circles of triangles $A_1BC$ and $AB_1C$. Then $\angle(AC, CD) = \angle(AB_1, B_1D)$ and $\angle(DC, CB) = \angle(DA_1, A_1B)$.

Hence,

$$\angle(A_1C_1, C_1B_1) = \angle(AC, CB) = \angle(AC, CD) + \angle(DC, CB)$$
$$= \angle(AB_1, B_1D) + \angle(DA_1, A_1B) = \angle(A_1D, DB_1),$$

i.e., points $A_1$, $B_1$, $C_1$ and $D$ lie on one circle. Therefore, $\angle(A_1C_1, C_1B) = \angle(A_1B_1, B_1D) = \angle(AC, CD)$. Taking into account that $A_1C_1 \parallel AC$, we get the desired statement.

**2.48.** Let point $M$ be symmetric to point $A_1$ through line $AC$. By Problem 1.57 point $M$ lies on line $B_1C_1$. Therefore,

$$\angle(LM, MA_1) = \angle(C_1B_1) = \angle(C_1C, CB) = \angle(LK, KA_1),$$

i.e., point $M$ lies on the circumscribed circle of triangle $A_1KL$. It follows that the center of this circle lies on line $AC$ — the midperpendicular to segment $A_1M$.

**2.49.** Let $PQ$ be the diameter perpendicular to $AB$ and such that $Q$ and $C$ lie on one side of $AB$; let $L$ be the intersection point of line $QO$ with the circumscribed circle; let $M'$ and $N'$ be the intersection points of lines $LB'$ and $LA'$ with sides $AC$ and $BC$, respectively. It suffices to verify that $M' = M$ and $N' = N$.

Since $\smile PA + \smile AB' + \smile B'Q = 180°$, it follows that $\smile B'Q = \angle A$ and, therefore, $\angle B'LQ = \angle M'AO$. Hence, quadrilateral $AM'OL$ is an inscribed one and $\angle M'OA = \angle M'LA = \frac{1}{2}\angle B$. Therefore, $\angle CMO = \frac{1}{2}(\angle A + \angle B)$, i.e., $M' = M$. Similarly, $N' = N$.

**2.50.** Since $\triangle ADM \sim \triangle CBM$ and $\triangle ACM \sim \triangle DBM$, it follows that $AD : CB = DM : BM$ and $AC : DB = AM : DM$. It remains to multiply these equalities.

**2.51.** Let $D_1$ be the intersection point of line $BD$ with the circle distinct from point $B$. Then $\smile AB = \smile AD_1$; hence, $\angle ACB = \angle AD_1B = \angle ABD_1$. Triangles $ACB$ and $ABD$ have a common angle, $\angle A$, and, moreover, $\angle ACB = \angle ABD$; hence, $\triangle ACB \sim \triangle ABD$. Therefore, $AB : AC = AD : AB$.

**2.52.** Let $O$ be the center of the circle. Since $\angle MAC = \angle ACO = \angle CAO$, it follows that $\triangle AMC = \triangle ADC$. Similarly, $\triangle CDB = \triangle CNB$. Since $\triangle ACD \sim \triangle CDB$, it follows that $CD^2 = AD \cdot DB = AM \cdot NB$.

**2.53.** Points $B_1$ and $H$ lie on the circle with diameter $AB$, hence,

$$\angle(AB, BC) = \angle(AB, BH) = \angle(AB_1, B_1H) = \angle(B_1C_1, B_1H).$$

Similarly, $\angle(AC, BC) = \angle(B_1C_1, C_1H)$.

**2.54.** On an extension of segment $BP$ beyond point $P$ take point $D$ such that $PD = CP$. Then triangle $CDP$ is an equilateral one and $CD \parallel QP$. Therefore, $BP : PQ = BD : DC = (BP + CP) : CP$, i.e., $\frac{1}{PQ} = \frac{1}{CP} + \frac{1}{BP}$.

**2.55.** Segment $QE$ subtends angles of $45°$ with vertices at points $A$ and $B$, hence, quadrilateral $ABEQ$ is an inscribed one. Since $\angle ABE = 90°$, it follows that $\angle AQE = 90°$. Therefore, triangle $AQE$ is an isosceles right triangle and $\frac{AE}{AQ} = \sqrt{2}$. Similarly, $\frac{AF}{AP} = \sqrt{2}$.

**2.56.** Since $\angle ANC = \angle ABC = \angle CAB$, it follows that $\triangle CAM \sim \triangle CNA$ and, therefore, $CA : CM = CN : CA$, i.e., $CM \cdot CN = AC^2$ and $AM : NA = CM : CA$. Similarly, $BM : NB = CM : CB$. Therefore,

$$\frac{AM \cdot BM}{AN \cdot BN} = \frac{CM^2}{CA^2} = \frac{CM^2}{CM \cdot CN} = \frac{CM}{CN}.$$

**2.57.** Since $AK = AB = CD$, $AD = BC = CH$ and $\angle KAD = \angle DCH$, it follows that $\triangle ADK = \triangle CHD$ and $DK = DH$. Let us show that points $A$, $K$, $H$, $C$ and $D$ lie on one circle. Let us circumscribe the circle about triangle $ADC$. Draw chord $CK_1$ in this circle parallel to $AD$ and chord $AH_1$ parallel to $DC$. Then $K_1A = DC$ and $H_1C = AD$. Hence, $K_1 = K$ and $H_1 = H$, i.e., the constructed circle passes through points $K$ and $H$ and angles

$\angle KAH$ and $\angle KDH$ are equal because they subtend the same arc. Moreover, as we have already proved, $KDH$ is an isosceles triangle.

**2.58.** a) $\angle PBA_1 = \angle PAC_1$ and $\angle PBC_1 = \angle PAB_1$ and, therefore, right triangles $PBA_1$ and $PAC_1$, $PAB_1$ and $PBC_1$ are similar, i.e., $PA_1 : PB = PC_1 : PA$ and $PB_1 : PA = PC_1 : PB$. By multiplying these equalities we get $PA_1 \cdot PB_1 = PC_1^2$.

b) According to heading a)

$$OA'' = \sqrt{OB' \cdot OC'}, \quad OB'' = \sqrt{OA' \cdot OC'}, \quad OC'' = \sqrt{OA' \cdot OB'}.$$

By multiplying these equalities we get the desired statement.

**2.59.** Let $K$, $L$, $M$ and $N$ be the bases of perpendiculars dropped from point $E$ to lines $AB$, $BC$, $CD$ and $DA$, respectively. Points $K$ and $N$ lie on the circle with diameter $AE$, hence, $\angle(EK, KN) = \angle(EA, AN)$. Similarly, $\angle(EL, LM) = \angle(EC, CM) = \angle(EA, AN)$ and, therefore, $\angle(EK, KN) = \angle(EL, LM)$. Similarly, $\angle(EN, NK) = \angle(EM, ML)$ and $\angle(KE, EN) = \angle(LE, EM)$. It follows that $\triangle EKN \sim \triangle ELM$ and, therefore, $EK : EN = EL : EM$, i.e., $EN = \frac{EK \cdot EM}{EL} = \frac{ac}{b}$.

**2.60.** Let $H$ be the intersection point of heights, $M$ the midpoint of side $BC$. Points $A_1$, $B_2$ and $C_2$ lie on the circle with diameter $MH$, hence, $\angle(B_2A_1, A_1C_2) = \angle(B_2M, MC_2) = \angle(AC, AB)$. Moreover, $\angle(A_1B_2, B_2C_2) = \angle(A_1H, HC_2) = \angle(BC, AB)$ and $\angle(A_1C_2, C_2B_2) = \angle(BC, AC)$.

**2.61.** Let $M$ be the intersection point of medians, $H$ the intersection point of heights of triangle $ABC$. Points $A_1$, $B_1$ and $C_1$ are the projections of point $M$ on the heights and, therefore, these points lie on the circle with diameter $MH$. Hence, $\angle(A_1B_1, B_1C_1) = \angle(AH, HC) = \angle(BC, AB)$. By writing similar equalities for the other angles we get the desired statement.

**2.62.** Let lines $BM$ and $DN$ meet $S_2$ at points $L$ and $C_1$, respectively. Let us prove that lines $DC_1$ and $CN$ are symmetric through line $AN$. Since $BN \perp NA$, it suffices to verify that $\angle CNB = \angle BND$. But arcs $\smile CB$ and $\smile BD$ are equal. Arcs $\smile C_1M$ and $\smile CL$ are symmetric through line $AN$, hence, they are equal and, therefore, $\angle MDC_1 = \angle CML$. Besides, $\angle CNM = \angle MND$. Thus, $\triangle MCN \sim \triangle DMN$, i.e., $CN : MN = MN : DN$.

**2.63.** Let us drop from point $Q$ perpendiculars $QK_1$ and $QN_1$ to $KL$ and $NM$, respectively, and from point $P$ perpendiculars $PM_1$ and $PL_1$ to $NM$ and $KL$, respectively. Clearly, $\frac{QC}{PC} = \frac{QK_1}{PL_1} = \frac{QN_1}{PM_1}$, i.e., $\frac{QC^2}{PC^2} = \frac{QK_1 \cdot QN_1}{PL_1 \cdot PM_1}$. Since $\angle KNC = \angle MLC$ and $\angle NKC = \angle LMC$, it follows that $QN_1 : PL_1 = QN : PL$ and $QK_1 : PM_1 = QK : PM$. Therefore,

$$\frac{QC^2}{PC^2} = \frac{QK \cdot QN}{PL \cdot PM} = \frac{AQ \cdot QB}{PB \cdot AP} = \frac{(AC - QC) \cdot (AC + QC)}{(AC - PC) \cdot (AC + PC)} = \frac{AC^2 - QC^2}{AC^2 - PC^2}.$$

This implies that $QC = PC$.

**2.64.** a) Since $\angle CAM = \angle CBM$ and $\angle CB_1M = \angle CA_1M$, it follows that $\angle B_1AM = \angle A_1BM$ and $\angle AB_1M = \angle BA_1M$.

b) Let $M_1$ be a point of the circle $S$ with diameter $CO$ such that $CM_1 \parallel A_1B_1$; let $M_2$ be an intersection point of circle $S$ with the circumscribed circle of triangle $ABC$; let $A_2$ and $B_2$ be the tangent points of of the inscribed circle with sides $BC$ and $AC$, respectively. It suffices to verify that $M_1 = M_2$. By Problem a) $\triangle AB_2M_2 \sim \triangle BA_2M_2$, hence, $B_2M_2 : A_2M_2 = AB_2 : BA_2$. Since $CA_1 = p - b - BA_2$ and $CB_1 = AB_2$, it follows that

$$\frac{B_2M_1}{A_2M_1} = \frac{\sin B_2CM_1}{\sin A_2CM_1} = \frac{\sin CA_1B_1}{\sin CB_1A_1} = \frac{CB_1}{CA_1} = \frac{AB_2}{BA_2}.$$

On arc $\smile A_2CB_2$ of circle $S$, there exists a unique point $X$ for which $B_2X : A_2X = k$ (Problem 7.14), hence, $M_1 = M_2$.

**2.65.** Let $O$ be the center of the circumscribed circle of the triangle, $M$ the midpoint of side $AB$, $H$ the base of height $CH$, $D$ the midpoint of the arc on which point $C$ does not lie and with endpoints $A$ and $B$. Since $OD \parallel CH$, it follows that $\angle DCH = \angle MDC$. The bisector divides the angle between the median and the height in halves if and only if $\angle MCD = \angle DCH = \angle MDC = \angle ODC = \angle OCD$, i.e., $M = O$ and $AB$ is the diameter of the circle.

**2.66.** Let $\alpha = \angle A < \angle B$. By the preceding problem $\angle C = 90°$. Median $CM$ divides triangle $ABC$ into two isosceles triangles. Since $\angle ACM = \angle A = \alpha$, $\angle MCB = 3\alpha$, it follows that $\alpha + 3\alpha = 90°$, i.e., $\alpha = 22.5°$. Therefore, $\angle A = 22.5°$, $\angle B = 67.5°$, $\angle C = 90°$.

**2.67.** Let $D$ be a point at which line $AE$ intersects the circumscribed circle. Point $D$ is the midpoint of arc $\smile BC$. Therefore, $MD \parallel AH$, moreover, points $A$ and $D$ lie on different sides of line $MH$. It follows that point $E$ lies on segment $MH$.

**2.68.** Clearly,

$$\angle(AQ, QP) = \angle(AN, NP) = \angle(PM, MB) = \angle(QP, QB).$$

Therefore, point $Q$ lies on the circle such that segment $AB$ subtends an angle of $2\angle(AC, CB)$ with vertex at $Q$ and line $QP$ divides arc $\smile AB$ of this circle in halves.

**2.69.** Points $P$ and $Q$ lie on the circle with diameter $AD$; this circle intersects side $BC$ at point $F$. (Observe that $F$ does not coincide with $D$ if $AB \neq AC$.) Clearly,

$$\angle(FC, CE) = \angle(BA, AE) = \angle(DA, AQ) = \angle(DF, FQ), \text{ i.e., } EC \parallel FQ.$$

Similarly, $BE \parallel FP$. To complete the proof it suffices to notice that the areas of triangles adjacent to the lateral sides of the trapezoid are equal.

**2.70.** Let $\angle AOB = \alpha$ and $\angle COD = \beta$. Then $\frac{\alpha}{2} + \frac{\beta}{2} = \angle ADP + \angle PAD = 90°$. Since $2S_{AOB} = R^2 \sin \alpha$ and $2S_{COD} = R^2 \sin \beta$, where $R$ is the radius of the circumscribed circle, it follows that $S_{AOB} = S_{COD}$. Similarly, $S_{BOC} = S_{AOD}$.

**2.71.** Let $\angle AOB = 2\alpha$ and $\angle COD = 2\beta$. Then $\alpha + \beta = \angle ADP + \angle PAD = 90°$. Hence,

$$(AP^2 + BP^2) + (CP^2 + DP^2) = AB^2 + CD^2 = 4R^2(\sin^2 \alpha + \cos^2 \alpha) = 4R^2.$$

Similarly, $BC^2 + AD^2 = 4R^2$.

**2.72.** Let $M$ be the midpoint of $AC$, $N$ the midpoint of $BD$. We have $AM^2 = AO^2 - OM^2$ and $BN^2 = BO^2 - ON^2$; hence,

$$AC^2 + BD^2 = 4(R^2 - OM^2) + 4(R^2 - ON^2) = 8R^2 - 4(OM^2 + ON^2) = 8R^2 - 4OP^2$$

since $OM^2 + ON^2 = OP^2$.

**2.73.** The correspondiong legs of acute angles $\angle BLP$ and $\angle BDC$ are perpendicular, hence, the angles are equal.

Therefore, $\angle BLP = \angle BDC = \angle BAP$. Moreover, $AK \parallel BL$ and $AL \perp BK$. It follows that $AKLB$ is a rhombus.

**2.74.** In the circumscribed circle take a point $D'$ so that $DD' \parallel AC$. Since $DD' \perp BD$, it follows that $BD'$ is a diameter and, therefore, $\angle D'AB = \angle D'CB = 90°$. Hence,

$$S_{ABCD} = S_{ABCD} = \frac{1}{2}(AD' \cdot AB + BC \cdot CD') = \frac{1}{2}(AB \cdot CD + BC \cdot AD).$$

**2.75.** Let us draw diameter $AE$. Since $\angle BEA = \angle BCP$ and $\angle ABE = \angle BPC = 90°$, it follows that $\angle EAB = \angle CBP$. The angles that intersect chords $EB$ and $CD$ are equal, hence, $EB = CD$. Since $\angle EBA = 90°$, the distance from point $O$ to $AB$ is equal to $\frac{1}{2}EB$.

**2.76.** Let the perpendicular dropped from point $P$ to $BC$ intersect $BC$ at point $H$ and $AD$ at point $M$ (Fig. 16).
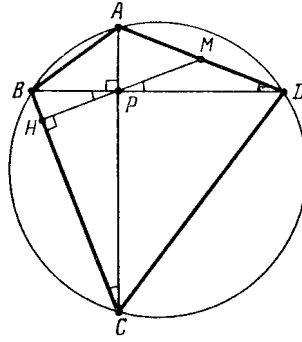
Figure 16 (Sol. 2.76)

Therefore, $\angle BDA = \angle BCA = \angle BPH = \angle MPD$. Since angles $MDP$ and $MPD$ are equal, $MP$ is a median of right triangle $APD$. Indeed,

$$\angle APM = 90° - \angle MPD = 90° - \angle MDP = \angle PAM,$$

i.e., $AM = PM = MD$.

**2.77.** The midpoints of the sides of quadrilateral $ABCD$ are vertices of a rectangle (cf. Problem 1.2), hence, they lie on one circle. Let $K$ and $L$ be the midpoints of sides $AB$ and $CD$, let $M$ be the intersection point of lines $KP$ and $CD$. By Problem 2.76 $PM \perp CD$; hence, $M$ is the projection of point $P$ on side $CD$ and point $M$ lies on the circle with diameter $KL$.

For the other projections the proof is similar.

**2.78.** a) It is worth to observe that since points $A$, $B$, $C$ and $D$ divide the circle into arcs smaller than $180°$ each, then the quadrilateral constructed contains this circle. The angle $\varphi$ between the tangents drawn through points $A$ and $B$ is equal to $180° - \angle AOB$ and the angle $\psi$ between the tangents drawn through points $C$ and $D$ is equal to $180° - \angle COD$. Since $\angle AOB + \angle COD = 180°$, it follows that $\varphi + \psi = 180°$.

REMARK. Conversely, the equality $\varphi + \psi = 180°$ implies that $\angle AOB + \angle COD = 180°$, i.e., $AC \perp BD$.

b) Let $O$ be the center of the inscribed circle. Since $\angle AKO + \angle BMO = 90°$, it follows that $\angle AKO = \angle BOM$ and $\triangle AKO \sim \triangle BOM$. Therefore, $AK \cdot BM = BO \cdot AO = r^2$.

**2.79.** First, let us suppose that the circumscribed circles of triangles $A'BC$ and $AB'C$ are not tangent to each other and $P$ is their common point distinct from $C$. Then

$$\angle(PA, PB) = \angle(PA, PC) + \angle(PC, PB)$$
$$= \angle(B'A, B'C) + \angle(A'C, A'B) = \angle(C'A, C'B),$$

i.e., point $P$ lies on the circumscribed circle of triangle $ABC'$.

If the the circumscribed circles of triangles $A'BC$ and $AB'C$ are tangent to each other, i.e., $P = C$, then our arguments require an insignificant modifications: instead of line $PC$ we have to take the common tangent.

**2.80.** a) By applying the statement of Problem 2.79 to triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$ constructed on the sides of triangle $A_1B_1C_1$ we get the desired statement.

b) Let $P$ be the intersection point of the indicated circles. Let us prove that the value of the angle $\angle(AP, PC)$ is a constant. Since

$$\angle(AP, PC) = \angle(AP, AB) + \angle(AB, BC) + \angle(BC, PC)$$

and angle $\angle(AB, BC)$ is a constant, it remains to verify that the sum $\angle(AP, AB) + \angle(BC, PC)$ is a constant. Clearly,

$$\begin{array}{rcl} \angle(AP, AB) + \angle(BC, CP) & = & \angle(AP, AC_1) + \angle(CA_1, CP) = \\ \angle(B_1P, B_1C_1) + \angle(B_1A_1, B_1P) & = & \angle(B_1A_1, B_1C_1) \end{array}$$

and the value of the latter angle is constant by hypothesis.

We similarly prove that the values of angles $\angle(AP, PB)$ and $\angle(BP, PC)$ are constants. Hence, point $P$ remains fixed.

**2.81.** As follows from Problem 2.80 b) it suffices to carry out the proof for one such triangle $A_1B_1C_1$ only; for instance, for the triangle with vertices in the midpoints of sides of triangle $ABC$. Let $H$ be the intersection point of heights of triangle $A_1B_1C_1$, i.e., the center of the circumscribed circle of triangle $ABC$. Since $A_1H \perp B_1C_1$ and $B_1H \perp A_1C_1$, it follows that $\angle(A_1H, HB_1) = \angle(B_1C_1, A_1C_1) = \angle(A_1C, CB_1)$, i.e., point $H$ lies on the circumscribed circle of triangle $A_1B_1C$.

A similar argument shows that point $H$ lies on the circumscribed circles of triangles $A_1BC_1$ and $AB_1C_1$.

**2.82.** a) Let $X$ be the intersection point of the circumscribed circles of triangles $ABC$ and $AB'C'$. Then

$$\angle(XB', XC) = \angle(XB', XA) + \angle(XA, XC) = \angle(C'B', C'A) + \angle(BA, BC).$$

Since $AC' = AP = AB'$, triangle $C'AB'$ is an isosceles one and $\angle C'AB' = 2\angle A$; hence, $\angle(C'B', C'A) = \angle A - 90°$. Therefore,

$$\angle(XB', XC) = \angle A - 90° + \angle B = 90° - \angle C = \angle(A'B', A'C),$$

i.e., point $X$ lies on the circumscribed circle of triangle $A'B'C$. For the circumscribed circle of triangle $A'BC'$ the proof is similar.

b) Let $X$ be the intersection point of the circumscribed circles of triangles $A'B'C'$ and $A'BC$. Let us prove that $X$ lies on the circumscribed circle of triangle $ABC'$. Clearly,

$$\angle(XB, XC') = \angle(XB, XA') + \angle(XA', XC') = \angle(CB, CA') + \angle(B'A', B'C').$$

Let $A_1$, $B_1$ and $C_1$ be the midpoints of segments $PA'$, $PB'$ and $PC'$. Then

$$\angle(CB, CA') = \angle(CP, CA_1) = \angle(B_1P, B_1A_1), \angle(B'A', B'C') = \angle(B_1A_1, B_1C_1)$$

and

$$\angle(AB, AC') = \angle(AP, AC_1) = \angle(B_1P, B_1C_1).$$

It follows that $\angle(XB, XC') = \angle(AB, AC')$.

We similarly prove that point $X$ lies on the circumscribed circle of triangle $AB'C$.

c) Since $QA'$ is the common chord of circles centered at $O$ and $I$, it follows that $QA' \perp OI$. Similarly, $QB' \perp OJ$ and $QC \perp IJ$. Therefore, sides of angles $OJI$ and $B'QC$, as well as sides of angles $OIJ$ and $A'QC$, are mutually perpendicular, hence, $\sin OJI = \sin B'QC$ and $\sin OIJ = \sin A'QC$. Therefore, $OI : OJ = \sin OJI : \sin OIJ = \sin B'QC : \sin A'QC$. It is also clear that

$$\frac{QI}{QJ} = \frac{\sin QJI}{\sin QIJ} = \frac{\sin(\frac{1}{2}QJC)}{\sin(\frac{1}{2}QIC)} = \frac{\sin QB'C}{\sin QA'C}.$$

Taking into account that $\sin B'QC : \sin QB'C = B'C : QC$ and $\sin A'QC : \sin QA'C = A'C : QC$ we get

$$\frac{OI}{OJ} : \frac{QI}{QJ} = \frac{B'C}{QC} : \frac{A'C}{QC} = 1.$$

**2.83.** a) The conditions of the problem imply that no three lines meet at one point. Let lines $AB$, $AC$ and $BC$ intersect the fourth line at points $D$, $E$, and $F$, respectively (Fig. 17).
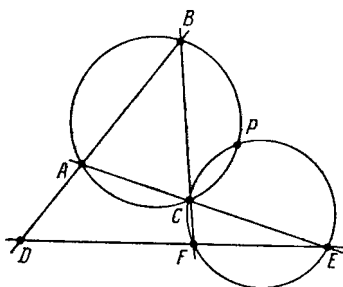


Figure 17 (Sol. 2.83)

Denote by $P$ the intersection point of circumscribed circles of triangles $ABC$ and $CEF$ distinct from point $C$. Let us prove that point $P$ belongs to the circumscribed circle of triangle $BDF$. For this it suffices to verify that $\angle(BP, PF) = \angle(BD, DF)$. Clearly,

$$\angle(BP, PF) = \angle(BP, PC) + \angle(PC, PF) = \angle(BA, AC) + \angle(EC, EF)$$
$$= \angle(BD, AC) + \angle(AC, DF) = \angle(BD, DF).$$

We similarly prove that point $P$ belongs to the circumscribed circle of triangle $ADE$.

b) Let us make use of notations of Fig. 17. Thanks to heading a), the circumscribed circles of triangles $ABC$, $ADE$ and $BDF$ pass through point $P$ and, therefore, we can consider them as the circumscribed circles of triangles $ABP$, $ADP$ and $BDP$ respectively. Therefore, their centers lie on a circle that passes through point $P$ (cf. Problem 5.86).

We similarly prove that the centers of any of the three of given circles lie on a circle that passes through point $P$. It follows that all the four centers lie on a circle that passes through point $P$.

**2.84.** a) Let $P$ be Michel's point for lines $AB$, $BC$, $CA$ and $A_1B_1$. The angles between rays $PA$, $PB$, $PC$ and the tangents to circles $S_a$, $S_b$, $S_c$ are equal to $\angle(PB_1, B_1A) = \angle(PC_1, C_1A)$, $\angle(PC_1, C_1B) = \angle(PA_1, A_1B)$, $\angle(PA_1, A_1C) = \angle(PB_1, B_1C)$, respectively. Since $\angle(PC_1, C_1A) = \angle(PC_1, C_1B) = \angle(PA_1, A_1C) = \varphi$, it follows that after a rotation through an angle of $\varphi$ about point $P$ lines $PA$, $PB$ and $PC$ turn into the tangents to circles $S_a$, $S_b$ and $S_c$, respectively, and, therefore, after a rotation through an angle of $90° - \varphi$ these lines turn into lines $PO_a$, $PO_b$ and $PO_c$ respectively. Moreover,

$$\frac{PO_a}{PA} = \frac{PO_b}{PB} = \frac{PO_c}{PC} = \frac{1}{2}\sin\varphi.$$

Therefore, the composition of the rotation through an angle of $90° - \varphi$ and the homothety (see ???) with center $P$ and coefficient $\frac{1}{2}\sin\varphi$ sends triangle $ABC$ to $O_aO_bO_c$.

b) The transformation considered in the solution of heading a) sends the center $O$ of the circumscribed circle of triangle $ABC$ into the center $O'$ of the circumscribed circle of triangle $O_aO_bO_c$ and the orthocenter $H$ of triangle $ABC$ to orthocenter $H'$ of triangle $O_aO_bO_c$. Let us complement triangle $OO'H'$ to parallelogram $OO'H'M$. Since $\frac{OH}{OM} = \frac{OH}{O'H'} = 2\sin\varphi$ and $\angle HOM = \angle(HO, O'H') = 90° - \varphi$, it follows that $MH = MO$, i.e., point $M$ lies on the midperpendicular of segment $OH$. It remains to notice that for the inscribed quadrilateral $OO_aO_bO_c$ point $M$ is uniquely determined: taking instead of point $O$ any of the points $O_a$, $O_b$ or $O_c$ we get the same point $M$ (cf. Problem 13.33).

**2.85.** We may assume that rays $AB$ and $DC$ meet at point $E$ and rays $BC$ and $AD$ meet at point $F$. Let $P$ be the intersection point of circumscribed circles of triangles $BCE$

and $CDF$. Then $\angle CPE = \angle ABC$ and $\angle CPF = \angle ADC$. Hence, $\angle CPE + \angle CPF = 180°$, i.e., point $P$ lies on segment $EF$.

**2.86.** a) Since

$$\angle(AP, PD) = \angle(AP, PE) + \angle(PE, PD) =$$
$$\angle(AC, CD) + \angle(AB, BD) + \angle(AO, OD),$$

points $A$, $P$, $D$ and $O$ lie on one circle.

b) Clearly,

$$\angle(EP, PO) = \angle(EP, PA) + \angle(PA, PO) = \angle(DC, CA) + \angle(DA, DO) = 90°,$$

because the arcs intersected by these angles constitute a half of the circle.

**2.87.** Let us make use of notations of Fig. 17. The projections of point $P$ on lines $CA$ and $CB$ coincide with its projection to $CE$ and $CF$, respectively. Therefore, Simson's lines of point $P$ relative triangles $ABC$ and $CEF$ coincide (cf. Problem 5.85 a).

**2.88.** Let point $A'$ be symmetric to point $A$ through the midperpendicular to segment $BC$. Then $\angle OAH = \frac{1}{2}\angle AOA' = \angle ABA' = |\angle B - \angle C|$.

**2.89.** Since $AA'$ is a diameter, $A'C \perp AC$; hence, $BH \parallel A'C$. Similarly, $CH \parallel A'B$. Therefore, $BA'CH$ is a parallelogram.

**2.90.** Let $l$ be a line parallel to the two given lines, $D$ the intersection point of lines $m$ and $n$. Then

$$\angle(AD, DB) = \angle(m, AB) + \angle(AB, n) = \angle(AC, l) + \angle(l, CB) = \angle(AC, CB)$$

and, therefore, point $D$ lies on the circumscribed circle of triangle $ABC$.

**2.91.** a) Let $O$ be the midpoint of the arc of circle $S$ that lies inside triangle $ABC$. Then $\angle CBO = \angle BCO$ and due to a property of the angle between a tangent and a chord, $\angle BCO = \angle ABO$. Therefore, $BO$ is the bisector of angle $ABC$, i.e., $O$ is the center of the inscribed circle of triangle $ABC$. We similarly prove that the midpoint of the arc of circle $S$ that lies outside triangle $ABC$ is the center of its escribed circle.

b) We have to prove that the center of the considered circle $S$ lies on the bisector of angle $BAC$. Let $D$ be the intersection point of the bisector of the angle with the circumscribed circle of triangle $ABC$. Then $DB = DO = DC$ (cf. Problem 2.4 a), i.e., $D$ is the center of circle $S$.

**2.92.** If angle $\angle C$ is a right one, then the solution of the problem is obvious: $C$ is the intersection point of lines $A_1B$, $A_2B_2$, $AB_1$. If $\angle C \neq 90°$, then the circumscribed circles of squares $ACA_1A_2$ and $BCB_1B_2$ have in addition to $C$ one more common point, $C_1$. Then

$$\angle(AC_1, A_2C_1) \quad = \quad \angle(A_2C_1, A_1C_1) = \angle(A_1C_1, C_1C) = \angle(C_1C, C_1B_1)$$
$$= \angle(C_1B_1, C_1B_2) \quad = \quad \angle(C_1B_2, C_1B) = 45°$$

(or $-45°$; it is only important that all the angles are of the same sign). Hence, $\angle(AC, C_1B_1) = 4 \cdot 45° = 180°$, i.e., line $AB_1$ passes through point $C_1$.

Similarly, $A_2B_2$ and $A_1B$ pass through point $C_1$.

**2.93.** Let $P$ and $O$ be the centers of circles $S_1$ and $S_2$, respectively; let $\alpha = \angle APC$, $\beta = \angle BPC$; lines $AC$ and $BC$ intersect $S_2$ at points $K$ and $L$, respectively. Since $\angle OAP = \angle OBP = 90°$, it follows that $\angle AOB = 180° - \alpha - \beta$. Furthermore,

$$\angle LOB = 180° - 2\angle LBO = 2\angle CBP = 180° - \beta.$$

Similarly, $\angle KOA = 180° - \alpha$. Therefore,

$$\angle LOK = \angle LOB + \angle KOA - \angle AOB = 180°,$$

i.e., $KL$ is a diameter.

**2.94.** Let us consider points $M'$, $P'$, $Q'$ and $R'$ symmetric to points $M$, $P$, $Q$ and $R$, respectively, through line $OA$. Since point $C$ is symmetric to point $B$ through $OA$, it follows that line $P'Q'$ passes through point $C$. The following equalities are easy to verify:

$$\angle(CS, NS) = \angle(Q'Q, NQ) = \angle(Q'P, NP') = \angle(CP', NP');$$
$$\angle(CR', P'R') = \angle(MM', P'M') = \angle(MN, P'N) = \angle(CN, P'N).$$

From these equalities we deduce that points $C$, $N$, $P'$, $S$ and $R'$ lie on one circle. But points $S$, $R'$ and $C$ lie on one line, therefore, $S = R'$.

# CHAPTER 3. CIRCLES

## Background

1. A line that has exactly one common point with a circle is called a line *tangent* to the circle. Through any point $A$ outside the circle exactly two tangents to the circle can be drawn.

Let $B$ and $C$ be the tangent points and $O$ the center of the circle. Then:

a) $AB = AC$;

b) $\angle BAO = \angle CAO$;

c) $OB \perp AB$.

(Sometimes the word "tangent" is applied not to the whole line $AB$ but to the segment $AB$. Then property a), for example, is formulated as: *the tangents to one circle drawn from one point are equal.*)

2. Let lines $l_1$ and $l_2$ that pass through point $A$ intersect a circle at points $B_1$, $C_1$ and $B_2$, $C_2$, respectively. Then $AB_1 \cdot AC_1 = AB_2 \cdot AC_2$. Indeed, $\triangle AB_1C_2 \sim \triangle AB_2C_1$ in three angles. (We advise the reader to prove this making use of the properties of the inscribed angles and considering two cases: $A$ lies outside the circle and $A$ lies inside the circle.)

If line $l_2$ is tangent to the circle, i.e., $B_2 = C_2$, then $AB_1 \cdot AC_1 = AB_2^2$. The proof runs along the same lines as in the preceding case except that now we have to make use of *the properties of the angle between a tangent and a chord.*

3. The line that connects the centers of tangent circles passes through their tangent point.

4. The *value of the angle between two intersecting circles* is the value of the angle between the tangents to these circles drawn through the intersection point. It does not matter which of the two of intersection points we choose: the corresponding angles are equal.

The angle between tangent circles is equal to $0°$.

5. In solutions of problems from §6 a property that has no direct relation to circles is used: *the heights of a triangle meet at one point.* The reader can find the proof of this fact in solutions of Problems 5.45 and 7.41 or can take it for granted for the time being.

6. It was already in the middle of the V century A.D. that **Hyppocratus** from island Chios (do not confuse him with the famous doctor Hyppocratus from island Kos who lived somewhat later) and Pythagoreans began to solve the *quadrature of the circle* problem. It is formulated as follows: *with the help of a ruler and compass construct a square of the same area as the given circle.*

In 1882 the German mathematician **Lindemann** proved that number $\pi$ is transcendental, i.e., is not a root of a polynomial with integer coefficients. This implies, in particular, that the problem on the quadrature of the circle is impossible to solve as stated (using other tools one can certainly solve it).

It seems that it was the problem on *Hyppocratus' crescents* (Problem 3.38) that induced in many a person great expectations to the possibility of squaring the circle: *the area of the figure formed by arcs of circles is equal to the area of a triangle.* Prove this statement and try to understand why such expectations were not grounded in this case.

## Introductory problems

**1.** Prove that from a point $A$ outside a circle it is possible to draw exactly two tangents to the circle and the lengths of these tangents (more exactly, the lengths from $A$ to the tangent points) are equal.

**2.** Two circles intersect at points $A$ and $B$. Point $X$ lies on line $AB$ but not on segment $AB$. Prove that the lengths of all the tangents drawn from point $X$ to the circles are equal.

**3.** Two circles whose radii are $R$ and $r$ are tangent from the outside (i.e., none of them lies inside the other one). Find the length of the common tangent to these circles.

**4.** Let $a$ and $b$ be the lengths of the legs of a right triangle, $c$ the length of its hypothenuse. Prove that:

a) the radius of the inscribed circle of this triangle is equal to $\frac{1}{2}(a + b - c)$;

b) the radius of the circle tangent to the hypothenuse and the extensions of the legs is equal to $\frac{1}{2}(a + b + c)$.

## §1. The tangents to circles

**3.1.** Lines $PA$ and $PB$ are tangent to a circle centered at $O$; let $A$ and $B$ be the tangent points. A third tangent to the circle is drawn; it intersects with segments $PA$ and $PB$ at points $X$ and $Y$, respectively. Prove that the value of angle $XOY$ does not depend on the choice of the third tangent.

**3.2.** The inscribed circle of triangle $ABC$ is tangent to side $BC$ at point $K$ and an escribed circle is tangent at point $L$. Prove that $CK = BL = \frac{1}{2}(a + b - c)$, where $a$, $b$, $c$ are the lengths of the triangle's sides.

**3.3.** On the base $AB$ of an isosceles triangle $ABC$ a point $E$ is taken and circles tangent to segment $CE$ at points $M$ and $N$ are inscribed into triangles $ACE$ and $ECB$, respectively. Find the length of segment $MN$ if the lengths of segments $AE$ and $BE$ are known.

**3.4.** Quadrilateral $ABCD$ is such that there exists a circle inscribed into angle $\angle BAD$ and tangent to the extensions of sides $BC$ and $CD$. Prove that $AB + BC = AD + DC$.

**3.5.** The common inner tangent to circles whose radii are $R$ and $r$ intersects their common outer tangents at points $A$ and $B$ and is tangent to one of the circles at point $C$. Prove that $AC \cdot CB = Rr$.

**3.6.** Common outer tangents $AB$ and $CD$ are drawn to two circles of distinct radii. Prove that quadrilateral $ABCD$ is a circumscribed one if and only if the circles are tangent to each other.

**3.7.** Consider parallelogram $ABCD$ such that the escribed circle of triangle $ABD$ is tangent to the extensions of sides $AD$ and $AB$ at points $M$ and $N$, respectively. Prove that the intersection points of segment $MN$ with $BC$ and $CD$ lie on the inscribed circle of triangle $BCD$.
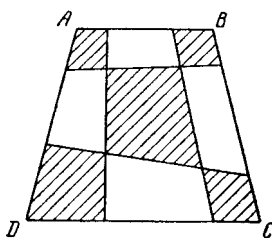


Figure 18 (3.7)

**3.8.** On each side of quadrilateral $ABCD$ two points are taken; these points are connected as shown on Fig. 18. Prove that if all the five dashed quadrilaterals are circumscribed ones, then the quadrilateral $ABCD$ is also a circumscribed one.

## §2. The product of the lengths of a chord's segments

**3.9.** Through a point $P$ lying on the common chord $AB$ of two intersecting circles chord $KM$ of the first circle and chord $LN$ of the second circle are drawn. Prove that quadrilateral $KLMN$ is an inscribed one.

**3.10.** Two circles intersect at points $A$ and $B$; let $MN$ be their common tangent. Prove that line $AB$ divides $MN$ in halves.

**3.11.** Line $OA$ is tangent to a circle at point $A$ and chord $BC$ is parallel to $OA$. Lines $OB$ and $OC$ intersect the circle for the second time at points $K$ and $L$, respectively. Prove that line $KL$ divides segment $OA$ in halves.

**3.12.** In parallelogram $ABCD$, diagonal $AC$ is longer than diagonal $BD$; let $M$ be a point on diagonal $AC$ such that quadrilateral $BCDM$ is an inscribed one. Prove that line $BD$ is a common tangent to the circumscribed circles of triangles $ABM$ and $ADM$.

**3.13.** Given circle $S$ and points $A$ and $B$ outside it. For each line $l$ that passes through point $A$ and intersects circle $S$ at points $M$ and $N$ consider the circumscribed circle of triangle $BMN$. Prove that all these circles have a common point distinct from point $B$.

**3.14.** Given circle $S$, points $A$ and $B$ on it and point $C$ on chord $AB$. For every circle $S'$ tangent to chord $AB$ at point $C$ and intersecting circle $S$ at points $P$ and $Q$ consider the intersection point $M$ of lines $AB$ and $PQ$. Prove that the position of point $M$ does not depend on the choice of circle $S'$.

## §3. Tangent circles

**3.15.** Two circles are tangent at point $A$. A common (outer) tangent line is drawn to them; it is tangent to the circles at points $C$ and $D$, respectively. Prove that $\angle CAD = 90°$.

**3.16.** Two circles $S_1$ and $S_2$ centered at $O_1$ and $O_2$ are tangent to each other at point $A$. A line that intersects $S_1$ at point $A_1$ and $S_2$ at point $A_2$ is drawn through point $A$. Prove that $O_1A_1 \parallel O_2A_2$.

**3.17.** Three circles $S_1$, $S_2$ and $S_3$ are pairwise tangent to each other at three distinct points. Prove that the lines that connect the tangent point of circles $S_1$ and $S_2$ with the other two tangent points intersect circle $S_3$ at points that are the endpoints of its diameter.

**3.18.** Two tangent circles centered at $O_1$ and $O_2$, respectively, are tangent from the inside to the circle of radius $R$ centered at $O$. Find the perimeter of triangle $OO_1O_2$.

**3.19.** Circles $S_1$ and $S_2$ are tangent to circle $S$ from the inside at points $A$ and $B$ so that one of the intersection points of circles $S_1$ and $S_2$ lies on segment $AB$. Prove that the sum of the radii of circles $S_1$ and $S_2$ is equal to the radius of circle $S$.

**3.20.** The radii of circles $S_1$ and $S_2$ tangent at point $A$ are equal to $R$ and $r$ $(R > r)$. Find the length of the tangent drawn to circle $S_2$ from point $B$ on circle $S_1$ if $AB = a$ (consider the cases of the inner and outer tangent).

**3.21.** A point $C$ is taken on segment $AB$. A line that passes through point $C$ intersects circles with diameters $AC$ and $BC$ at points $K$ and $L$ and the circle with diameter $AB$ at points $M$ and $N$, respectively. Prove that $KM = LN$.

**3.22.** Given four circles $S_1$, $S_2$, $S_3$ and $S_4$ such that $S_i$ and $S_{i+1}$ are tangent from the outside for $i = 1, 2, 3, 4$ $(S_5 = S_1)$. Prove that the tangent points are the vertices of an inscribed quadrilateral.

**3.23.** a) Three circles centered at $A$, $B$ and $C$ are tangent to each other and line $l$; they are placed as shown on Fig. 19. Let $a$, $b$ and $c$ be radii of circles centered at $A$, $B$ and $C$, respectively. Prove that $\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$.
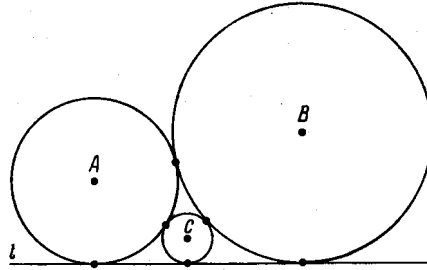


Figure 19 (3.23)

b) Four circles are pairwise tangent from the outside (at 6 distinct points). Let $a$, $b$, $c$ and $d$ be their radii; $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$ and $\delta = \frac{1}{d}$. Prove that

$$2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (\alpha + \beta + \gamma + \delta)^2.$$

### §4. Three circles of the same radius

**3.24.** Three circles of radius $R$ pass through point $H$; let $A$, $B$ and $C$ be points of their pairwise intersection distinct from $H$. Prove that
a) $H$ is the intersection point of heights of triangle $ABC$;
b) the radius of the circumscribed circle of the triangle $ABC$ is also equal to $R$.
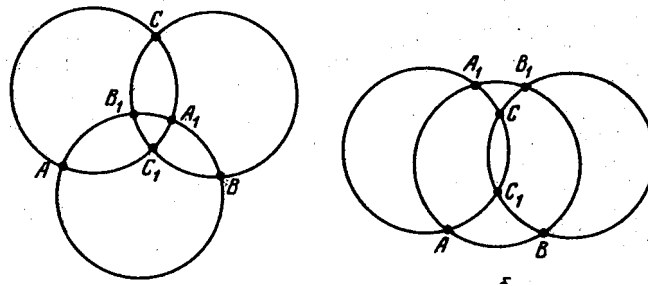


Figure 20 (3.24)

**3.25.** Three equal circles intersect as shown on Fig. 20 a) or b). Prove that $\smile AB_1 + \smile BC_1 \pm \smile CA_1 = 180°$, where the minus sign is taken in case b) and plus in case a).
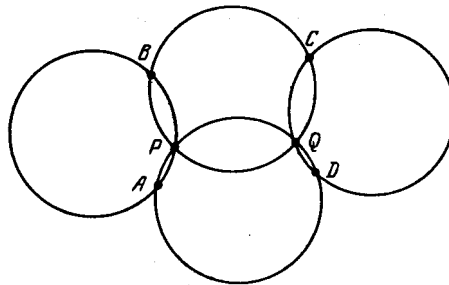


Figure 21 (3.26)

**3.26.** Three circles of the same radius pass through point $P$; let $A$, $B$ and $Q$ be points of their pairwise intersections. A fourth circle of the same radius passes through point $Q$

and intersects the other two circles at points $C$ and $D$. The triangles $ABQ$ and $CDP$ thus obtained are acute ones and quadrilateral $ABCD$ is a convex one (Fig. 21). Prove that $ABCD$ is a parallelogram.

## §5. Two tangents drawn from one point

**3.27.** Tangents $AB$ and $AC$ are drawn from point $A$ to a circle centered at $O$. Prove that if segment $AO$ subtends a right angle with vertex at point $M$, then segments $OB$ and $OC$ subtend equal angles with vertices at $M$.

**3.28.** Tangents $AB$ and $AC$ are drawn from point $A$ to a circle centered at $O$. Through point $X$ on segment $BC$ line $KL$ perpendicular to $XO$ is drawn so that points $K$ and $L$ lie on lines $AB$ and $AC$, respectively. Prove that $KX = XL$.

**3.29.** On the extension of chord $KL$ of a circle centered at $O$ a point $A$ is taken and tangents $AP$ and $AQ$ to the circle are drawn from it; let $M$ be the midpoint of segment $PQ$. Prove that $\angle MKO = \angle MLO$.

**3.30.** From point $A$ tangents $AB$ and $AC$ to a circle and a line that intersects the circle at points $D$ and $E$ are drawn; let $M$ be the midpoint of segment $BC$. Prove that $BM^2 = DM \cdot ME$ and either $\angle DME = 2\angle DBE$ or $\angle DME = 2\angle DCE$; moreover, $\angle BEM = \angle DEC$.

**3.31.** Quadrilateral $ABCD$ is inscribed in a circle so that tangents to this circle at points $B$ and $D$ intersect at a point $K$ that lies on line $AC$.

a) Prove that $AB \cdot CD = BC \cdot AD$.

b) A line parallel to $KB$ intersects lines $BA$, $BD$ and $BC$ at points $P$, $Q$ and $R$, respectively. Prove that $PQ = QR$.

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**3.32.** A circle $S$ and a line $l$ that has no common points with $S$ are given. From point $P$ that moves along line $l$ tangents $PA$ and $PB$ to circle $S$ are drawn. Prove that all chords $AB$ have a common point.

Let point $P$ lie outside circle $S$; let $PA$ and $PB$ be tangents to the circle. Then line $AB$ is called the *polar line* of point $P$ relative circle $S$.

**3.33.** Circles $S_1$ and $S_2$ intersect at points $A$ and $B$ so that the center $O$ of circle $S_1$ lies on $S_2$. A line that passes through point $O$ intersects segment $AB$ at point $P$ and circle $S_2$ at point $C$. Prove that point $P$ lies on the polar line of point $C$ relative circle $S_1$.

## §6. Application of the theorem on triangle's heights

**3.34.** Points $C$ and $D$ lie on the circle with diameter $AB$. Lines $AC$ and $BD$, $AD$ and $BC$ meet at points $P$ and $Q$, respectively. Prove that $AB \perp PQ$.

**3.35.** Lines $PC$ and $PD$ are tangent to the circle with diameter $AB$ so that $C$ and $D$ are tangent points. Prove that the line that connects $P$ with the intersection point of lines $AC$ and $BD$ is perpendicular to $AB$.

**3.36.** Given diameter $AB$ of a circle and point $C$ outside $AB$. With the help of the ruler alone (no compasses) drop the perpendicular from $C$ to $AB$ if:

a) point $C$ does not lie on the circle;

b) point $C$ lies on the circle.

**3.37.** Let $O_a$, $O_b$ and $O_c$ be the centers of circumscribed circles of triangles $PBC$, $PCA$ and $PAB$. Prove that if points $O_a$ and $O_b$ lie on lines $PA$ and $PB$, then point $O_c$ lies on line $PC$.

## §7. Areas of curvilinear figures

**3.38.** On the hypothenuse and legs of a rectangular triangle semicircles are constructed as shown on Fig. 22. Prove that the sum of the areas of the crescents obtained (shaded) is equal to the area of the given triangle.
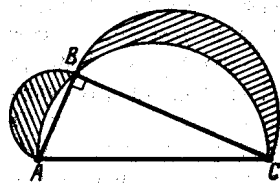


Figure 22 (3.38)

**3.39.** In a disc two perpendicular diameters, i.e., four radii, are constructed. Then there are constructed four disks whose diameters are these radii. Prove that the total area of the pairwise common parts of these four disks is equal to the area of the initial (larger) disk that lies outside the considered four disks (Fig. 23).
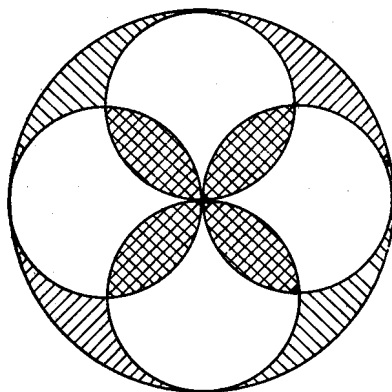


Figure 23 (3.39)

**3.40.** On three segments $OA$, $OB$ and $OC$ of the same length (point $B$ lies outside angle $AOC$) circles are constructed as on diameters. Prove that the area of the curvilinear triangle bounded by the arcs of these circles and not containing point $O$ is equal to a half area of the (common) triangle $ABC$.

**3.41.** On sides of an arbitrary acute triangle $ABC$ as on diameters circles are constructed. They form three "outer" curvilinear triangles and one "inner" triangle (Fig. 24). Prove that if we subtract the area of the "inner" triangle from the sum of the areas of "outer" triangles we get the doubled area of triangle $ABC$.

## §8. Circles inscribed in a disc segment

In this section a *segment* is always a disc segment.

**3.42.** Chord $AB$ divides circle $S$ into two arcs. Circle $S_1$ is tangent to chord $AB$ at point $M$ and one of the arcs at point $N$. Prove that:

a) line $MN$ passes through the midpoint $P$ of the second arc;

b) the length of tangent $PQ$ to circle $S_1$ is equal to that of $PA$.

**3.43.** From point $D$ of circle $S$ the perpendicular $DC$ is dropped to diameter $AB$. Circle $S_1$ is tangent to segment $CA$ at point $E$ and also to segment $CD$ and to circle $S$. Prove that $DE$ is a bisector of triangle $ADC$.
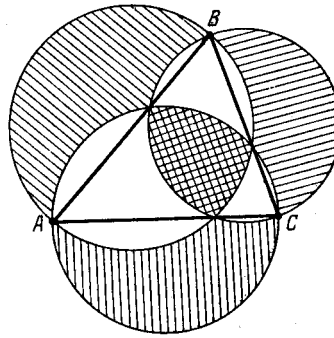
Figure 24 (3.41)

**3.44.** Two circles inscribed in segment $AB$ of the given circle intersect at points $M$ and $N$. Prove that line $MN$ passes through the midpoint $C$ of arc $AB$ complementary for the given segment.

**3.45.** A circle tangent to sides $AC$ and $BC$ of triangle $ABC$ at points $M$ and $N$, respectively is also tangent to its circumscribed circle (from the inside). Prove that the midpoint of segment $MN$ coincides with the center of the inscribed circle of triangle $ABC$.

**3.46.** Triangles $ABC_1$ and $ABC_2$ are inscribed in circle $S$ so that chords $AC_2$ and $BC_1$ intersect. Circle $S_1$ is tangent to chord $AC_2$ at point $M_2$, to chord $BC_1$ at point $N_1$ and to circle $S$ (???where?). Prove that the centers of the inscribed circles of triangles $ABC_1$ and $ABC_2$ lie on segment $M_2N_1$.

## §9. Miscellaneous problems

**3.47.** The radii of two circles are equal to $R_1$ and $R_2$ and the distance between the centers of the circles is equal to $d$. Prove that these circles are orthogonal if and only if $d^2 = R_1^2 + R_2^2$.

**3.48.** Three circles are pairwise tangent from the outside at points $A$, $B$ and $C$. Prove that the circumscribed circle of triangle $ABC$ is perpendicular to all the three circles.

**3.49.** Two circles centered at $O_1$ and $O_2$ intersect at points $A$ and $B$. A line is drawn through point $A$; the line intersects the first circle at point $M_1$ and the second circle at point $M_2$. Prove that $\angle BO_1M_1 = \angle BO_2M_2$.

## §10. The radical axis

**3.50.** Circle $S$ and point $P$ are given on the plane. A line drawn through point $P$ intersects the circle at points $A$ and $B$. Prove that the product $PA \cdot PB$ does not depend on the choice of a line.

This product taken with the plus sign if point $P$ is outside the circle and with minus sign if $P$ is inside of the circle is called the *degree of point $P$ with respect to circle $S$*.

**3.51.** Prove that for a point $P$ outside circle $S$ its degree with respect to $S$ is equal to the square of the length of the tangent drawn to the circle from point $P$.

**3.52.** Prove that the degree of point $P$ with respect to circle $S$ is equal to $d^2 - R^2$, where $R$ is the radius of $S$ and $d$ is the distance from $P$ to the center of $S$.

**3.53.** Two nonconcentric circles $S_1$ and $S_2$ are given in plane. Prove that the locus of points whose degree with respect to $S_1$ is equal to the degree with respect to $S_2$ is a line.

This line is called *the radical axis* of circles $S_1$ and $S_2$.

**3.54.** Prove that the radical axis of two intersecting circles passes through the intersection points.

**3.55.** Given three circles in plane whose centers do not lie on one line. Let us draw radical axes for each pair of these circles. Prove that all the three radical axes meet at one point.

This point is called the *radical center* of the three circles.

**3.56.** Consider three pairwise intersecting circles in plane. Through the intersection points of any two of them a line is drawn. Prove that either these three lines meet at one point or are parallel.

**3.57.** Two nonconcentric circles $S_1$ and $S_2$ are given. Prove that the set of centers of circles that intersect both these circles at a right angle is their radical axis (without their common chord if the given circles intersect).

**3.58.** a) Prove that the midpoints of the four common tangents to two nonintersecting circles lie on one line.

b) Through two of the tangent points of common exterior tangents with two circles a line is drawn, see Fig. . Prove that the circles cut on this line equal chords.

**3.59.** On sides $BC$ and $AC$ of triangle $ABC$, points $A_1$ and $B_1$, respectively, are taken; let $l$ be the line that passes through the common points of circles with diameters $AA_1$ and $BB_1$. Prove that:

a) Line $l$ passes through the intersection point $H$ of heights of triangle $ABC$;

b) line $l$ passes through point $C$ if and only if $AB_1 : AC = BA_1 : BC$.

**3.60.** The extensions of sides $AB$ and $CD$ of quadrilateral $ABCD$ meet at point $F$ and the extensions of sides $BC$ and $AD$ meet at point $E$. Prove that the circles with diameters $AC$, $BD$ and $EF$ have a common radical axis and the orthocenters of triangles $ABE$, $CDE$, $ADF$ and $BCF$ lie on it.

**3.6l.** Three circles intersect pairwise at points $A_1$ and $A_2$, $B_1$ and $B_2$, $C_1$ and $C_2$. Prove that $A_1B_2 \cdot B_1C_2 \cdot C_1A_2 = A_2B_1 \cdot B_2C_1 \cdot C_2A_1$.

**3.62.** On side $BC$ of triangle $ABC$ point $A'$ is taken. The midperpendicular to segment $A'B$ intersects side $AB$ at point $M$ and the midperpendicular to segment $A'C$ intersects side $AC$ at point $N$. Prove that point symmetric to point $A'$ through line $MN$ lies on the circumscribed circle of triangle $ABC$.

**3.63.** Solve Problem 1.66 making use of the properties of the radical axis.

**3.64.** Inside a convex polygon several pairwise nonintersecting disks of distinct radii are placed. Prove that it is possible to cut the polygon into smaller polygons so that all these small polygons are convex and each of them contains exactly one of the given disks.

**3.65.** a) In triangle $ABC$, heights $AA_1$, $BB_1$ and $CC_1$ are drawn. Lines $AB$ and $A_1B_1$, $BC$ and $B_1C_1$, $CA$ and $C_1A_1$ intersect at points $C'$, $A'$ and $B'$, respectively. Prove that points $A'$, $B'$ and $C'$ lie on the radical axis of the circle of nine points (cf. Problem 5.106) and on that of the circumscribed circle.

b) The bisectors of the outer angles of triangle $ABC$ intersect the extensions of the opposite sides at points $A'$, $B'$ and $C'$. Prove that points $A'$, $B'$ and $C'$ lie on one line and this line is perpendicular to the line that connects the centers of the inscribed and circumscribed circles of triangle $ABC$.

**3.66.** Prove that diagonals $AD$, $BE$ and $CF$ of the circumscribed hexagon $ABCDEF$ meet at one point. (*Brianchon's theorem.*)

**3.67.** Given four circles $S_1$, $S_2$, $S_3$ and $S_4$ such that the circles $S_i$ and $S_{i+1}$ are tangent from the outside for $i = 1, 2, 3, 4$, where $S_5 = S_1$. Prove that the radical axis of circles $S_1$ and $S_3$ passes through the intersection point of common outer tangents to $S_2$ and $S_4$.

**3.68.** a) Circles $S_1$ and $S_2$ intersect at points $A$ and $B$. The degree of point $P$ of circle $S_1$ with respect to circle $S_2$ is equal to $p$, the distance from point $P$ to line $AB$ is equal to $h$ and the distance between the centers of circles is equal to $d$. Prove that $|p| = 2dh$.

b) The degrees of points $A$ and $B$ with respect to the circumscribed circles of triangles $BCD$ and $ACD$ are equal to $p_a$ and $p_b$, respectively. Prove that $|p_a|S_{BCD} = |p_b|S_{ACD}$.

## Problems for independent study

**3.69.** An easy chair of the form of a disc sector of radius $R$ is swinging on a horizontal table. What is the trajectory of the vertex of the sector?

**3.70.** From a point $A$ outside a circle of radius $R$ two tangents $AB$ and $AC$ are drawn, $B$ and $C$ are tangent points. Let $BC = a$. Prove that $4R^2 = r^2 + r_a^2 + \frac{1}{2}a^2$, where $r$ and $r_a$ are the radii of the inscribed and escribed circles of triangle $ABC$.

**3.71.** Two circles have an inner tangent. The line that passes through the center of a smaller circle intersects the greater one at points $A$ and $D$ and the smaller one at points $B$ and $C$. Find the ratio of the radii of the circles if $AB : BC : CD = 2 : 3 : 4$.

**3.72.** The centers of three circles each of radius $R$, where $1 < R < 2$, form an equilateral triangle with side 2. What is the distance between the intersection points of these circles that lie outside the triangle?

**3.73.** A point $C$ is taken on segment $AB$ and semicircles with diameters $AB$, $AC$ and $BC$ are constructed (on one side of line $AB$). Find the ratio of the area of the curvilinear triangle bounded by these semicircles to the area of the triangle formed by the midpoints of the arcs of these semicircles.

**3.74.** A circle intersects side $BC$ of triangle $ABC$ at points $A_1$ and $A_2$, side $AC$ at points $B_1$ and $B_2$, side $AB$ at points $C_1$ and $C_2$. Prove that

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \left(\frac{AC_2}{C_2B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A}\right)^{-1}.$$

**3.75.** From point $A$ tangents $AB$ and $AC$ to a circle are drawn ($B$ and $C$ are tangent points); $PQ$ is a diameter of the circle; line $l$ is tangent to the circle at point $Q$. Lines $PA$, $PB$ and $PC$ intersect line $l$ at points $A_1$, $B_1$ and $C_1$. Prove that $A_1B_1 = A_1C_1$.

## Solutions

**3.1.** Let line $XY$ be tangent to the given circle at point $Z$. The corresponding sides of triangles $XOA$ and $XOZ$ are equal and, therefore, $\angle XOA = \angle XOZ$. Similarly, $\angle ZOY = \angle BOY$. Therefore,

$$\angle XOY = \angle XOZ + \angle ZOY = \frac{\angle AOZ + \angle ZOB}{2} = \frac{\angle AOB}{2}.$$

**3.2.** Let $M$ and $N$ be the tangent points of the inscribed circle with sides $AB$ and $BC$. Then $BK + AN = BM + AM = AB$, hence, $CK + CN = a + b - c$.

Let $P$ and $Q$ be the tangent points of the escribed circle with the extensions of sides $AB$ and $BC$. Then $AP = AB + BP = AB + BL$ and $AQ = AC + CQ = AC + CL$. Hence, $AP + AQ = a + b + c$. Therefore, $BL = BP = AP - AB = \frac{1}{2}(a + b - c)$.

**3.3.** By Problem 3.2 $CM = \frac{1}{2}(AC + CE - AE)$ and $CN = \frac{1}{2}(BC + CE - BE)$. Taking into account that $AC = BC$ we get $MN = |CM - CN| = \frac{1}{2}|AE - BE|$.

**3.4.** Let lines $AB$, $BC$, $CD$ and $DA$ be tangent to the circle at points $P$, $Q$, $R$ and $S$, respectively. Then $CQ = CR = x$, hence, $BP = BC + CQ = BC + x$ and $DS = DC + CR = DC + x$. Therefore, $AP = AB + BP = AB + BC + x$ and $AS = AD + DS = AD + DC + x$. Taking into account that $AP = AS$, we get the statement desired.

**3.5.** Let line $AB$ be tangent to the circles centered at $O_1$ and $O_2$ at points $C$ and $D$, respectively. Since $\angle O_1AO_2 = 90°$, the right triangles $AO_1C$ and $O_2AD$ are similar. Therefore, $O_1C : AC = AD : DO_2$. Moreover, $AD = CB$ (cf. Problem 3.2). Therefore, $AC \cdot CB = Rr$.

**3.6.** Let lines $AB$ and $CD$ intersect at point $O$. Let us assume for definiteness that points $A$ and $D$ lie on the first circle while points $B$ and $C$ lie on the second one. Suppose also that $OB < OA$ (Fig. 25).
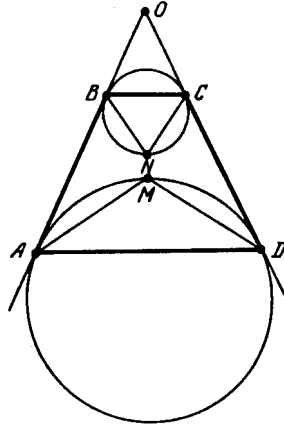


Figure 25 (Sol. 3.6)

The intersection point $M$ of bisectors of angles $\angle A$ and $\angle D$ of quadrilateral $ABCD$ is the midpoint of the arc of the first circle that lies inside triangle $AOD$ and the intersection point $N$ of bisectors of angles $\angle B$ and $\angle C$ is the midpoint of the arc of the second circle that lies outside triangle $BOC$, cf. Problem 2.91 a). Quadrilateral $ABCD$ is a circumscribed one if and only if points $M$ and $N$ coincide.

**3.7.** Let $R$ be the tangent point of the escribed circle with side $BD$, let $P$ and $Q$ be the intersection points of segment $MN$ with $BC$ and $CD$, respectively (Fig. 26).
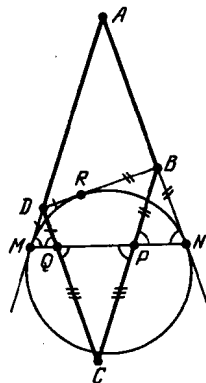


Figure 26 (Sol. 3.7)

Since $\angle DMQ = \angle BPN$, $\angle DQM = \angle BNP$ and $\angle DMQ = \angle BNP$, it follows that triangles $MDQ$, $PBN$ and $PCQ$ are isosceles ones. Therefore, $CP = CQ$, $DQ = DM = DR$ and $BP = BN = BR$. Therefore, $P$, $Q$ and $R$ are the tangent points of the inscribed circle of triangle $BCD$ with its sides (cf. Problem 5.1).

**3.8.** Denote some of the tangent points as shown on Fig. 27. The sum of the lengths of one pair of the opposite sides of the inner quadrilateral is equal to the sum of the lengths

of the pair of its other sides. Let us extend the sides of this quadrilateral to tangent points with inscribed circles of the other quadrilaterals ($ST$ is one of the obtained segments).
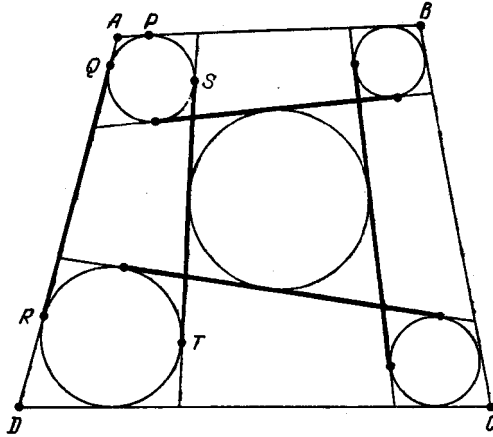


Figure 27 (Sol. 3.8)

Then both sums of lengths of pairs of opposite segments increase by the same number. Each of the obtained segments is the common tangent to a pair of "corner" circles; each segment can be replaced with another common outer tangent of equal length (i.e., replace $ST$ with $QR$). To prove the equality $AB + CD = BC + AD$, it remains to make use of equalities of the form $AP = AQ$.

**3.9.** Let $P$ be the intersection point of diagonals of convex quadrilateral $ABCD$. Quadrilateral $ABCD$ is an inscribed one if and only if $\triangle APB \sim \triangle DPC$, i.e., $PA \cdot PC = PB \cdot PD$. Since quadrilaterals $ALBN$ and $AMBK$ are inscribed ones, $PL \cdot PN = PA \cdot PB = PM \cdot PK$. Hence, quadrilateral $KLMN$ is an inscribed one.

**3.10.** Let $O$ be the intersection point of line $AB$ and segment $MN$. Then $OM^2 = OA \cdot OB = ON^2$, i.e., $OM = ON$.

**3.11.** Let, for definiteness, rays $OA$ and $BC$ be codirected, $M$ the intersection point of lines $KL$ and $OA$. Then $\angle LOM = \angle LCB = \angle OKM$ and, therefore, $\triangle KOM \sim \triangle OLM$. Hence, $OM : KM = LM : OM$, i.e., $OM^2 = KM \cdot LM$. Moreover, $MA^2 = MK \cdot ML$. Therefore, $MA = OM$.

**3.12.** Let $O$ be the intersection point of diagonals $AC$ and $BD$. Then $MO \cdot OC = BO \cdot OD$. Since $OC = OA$ and $BO = OD$, we have $MO \cdot OA = BO^2$ and $MO \cdot OA = DO^2$. These equalities mean that $OB$ is tangent to the circumscribed circle of triangle $ABM$ and $OD$ is tangent to the circumscribed circle of triangle $ADM$.

**3.13.** Let $C$ be the intersection point of line $AB$ with the circumscribed circle of triangle $BMN$ distinct from point $B$; let $AP$ be the tangent to circle $S$. Then $AB \cdot AC = AM \cdot AN = AP^2$ and, therefore, $AC = \frac{AP^2}{AB}$, i.e., point $C$ is the same for all lines $l$.

REMARK. We have to exclude the case when the length of the tangent drawn to $S$ from $A$ is equal to $AB$.

**3.14.** Clearly, $MC^2 = MP \cdot MQ = MA \cdot MB$ and point $M$ lies on ray $AB$ if $AC > BC$ and on ray $BA$ if $AC < BC$. Let, for definiteness sake, point $M$ lie on ray $AB$. Then $(MB + BC)^2 = (MB + BA) \cdot MB$. Therefore, $MB = \frac{BC^2}{AB - 2BC}$ and we deduce that the position of point $M$ does not depend on the choice of circle $S'$.

**3.15.** Let $M$ be the intersection point of line $CD$ and the tangent to circles at point $A$. Then $MC = MA = MD$. Therefore, point $A$ lies on the circle with diameter $CD$.

**3.16.** Points $O_1$, $A$ and $O_2$ lie on one line, hence, $\angle A_2 A O_2 = \angle A_1 A O_1$. Triangles $A O_2 A_2$ and $A O_1 A_1$ are isosceles ones, hence, $\angle A_2 A O_2 = \angle A A_2 O_2$ and $\angle A_1 A O_1 = \angle A A_1 O_1$. Therefore, $\angle A A_2 O_2 = \angle A A_1 O_1$, i.e., $O_1 A_1 \parallel O_2 A_2$.

**3.17.** Let $O_1$, $O_2$ and $O_3$ be the centers of circles $S_1$, $S_2$ and $S_3$; let $A$, $B$, $C$ be the tangent points of circles $S_2$ and $S_3$, $S_3$ and $S_1$, $S_1$ and $S_2$, respectively; $A_1$ and $B_1$ the intersection points of lines $CA$ and $CB$, respectively, with circle $S_3$. By the previous problem $B_1 O_3 \parallel C O_1$ and $A_1 O_3 \parallel C O_2$. Points $O_1, C$ and $O_2$ lie on one line and, therefore, points $A_1$, $O_3$ and $B_1$ also lie on one line, i.e., $A_1 B_1$ is a diameter of cicle $S_3$.

**3.18.** Let $A_1$, $A_2$ and $B$ be the tangent points of the circles centered at $O$ and $O_1$, $O$ and $O_2$, $O_1$ and $O_2$, respectively. Then $O_1 O_2 = O_1 B + B O_2 = O_1 A_1 + O_2 A_2$. Therefore,

$$OO_1 + OO_2 + O_1 O_2 = (OO_1 + O_1 A_1) + (OO_2 + O_2 A_2) = OA_1 + OA_2 = 2R.$$

**3.19.** Let $O$, $O_1$ and $O_2$ be centers of circles $S$, $S_1$ and $S_2$; let $C$ be the common point of circles $S_1$ and $S_2$ that lies on segment $AB$. Triangles $AOB$, $AO_1C$ and $CO_2B$ are isosceles ones; consequently, $OO_1CO_2$ is a parallelogram and $OO_1 = O_2C = O_2B$; hence, $AO = AO_1 + O_1O = AO_1 + O_2B$.

**3.20.** Let $O_1$ and $O_2$ be the centers of circles $S_1$ and $S_2$; let $X$ be the other intersection point of line $AB$ with circle $S_2$. The square of the length of the tangent in question is equal to $BA \cdot BX$. Since $AB : BX = O_1A : O_1O_2$, it follows that $AB \cdot BX = \frac{AB^2 \cdot O_1O_2}{R} = \frac{a^2(R \pm r)}{R}$, where the minus sign is taken for the inner tangent and the plus sign for the outer tangent.

**3.21.** Let $O$, $O_1$ and $O_2$ be the centers of the circles with diameters $AB$, $AC$ and $BC$, respectively. It suffices to verify that $KO = OL$. Let us prove that $\triangle O_1 KO = \triangle O_2 OL$. Indeed, $O_1 K = \frac{1}{2}AC = O_2O$, $O_1O = \frac{1}{2}BC = O_2L$ and $\angle KO_1O = \angle OO_2L = 180° - 2\alpha$, where $\alpha$ is the value of the angle between lines $KL$ and $AB$.

**3.22.** Let $O_i$ be the center of circle $S_i$ and $A_i$ the tangent point of circles $S_i$ and $S_{i+1}$. Quadrilateral $O_1O_2O_3O_4$ is a convex one; let $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ be the values of its angles. It is easy to verify that $\angle A_{i-1}A_iA_{i+1} = \frac{1}{2}(\alpha_i + \alpha_{i+1})$ and, therefore,

$$\angle A_1 + \angle A_3 = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \angle A_2 + \angle A_4.$$

**3.23.** a) Let $A_1$, $B_1$ and $C_1$ be the projections of points $A$, $B$ and $C$, respectively, to line $l$; let $C_2$ be the projection of point $C$ to line $AA_1$. By Pythagorus theorem $CC_2^2 = AC^2 - AC_2^2$, i.e., $A_1C_1^2 = (a + c)^2 - (a - c)^2 = 4ac$. Similarly, $B_1C_1^2 = 4bc$ and $A_1B_1^2 = 4ab$. Since $A_1C_1 + C_1B_1 = A_1B_1$, it follows that $\sqrt{ac} + \sqrt{bc} = \sqrt{ab}$, i.e., $\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{a}} = \frac{1}{\sqrt{c}}$.
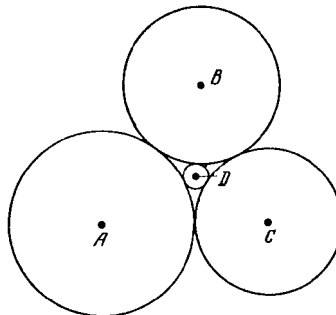


Figure 28 (Sol. 3.23 b))

b) Let $A$, $B$, $C$ be the centers of "outer" circles, $D$ the center of the "inner" circle (Fig. 28). The semiperimeter of triangle $BDC$ is equal to $b + c + d$, and, therefore,

$$\cos^2\left(\frac{\angle BDC}{2}\right) = \frac{d(b+c+d)}{(b+d)(c+d)}, \quad \sin^2\left(\frac{\angle BDC}{2}\right) = \frac{bc}{(b+d)(c+d)}$$

(cf. Problem 12.13). As is easy to see the law of cosines is equivalent to the statement:

$$\alpha' + \beta' + \gamma' = 180° \Longrightarrow \sin^2\alpha' - \sin^2\beta' - \sin^2\gamma' + 2\sin\beta'\sin\gamma'\cos\alpha' = 0. \qquad (*)$$

Substituting the values $\alpha' = \frac{1}{2}\angle BDC$, $\beta' = \frac{1}{2}\angle ADC$ and $\gamma' = \frac{1}{2}\angle ADB$ into formula $(*)$, we get

$$\frac{bc}{(b+d)(c+d)} - \frac{ac}{(a+d)(c+d)} - \frac{ab}{(a+d)(b+d)} + 2\frac{a\sqrt{bcd(b+c+d)}}{(a+d)(b+d)(c+d)} = 0,$$

i.e.,

$$\frac{a+d}{a} - \frac{b+d}{b} - \frac{c+d}{c} + 2\sqrt{\frac{d(b+c+d)}{bc}} = 0.$$

Dividing this by $d$ we get

$$\alpha - \beta - \gamma - \delta + 2\sqrt{\beta\gamma + \gamma\delta + \delta\beta} = 0.$$

Therefore,

$$(\alpha + \beta + \gamma + \delta)^2 = (\alpha - \beta - \gamma - \delta)^2 + 4(\alpha\beta + \alpha\gamma + \alpha\delta) +$$
$$4(\beta\gamma + \gamma\delta + \delta\beta) + 4(\alpha\beta + \alpha\gamma + \alpha\delta) =$$
$$2(\alpha + \beta + \gamma + \delta)^2 - 2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2),$$

i.e.,

$$2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (\alpha + \beta + \gamma + \delta)^2.$$

**3.24.** Let $A_1$, $B_1$ and $C_1$ be the centers of the given circles (Fig. 29). Then $A_1BC_1H$ is a rhombus and, therefore, $BA_1 \parallel HC_1$. Similarly, $B_1A \parallel HC_1$; hence, $B_1A \parallel BA_1$ and $B_1ABA_1$ is a parallelogram.
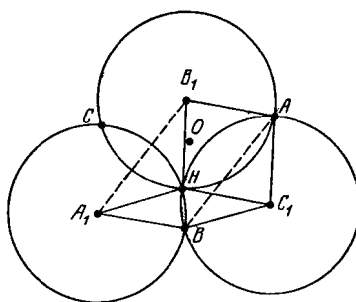


Figure 29 (Sol. 3.24)

a) Since $A_1B_1 \perp CH$ and $A_1B_1 \parallel AB$, it follows that $AB \perp CH$. We similarly prove that $BC \perp AH$ and $CA \perp BH$.

b) In the same way as we have proved that $B_1A \parallel BA_1$, we can prove that $B_1C \parallel BC_1$ and $A_1C \parallel AC_1$; moreover, the lengths of all these six segments are equal to $R$. Let us complement the triangle $BA_1C$ to a rhombus $BA_1CO$. Then $AB_1CO$ is also a rhombus. Therefore, $AO = BO = CO = R$, i.e., $O$ is the center of the circumscribed circle of triangle $ABC$ and its radius is equal to $R$.

**3.25.** It is easy to verify that

$$
\begin{array}{rcl}
\smile AB_1 \pm \smile B_1A_1 & = & \smile AC_1 + \smile C_1A_1, \\
\smile BC_1 + \smile C_1B_1 & = & \smile BA_1 \pm \smile B_1A_1 \\
\smile C_1A_1 \pm \smile CA_1 & = & \smile C_1B_1 \pm \smile B_1C,
\end{array}
$$

where the minus sign is only taken in case b). Adding up these equalities we get

$$
\smile AB_1 + \smile BC_1 \pm \smile CA_1 = \smile AC_1 + \smile BA \pm \smile CB_1.
$$

On the other hand, the doubled values of the angles of triangle $ABC$ are equal to $\smile BA_1 \pm \smile CA_1$, $\smile AB_1 \pm \smile CB_1$ and $\smile BC_1 + \smile AC_1$, and their sum is equal to $360°$.

**3.26.** Since $\smile AP + \smile BP + \smile PQ = 180°$ (cf. Problem 3.25), it follows that $\smile AB = 180° - \smile PQ$. Similarly, $\smile CD = 180° - \smile PQ$, i.e., $\smile AB = \smile CD$ and, therefore, $AB = CD$. Moreover, $PQ \perp AB$ and $PQ \perp CD$ (cf. Problem 3.24) and, therefore, $AB \parallel CD$.

**3.27.** Points $M$, $B$ and $C$ lie on the circle with diameter $AO$. Moreover, chords $OB$ and $OC$ of the circle are equal.

**3.28.** Points $B$ and $X$ lie on the circle with diameter $KO$, and, therefore, $\angle XKO = \angle XBO$. Similarly, $\angle XLO = \angle XCO$. Since $\angle XBO = \angle XCO$, triangle $KOL$ is an isosceles one and $OX$ is its height.

**3.29.** It suffices to verify that $AK \cdot AL = AM \cdot AO$. Indeed, if such is the case, then points $K$, $L$, $M$ and $O$ lie on one circle and, therefore, $\angle MKO = \angle MLO$. Since $\triangle AOP \sim \triangle APM$, it follows that $AM \cdot AO = AP^2$; it is also clear that $AK \cdot AL = AP^2$.

**3.30.** Let $O$ be the center of the circle; let points $D'$ and $E'$ be symmetric to points $D$ and $E$ through line $AO$. By Problem 28.7 the lines $ED'$ and $E'D$ meet at point $M$. Hence, $\angle BDM = \angle EBM$ and $\angle BEM = \angle DBM$ and, therefore, $\triangle BDM \sim \triangle EBM$. It follows that $BM : DM = EM : BM$. Moreover, if line $ED$ separates points $B$ and $M$, then $\angle DME = \smile DE = 2\angle DCE$.

The equality $\angle BEM = \angle DBM$ implies that $\angle BEM = \angle DBC = \angle DEC$.

**3.31.** a) Since $\triangle KAB \sim \triangle KBC$, we have $AB : BC = KB : KC$. Similarly, $AD : DC = KD : KC$. Taking into account that $KB = KD$ we get the desired statement.

b) The problem of this heading reduces to that of the previous one, since

$$
\frac{PQ}{BQ} = \frac{\sin \angle PBQ}{\sin \angle BPQ} = \frac{\sin \angle ABD}{\sin \angle KBA} = \frac{\sin \angle ABD}{\sin \angle ADB} = \frac{AD}{AB}, \quad \frac{QR}{BQ} = \frac{CD}{CB}.
$$

**3.32.** Let us drop perpendicular $OM$ to line $l$ from center $O$ of circle $S$. Let us prove that point $X$ at which $AB$ and $OM$ intersect remains fixed. Points $A$, $B$ and $M$ lie on the circle with diameter $PO$. Hence, $\angle AMO = \angle ABO = \angle BAO$ and, therefore, $\triangle AMO \sim \triangle XAO$, because these triangles have a common angle at vertex $O$. It follows that $AO : MO = XO : AO$, i.e., $OX = \frac{OA^2}{MO}$ is a constant.

**3.33.** Since $\angle OBP = \angle OAB = \angle OCB$, we deduce that $\triangle OBP \sim \triangle OCB$ and, therefore, $OB^2 = OP \cdot OC$. Let us draw tangent $CD$ to circle $S_1$ from point $C$. Then $OD^2 = OB^2 = OP \cdot OC$. Therefore, $\triangle ODC \sim \triangle OPD$ and $\angle OPD = \angle ODC = 90°$.

**3.34.** Lines $BC$ and $AD$ are heights of triangle $APB$ and, therefore, line $PQ$ that passes through their intersection point $Q$ is perpendicular to line $AB$.

**3.35.** Denote the intersection points of lines $AC$ and $BD$, $BC$ and $AD$ by $K$ and $K_1$, respectively. Thanks to the above problem, $KK_1 \perp AB$ and, therefore, it suffices to show that the intersection point of tangents at points $C$ and $D$ lies on line $KK_1$.

Let us prove that the tangent at point $C$ passes through the midpoint of segment $KK_1$. Let $M$ be the intersection point of the tangent at point $C$ and segment $KK_1$. The respective sides of acute angles $\angle ABC$ and $\angle CKK_1$ are perpendicular and, therefore, the angles are

equal. Similarly, $\angle CAB = \angle CK_1K$. It is also clear that $\angle KCM = \angle ABC$ and, therefore, triangle $CMK$ is an isosceles one. Similarly, triangle $CMK_1$ is an isosceles one and $KM = CM = K_1M$, i.e., $M$ is the midpoint of segment $KK_1$.

We similarly prove that the tangent at point $D$ passes through the midpoint of segment $KK_1$.

**3.36.** a) Line $AC$ intersects the circle at points $A$ and $A_1$, line $BC$ does same at points $B$ and $B_1$. If $A = A_1$ (or $B = B_1$), then line $AC$ (or $BC$) is the perpendicular to be constructed. If this is not the case, then $AB_1$ and $BA_1$ are heights of triangle $ABC$ and the line to be constructed is the line that passes through point $C$ and the intersection point of lines $AB_1$ and $BA_1$.

b) Let us take point $C_1$ that does not lie on the circle and drop from it perpendicular to $AB$. Let the perpendicular intersect the circle at points $D$ and $E$. Let us construct the intersection point $P$ of lines $DC$ and $AB$ and then the intersection point $F$ of line $PE$ with the circle. The symmetry through $AB$ sends point $C$ to point $F$. Therefore, $CF$ is the perpendicular to be constructed.

**3.37.** Since $PA \perp O_bO_c$, line $PA$ passes through point $O_a$ if and only if line $PO_a$ passes through the intersection point of heights of triangle $O_aO_bO_c$. Similar statements are true for points $B$ and $C$ as well.

The hypothesis of the problem implies that $P$ is the intersection point of heights of triangle $O_aO_bO_c$ and, therefore, $PO_c \perp O_aO_b$.

**3.38.** Let $2a$ and $2b$ be the lengths of the legs, $2c$ the length of the hypothenuse. The sum of the areas of the "crescents" is equal to $\pi a^2 + \pi b^2 + S_{ABC} - \pi c^2$. But $\pi(a^2+b^2-c^2) = 0$.

**3.39.** It suffices to carry out the proof for each of the four parts into which the diameters divide the initial disc (Fig. 30).
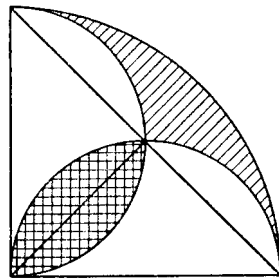


Figure 30 (Sol. 3.39)

In the disc, consider the segment cut off by the chord intercepted by the central angle of $90°$; let $S$ and $s$ be the areas of such segments for the initial disc and any of the four constructed disks, respectively. Clearly, $S = 4s$. It remains to observe that the area of the part shaded once is equal to $S - 2s = 2s$ and the area of the part shaded twice is equal to $2s$.

**3.40.** Denote the intersection points of circles constructed on segments $OB$ and $OC$, $OA$ and $OC$, $OA$ and $OB$ as on diameters by $A_1$, $B_1$, $C_1$, respectively (Fig. 31). Since $\angle OA_1B = \angle OA_1C = 90°$, it follows that points $B$, $A_1$ and $C$ lie on one line and since all the circles have equal radii, $BA_1 = A_1C$.

Points $A_1$, $B_1$, $C_1$ are the midpoints of sides of triangle $ABC$, therefore, $BA_1 = C_1B_1$ and $BC = A_1B_1$. Since the disks are of the same radius, the equal chords $BA_1$ and $C_1B_1$ cut off the disks parts of equal area and equal chords $C_1B$ and $B_1A_1$ also cut off the disc's parts of equal area. Therefore, the area of curvilinear triangle $A_1B_1C_1$ is equal to the area of parallelogram $A_1B_1C_1B$, i.e., is equal to half the area of triangle $ABC$.
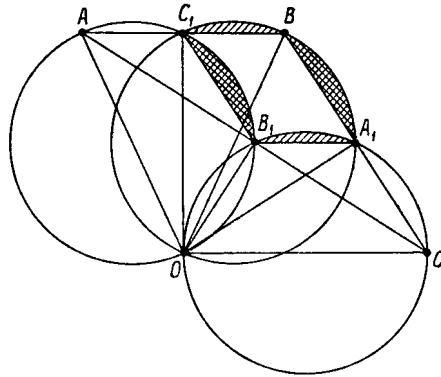
Figure 31 (Sol. 3.40)
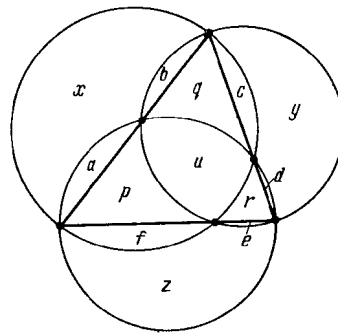


Figure 32 (Sol. 3.41)

**3.41.** The considered circles pass through the bases of the triangle's heights and, therefore, their intersection points lie on the triangle's sides. Let $x$, $y$, $z$ and $u$ be the areas of the considered curvilinear triangles; let $a$, $b$, $c$, $d$, $e$ and $f$ be the areas of the segments cut off the circles by the sides of the triangle; let $p$, $q$ and $r$ be the areas of the parts of the triangle that lie outside the inner curvilinear triangle (see Fig. 32). Then

$$\begin{array}{rcl} x + (a+b) & = & u + p + q + (c+f), \\ y + (c+d) & = & u + q + r + (e+b), \\ z + (e+f) & = & u + r + p + (a+d) \end{array}$$

By adding up these equalities we get

$$x + y + z = 2(p + q + r + u) + u.$$

**3.42.** a) Let $O$ and $O_1$ be the centers of circles $S$ and $S_1$. The triangles $MO_1N$ and $PON$ are isosceles ones and $\angle MO_1N = \angle PON$. Therefore, points $P$, $M$ and $N$ lie on one line.

b) It is clear that $PQ^2 = PM \cdot PN = PM \cdot (PM + MN)$. Let $K$ be the midpoint of chord $AB$. Then

$$PM^2 = PK^2 + MK^2 \quad \text{and } PM \cdot MN = AM \cdot MB = AK^2 - MK^2.$$

Therefore, $PQ^2 = PK^2 + AK^2 = PA^2$.

**3.43.** By Problem 3.42 b) $BE = BD$. Hence,

$$\angle DAE + \angle ADE = \angle DEB = \angle BDE = \angle BDC + \angle CDE.$$

Since $\angle DAB = \angle BDC$, it follows that $\angle ADE = \angle CDE$.

**3.44.** Let $O_1$ and $O_2$ be the centers of the inscribed circles, $CP$ and $CQ$ the tangents to them. Then $CO_1^2 = CP^2 + PO_1^2 = CP^2 + O_1M^2$ and since $CQ = CA = CP$ (by Problem 3.42

b), we have $CO_2^2 = CQ^2 + QO_2^2 = CP^2 + O_2M^2$. It follows that $CO_1^2 - CO_2^2 = MO_1^2 - MO_2^2$ and, therefore, line $CM$ is perpendicular to $O_1O_2$ (see Problem 7.6). Therefore, line $MN$ passes through point $C$.

REMARK. If the circles do not intersect but are tangent to each other the statement is still true; in this case, however, one should replace line $MN$ with the tangent to the circles at their common point.

**3.45.** Let $A_1$ and $B_1$ be the midpoints of arcs $\smile BC$ and $\smile AC$; let $O$ the center of the inscribed circle. Then $A_1B_1 \perp CO$ (cf. Problem 2.19 a) and $MN \perp CO$, consequently, $MN \parallel A_1B_1$. Let us move points $M'$ and $N'$ along rays $CA$ and $CB$, respectively, so that $M'N' \parallel A_1B_1$. Only for one position of points $M'$ and $N'$ does point $L$ at which lines $B_1M'$ and $A_1N'$ intersect lie on the circumscribed circle of triangle $ABC$.

On the other hand, if segment $MN$ passes through point $O$, then point $L$ lies on this circle (cf. Problem 2.49).

**3.46.** The solution of this problem generalizes the solution of the preceding problem. It suffices to prove that the center $O_1$ of the inscribed circle of triangle $ABC_1$ lies on segment $M_2N_1$. Let $A_1$ and $A_2$ be the midpoints of arcs $\smile BC_1$ and $\smile BC_2$; let $B_1$ and $B_2$ be the midpoints of arcs $\smile AC_1$ and $\smile AC_2$; let $PQ$ be the diameter of circle $S$ perpendicular to chord $AB$ and let points $Q$ and $C_1$ lie on one side of line $AB$. Point $O_1$ is the intersection point of chords $AA_1$ and $BB_1$ and point $L$ of tangent of circles $S$ and $S_1$ is the intersection point of lines $A_1N_1$ and $B_2M_2$ (Fig. 33).
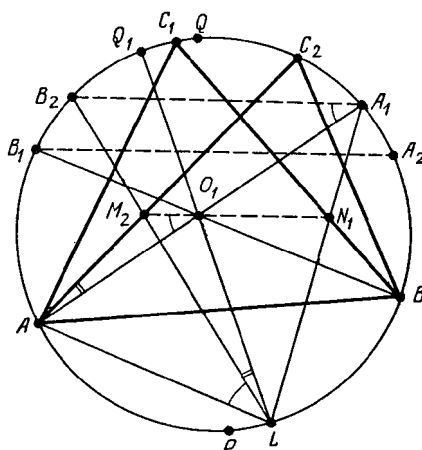


Figure 33 (Sol. 3.46)

Let $\angle C_1AB = 2\alpha$, $\angle C_1BA = 2\beta$, $\angle C_1AC_2 = 2\varphi$. Then $\smile A_1A_2 = 2\varphi = \smile B_1B_2$, i.e., $A_1B_2 \parallel B_1A_2$. For the angles between chords we have:

$$\angle(A_1B_2, BC_1) = \tfrac{1}{2}(\smile B_2C_1 + \smile A_1B) = \beta - \varphi + \alpha,$$
$$\angle(BC_1, AC_2) = \tfrac{1}{2}(\smile C_1C_2 + \smile AB) = 2\varphi + 180° - 2\alpha - 2\beta.$$

Consequently, chord $M_2N_1$ constitutes equal angles with tangents $BC_1$ and $AC_2$, each angle equal to $\alpha + \beta - \varphi$. Therefore, $M_2N_1 \parallel A_1B_2$.

Now, suppose that points $M_2'$ and $N_1'$ are moved along chords $AC_2$ and $BC_1$ so that $M_2'N_1' \parallel A_1B_2$. Let lines $A_1N_1'$ and $B_2M_2'$ meet at point $L'$. Point $L'$ lies on circle $S$ for one position of points $M_2'$ and $N_1'$ only. Therefore, it suffices to indicate on arc $\smile AB$ a point $L_1$ such that if $M_2''$, $N_1''$ are the intersection points of chords $AC_2$ and $L_1B_2$, $BC_1$ and $L_1A_1$,

respectively, then $M_2''N_1'' \parallel A_1B_2$ and point $O_1$ lies on segment $M_2''N_1''$. Let $Q_1$ be a point on circle $S$ such that $2\angle(PQ, PQ_1) = \angle(PC_2, PC_1)$ and $L_1$ the intersection point of line $Q_1O_1$ with $S$.

Let us prove that $L_1$ is the desired point. Since $\smile B_1Q = 2\alpha$, it follows that $\smile B_2Q_1 = 2(\alpha - 2\varphi) = \smile C_2A_1$. Hence, quadrilateral $AM_2''O_1L_1$ is an inscribed one and, therefore, $\angle M_2''O_1A = \angle M_2''L_1A = \angle B_2A_1A$, i.e., $M_2''O_1 \parallel B_2A_1$.

Similarly, $N_1''O_1 \parallel B_2A_1$.

**3.47.** Let circles centered at $O_1$ and $O_2$ pass through point $A$. The radii $O_1A$ and $O_2A$ are perpendicular to the tangents to circles at point $A$ and, therefore, the circles are orthogonal if and only if $\angle O_1AO_2 = 90°$, i.e., $\angle O_1O_2^2 = O_1A^2 + O_2A^2$.

**3.48.** Let $A_1$, $B_1$ and $C_1$ be the centers of the given circles so that points $A$, $B$ and $C$ lie on segments $B_1C_1$, $C_1A_1$ and $A_1B_1$, respectively. Since $A_1B = A_1C$, $B_1A = B_1C$ and $CA = C_1B$, it follows that $A$, $B$ and $C$ are the tangent points of the inscribed circle of triangle $A_1B_1C_1$ with its sides (cf. Problem 5.1). Therefore, the radii $A_1B$, $BC$ and $C_1A$ of the given circles are tangent to the circumscribed circle of triangle $ABC$.

**3.49.** It is easy to verify that the angle of rotation from vector $\overrightarrow{O_iB}$ to vector $\overrightarrow{O_iM_i}$ (counterclockwise) is equal to $2\angle(AB, AM_i)$. It is also clear that $\angle(AB, AM_1) = \angle(AB, AM_2)$.

**3.50.** Let us draw through point $P$ another line that intersects the circle at points $A_1$ and $B_1$. Then $\triangle PAA_1 \sim \triangle PB_1B$ and, therefore, $PA : PA_1 = PB_1 : PB$.

**3.51.** Let us draw through point $P$ tangent $PC$. Since $\triangle PAC \sim \triangle PCB$, it follows that $PA : PC = PC : PB$.

**3.52.** Let the line that passes through point $P$ and the center of the circle intersect the circle at points $A$ and $B$. Then $PA = d + R$ and $PB = |d - R|$. Therefore, $PA \cdot PB = |d^2 - R^2|$. It is also clear that the signs of the expression $d^2 - R^2$ and of the degree of point $P$ with respect to to $S$ are the same.

**3.53.** Let $R_1$ and $R_2$ be the radii of the circles. Let us consider the coordinate system in which the coordinates of the centers of the circles are $(-a, 0)$ and $(a, 0)$. By Problem 3.52 the degrees of the point with coordinates $(x, y)$ with respect to the given circles are equal to $(x + a)^2 + y^2 - R_1^2$ and $(x - a)^2 + y^2 - R_2^2$, respectively. By equating these expressions we get $x = \frac{R_1^2 - R_2^2}{4a}$. This equation determines the perpendicular to the segment that connects the centers of the circles.

**3.54.** The degrees of the intersection point of the circles with respect to each one of the cicles are equal to zero and, therefore, the point belongs to the radical axis. If there are two intersection points, then they uniquely determine the radical axis.

**3.55.** Since the centers of the circles do not lie on one line, the radical axis of the first and the second circles intersects with the radical axis of the second and third circles. The degrees of the intersection point with respect to all three circles are equal and, therefore, this intersection point lies on the radical axis of the first and third circles.

**3.56.** By Problem 3.54 the lines that contain chords are radical axes. By Problem 3.55 the radical axes meet at one point if the centers of the circles do not lie on one line. Otherwise they are perpendicular to this line.

**3.57.** Let $O_1$ and $O_2$ be the centers of given circles, $r_1$ and $r_2$ their radii. The circle $S$ of radius $r$ centered at $O$ is orthogonal to circle $S_i$ if and only if $r^2 = OO_i^2 - r_i^2$, i.e., the squared radius of $S$ is equal to the degree of point $O$ with respect to circle $S_i$. Therefore, the locus of the centers of the circles to be found is the set of the points of the radical axis whose degrees with respect to the given circles are positive.

**3.58.** a) The indicated points lie on the radical axis.

b) The tangent points of the outer tangents with the circles are vertices of trapezoid $ABCD$ with base $AB$. The midpoints of lateral sides $AD$ and $BC$ belong to the radical axis and, therefore, the midpoint $O$ of diagonal $AC$ also belongs to the radical axis. If line $AC$ intersects the circles at points $A_1$ and $C_1$, then $OA_1 \cdot OA = OC_1 \cdot OC$; consequently, $OA_1 = OC_1$ and $AA_1 = CC_1$.

**3.59.** a) Let $S_A$ and $S_B$ be circles with diameters $AA_1$ and $BB_1$; let $S$ be the circle with diameter $AB$. The common chords of circles $S$ and $S_A$, $S$ and $S_B$ are heights $AH_a$ and $BH_b$ and, therefore, these heights (or their extensions) intersect at point $H$. By Problem 3.56 the common chord of circles $S_A$ and $S_B$ passes through the intersection point of chords $AH_a$ and $BH_b$.

b) The common chord of circles $S_A$ and $S_B$ passes through the intersection point of lines $A_1H_a$ and $B_1H_b$ (i.e., through point $C$) if and only if $CB_1 \cdot CH_b = CA_1 \cdot CH_a$ (here we should consider the lengths of segments as oriented). Since

$$CH_b = \frac{a^2 + b^2 - c^2}{2b} \quad \text{and} \quad CH_a = \frac{a^2 + b^2 - c^2}{2a},$$

we deduce that $\frac{CB_1}{b} = \frac{CA_1}{a}$.

**3.60.** In triangle $CDE$, draw heights $CC_1$ and $DD_1$; let $H$ be their intersection point. The circles with diameters $AC$ and $BD$ pass through points $C_1$ and $D_1$, respectively, therefore, the degree of point $H$ with respect to each of these circles is equal to its degree with respect to the circle with diameter $CD$ (this circle passes through points $C_1$ and $D_1$). We similarly prove that the degrees of point $H$ with respect to to circles with diameters $AC$, $BD$ and $EF$ are equal, i.e., the radical axes of these circles pass through point $H$.

For the intersection points of heights of the other three triangles the proof is carried out in a similar way.

**Remark.** The centers of the considered circles lie on the Gauss' line (cf. Problem 4.55) and, therefore, their common radical axis is perpendicular to the Gauss line.

**3.61.** Lines $A_1A_2$, $B_1B_2$ and $C_1C_2$ meet at a point $O$ (cf. Problem 3.56). Since $\triangle A_1OB_2 \sim \triangle B_1OA_2$, it follows that $A_1B_2 : A_2B_1 = OA_1 : OB_1$. Similarly, $B_1C_2 : B_2C_1 = OB_1 : OC_1$ and $C_1A_2 : C_2A_1 = OC_1 : OA_1$. By multiplying these equalities we get the statement desired.

**3.62.** Denote by $B'$ and $C'$ the intersection points of lines $A'M$ and $A'N$, respectively, with the line drawn through point $A$ parallel to $BC$ (Fig. 34).
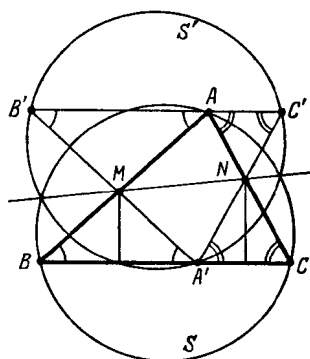


Figure 34 (Sol. 3.62)

Since triangles $A'BM$ and $A'NC$ are isosceles ones, $\triangle ABC = \triangle A'B'C'$. Since $AM \cdot BM = A'M \cdot B'M$, the degrees of point $M$ with respect to circles $S$ and $S'$ circumscribed about triangles $ABC$ and $A'B'C'$, respectively, are equal. This is true for point $N$ as well

and, therefore, line $MN$ is the radical axis of circles $S$ and $S'$. Circles $S$ and $S'$ have equal radii and, therefore, their radical axis is their axis of symmetry. The symmetry through line $MN$ sends a point $A'$ that lies on circle $S'$ into a point that lies on circle $S$.

**3.63.** Let $AC$ and $BD$ be the tangents; $E$ and $K$ the intersection points of lines $AC$ and $BD$, $AB$ and $CD$, respectively; $O_1$ and $O_2$ the centers of the circles (Fig. 35).
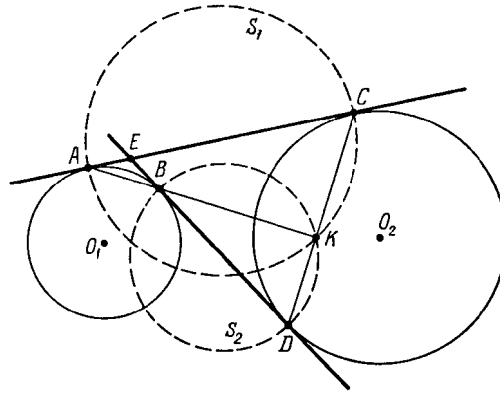


Figure 35 (Sol. 3.63)

Since $AB \perp O_1E$, $O_1E \perp O_2E$ and $O_2E \perp CD$, it follows that $AB \perp CD$ and, therefore, $K$ is the intersection point of circles $S_1$ and $S_2$ with diameters $AC$ and $BD$, respectively. Point $K$ lies on the radical axis of circles $S_1$ and $S_2$; it remains to verify that line $O_1O_2$ is this radical axis. The radii $O_1A$ and $O_1B$ are tangent to $S_1$ and $S_2$, respectively, and, therefore, point $O_1$ lies on the radical axis. Similarly, point $O_2$ also lies on the radical axis.

**3.64.** Denote the given circles by $S_1, \ldots, S_n$. For each circle $S_i$ consider the set $M_i$ that consists of all the points $X$ whose degree with respect to $S_i$ does not exceed their degrees with respect to the other circles $S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n$.

The set $M_i$ is a convex one. Indeed, let $M_{ij}$ be the set of points $X$ whose degree with respect to $S_i$ does not exceed the degree with respect to $S_j$. The set $M_{ij}$ is a half plane that consists of the points that lie on the same side of the radical axis of circles $S_i$ and $S_j$ as $S_i$ does. The set $M_i$ is the intersection of the convex sets $M_{ij}$ for all $j$ and, therefore, is a convex set itself. Moreover, since each of the sets $M_{ij}$ contains circle $S_i$, then $M_i$ also contains $S_i$. Since for each point of the plane at least one of the degrees with respect to $S_1, \ldots, S_n$ is the least one, the sets $M_i$ cover the whole plane.

Now, by considering the parts of the sets $M_i$ that lie inside the initial polygon we get the partition statement desired.

**3.65.** a) Points $B_1$ and $C_1$ lie on the circle with diameter $BC$ and, therefore, the degrees of point $A'$ with respect to the circumscribed circles of triangles $A_1B_1C_1$ and $ABC$ are equal to the degrees of point $A'$ with respect to this circle. This means that point $A'$ lies on the radical axis of the Euler circle and the circumscribed circle of triangle $ABC$. For points $B'$ and $C'$ the proof is similar.

b) Let us consider triangle $A_1B_1C_1$ formed by the outer bisectors of triangle $ABC$ (triangle $A_1B_1C_1$ is an acute one). Thanks to heading a) points $A'$, $B'$ and $C'$ lie on the radical axis of the circumscribed circles of triangles $ABC$ and $A_1B_1C_1$. The radical axis of these circles is perpendicular to the line that connects their centers, i.e., the Euler line of triangle $A_1B_1C_1$. It remains to notice that the intersection point of the heights of triangle $A_1B_1C_1$ is the intersection point of the bisectors of triangle $ABC$, cf. Problem 1.56 a).

**3.66.** Let a convex hexagon $ABCDEF$ be tangent to the circle at points $R$, $Q$, $T$, $S$, $P$, $U$ (point $R$ lies on $AB$, point $Q$ lies on $BC$, etc.).

Take a number $a > 0$ and construct points $Q'$ and $P'$ on lines $BC$ and $EF$ so that $QQ' = PP' = a$ and vectors $\overrightarrow{QQ'}$ and $\overrightarrow{PP'}$ are codirected with vectors $\overrightarrow{CB}$ and $\overrightarrow{EF}$.

Let us similarly construct points $R'$, $S'$, $T'$, $U'$ (see Fig. 36, where $RR' = SS' = TT' = UU' = a$). Let us construct circle $S_1$ tangent to lines $BC$ and $EF$ at points $Q'$ and $P'$. Let us similarly construct circles $S_2$ and $S_3$.
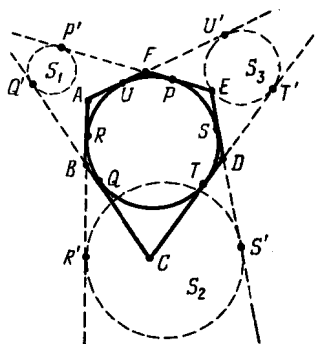


Figure 36 (Sol. 3.66)

Let us prove that points $B$ and $E$ lie on the radical axis of circles $S_1$ and $S_2$. We have

$$BQ' = QQ' - BQ = RR' - BR = BR'$$

(if $QQ' < BQ$, then $BQ' = BQ - QQ' = BR - RR' = BR'$) and

$$EP' = EP + PP' = ES + SS' = ES'.$$

We similarly prove that lines $FC$ and $AD$ are the radical axes of circles $S_1$ and $S_3$, $S_2$ and $S_3$, respectively. Since the radical axes of three circles meet at one point, lines $AD$, $BE$ and $CF$ meet at one point.

**3.67.** Let $A_i$ be the tangent point of circles $S_i$ and $S_{i+1}$ and $X$ be the intersection point of lines $A_1A_4$ and $A_2A_3$. Then $X$ is the intersection point of the common outer tangents to circles $S_2$ and $S_4$ (cf. Problem 5.60). Since quadrilateral $A_1A_2A_3A_4$ is an inscribed one (by Problem 3.22), $XA_1 \cdot XA_4 = XA_2 \cdot XA_3$; consequently, point $X$ lies on the radical axis of circles $S_1$ and $S_3$.

**3.68.** a) Let us consider the coordinate system whose origin $O$ is at the center of the segment that connects the centers of the circles and the $Ox$-axis is directed along this segment. Let $(x, y)$ be the coordinates of point $P$; let $R$ and $r$ be the radii of circles $S_1$ and $S_2$, respectively; $a = \frac{1}{2}d$. Then $(x + a)^2 + y^2 = R^2$ and

$$p = (x - a)^2 + y^2 - r^2 = ((x + a)^2 + y^2 - R^2) - 4ax - r^2 + R^2 = R^2 - r^2 - 4ax.$$

Let $(x_0, y_0)$ be the coordinates of point $A$. Then

$$(x_0 + a)^2 + y_0^2 - R^2 = (x_0 - a)^2 + y_0^2 - r^2, \quad \text{i.e.,} \quad x_0 = \frac{R^2 - r^2}{4a}.$$

Therefore,

$$2dh = 4a|x_0 - x| = |R^2 - r^2 - 4ax| = |p|.$$

b) Let $d$ be the distance between the centers of the circumscribed circles of triangles $ACD$ and $BCD$; let $h_a$ and $h_b$ be the distances from points $A$ and $B$ to line $CD$. By heading a) $|p_a| = 2dh_a$ and $|p_b| = 2dh_b$. Taking into account that $S_{BCD} = \frac{1}{2}h_bCD$ and $S_{ACD} = \frac{1}{2}h_aCD$ we get the statement desired.

# CHAPTER 4. AREA

## Background

1. One can calculate the area $S$ of triangle $ABC$ with the help of the following formulas:

a) $S = \frac{1}{2}ah_a$, where $a = BC$ and $h_a$ is the length of the height dropped to $BC$;

b) $S = \frac{1}{2}bc\sin\angle A$, where $b$, $c$ are sides of the triangle, $\angle A$ the angle between these sides;

c) $S = pr$, where $p$ is a semiperimeter, $r$ the radius of the inscribed circle. Indeed, if $O$ is the center of the inscribed circle, then

$$S = S_{ABO} + S_{AOC} + S_{OBC} = \frac{1}{2}(c + b + a)r = pr.$$

2. If a polygon is cut into several polygons, then the sum of their areas is equal to the area of the initial polygon.

## Introductory problems

**1.** Prove that the area of a convex quadrilateral is equal to $\frac{1}{2}d_1 d_2 \sin\varphi$, where $d_1$ and $d_2$ are the lengths of the diagonals and $\varphi$ is the angle between them.

**2.** Let $E$ and $F$ be the midpoints of sides $BC$ and $AD$ of parallelogram $ABCD$. Find the area of the quadrilateral formed by lines $AE$, $ED$, $BF$ and $FC$ if it is known that the area of $ABCD$ is equal to $S$.

**3.** A polygon is circumscribed about a circle of radius $r$. Prove that the area of the polygon is equal to $pr$, where $p$ is the semiperimeter of the polygon.

**4.** Point $X$ is inside parallelogram $ABCD$. Prove that $S_{ABX} + S_{CDX} = S_{BCX} + S_{ADX}$.

**5.** Let $A_1$, $B_1$, $C_1$ and $D_1$ be the midpoints of sides $CD$, $DA$, $AB$, $BC$, respectively, of square $ABCD$ whose area is equal to $S$. Find the area of the quadrilateral formed by lines $AA_1$, $BB_1$, $CC_1$ and $DD_1$.

## §1. A median divides the triangle into triangles of equal areas

**4.1.** Prove that the medians divide any triangle into six triangles of equal area.

**4.2.** Given triangle $ABC$, find all points $P$ such that the areas of triangles $ABP$, $BCP$ and $ACP$ are equal.

**4.3.** Inside given triangle $ABC$ find a point $O$ such that the areas of triangles $BOL$, $COM$ and $AON$ are equal (points $L$, $M$ and $N$ lie on sides $AB$, $BC$ and $CA$ so that $OL \parallel BC$, $OM \parallel AC$ and $ON \parallel AB$; see Fig. 37).

**4.4.** On the extensions of the sides of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ are taken so that $\overrightarrow{AB_1} = 2\overrightarrow{AB}$, $\overrightarrow{BC_1} = 2\overrightarrow{BC}$ and $\overrightarrow{CA_1} = 2\overrightarrow{AC}$. Find the area of triangle $A_1B_1C_1$ if it is known that the area of triangle $ABC$ is equal to $S$.

**4.5.** On the extensions of sides $DA$, $AB$, $BC$, $CD$ of convex quadrilateral $ABCD$ points $A_1$, $B_1$, $C_1$, $D_1$ are taken so that $\overrightarrow{DA_1} = 2\overrightarrow{DA}$, $\overrightarrow{AB_1} = 2\overrightarrow{AB}$, $\overrightarrow{BC_1} = 2\overrightarrow{BC}$ and $\overrightarrow{CD_1} = 2\overrightarrow{CD}$. Find the area of the obtained quadrilateral $A_1B_1C_1D_1$ if it is known that the area of quadrilateral $ABCD$ is equal to $S$.
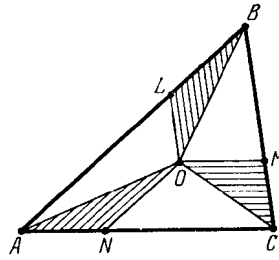
Figure 37 (4.3)

**4.6.** Hexagon $ABCDEF$ is inscribed in a circle. Diagonals $AD$, $BE$ and $CF$ are diameters of this circle. Prove that $S_{ABCDEF} = 2S_{ACE}$.

**4.7.** Inside a convex quadrilateral $ABCD$ there exists a point $O$ such that the areas of triangles $OAB$, $OBC$, $OCD$ and $ODA$ are equal. Prove that one of the diagonals of the quadrilateral divides the other diagonal in halves.

## §2. Calculation of areas

**4.8.** The height of a trapezoid whose diagonals are mutually perpendicular is equal to 4. Find the area of the trapezoid if it is known that the length of one of its diagonals is equal to 5.

**4.9.** Each diagonal of convex pentagon $ABCDE$ cuts off it a triangle of unit area. Calculate the area of pentagon $ABCDE$.

**4.10.** In a rectangle $ABCD$ there are inscribed two distinct rectangles with a common vertex $K$ lying on side $AB$. Prove that the sum of their areas is equal to the area of rectangle $ABCD$.

**4.11.** In triangle $ABC$, point $E$ is the midpoint of side $BC$, point $D$ lies on side $AC$; let $AC = 1$, $\angle BAC = 60°$, $\angle ABC = 100°$, $\angle ACB = 20°$ and $\angle DEC = 80°$ (Fig. 38). Find $S_{\triangle ABC} + 2S_{\triangle CDE}$.
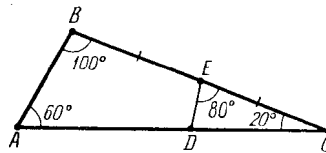


Figure 38 (4.11)

**4.12.** Triangle $T_a = \triangle A_1 A_2 A_3$ is inscribed in triangle $T_b = \triangle B_1 B_2 B_3$ and triangle $T_b$ is inscribed in triangle $T_c = \triangle C_1 C_2 C_3$ so that the sides of triangles $T_a$ and $T_c$ are pairwise parallel. Express the area of triangle $T_b$ in terms of the areas of triangles $T_a$ and $T_c$.
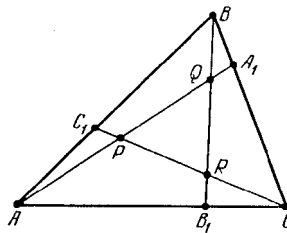


Figure 39 (4.12)

**4.13.** On sides of triangle $ABC$, points $A_1$, $B_1$ and $C_1$ that divide its sides in ratios $BA_1 : A_1C = p$, $CB_1 : B_1A = q$ and $AC_1 : C_1B = r$, respectively, are taken. The

intersection points of segments $AA_1$, $BB_1$ and $CC_1$ are situated as depicted on Fig. 39. Find the ratio of areas of triangles $PQR$ and $ABC$.

## §3. The areas of the triangles into which a quadrilateral is divided

**4.14.** The diagonals of quadrilateral $ABCD$ meet at point $O$. Prove that $S_{AOB} = S_{COD}$ if and only if $BC \parallel AD$.

**4.15.** a) The diagonals of convex quadrilateral $ABCD$ meet at point $P$. The areas of triangles $ABP$, $BCP$, $CDP$ are known. Find the area of triangle $ADP$.

b) A convex quadrilateral is divided by its diagonals into four triangles whose areas are expressed in integers. Prove that the product of these integers is a perfect square.

**4.16.** The diagonals of quadrilateral $ABCD$ meet at point $P$ and $S_{ABP}^2 + S_{CDP}^2 = S_{BCP}^2 + S_{ADP}^2$. Prove that $P$ is the midpoint of one of the diagonals.

**4.17.** In a convex quadrilateral $ABCD$ there are three inner points $P_1$, $P_2$, $P_3$ not on one line and with the property that

$$S_{ABP_i} + S_{CDP_i} = S_{BCP_i} + S_{ADP_i}$$

for $i = 1$, 2, 3. Prove that $ABCD$ is a parallelogram.

## §4. The areas of the parts into which a quadrilateral is divided

**4.18.** Let $K$, $L$, $M$ and $N$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$, respectively, of convex quadrilateral $ABCD$; segments $KM$ and $LN$ intersect at point $O$. Prove that

$$S_{AKON} + S_{CLOM} = S_{BKOL} + S_{DNOM}.$$

**4.19.** Points $K$, $L$, $M$ and $N$ lie on sides $AB$, $BC$, $CD$ and $DA$, respectively, of parallelogram $ABCD$ so that segments $KM$ and $LN$ are parallel to the sides of the parallelogram. These segments meet at point $O$. Prove that the areas of parallelograms $KBLO$ and $MDNO$ are equal if and only if point $O$ lies on diagonal $AC$.

**4.20.** On sides $AB$ and $CD$ of quadrilateral $ABCD$, points $M$ and $N$ are taken so that $AM : MB = CN : ND$. Segments $AN$ and $DM$ meet at point $K$, and segments $BN$ and $CM$ meet at point $L$. Prove that $S_{KMLN} = S_{ADK} + S_{BCL}$.

**4.21.** On side $AB$ of quadrilateral $ABCD$, points $A_1$ and $B_1$ are taken, on side $CD$ points $C_1$ and $D_1$ are taken so that $AA_1 = BB_1 = pAB$ and $CC_1 = DD_1 = pCD$, where $p < 0.5$. Prove that $\frac{S_{A_1B_1C_1D_1}}{S_{ABCD}} = 1 - 2p$.

**4.22.** Each of the sides of a convex quadrilateral is divided into five equal parts and the corresponding points of the opposite sides are connected as on Fig. 40. Prove that the area of the middle (shaded) quadrilateral is 25 times smaller than the area of the initial quadrilateral.
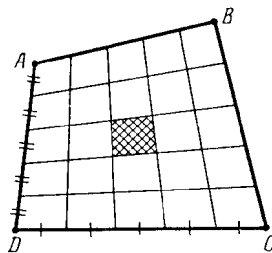


Figure 40 (4.22)

**4.23.** On each side of a parallelogram a point is taken. The area of the quadrilateral with vertices at these points is equal to half the area of the parallelogram. Prove that at least one of the diagonals of the quadrilateral is parallel to a side of the parallelogram.

**4.24.** Points $K$ and $M$ are the midpoints of sides $AB$ and $CD$, respectively, of convex quadrilateral $ABCD$, points $L$ and $N$ lie on sides $BC$ and $AD$ so that $KLMN$ is a rectangle. Prove that $S_{ABCD} = S_{KLMN}$.

**4.25.** A square is divided into four parts by two perpendicular lines whose intersection point lies inside the square. Prove that if the areas of three of these parts are equal, then the area of all four parts are equal.

## §5. Miscellaneous problems

**4.26.** Given parallelogram $ABCD$ and a point $M$, prove that

$$S_{ACM} = |S_{ABM} \pm S_{ADM}|.$$

**4.27.** On sides $AB$ and $BC$ of triangle $ABC$, parallelograms are constructed outwards; let $P$ be the intersection point of the extensions of the sides of these parallelograms parallel to $AB$ and $BC$. On side $AC$, a parallelogram is constructed whose other side is equal and parallel to $BP$. Prove that the area of this parallelogram is equal to the sum of areas of the first two parallelograms.

**4.28.** Point $O$ inside a regular hexagon is connected with the vertices. The six triangles obtained in this way are alternately painted red and blue. Prove that the sum of areas of red triangles is equal to the sum of areas of blue ones.

**4.29.** The extensions of sides $AD$ and $BC$ of convex quadrilateral $ABCD$ meet at point $O$; let $M$ and $N$ be the midpoints of sides $AB$ and $CD$; let $P$ and $Q$ be the midpoints of diagonals $AC$ and $BD$. Prove that:

a) $S_{PMQN} = \frac{1}{2}|S_{ABD} - S_{ACD}|$;

b) $S_{OPQ} = \frac{1}{2}S_{ABCD}$.

**4.30.** On sides $AB$ and $CD$ of a convex quadrilateral $ABCD$ points $E$ and $F$ are taken. Let $K$, $L$, $M$ and $N$ be the midpoints of segments $DE$, $BF$, $CE$ and $AF$, respectively. Prove that quadrilateral $KLMN$ is a convex one and its area does not depend on the choice of points $E$ and $F$.

**4.31.** The midpoints of diagonals $AC$, $BD$, $CE$, ... of convex hexagon $ABCDEF$ are vertices of a convex hexagon. Prove that the area of the new hexagon is $\frac{1}{4}$ of that of the initial one.

**4.32.** The diameter $PQ$ and the chord $RS$ perpendicular to it intersect in point $A$. Point $C$ lies on the circle, point $B$ lies inside the circle and we know that $BC \parallel PQ$ and $BC = RA$. From points $A$ and $B$ perpendiculars $AK$ and $BL$ are dropped to line $CQ$. Prove that $S_{ACK} = S_{BCL}$.

## * * *

**4.33.** Through point $O$ inside triangle $ABC$ segments are drawn parallel to its sides (Fig. 41). Segments $AA_1$, $BB_1$ and $CC_1$ divide triangle $ABC$ into four triangles and three quadrilaterals. Prove that the sum of areas of the triangles adjacent to vertices $A$, $B$ and $C$ is equal to the area of the fourth triangle.

**4.34.** On the bisector of angle $\angle A$ of triangle $ABC$ a point $A_1$ is taken so that $AA_1 = p - a = \frac{1}{2}(b + c - a)$ and through point $A_1$ line $l_a$ perpendicular to the bisector is drawn. If we similarly construct lines $l_b$ and $l_c$, then triangle $ABC$ will be divided into parts among
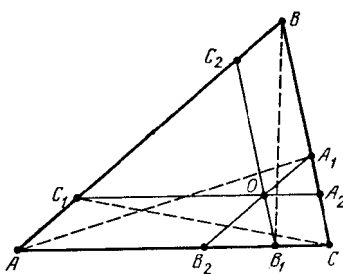
Figure 41 (4.33)

which there are four triangles. Prove that the area of one of these triangles is equal to the sum of areas of the three other triangles.

See also problems 3.38–3.41, 13.52–13.56, 16.5, 24.5.

## §6. Lines and curves that divide figures into parts of equal area

**4.35.** Segment $MN$ parallel to side $CD$ of quadrilateral $ABCD$ divides its area in halves (points $M$ and $N$ lie on sides $BC$ and $AD$). The lengths of segments drawn from points $A$ and $B$ parallel to $CD$ till they intersect with lines $BC$ and $AD$ are equal to $a$ and $b$, respectively. Prove that $MN^2 = \frac{1}{2}(ab + c^2)$, where $c = CD$.

**4.36.** Each of certain three lines divides the area of a figure in halves. Prove that the area of the part of the figure confined inside the triangle formed by these lines does not exceed $\frac{1}{4}$ of the area of the whole figure.

**4.37.** Line $l$ divides the area of a convex polygon in halves. Prove that this line divides the projection of the given polygon onto a line perpendicular to $l$ in the ratio that does not exceed $1 + \sqrt{2}$.

**4.38.** Prove that any convex polygon can be cut by two mutually perpendicular lines in four figures of equal area.

**4.39.** a) Prove that any line that divides the area and the perimeter of the triangle in halves passes through the center of the inscribed circle.

b) Prove a similar statement for any circumscribed polygon.

**4.40.** Points $A$ and $B$ of circle $S_1$ are connected by an arc of circle $S_2$ that divides the area of the disk bounded by $S_1$ into equal parts. Prove that the length of the arc of $S_2$ that connects $A$ and $B$ is greater than that of the diameter of $S_1$.

**4.41.** Curve $\Gamma$ divides a square into two parts of equal area. Prove that on $\Gamma$ we can select two points $A$ and $B$ so that line $AB$ passes through the center $O$ of the square.

See also problems 6.51, 6.52, 16.8, 18.29.

## §7. Formulas for the area of a quadrilateral

**4.42.** The diagonals of quadrilateral $ABCD$ meet at point $P$. The distances from points $A$, $B$ and $P$ to line $CD$ are equal to $a$, $b$ and $p$, respectively. Prove that the area of quadrilateral $ABCD$ is equal to $\frac{ab \cdot CD}{2p}$.

**4.43.** Quadrilateral $ABCD$ is inscribed into a circle of radius $R$; let $\varphi$ be the angle between the diagonals of $ABCD$. Prove that the area $S$ of $ABCD$ is equal to $2R^2 \cdot \sin \angle A \cdot \sin \angle B \cdot \sin \varphi$.

**4.44.** Prove that the area of a quadrilateral whose diagonals are not perpendicular is equal to $\frac{1}{4}\tan\varphi \cdot |a^2 + c^2 - b^2 - d^2|$, where $a$, $b$, $c$ and $d$ are the lengths of the consecutive sides and $\varphi$ is the angle between the diagonals.

**4.45.** a) Prove that the area of a convex quadrilatral $ABCD$ can be computed with the help of the formula

$$S^2 = (p-a)(p-b)(p-c)(p-d) - abcd\cos^2\left(\frac{\angle B + \angle D}{2}\right),$$

where $p$ is the semiperimeter, $a$, $b$, $c$, $d$ are the lengths of the quadrilateral's sides.

b) Prove that if quadrilateral $ABCD$ is an inscribed one, then

$$S^2 = (p-a)(p-b)(p-c)(p-d).$$

c) Prove that if quadrilateral $ABCD$ is a circumscribed one, then

$$S^2 = abcd\sin^2\left(\frac{\angle B + \angle D}{2}\right).$$

See also Problem 11.34.

## §8. An auxiliary area

**4.46.** Prove that the sum of distances from an arbitrary point within an equilateral triangle to the triangle's sides is constant (equal to the length of the triangle's height).

**4.47.** Prove that the length of the bisector $AD$ of triangle $ABC$ is equal to $\frac{2bc}{b+c}\cos\frac{1}{2}\alpha$.

**4.48.** Inside triangle $ABC$, point $O$ is taken; lines $AO$, $BO$ and $CO$ meet the sides of the triangle at points $A_1$, $B_1$ and $C_1$, respectively. Prove that:
   a) $\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} = 1$;
   b) $\frac{AC}{CB} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$.

**4.49.** A $(2n-1)$-gon $A_1 \ldots A_{2n-1}$ and a point $O$ are given. Lines $A_kO$ and $A_{n+k-1}A_{n+k}$ meet at point $B_k$. Prove that the product of ratios $\frac{A_{n+k-1}B_k}{A_{n+k}B_k}$ for $k = 1, \ldots, n$ is equal to 1.

**4.50.** A convex polygon $A_1A_2\ldots A_n$ is given. On side $A_1A_2$ points $B_1$ and $D_2$ are taken, on side $A_2A_3$ points $B_2$ and $D_3$, etc. so that if we construct parallelograms $A_1B_1C_1D_1, \ldots, A_nB_nC_nD_n$, then lines $A_1C_1, \ldots, A_nC_n$ would meet at one point $O$. Prove that

$$A_1B_1 \cdot A_2B_2 \cdot \cdots \cdot A_nB_n = A_1D_1 \cdot A_2D_2 \cdot \cdots \cdot A_nD_n.$$

**4.51.** The lengths of the sides of a triangle form an arithmetic progression. Prove that the (length of the) radius of the inscribed circle is equal to one third of the length of one of the triangle's heights.

**4.52.** The distances from point $X$ on side $BC$ of triangle $ABC$ to lines $AB$ and $AC$ are equal to $d_b$ and $d_c$, respectively. Prove that $\frac{d_b}{d_c} = \frac{BX \cdot AC}{CX \cdot AB}$.

**4.53.** A polygon circumscribed about a circle of radius $r$ is divided into triangles (in an arbitrary way). Prove that the sum of radii of the inscribed circles of these triangles is greater than $r$.

**4.54.** Through point $M$ inside parallelogram $ABCD$ lines $PR$ and $QS$ parallel to sides $BC$ and $AB$ are drawn (points $P$, $Q$, $R$ and $S$ lie on sides $AB$, $BC$, $CD$ and $DA$, respectively). Prove that lines $BS$, $PD$ and $MC$ meet at one point.

**4.55.** Prove that if no side of a quadrilateral is parallel to any other side, then the midpoint of the segment that connects the intersection points of the opposite sides lies on the line that connects the midpoints of the diagonals. (The *Gauss line*.)

**4.56.** In an acute triangle $ABC$ heights $BB_1$ and $CC_1$ are drawn and points $K$ and $L$ are taken on sides $AB$ and $AC$ so that $AK = BC_1$ and $AL = CB_1$. Prove that line $AO$,

where $O$ is the center of the circumscribed circle of triangle $ABC$, divides segment $KL$ in halves.

**4.57.** Medians $AA_1$ and $CC_1$ of triangle $ABC$ meet at point $M$. Prove that if quadrilateral $A_1BC_1M$ is a circumscribed one, then $AB = BC$.

**4.58.** Inside triangle $ABC$ a point $O$ is taken. Denote the distances from $O$ to sides $BC$, $CA$, $AB$ of the triangle by $d_a$, $d_b$, $d_c$, respectively, and the distances from point $O$ to vertices $A$, $B$, $C$ by $R_a$, $R_b$, $R_c$, respectively. Prove that:

a) $aR_a \geq cd_c + bd_b$;
b) $d_aR_a + d_bR_b + d_cR_c \geq 2(d_ad_b + d_bd_c + d_cd_a)$;
c) $R_a + R_b + R_c \geq 2(d_a + d_b + d_c)$;
d) $R_aR_bR_c \geq \frac{R}{2r}(d_a + d_b)(d_b + d_c)(d_c + d_a)$.

See also problems 5.5, 10.6.

## §9. Regrouping areas

**4.59.** Prove that the area of a regular octagon is equal to the product of the lengths of its greatest and smallest diagonals.

**4.60.** From the midpoint of each side of an acute triangle perpendiculars are dropped to two other sides. Prove that the area of the hexagon bounded by these perpendiculars is equal to a half area of the initial triangle.

**4.61.** Sides $AB$ and $CD$ of parallelogram $ABCD$ of unit area are divided into $n$ equal parts; sides $AD$ and $BC$ are divided into $m$ equal parts. The division points are connected as indicated on a) Fig. 42 a); b) Fig. 42 b).
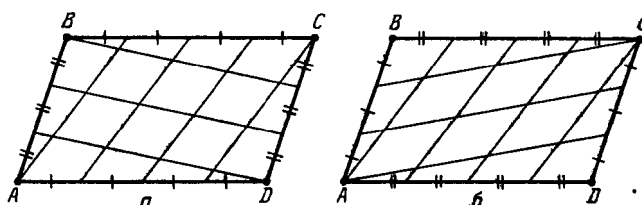


Figure 42 (4.61)

What are the areas of small parallelograms obtained in this way?

**4.62.** a) Four vertices of a regular 12-gon lie in the midpoints of a square (Fig. 43). Prove that the area of the shaded part is equal to $\frac{1}{12}$ that of the 12-gon.
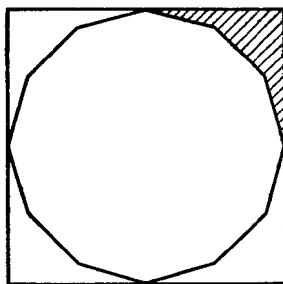


Figure 43 (4.62)

b) Prove that the area of a 12-gon inscribed in the unit circle is equal to 3.

## Problems for independent study

**4.63.** The sides of an inscribed quadrilateral $ABCD$ satisfy the relation $AB \cdot BC = AD \cdot DC$. Prove that the areas of triangles $ABC$ and $ADC$ are equal.

**4.64.** Is it possible to use two straight cuts passing through two vertices of a triangle to divide the triangle into four parts so that three triangles (of these parts) were of equal area?

**4.65.** Prove that all the convex quadrilaterals with common midpoints of sides are of equal area.

**4.66.** Prove that if two triangles obtained by extention of sides of a convex quadrilateral to their intersection are of equal area, then one of the diagonals divides the other one in halves.

**4.67.** The area of a triangle is equal to $S$, its perimeter is equal to $P$. Each of the lines on which the sides of the triangle lie are moved (outwards) by a distance of $h$. Find the area and the perimeter of the triangle formed by the three obtained lines.

**4.68.** On side $AB$ of triangle $ABC$, points $D$ and $E$ are taken so that $\angle ACD = \angle DCE = \angle ECB = \varphi$. Find the ratio $CD : CE$ if the lengths of sides $AC$ and $BC$ and angle $\varphi$ are known.

**4.69.** Let $AA_1$, $BB_1$ and $CC_1$ be the bisectors of triangle $ABC$. Prove that

$$\frac{S_{A_1 B_1 C_1}}{S_{ABC}} = \frac{2abc}{(a+b) \cdot (b+c) \cdot (c+a)}.$$

**4.70.** Points $M$ and $N$ are the midpoints of lateral sides $AB$ and $CD$ of trapezoid $ABCD$. Prove that if the doubled area of the trapezoid is equal to $AN \cdot NB + CM \cdot MD$, then $AB = CD = BC + AD$.

**4.71.** If a quadrilateral with sides od distinct lengths is inscribed into a circle of radius $R$, then there exist two more quadrilaterals not equal to it with the same lengths of sides inscribed in the same circle. These quadrilaterals have not more than three distinct lengths of diagonals: $d_1$, $d_2$ and $d_3$. Prove that the area of the quadrilateral is equal to $\frac{d_1 d_2 d_3}{4R}$.

**4.72.** On sides $AB$, $BC$ and $CA$ of triangle $ABC$ points $C_1$, $A_1$ and $B_1$ are taken; points $C_2$, $A_2$ and $B_2$ are symmetric to these points through the midpoints of the corresponding sides. Prove that $S_{A_1 B_1 C_1} = S_{A_2 B_2 C_2}$.

**4.73.** Inside triangle $ABC$, point $P$ is taken. The lines that pass through $P$ and vertices of the triangle intersect the sides at points $A_1$, $B_1$ and $C_1$. Prove that the area of the triangle determined by the midpoints of segments $AA_1$, $BB_1$ and $CC_1$ is equal to $\frac{1}{4}$ of the area of triangle $A_1 B_1 C_1$.

## Solutions

**4.1.** The triangles adjacent to one side have equal bases and a common height and, therefore, are of equal area. Let $M$ be the intersection point of the medians of triangle $ABC$. Line $BM$ divides each of the triangles $ABC$ and $AMC$ into two triangles of equal area; consequently, $S_{ABM} = S_{BCM}$. Similarly, $S_{BCM} = S_{CAM}$.

**4.2.** The equality of areas of triangles $ABP$ and $BCP$ implies that the distances from points $A$ and $C$ to line $BP$ are equal. Therefore, either line $BP$ passes through the midpoint of segment $AC$ or it is parallel to it. The points to be found are depicted on Fig. 44.

**4.3.** Denote the intersection point of line $LO$ with side $AC$ by $L_1$. Since $S_{LOB} = S_{MOC}$ and $\triangle MOC = \triangle L_1 OC$, it follows that $S_{LOB} = S_{L_1 CO}$. The heights of triangles $LOB$ and $L_1 OC$ are equal and, therefore, $LO = L_1 O$, i.e., point $O$ lies on the median drawn from vertex $A$.
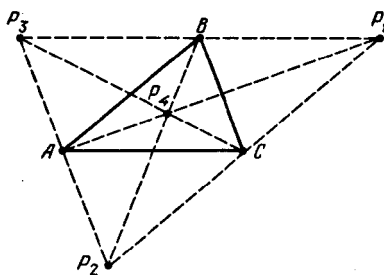
Figure 44 (Sol. 4.2)

We similarly prove that point $O$ lies on the medians drawn from vertices $B$ and $C$, i.e., $O$ is the intersection point of the medians of the triangle. These arguments also demonstrate that the intersection point of the medians of the triangle possesses the necessary property.

**4.4.** Since $S_{A_1BB_1} = S_{A_1AB} = S_{ABC}$, it follows that $S_{AA_1B_1} = 2S$. Similarly, $S_{BB_1C_1} = S_{CC_1A_1} = 2S$. Therefore, $S_{ABC} = 7S$.

**4.5.** Since $AB = BB_1$, it follows that $S_{BB_1C} = S_{BAC}$. Since $BC = CC_1$, we have $S_{B_1C_1C} = S_{BB_1C} = S_{BAC}$ and $S_{BB_1C_1} = 2S_{BAC}$. Similarly, $S_{DD_1A_1} = 2S_{ACD}$ and, consequently,

$$S_{BB_1C_1} + S_{DD_1A_1} = 2S_{ABC} + 2S_{ACD} = 2S_{ABCD}.$$

Similarly, $S_{AA_1B_1} + S_{CC_1D_1} = 2S_{ABCD}$, consequently,

$$S_{A_1B_1C_1D_1} = S_{ABCD} + S_{AA_1B_1} + S_{BB_1C_1} + S_{CC_1D_1} + S_{DD_1A_1} = 5S_{ABCD}.$$

**4.6.** Let $O$ be the center of the circumscribed circle. Since $AD$, $BE$ and $CF$ are diameters,

$$S_{ABO} = S_{DEO} = S_{AEO}, \quad S_{BCO} = S_{EFO} = S_{CEO}, \quad S_{CDO} = S_{AFO} = S_{ACO}.$$

It is also clear that $S_{ABCDEF} = 2(S_{ABO} + S_{BCO} + S_{CDO})$ and $S_{ACE} = S_{AEO} + S_{CEO} + S_{ACO}$. Therefore, $S_{ABCDEF} = 2S_{ACE}$.

**4.7.** Let $E$ and $F$ be the midpoints of diagonals $AC$ and $BD$, respectively. Since $S_{AOB} = S_{AOD}$, point $O$ lies on line $AF$. Similarly, point $O$ lies on line $CF$. Suppose that the intersection point of the diagonals is not the midpoint of either of them. Then the lines $AF$ and $CF$ have a unique common point, $F$; hence, $O = F$. We similarly prove that $O = E$. Contradiction.

**4.8.** Let the length of diagonal $AC$ of trapezoid $ABCD$ with base $AD$ be equal to 5. Let us complement triangle $ACB$ to parallelogram $ACBE$. The area of trapezoid $ABCD$ is equal to the area of the right triangle $DBE$. Let $BH$ be a height of triangle $DBE$. Then $EH^2 = BE^2 - BH^2 = 5^2 - 4^2 = 3^2$ and $ED = \frac{BE^2}{EH} = \frac{25}{3}$. Therefore, $S_{DBE} = \frac{1}{2}ED \cdot BH = \frac{50}{3}$.

**4.9.** Since $S_{ABE} = S_{ABC}$, it follows that $EC \parallel AB$. The remaining diagonals are also parallel to the corresponding sides. Let $P$ be the intersection point of $BD$ and $EC$. If $S_{BPC} = x$, then

$$S_{ABCDE} = S_{ABE} + S_{EPB} + S_{EDC} + S_{BPC} = 3 + x.$$

(we have $S_{EPB} = S_{ABE} = 1$ because $ABPE$ is a parallelogram). Since $S_{BPC} : S_{DPC} = BP : DP = S_{EPB} : S_{EPD}$, it follows that $x : (1 - x) = 1 : x$ and, therefore, $x = \frac{\sqrt{5}-1}{2}$ and $S_{ABCDE} = \frac{\sqrt{5}+5}{2}$.

**4.10.** The centers of all the three rectangles coincide (see Problem 1.7) and, therefore, two smaller rectangles have a common diagonal, $KL$. Let $M$ and $N$ be the vertices of these rectangles that lie on side $BC$. Points $M$ and $N$ lie on the circle with diameter $KL$. Let $O$ be

the center of the circle, $O_1$ the projection of $O$ to $BC$. Then $BO_1 = CO_1$ and $MO_1 = NO_1$ and, therefore, $BM = NC$. To prove that $S_{KLM} + S_{KLN} = S_{KBCL}$ it suffices to verify that

$$(S_{KBM} + S_{LCM}) + (S_{KBN} + S_{LCN}) = S_{KBCL} = \frac{1}{2}BC(KB + CL) = \frac{1}{2}BC \cdot AB.$$

It remains to observe that

$$KB \cdot BM + KB \cdot BN = KB \cdot BC,$$
$$LC \cdot CM + LC \cdot CN = LC \cdot BC,$$
$$KB \cdot BC + LC \cdot BC = AB \cdot BC.$$

**4.11.** Let us drop perpendicular $l$ from point $C$ to line $AB$. Let points $A'$, $B'$ and $E'$ be symmetric to points $A$, $B$ and $E$, respectively, through line $l$. Then triangle $AA'C$ is an equilateral one and $\angle ACB = \angle BCB' = \angle B'CA' = 20°$. Triangles $EE'C$ and $DEC$ are isosceles ones with the angle of $20°$ at the vertex and a common lateral side $EC$. Therefore, $S_{ABC} + 2S_{EDC} = S_{ABC} + 2S_{EE'C}$. Since $E$ is the midpoint of $BC$, it follows that $2S_{EE'C} = S_{BE'C} = \frac{1}{2}S_{BB'C}$. Hence,

$$S_{ABC} + 2S_{EDC} = \frac{S_{AA'C}}{2} = \frac{\sqrt{3}}{8}.$$

**4.12.** Let the areas of triangles $T_a$, $T_b$ and $T_c$ be equal to $a$, $b$ and $c$, respectively. Triangles $T_a$ and $T_c$ are homothetic and, therefore, the lines that connect their respective vertices meett at one point, $O$. The similarity coefficient $k$ of these triangles is equal to $\sqrt{\frac{a}{c}}$. Clearly, $S_{A_1B_3O} : S_{C_1B_3O} = A_1O : C_1O = k$. Writing similar equations for \$??? and adding them, we get $a : b = k$ and, therefore, $b = \sqrt{ac}$.

**4.13.** Making use of the result of Problem 1.3 it is easy to verify that

$$\frac{BQ}{BB_1} = \frac{p + pq}{1 + p + pq}, \qquad \frac{B_1R}{BB_1} = qr1 + q + qr,$$

(1)
$$\frac{CR}{CC_1} = \frac{q + qr}{1 + q + qr}, \qquad \frac{CP}{CC_1} = \frac{pr}{1 + r + pr}.$$

It is also clear that $\frac{S_{PQR}}{S_{RB_1C}} = \frac{QR}{RB_1} \cdot \frac{PR}{RC}$ and $\frac{S_{RB_1C}}{S_{ABC}} = \frac{B_1C}{AC} \cdot \frac{B_1R}{BB_1}$. Hence,

$$\frac{S_{PQR}}{S_{ABC}} = \frac{QR}{BB_1} \cdot \frac{PR}{RC} \cdot \frac{B_1C}{AC} = \frac{QR}{BB_1} \cdot \frac{PR}{CC_1} \cdot \frac{CC_1}{CR} \cdot \frac{B_1C}{AC}.$$

Taking into account that

$$\frac{QR}{BB_1} = 1 - \frac{p+pq}{1+p+pq} - \frac{qr}{1+q+rq} = \frac{1}{1+p+pq} - \frac{rq}{1+q+rq}$$
$$= \frac{(1+q)(1-pqr)}{(1+p+pq)(1+q+qr)}$$

and

$$\frac{PR}{CC_1} = \frac{(1 + r)(1 - pqr)}{(1 + q + qr)(1 + r + pr)}$$

we get

$$\frac{S_{PQR}}{S_{ABC}} = \frac{(1 - pqr)^2}{(1 + p + pq)(1 + q + qr)(1 + r + pr)}.$$

**4.14.** If $S_{AOB} = S_{COD}$, then $AO \cdot BO = CO \cdot DO$. Hence, $\triangle AOD \sim \triangle COB$ and $AD \parallel BC$. These arguments are invertible.

**4.15.** a) Since $S_{ADP} : S_{ABP} = DP : BP = S_{CDP} : S_{BCP}$, we have

$$S_{ADP} = \frac{S_{ABP} \cdot S_{CDP}}{S_{BCP}}.$$

b) Thanks to heading a) $S_{ADP} \cdot S_{CBP} = S_{ABP} \cdot S_{CDP}$. Therefore,

$$S_{ABP} \cdot S_{CBP} \cdot S_{CDP} \cdot S_{ADP} = (S_{ADP} \cdot S_{CBP})^2.$$

**4.16.** After division by $\frac{1}{4}\sin^2\varphi$, where $\varphi$ is the angle between the diagonals, we rewrite the given equality of the areas in the form

$$(AP \cdot BP)^2 + (CP \cdot DP)^2 = (BP \cdot CP)^2 + (AP \cdot DP)^2,$$

i.e.,

$$(AP^2 - CP^2)(BP^2 - DP^2) = 0.$$

**4.17.** Suppose that quadrilateral $ABCD$ is not a parallelogram; for instance, let lines $AB$ and $CD$ intersect. By Problem 7.2 the set of points $P$ that lie inside quadrilateral $ABCD$ for which

$$S_{ABP} + S_{CDP} = S_{BCP} + S_{ADP} = \frac{1}{2}S_{ABCD}$$

is a segment. Therefore, points $P_1$, $P_2$ and $P_3$ lie on one line. Contradiction.

**4.18.** Clearly,

$$S_{AKON} = S_{AKO} + S_{ANO} = \frac{1}{2}(S_{AOB} + S_{AOD}).$$

Similarly, $S_{CLOM} = \frac{1}{2}(S_{BCO} + S_{COD})$. Hence, $S_{AKON} + S_{CLOM} = \frac{1}{2}S_{ABCD}$.

**4.19.** If the areas of the parallelograms $KBLO$ and $MDNO$ are equal, then $OK \cdot OL = OM \cdot ON$. Taking into account that $ON = KA$ and $OM = LC$, we get $KO : KA = LC : LO$. Therefore, $\triangle KOA \sim \triangle LCO$ which means that point $O$ lies on diagonal $AC$. These arguments are invertible.

**4.20.** Let $h_1$, $h$ and $h_2$ be the distances from points $A$, $M$ and $B$ to line $CD$, respectively. By Problem 1.1 b) we have $h = ph_2 + (1-p)h_1$, where $p = \frac{AM}{AB}$. Therefore,

$$S_{DMC} = \frac{h \cdot DC}{2} = \frac{h_2 p \cdot DC + h_1(1-p) \cdot DC}{2} = S_{BCN} + S_{ADN}.$$

Subtracting $S_{DKN} + S_{CLN}$ from both sides of this equality we get the desired statement.

**4.21.** Thanks to Problem 4.20,

$$S_{ABD_1} + S_{CDB_1} = S_{ABCD}.$$

Hence,

$$S_{A_1B_1C_1D_1} = S_{A_1B_1D_1} + S_{C_1D_1B_1}$$
$$= (1-2p)S_{ABD_1} + (1-2p)S_{CDB_1} = (1-2p)S_{ABCD}.$$

**4.22.** By Problem 4.21 the area of the middle quadrilateral of those deteremined by segments that connect points of sides $AB$ and $CD$ is $\frac{1}{5}$ of the area of the initial quadrilateral. Since each of the considered segments is divided by segments that connect the corresponding points of the other pair of opposite sides into 5 equal parts (see Problem 1.16). By making use once again of the result of Problem 4.21, we get the desired statement.

**4.23.** On sides $AB$, $BC$, $CD$ and $AD$ points $K$, $L$, $M$ and $N$, respectively, are taken. Suppose that diagonal $KM$ is not parallel to side $AD$. Fix points $K$, $M$ and $N$ and let us move point $L$ along side $BC$. In accordance with this movement the area of triangle $KLM$ varies strictly monotonously. Moreover, if $LN \parallel AB$, then the equality $S_{AKN} + S_{BKL} + S_{CLM} + S_{DMN} = \frac{1}{2}S_{ABCD}$ holds, i.e., $S_{KLMN} = \frac{1}{2}S_{ABCD}$.

**4.24.** Let $L_1$ and $N_1$ be the midpoints of sides $BC$ and $AD$, respectively. Then $KL_1MN_1$ is a parallelogram and its area is equal to a half area of quadrilateral $ABCD$, cf. Problem 1.37 a). Therefore, it suffices to prove that the areas of parallelograms $KLMN$ and $KL_1MN_1$ are equal. If these parallelograms coincide, then there is nothing more to prove and if they do not coincide, then $LL_1 \parallel NN_1$ and $BC \parallel AD$ because the midpoint of segment $KM$ is

their center of symmetry. In this case the midline $KM$ of trapezoid $ABCD$ is parallel to bases $BC$ and $AD$ and therefore, heights of triangles $KLM$ and $KL_1M$ dropped to side $KM$ are equal, i.e., the areas of parallelograms $KLMN$ and $KL_1MN_1$ are equal.

**4.25.** Let the given lines $l_1$ and $l_2$ divide the square into four parts whose areas are equal to $S_1$, $S_2$, $S_3$ and $S_4$ so that for the first line the areas of the parts into which it divides the square are equal to $S_1 + S_2$ and $S_3 + S_4$ and for the second line they are equal to $S_2 + S_3$ and $S_1 + S_4$. Since by assumption $S_1 = S_2 = S_3$, it follows that $S_1 + S_2 = S_2 + S_3$. This means that the image of line $l_1$ under the rotation about the center of the square through an angle of $+90°$ or $-90°$ is not just parallel to line $l_2$ but coincides with it.

It remains to prove that line $l_1$ (hence, line $l_2$) passes through the center of the square. Suppose that this is not true. Let us consider the images of lines $l_1$ and $l_2$ under rotations through an angle of $\pm90°$ and denote the areas of the parts into which they divide the square as plotted on Fig. 45 (on this figure both distinct variants of the disposition of the lines are plotted).
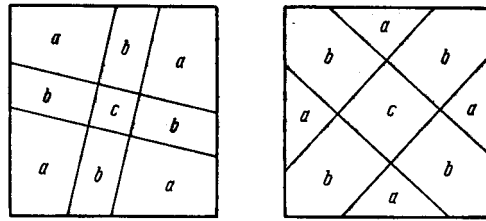


Figure 45 (Sol. 4.25)

Lines $l_1$ and $l_2$ divide the square into four parts whose areas are equal to $a$, $a+b$, $a+2b+c$ and $a + b$, where numbers $a$, $b$ and $c$ are nonzero. It is clear that three of the four numbers indicated cannot be equal. Contradiction.

**4.26.** All the three triangles considered have a common base $AM$. Let $h_b$, $h_c$ and $h_d$ be the distances from points $B$, $C$ and $D$, respectively, to line $AM$. Since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, it follows that $h_c = |h_b \pm h_d|$.

**4.27.** We may assume that $P$ is a common point of parallelograms constructed on sides $AB$ and $BC$, i.e., these parallelograms are of the form $ABPQ$ and $CBPR$. It is clear that $S_{ACRQ} = S_{ABPQ} + S_{CBPR}$.

**4.28.** Let the length of the hexagon's side be equal to $a$. The extensions of red sides of the hexagon form an equilateral triangle with side $3a$ and the sum of areas of red triangles is equal to a half product of $a$ by the sum of distances from point $O$ to a side of this triangle and, therefore, it is equal to $\frac{3\sqrt{3}}{4}a^2$, cf. Problem 4.46.

The sum of areas of blue triangles is similarly calculated.

**4.29.** a) The area of parallelogram $PMQN$ is equal to $\frac{1}{4}BC \cdot AD \sin \alpha$, where $\alpha$ is the angle between lines $AD$ and $BC$. The heights of triangles $ABD$ and $ACD$ dropped from vertices $B$ and $C$ are equal to $OB \sin \alpha$ and $OC \sin \alpha$, respectively; hence,

$$|S_{ABD} - S_{ACD}| = \frac{|OB - OC| \cdot AD \sin \alpha}{2} = \frac{BC \cdot AD \sin \alpha}{2}.$$

b) Let, for definiteness, rays $AD$ and $BC$ intersect. Since $PN \parallel AO$ and $QN \parallel CO$, point $N$ lies inside triangle $OPQ$. Therefore,

$$S_{OPQ} = S_{PQN} + S_{PON} + S_{QON} = \frac{1}{2}S_{PMQN} + \frac{1}{4}S_{ACD} + \frac{1}{4}S_{BCD}$$
$$= \frac{1}{4}(S_{ABD} - S_{ACD} + S_{ACD} + S_{BCD} = \frac{1}{4}S_{ABCD}.$$

**4.30.** Segments $KM$ and $LN$ are the midlines of triangles $CED$ and $AFB$ and, therefore, they have a common point — the midpoint of segment $EF$. Moreover, $KM = \frac{1}{2}CD$, $LN = \frac{1}{2}AB$ and the angle between lines $KM$ and $LN$ is equal to the angle $\alpha$ between lines $AB$ and $CD$. Therefore, the area of quadrilateral $KLMN$ is equal to $\frac{1}{8}AB \cdot CD \sin \alpha$.
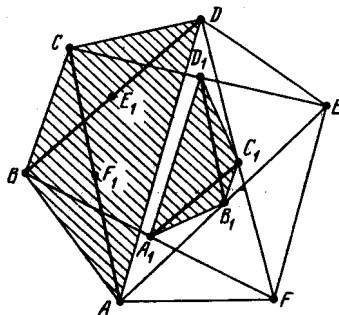


Figure 46 (Sol. 4.31)

**4.31.** Denote the midpoints of the diagonals of hexagon $ABCDEF$ as shown on Fig. 46. Let us prove that the area of quadrilateral $A_1B_1C_1D_1$ is $\frac{1}{4}$ of the area of quadrilateral $ABCD$. To this end let us make use of the fact that the area of the quadrilateral is equal to a half product of the lengths of the diagonals by the sine of the angle between them. Since $A_1C_1$ and $B_1D_1$ are the midlines of triangles $BDF$ and $ACE$, we get the desired statement.

We similarly prove that the area of quadrilateral $D_1E_1F_1A_1$ is $\frac{1}{4}$ of the area of quadrilateral $DEFA$.

**4.32.** Let $\alpha = \angle PQC$. Then

$$2S_{ACK} = CK \cdot AK = (AP \cos \alpha) \cdot (AQ \sin \alpha) = AR^2 \sin \alpha \cdot \cos \alpha$$
$$= BC^2 \sin \alpha \cdot \cos \alpha = BL \cdot CL = 2S_{BCL}.$$

**4.33.** Let $S_a$, $S_b$ and $S_c$ be the areas of the triangles adjacent to vertices $A$, $B$ and $C$; let $S$ be the area of the fourth of the triangles considered. Clearly,

$$S_{ACC_1} + S_{BAA_1} + S_{CBB_1} = S_{ABC} - S + S_a + S_b + S_c.$$

Moreover,

$$S_{ABC} = S_{AOC} + S_{AOB} + S_{BOC} = S_{ACC_1} + S_{BAA_1} + S_{CBB_1}.$$

**4.34.** Let $O$ be the center of the inscribed circle of triangle $ABC$, let $B_1$ be the tangent point of the inscribed circle and side $AC$. Let us cut off triangle $ABC$ triangle $AOB_1$ and reflect $AOB_1$ symmetrically through the bisector of angle $OAB_1$. Under this reflection line $OB_1$ turns into line $l_a$. Let us perform a similar operation for the remaining triangles. The common parts of the triangles obtained in this way are three triangles of the considered partition and the uncovered part of triangle $ABC$ is the fourth triangle. It is also clear that the area of the uncovered part is equal to the sum of areas of the parts covered twice.

**4.35.** Let, for definiteness, rays $AD$ and $BC$ meet at point $O$. Then $S_{CDO} : S_{MNO} = c^2 : x^2$, where $x = MN$ and $S_{ABO} : S_{MNO} = ab : x^2$ because $OA : ON = a : x$ and $OB : OM = b : x$. It follows that $x^2 - c^2 = ab - x^2$, i.e., $2x^2 = ab + c^2$.

**4.36.** Denote the areas of the parts of the figure into which it is divided by lines as shown on Fig. 47. Let us denote by $S$ the area of the whole figure. Since

$$S_3 + (S_2 + S_7) = \frac{1}{2}S = S_1 + S_6 + (S_2 + S_7),$$
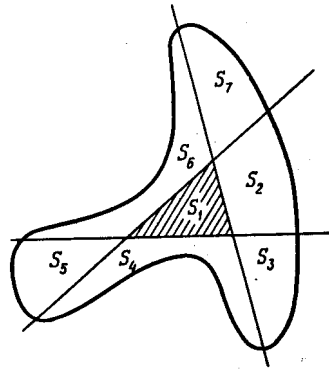
Figure 47 (Sol. 4.36)

it follows that $S_3 = S_1 + S_6$. Adding this equality to the equality $\frac{1}{2}S = S_1 + S_2 + S_3 + S_4$ we get

$$\frac{1}{2}S = 2S_1 + S_2 + S_4 + S_6 \geq 2S_1, \quad \text{i.e.,} \quad S_1 \leq \frac{1}{4}S.$$

**4.37.** Let us denote the projection of line $l$ by $B$ and the endpoints of the projection of the polygon by $A$ and $C$. Let $C_1$ be a point of the polygon whose projection is $C$. Then line $l$ intersects the polygon at points $K$ and $L$; let points $K_1$ and $L_1$ be points on lines $C_1K$ and $C_1L$ that have point $A$ as their projection (Fig. 48).
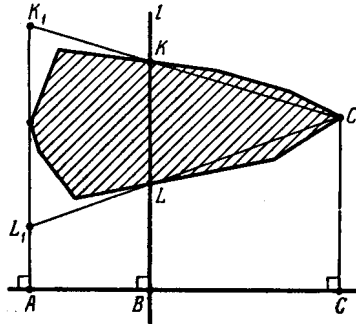


Figure 48 (Sol. 4.37)

One of the parts into which line $l$ divides the polygon is contained in trapezoid $K_1KLL_1$, the other part contains triangle $C_1KL$. Therefore, $S_{K_1KLL_1} \geq S_{C_1KL}$, i.e., $AB \cdot (KL + K_1L_1) \geq BC \cdot KL$. Since $K_1L_1 = KL \cdot \frac{AB+BC}{BC}$, we have $AB \cdot \left(2 + \frac{AB}{BC}\right) \geq BC$. Solving this inequality we get $\frac{BC}{AB} \leq 1 + \sqrt{2}$.

Similarly, $\frac{AB}{BC} \leq 1 + \sqrt{2}$. (We have to perform the same arguments but interchange $A$ and $C$.)

**4.38.** Let $S$ denote the area of the polygon, $l$ an arbitrary line. Let us introduce a coordinate system in which line $l$ is $Ox$-axis. Let $S(a)$ be the area of the part of the polygon below the line $y = a$. Clearly, $S(a)$ varies continuously from 0 to $S$ as $a$ varies from $-\infty$ to $+\infty$ and, therefore (by Calculus, see, e.g., ??), $S(a) = \frac{1}{2}S$ for some $a$, i.e., the line $y = a$ divides the area of the polygon in halves.

Similarly, there exists a line perpendicular to $l$ and this perpendicular also divides the area of the polygon in halves. These two lines divide the polygon into parts whose areas are equal to $S_1$, $S_2$, $S_3$ and $S_4$ (see Fig. 49). Since $S_1 + S_2 = S_3 + S_4$ and $S_1 + S_4 = S_2 + S_3$, we have $S_1 = S_3 = A$ and $S_2 = S_4 = B$. The rotation of line $l$ by $90°$ interchanges points $A$
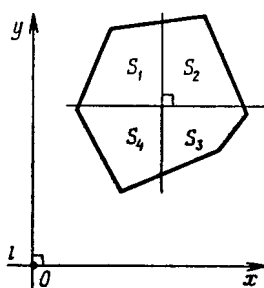
Figure 49 (Sol. 4.38)

and $B$. Since $A$ and $B$ vary continuously under the rotation of $l$, it follows that $A = B$ for a certain position of $l$, i.e., the areas of all the four figures are equal at this moment.

**4.39.** a) Let the line that divides the area and the perimeter of triangle $ABC$ in halves intersect sides $AC$ and $BC$ at points $P$ and $Q$, respectively. Denote the center of the inscribed circle of triangle $ABC$ by $O$ and the radius of the inscribed circle by $r$. Then $S_{ABQOP} = \frac{1}{2}r(AP + AB + BQ)$ and $S_{OQCP} = \frac{1}{2}r(QC + CP)$. Since line $PQ$ divides the perimeter in halves, $AP+AB+BQ = QC+CP$ and, therefore, $S_{ABQOP} = S_{OQCP}$. Moreover, $S_{ABQP} = S_{QCP}$ by the hypothesis. Therefore, $S_{OQP} = 0$, i.e., line $QP$ passes through point $O$.

b) Proof is carried out similarly to that of heading a).

**4.40.** By considering the image of circle $S_2$ under the symmetry through the center of circle $S_1$ and taking into account the equality of areas, it is possible to prove that diameter $AA_1$ of circle $S_1$ intersects $S_2$ at a point $K$ distinct from $A$ and so that $AK > A_1K$. The circle of radius $KA_1$ centered at $K$ is tangent to $S_1$ at point $A_1$ and, therefore, $BK > A_1K$, i.e., $BK + KA > A_1A$. It is also clear that the sum of the lengths of segments $BK$ and $KA$ is smaller than the length of the arc of $S_2$ that connects points $A$ and $B$.

**4.41.** The case when point $O$ belongs to $\Gamma$ is obvious; therefore, let us assume that $O$ does not belong to $\Gamma$. Let $\Gamma'$ be the image of the curve $\Gamma$ under the symmetry through point $O$. If curves $\Gamma$ and $\Gamma'$ do not intersect, then the parts into which $\Gamma$ divides the square cannot be of equal area. Let $X$ be the intersection point of $\Gamma$ and $\Gamma'$; let $X'$ be symmetric to $X$ through point $O$. Since under the symmetry through point $O$ curve $\Gamma'$ turns into $\Gamma$, it follows that $X'$ belongs to $\Gamma$. Hence, line $XX'$ is the desired one.

**4.42.** Let the areas of triangles $APB$, $BPC$, $CPD$ and $DPA$ be equal to $S_1$, $S_2$, $S_3$ and $S_4$, respectively. Then $\frac{a}{p} = \frac{S_3+S_4}{S_3}$ and $\frac{b \cdot CD}{2} = S_3 + S_2$; consequently,

$$\frac{ab \cdot CD}{2p} = \frac{(S_3 + S_4)(S_3 + S_2)}{S_3}.$$

Taking into account that $S_2 S_4 = S_1 S_3$ we get the desired statement.

**4.43.** By applying the law of sines to triangles $ABC$ and $ABD$ we get $AC = 2R\sin \angle B$ and $BD = 2R \cdot \sin \angle A$. Therefore,

$$S = \frac{1}{2}AC \cdot BD \sin \varphi = 2R^2 \sin \angle A \cdot \sin \angle B \cdot \sin \varphi.$$

**4.44.** Since the area of the quadrilateral is equal to $\frac{1}{2}d_1 d_2 \sin \varphi$, where $d_1$ and $d_2$ are the lengths of the diagonals, it remains to verify that $2d_1 d_2 \cos \varphi = |a^2 + c^2 - b^2 - d^2|$. Let $O$ be the intersection point of the diagonals of quadrilateral $ABCD$ and $\varphi = \angle AOB$. Then

$$AB^2 = AO^2 + BO^2 - 2AO \cdot OB \cos \varphi; \quad BC^2 = BO^2 + CO^2 + 2BO \cdot CO \cos \varphi.$$

Hence,

$$AB^2 - BC^2 = AO^2 - CO^2 - 2BO \cdot AC \cos\varphi.$$

Similarly,

$$CD^2 - AD^2 = CO^2 - AO^2 - 2DO \cdot AC \cos\varphi.$$

By adding these equalities we get the desired statement.

REMARK. Since

$$16S^2 = 4d_1^2 d_2^2 \sin^2\varphi = 4d_1^2 d_2^2 - (2d_1 d_2 \cos\varphi)^2,$$

it follows that $16S^2 = 4d_1^2 d_2^2 - (a^2 + c^2 - b^2 - d^2)^2$.

**4.45.** a) Let $AB = a$, $BC = b$, $CD = c$ and $AD = d$. Clearly,

$$S = S_{ABC} + S_{ADC} = \tfrac{1}{2}ab\sin\angle B + cd\sin\angle D;$$
$$a^2 + b^2 - 2ab\cos\angle B = AC^2 = c^2 + d^2 - 2cd\cos\angle D.$$

Therefore,

$$16S^2 = 4a^2b^2 - 4a^2b^2\cos^2\angle B + 8abcd\sin\angle B\sin\angle D + 4c^2d^2 - 4c^2d^2\cos^2\angle D,$$
$$(a^2 + b^2 - c^2 - d^2)^2 + 8abcd\cos\angle B\cos\angle D = 4a^2b^2 \cdot \cos^2\angle B + 4c^2d^2\cos^2\angle D.$$

By inserting the second equality into the first one we get

$$16S^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 8abcd(1 + \cos\angle B\cos\angle D - \sin\angle B\sin\angle D).$$

Clearly,

$$4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 = 16(p - a)(p - b)(p - c)(p - d);$$
$$1 + \cos\angle B\cos\angle D - \sin\angle B\sin\angle D = 2\cos^2\tfrac{\angle B + \angle D}{2}.$$

b) If $ABCD$ is an inscribed quadrilateral, then $\angle B + \angle D = 180°$ and, therefore, $\cos^2\frac{\angle B + \angle D}{2} = 0$.

c) If $ABCD$ is a circumscribed quadrilateral, then $a + c = b + d$ and, therefore, $p = a + c = b + d$ and $p - a = c$, $p - b = d$, $p - c = a$, $p - d = b$. Hence,

$$S^2 = abcd\left(1 - \cos^2\frac{\angle B + \angle D}{2}\right) = abcd\sin^2\frac{\angle B + \angle D}{2}.$$

If $ABCD$ is simultaneously an inscribed and circumscribed quadrilateral, then $S^2 = abcd$.

**4.46.** Let us drop perpendiculars $OA_1$, $OB_1$ and $OC_1$ to sides $BC$, $AC$ and $AB$, respectively, of an equilateral triangle $ABC$ from a point $O$ inside it. In triangle $ABC$, let $a$ be the length of the side, $h$ the length of the height. Clearly, $S_{ABC} = S_{BCO} + S_{ACO} + S_{ABO}$. Therefore, $ah = a \cdot OA_1 + a \cdot OB_1 + a \cdot OC_1$, i.e., $h = OA_1 + OB_1 + OC_1$.

**4.47.** Let $AD = l$. Then

$$2S_{ABD} = cl\sin\frac{\alpha}{2}, \quad 2S_{ACD} = bl\sin\frac{\alpha}{2}, \quad 2S_{ABD} = bc\sin\alpha.$$

Hence,

$$cl\sin\frac{\alpha}{2} + bl\sin\frac{\alpha}{2} = bc\sin\alpha = 2bc\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.$$

**4.48.** a) Let the distances from points $A$ and $O$ to line $BC$ be equal to $h$ and $h_1$, respectively. Then $S_{OBC} : S_{ABC} = h_1 : H = OA_1 : AA_1$. Similarly, $S_{OAC} : S_{ABC} = OB_1 : BB_1$ and $S_{OAB} : S_{ABC} = OC_1 : CC_1$. By adding these equalities and taking into account that $S_{OBC} + S_{OAC} + S_{OAB} = S_{ABC}$ we get the desired statement.

b) Let the distances from points $B$ and $C$ to line $AA_1$ be equal to $d_b$ and $d_c$, respectively. Then $S_{ABO} : S_{ACO} = d_b : d_c = BA_1 : A_1C$. Similarly, $S_{ACO} : S_{BCO} = AC_1 : C_1B$ and $S_{BCO} : S_{ABO} = CB_1 : B_1A$. It remains to multiply these equalities.

**4.49.** It is easy to verify that the ratio of the lengths of segments $A_{n+k-1}B_k$ and $A_{n+k}B_k$ is equal to the ratio of areas of triangles $A_{n+k-1}OA_k$ and $A_kOA_{n+k}$. By multiplying these equalities we get the desired statement.

**4.50.** Since $A_iB_iC_iD_i$ is a parallelogram and point $O$ lies on the extension of its diagonal $A_iC_i$, it follows that $S_{A_iB_iO} = S_{A_iD_iO}$ and, therefore, $A_iB_i : A_iD_i = h_i : h_{i-1}$, where $h_i$ is the distance from point $O$ to side $A_iA_{i+1}$. It remains to multiply these equalities for $i = 1, \dots, n$.

**4.51.** Let the lengths of sides of triangle $ABC$ be equal to $a$, $b$ and $c$, where $a \le b \le c$. Then $2b = a + c$ and $2S_{ABC} = r(a + b + c) = 3rb$, where $r$ is the radius of the inscribed circle. On the other hand, $2S_{ABC} = h_b b$. Therefore, $r = \frac{1}{3}h_b$.

**4.52.** It suffices to observe that

$$d_b \cdot AB = 2S_{AXB} = BX \cdot AX \sin\varphi,$$

where $\varphi = \angle AXB$ and $d_c \cdot AC = 2S_{AXC} = CX \cdot AX \sin\varphi$.

**4.53.** Let $r_1, \dots, r_n$ be the radii of the inscribed circles of the obtained triangles, let $P_1, \dots, P_n$ their perimeters and $S_1, \dots, S_n$ their areas. Let us denote the area and the perimeter of the initial polygon by $S$ and $P$, respectively.

It is clear that $P_i < P$ (cf. Problem 9.27, b). Hence,

$$r_1 + \cdots + r_n = 2\frac{S_1}{P_1} + \cdots + 2\frac{S_n}{P_n} > 2\frac{S_1}{P} + \cdots + 2\frac{S_n}{P} = 2\frac{S}{P} = r.$$

**4.54.** Let us draw lines $Q_1S_1$ and $P_1R_1$ parallel to lines $QS$ and $PR$ through the intersection point $N$ of lines $BS$ and $CM$ (points $P_1$, $Q_1$, $R_1$ and $S_1$ lie on sides $AB$, $BC$, $CD$ and $DA$, respectively). Let $F$ and $G$ be the intersection points of lines $PR$ and $Q_1S_1$, $P_1R_1$ and $QS$, respectively. Since point $M$ lies on diagonal $NC$ of parallelogram $NQ_1CR_1$, it follows that $S_{FQ_1QM} = S_{MRR_1G}$ (by Problem 4.19) and, therefore, $S_{NQ_1QG} = S_{NFRR_1}$. Point $N$ lies on diagonal $BS$ of parallelogram $ABQS$ and, therefore, $S_{AP_1NS_1} = S_{NQ_1QG} = S_{NFRR_1}$. It follows that point $N$ lies on diagonal $PD$ of parallelogram $APRD$.

**4.55.** Let $E$ and $F$ be the intersection points of the extensions of sides of the given quadrilateral. Denote the vertices of the quadrilateral so that $E$ is the intersection point of the extensions of sides $AB$ and $CD$ beyond points $B$ and $C$ and $F$ is the intersection point of rays $BC$ and $AD$. Let us complement triangles $AEF$ and $ABD$ to parallelograms $AERF$ and $ABLD$, respectively.

The homothety with center $A$ and coefficient 2 sends the midpoint of the diagonal $BD$, the midpoint of the diagonal $AC$ and the midpoint of segment $EF$ to points $L$, $C$ and $R$, respectively. Therefore, it suffices to prove that points $L$, $C$ and $R$ lie on one line. This is precisely the fact proved in the preceding problem.

**4.56.** It suffices to verify that $S_{AKO} = S_{ALO}$, i.e., $AO{\cdot}AL \sin\angle OAL = AO{\cdot}AK \sin\angle OAK$. Clearly,

$$AL = CB_1 = BC\cos\angle C, \quad \sin\angle OAL = \cos\angle B,$$
$$AK = BC_1 = BC\cos\angle B, \quad \sin\angle OAK = \cos\angle C.$$

**4.57.** Since quadrilateral $A_1BC_1M$ is a circumscribed one, then, first, the sums of the lengths of its opposite sides are equal:

$$\frac{a}{2} + \frac{m_c}{3} = \frac{c}{2} + \frac{m_a}{3}$$

and, second, its inscribed circle is simultaneously the inscribed circle of triangles $AA_1B$ and $CC_1B$. Since these triangles have equal areas, their perimeters are equal:

$$c + m_a + \frac{a}{2} = a + m_c + \frac{c}{2}.$$

By multiplying the first equality by 3 and adding to the second one we get the desired statement.

**4.58.** First, let us prove a general inequality that will be used in the proof of headings a)–d):

$$(*) \qquad BC_1 \cdot R_a \geq B_1K \cdot R_a + C_1L \cdot R_a = 2S_{AOB_1} + 2S_{AOC_1} = AB_1 \cdot d_c + AC_1 \cdot d_b.$$

On rays $AB$ and $AC$ take arbitrary points $B_1$ and $C_1$ and drop from them perpendiculars $B_1K$ and $C_1L$ to line $AO$. Since $B_1C_1 \geq B_1K + C_1L$, inequality $(*)$ follows.

a) Setting $B_1 = B$ and $C_1 = C$ we get the desired statement.

b) By multiplying both sides of the inequality $aR_a \geq cd_c + bd_b$ by $\frac{d_a}{a}$ we get

$$d_aR_a \geq \frac{c}{a}d_ad_c + \frac{b}{a}d_ad_b.$$

Taking the sum of this inequality with the similar inequalities for $d_bR_b$ and $d_cR_c$ and taking into account that $\frac{x}{y} + \frac{y}{x} \geq 2$ we get the desired statement.

c) Take points $B_1$ and $C_1$ such that $AB_1 = AC$ and $AC_1 = AB$. Then $aR_a \geq bd_c + cd_b$, i.e., $R_a \geq \frac{b}{a}d_c + \frac{c}{a}d_b$. Taking the sum of this inequality with similar inequalities for $R_b$ and $R_c$ and taking into account that $\frac{x}{y} + \frac{y}{x} \geq 2$ we get the desired statement.

d) Take points $B_1$ and $C_1$ such that $AB_1 = AC_1 = 1$; then $B_1C_1 = 2\sin\frac{1}{2}\angle A$ and, therefore, $2\sin\frac{1}{2}R_a \geq d_c + d_b$. By multiplying this inequality by similar inequalities for $R_b$ and $R_c$ and taking into account that $\sin\frac{1}{2}\angle A \sin\frac{1}{2}\angle B \sin\frac{1}{2}\angle C = \frac{r}{4R}$ (by Problem 12.36 a)) we get the desired statement.

**4.59.** Let us cut triangles off a regular octagon and replace the triangles as shown on Fig. 50. As a result we get a rectangle whose sides are equal to the longest and shortest diagonals of the octagon.
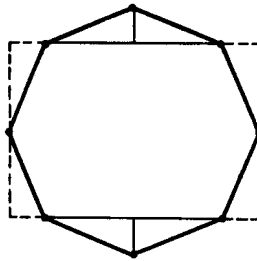


Figure 50 (Sol. 4.59)

**4.60.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$, respectively, of triangle $ABC$. The drawn segments are heights of triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$, respectively. Let $P$, $Q$ and $R$ be the respective intersection points of the heights of these triangles and $O$ the intersection point of the heights of triangle $A_1B_1C_1$ (Fig. 51).

The considered hexagon consists of triangle $A_1B_1C_1$ and triangles $B_1C_1P$, $C_1A_1Q$ and $A_1BR$. Clearly, $\triangle B_1C_1P = \triangle C_1B_1O$, $\triangle C_1A_1Q = \triangle A_1C_1O$ and $\triangle A_1B_1R = \triangle B_1A_1O$. Therefore, the area of the considered hexagon is equal to the doubled area of triangle $A_1B_1C_1$. It remains to observe that $S_{ABC} = 4S_{A_1B_1C_1}$.

**4.61.** a) Let us cut two parts off the parallelogram (Fig. 52 a)) and replace these parts as shown on Fig. 52 b). We get a figure composed of $mn + 1$ small parallelograms. Therefore, the area of a small parallelogram is equal to $\frac{1}{mn+1}$.

b) Let us cut off the parallelogram three parts (Fig. 53 a)) and replace these parts as indicated on Fig. 53 b). We get a figure that consists of $mn - 1$ small parallelograms. Therefore, the area of a small parallelogram is equal to $\frac{1}{mn-1}$.
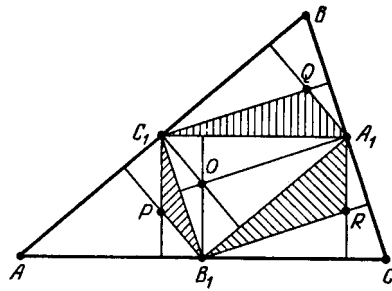
Figure 51 (Sol. 4.60)
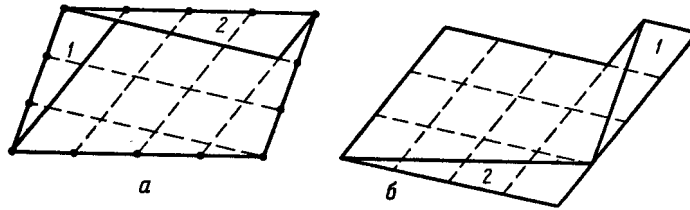


Figure 52 (Sol. 4.61 a))
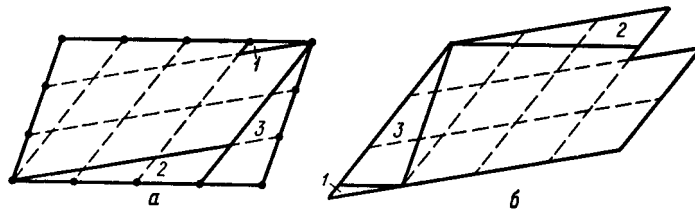


Figure 53 (Sol. 4.61 b))

**4.62.** a) Let us cut the initial square into four squares and consider one of them (Fig. 54). Let point $B'$ be symmetric to point $B$ through line $PQ$. Let us prove that $\triangle APB = \triangle OB'P$. Indeed, triangle $APB$ is an isosceles one and angle at its base is equal to $15°$ (Problem 2.26), hence, triangle $BPQ$ is an isosceles one. Therefore,

$$\angle OPB' = \angle OPQ - \angle B'PQ = 75° - 60° = 15°$$

and $\angle POB' = \frac{1}{2}\angle OPQ = 15°$. Moreover, $AB = OP$. We similarly prove that $\triangle BQC = \triangle OB'Q$. It follows that the area of the shaded part on Fig. 43 is equal to the area of triangle $OPQ$.
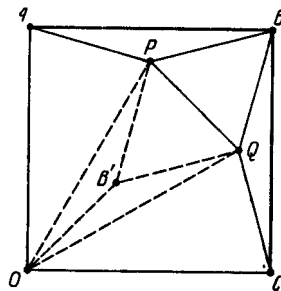


Figure 54 (Sol. 4.62)

b) Let the area of the regular 12-gon inscribed in a circle of radius 1 be equal to $12x$. Thanks to heading a) the area of the square circumscribed around this circle is equal to

$12x + 4x = 16x$; on the other hand, the area of the square is equal to 4; hence, $x = \frac{1}{4}$ and $12x = 3$.

# CHAPTER 5. TRIANGLES

## Background

1) The *inscribed circle* of a triangle is the circle tangent to all its sides. The *center* of an inscribed circle is the intersection point of the bisectors of the triangle's angles.

An *escribed circle* of triangle $ABC$ is the circle tangent to one side of the triangle and extensions of the other two sides. For each triangle there are exactly three escribed circles. The *center* of an escribed circle tangent to side $AB$ is the intersection point of the bisector of angle $C$ and the bisectors of the *outer* angles $A$ and $B$.

The *circumscribed circle* of a triangle is the circle that passes through the vertices of the triangle. The *center* of the circumscribed circle of a triangle is the intersection point of the midperpendiculars to the triangle's sides.

2) For elements of a triangle $ABC$ the following notations are often used:

$a$, $b$ and $c$ are the lengths of sides $BC$, $CA$ and $AB$, respectively;

$\alpha$, $\beta$ and $\gamma$ are the values of angles at vertices $A$, $B$, $C$;

$R$ is the radius of the circumscribed cirlce;

$r$ is the radius of the inscribed circle;

$r_a$, $r_b$ and $r_c$ are the radii of the escribed circles tangent to sides $BC$, $CA$ and $AB$, respectively;

$h_a, h_b$ and $h_c$ the lengths of the heights dropped from vertices $A, B$ and $C$, respectively.

3) If $AD$ is the bisector of angle $A$ of triangle $ABC$ (or the bisector of the outer angle $A$), then $BD : CD = AB : AC$ (cf. Problem 1.17).

4) In a right triangle, the median drawn from the vertex of the right angle is equal to a half the hypothenuse (cf. Problem 5.16).

5) To prove that the intersection points of certain lines lie on one line *Menelaus's theorem* (Problem 5.58) is often used.

6) To prove that certain lines intersect at one point *Ceva's theorem* (Problem 5.70) is often used.

## Introductory problems

**1.** Prove that the triangle is an isosceles one if a) one of its medians coincides with a height;

b) if one of its bisectors coincides with a height.

**2.** Prove that the bisectors of a triangle meet at one point.

**3.** On height $AH$ of triangle $ABC$ a point $M$ is taken. Prove that $AB^2 - AC^2 = MB^2 - MC^2$.

**4.** On sides $AB$, $BC$, $CA$ of an equilateral triangle $ABC$ points $P$, $Q$ and $R$, respectively, are taken so that

$$AP : PB = BQ : QC = CR : RA = 2 : 1.$$

Prove that the sides of triangle $PQR$ are perpendicular to the respective sides of triangle $ABC$.

## 1. The inscribed and the circumscribed circles

**5.1.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$, points $A_1$, $B_1$ and $C_1$, respectively, are taken so that $AC_1 = AB_1$, $BA_1 = BC_1$ and $CA_1 = CB_1$. Prove that $A_1$, $B$ and $C_1$ are the points at which the inscribed circle is tangent to the sides of the triangle.

**5.2.** Let $O_a$, $O_b$ and $O_c$ be the centers of the escribed circles of triangle $ABC$. Prove that points $A$, $B$ and $C$ are the bases of heights of triangle $O_aO_bO_c$.

**5.3.** Prove that side $BC$ of triangle $ABC$ subtends (1) an angle with the vertex at the center $O$ of the inscribed circle; the value of the angle is equal to $90° + \frac{1}{2}\angle A$ and (2) an angle with the vertex at the center $O_a$ of the escribed circle; the value of the angle is equal to $90° - \frac{1}{2}\angle A$.

**5.4.** Inside triangle $ABC$, point $P$ is taken such that

$$\angle PAB : \angle PAC = \angle PCA : \angle PCB = \angle PBC : \angle PBA = x.$$

Prove that $x = 1$.

**5.5.** Let $A_1$, $B_1$ and $C_1$ be the projections of an inner point $O$ of triangle $ABC$ to the heights. Prove that if the lengths of segments $AA_1$, $BB_1$ and $CC_1$ are equal, then they are equal to $2r$.

**5.6.** An angle of value $\alpha = \angle BAC$ is rotated about its vertex $O$, the midpoint of the basis $AC$ of an isosceles triangle $ABC$. The legs of this angle meet the segments $AB$ and $BC$ at points $P$ and $Q$, respectively. Prove that the perimeter of triangle $PBQ$ remains constant under the rotation.

**5.7.** In a scalene triangle $ABC$, line $MO$ is drawn through the midpoint $M$ of side $BC$ and the center $O$ of the inscribed circle. Line $MO$ intersects height $AH$ at point $E$. Prove that $AE = r$.

**5.8.** A circle is tangent to the sides of an angle with vertex $A$ at points $P$ and $Q$. The distances from points $P$, $Q$ and $A$ to a tangent to this circle are equal to $u$, $v$ and $w$, respectively. Prove that $\frac{uv}{w^2} = \sin^2 \frac{1}{2}\angle A$.

\* \* \*

**5.9.** Prove that the points symmetric to the intersection point of the heights of triangle $ABC$ through its sides lie on the circumscribed circle.

**5.10.** From point $P$ of arc $BC$ of the circumscribed circle of triangle $ABC$ perpendiculars $PX$, $PY$ and $PZ$ are dropped to $BC$, $CA$ and $AB$, respectively. Prove that $\frac{BC}{PX} = \frac{AC}{PY} + \frac{AB}{PZ}$.

\* \* \*

**5.11.** Let $O$ be the center of the circumscribed circle of triangle $ABC$, let $I$ be the center of the inscribed circle, $I_a$ the center of the escribed circle tangent to side $BC$. Prove that

a) $d^2 = R^2 - 2Rr$, where $d = OI$;

b) $d_a^2 = R^2 + 2Rr_a$, where $d_a = OI_a$.

**5.12.** The extensions of the bisectors of the angles of triangle $ABC$ intersect the circumscribed circle at points $A_1$, $B_1$ and $C_1$; let $M$ be the intersection point of bisectors. Prove that a) $\frac{MA \cdot MC}{MB_1} = 2r$; b) $\frac{MA_1 \cdot MC_1}{MB} = R$.

**5.13.** The lengths of the sides of triangle $ABC$ form an arithmetic progression: $a$, $b$, $c$, where $a < b < c$. The bisector of angle $\angle B$ intersects the circumscribed circle at point $B_1$. Prove that the center $O$ of the inscribed circle divides segment $BB_1$ in halves.

**5.14.** In triangle $ABC$ side $BC$ is the shortest one. On rays $BA$ and $CA$, segments $BD$ and $CE$, respectively, each equal to $BC$, are marked. Prove that the radius of the

circumscribed circle of triangle $ADE$ is equal to the distance between the centers of the inscribed and circumscribed circles of triangle $ABC$.

## §2. Right triangles

**5.15.** In triangle $ABC$, angle $\angle C$ is a right one. Prove that $r = \frac{a+b-c}{2}$ and $r_c = \frac{a+b+c}{2}$.

**5.16.** In triangle $ABC$, let $M$ be the midpoint of side $AB$. Prove that $CM = \frac{1}{2}AB$ if and only if $\angle ACB = 90°$.

**5.17.** Consider trapezoid $ABCD$ with base $AD$. The bisectors of the outer angles at vertices $A$ and $B$ meet at point $P$ and the bisectors of the angles at vertices $C$ and $D$ meet at point $Q$. Prove that the length of segment $PQ$ is equal to a half perimeter of the trapezoid.

**5.18.** In an isosceles triangle $ABC$ with base $AC$ bisector $CD$ is drawn. The line that passes through point $D$ perpendicularly to $DC$ intersects $AC$ at point $E$. Prove that $EC = 2AD$.

**5.19.** The sum of angles at the base of a trapezoid is equal to $90°$. Prove that the segment that connects the midpoints of the bases is equal to a half difference of the bases.

**5.20.** In a right triangle $ABC$, height $CK$ from the vertex $C$ of the right angle is drawn and in triangle $ACK$ bisector $CE$ is drawn. Prove that $CB = BE$.

**5.21.** In a right triangle $ABC$ with right angle $\angle C$, height $CD$ and bisector $CF$ are drawn; let $DK$ and $DL$ be bisectors in triangles $BDC$ and $ADC$. Prove that $CLFK$ is a square.

**5.22.** On hypothenuse $AB$ of right triangle $ABC$, square $ABPQ$ is constructed outwards. Let $\alpha = \angle ACQ$, $\beta = \angle QCP$ and $\gamma = \angle PCB$. Prove that $\cos \beta = \cos \alpha \cdot \cos \gamma$.

See also Problems 2.65, 5.62.

## §3. The equilateral triangles

**5.23.** From a point $M$ inside an equilateral triangle $ABC$ perpendiculars $MP$, $MQ$ and $MR$ are dropped to sides $AB$, $BC$ and $CA$, respectively. Prove that

$$AP^2 + BQ^2 + CR^2 = PB^2 + QC^2 + RA^2,$$
$$AP + BQ + CR = PB + QC + RA.$$

**5.24.** Points $D$ and $E$ divide sides $AC$ and $AB$ of an equilateral triangle $ABC$ in the ratio of $AD : DC = BE : EA = 1 : 2$. Lines $BD$ and $CE$ meet at point $O$. Prove that $\angle AOC = 90°$.

* * *

**5.25.** A circle divides each of the sides of a triangle into three equal parts. Prove that this triangle is an equilateral one.

**5.26.** Prove that if the intersection point of the heights of an acute triangle divides the heights in the same ratio, then the triangle is an equilateral one.

**5.27.** a) Prove that if $a + h_a = b + h_b = c + h_c$, then triangle $ABC$ is a equilateral one.

b) Three squares are inscribed in triangle $ABC$: two vertices of one of the squares lie on side $AC$, those of another one lie on side $BC$, and those of the third lie one on $AB$. Prove that if all the three squares are equal, then triangle $ABC$ is an equilateral one.

**5.28.** The circle inscribed in triangle $ABC$ is tangent to the sides of the triangle at points $A_1$, $B_1$, $C_1$. Prove that if triangles $ABC$ and $A_1B_1C_1$ are similar, then triangle $ABC$ is an equilateral one.

**5.29.** The radius of the inscribed circle of a triangle is equal to 1, the lengths of the heights of the triangle are integers. Prove that the triangle is an equilateral one.

See also Problems 2.18, 2.26, 2.36, 2.44, 2.54, 4.46, 5.56, 7.45, 10.3, 10.77, 11.3, 11.5, 16.7, 18.9, 18.12, 18.15, 18.17-18.20, 18.22, 18.38, 24.1.

## §4. Triangles with angles of $60°$ and $120°$

**5.30.** In triangle $ABC$ with angle $A$ equal to $120°$ bisectors $AA_1$, $BB_1$ and $CC_1$ are drawn. Prove that triangle $A_1B_1C_1$ is a right one.

**5.31.** In triangle $ABC$ with angle $A$ equal to $120°$ bisectors $AA_1$, $BB_1$ and $CC_1$ meet at point $O$. Prove that $\angle A_1C_1O = 30°$.

**5.32.** a) Prove that if angle $\angle A$ of triangle $ABC$ is equal to $120°$ then the center of the circumscribed circle and the orthocenter are symmetric through the bisector of the outer angle $\angle A$.

b) In triangle $ABC$, the angle $\angle A$ is equal to $60°$; $O$ is the center of the circumscribed circle, $H$ is the orthocenter, $I$ is the center of the inscribed circle and $I_a$ is the center of the escribed circle tangent to side $BC$. Prove that $IO = IH$ and $I_aO = I_aH$.

**5.33.** In triangle $ABC$ angle $\angle A$ is equal to $120°$. Prove that from segments of lengths $a$, $b$ and $b + c$ a triangle can be formed.

**5.34.** In an acute triangle $ABC$ with angle $\angle A$ equal to $60°$ the heights meet at point $H$.

a) Let $M$ and $N$ be the intersection points of the midperpendiculars to segments $BH$ and $CH$ with sides $AB$ and $AC$, respectively. Prove that points $M$, $N$ and $H$ lie on one line.

b) Prove that the center $O$ of the circumscribed circle lies on the same line.

**5.35.** In triangle $ABC$, bisectors $BB_1$ and $CC_1$ are drawn. Prove that if $\angle CC_1B_1 = 30°$, then either $\angle A = 60°$ or $\angle B = 120°$.

See also Problem 2.33.

## §5. Integer triangles

**5.36.** The lengths of the sides of a triangle are consecutive integers. Find these integers if it is known that one of the medians is perpendicular to one of the bisectors.

**5.37.** The lengths of all the sides of a right triangle are integers and the greatest common divisor of these integers is equal to 1. Prove that the legs of the triangle are equal to $2mn$ and $m^2 - n^2$ and the hypothenuse is equal to $m^2 + n^2$, where $m$ and $n$ are integers.

A right triangle the lengths of whose sides are integers is called a *Pythagorean triangle*.

**5.38.** The radius of the inscribed circle of a triangle is equal to 1 and the lengths of its sides are integers. Prove that these integers are equal to 3, 4, 5.

**5.39.** Give an example of an inscribed quadrilateral with pairwise distinct integer lengths of sides and the lengths of whose diagonals, the area and the radius of the circumscribed circle are all integers. (*Brakhmagupta.*)

**5.40.** a) Indicate two right triangles from which one can compose a triangle so that the lengths of the sides and the area of the composed triangle would be integers.

b) Prove that if the area of a triangle is an integer and the lengths of the sides are consecutive integers then this triangle can be composed of two right triangles the lengths of whose sides are integers.

**5.41.** a) In triangle $ABC$, the lengths of whose sides are rational numbers, height $BB_1$ is drawn.

Prove that the lengths of segments $AB_1$ and $CB_1$ are rational numbers.

b) The lengths of the sides and diagonals of a convex quadrilateral are rational numbers. Prove that the diagonals cut it into four triangles the lengths of whose sides are rational numbers.

See also Problem 26.7.

## §6. Miscellaneous problems

**5.42.** Triangles $ABC$ and $A_1B_1C_1$ are such that either their corresponding angles are equal or their sum is equal to $180°$. Prove that the corresponding angles are equal, actually.

**5.43.** Inside triangle $ABC$ an arbitrary point $O$ is taken. Let points $A_1$, $B_1$ and $C_1$ be symmetric to $O$ through the midpoints of sides $BC$, $CA$ and $AB$, respectively. Prove that $\triangle ABC = \triangle A_1B_1C_1$ and, moreover, lines $AA_1$, $BB_1$ and $CC_1$ meet at one point.

**5.44.** Through the intersection point $O$ of the bisectors of triangle $ABC$ lines parallel to the sides of the triangle are drawn. The line parallel to $AB$ meets $AC$ and $BC$ at points $M$ and $N$, respectively, and lines parallel to $AC$ and $BC$ meet $AB$ at points $P$ and $Q$, respectively. Prove that $MN = AM + BN$ and the perimeter of triangle $OPQ$ is equal to the length of segment $AB$.

**5.45.** a) Prove that the heigths of a triangle meet at one point.

b) Let $H$ be the intersection point of heights of triangle $ABC$ and $R$ the radius of the circumscribed circle. Prove that

$$AH^2 + BC^2 = 4R^2 \quad \text{and} \quad AH = BC|\cot \alpha|.$$

**5.46.** Let $x = \sin 18°$. Prove that $4x^2 + 2x = 1$.

**5.47.** Prove that the projections of vertex $A$ of triangle $ABC$ on the bisectors of the outer and inner angles at vertices $B$ and $C$ lie on one line.

**5.48.** Prove that if two bisectors in a triangle are equal, then the triangle is an isosceles one.

**5.49.** a) In triangles $ABC$ and $A'B'C'$, sides $AC$ and $A'C'$ are equal, the angles at vertices $B$ and $B'$ are equal, and the bisectors of angles $\angle B$ and $\angle B'$ are equal. Prove that these triangles are equal. (More precisely, either $\triangle ABC = \triangle A'B'C'$ or $\triangle ABC = \triangle C'B'A'$.)

b) Through point $D$ on the bisector $BB_1$ of angle $ABC$ lines $AA_1$ and $CC_1$ are drawn (points $A_1$ and $C_1$ lie on sides of triangle $ABC$). Prove that if $AA_1 = CC_1$, then $AB = BC$.

**5.50.** Prove that a line divides the perimeter and the area of a triangle in equal ratios if and only if it passes through the center of the inscribed circle.

**5.51.** Point $E$ is the midpoint of arc $\smile AB$ of the circumscribed circle of triangle $ABC$ on which point $C$ lies; let $C_1$ be the midpoint of side $AB$. Perpendicular $EF$ is dropped from point $E$ to $AC$. Prove that:

a) line $C_1F$ divides the perimeter of triangle $ABC$ in halves;

b) three such lines constructed for each side of the triangle meet at one point.

**5.52.** On sides $AB$ and $BC$ of an acute triangle $ABC$, squares $ABC_1D_1$ and $A_2BCD_2$ are constructed outwards. Prove that the intersection point of lines $AD_2$ and $CD_1$ lies on height $BH$.

**5.53.** On sides of triangle $ABC$ squares centered at $A_1$, $B_1$ and $C_1$ are constructed outwards. Let $a_1$, $b_1$ and $c_1$ be the lengths of the sides of triangle $A_1B_1C_1$; let $S$ and $S_1$ be the areas of triangles $ABC$ and $A_1B_1C_1$, respectively. Prove that:

a) $a_1^2 + b_1^2 + c_1^2 = a^2 + b^2 + c^2 + 6S$.

b) $S_1 - S = \frac{1}{8}(a^2 + b^2 + c^2)$.

**5.54.** On sides $AB$, $BC$ and $CA$ of triangle $ABC$ (or on their extensions), points $C_1$, $A_1$ and $B_1$, respectively, are taken so that $\angle(CC_1, AB) = \angle(AA_1, BC) = \angle(BB_1, CA) =$

$\alpha$. Lines $AA_1$ and $BB_1$, $BB_1$ and $CC_1$, $CC_1$ and $AA_1$ intersect at points $C'$, $A'$ and $B'$, respectively. Prove that:

a) the intersection point of heights of triangle $ABC$ coincides with the center of the circumscribed circle of triangle $A'B'C'$;

b) $\triangle A'B'C' \sim \triangle ABC$ and the similarity coefficient is equal to $2\cos\alpha$.

**5.55.** On sides of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ are taken so that $AB_1 : B_1C = c^n : a^n$, $BC_1 : CA = a^n : b^n$ and $CA_1 : A_1B = b^n : c^n$ (here $a$, $b$ and $c$ are the lengths of the triangle's sides). The circumscribed circle of triangle $A_1B_1C_1$ singles out on the sides of triangle $ABC$ segments of length $\pm x, \pm y$ and $\pm z$, where the signs are chosen in accordance with the *orientation* of the triangle. Prove that

$$\frac{x}{a^{n-1}} + \frac{y}{b^{n-1}} + \frac{z}{c^{n-1}} = 0.$$

**5.56.** In triangle $ABC$ *trisectors* (the rays that divide the angles into three equal parts) are drawn. The nearest to side $BC$ trisectors of angles $B$ and $C$ intersect at point $A_1$; let us define points $B_1$ and $C_1$ similarly, (Fig. 55). Prove that triangle $A_1B_1C_1$ is an equilateral one. (*Morlie's theorem.*)
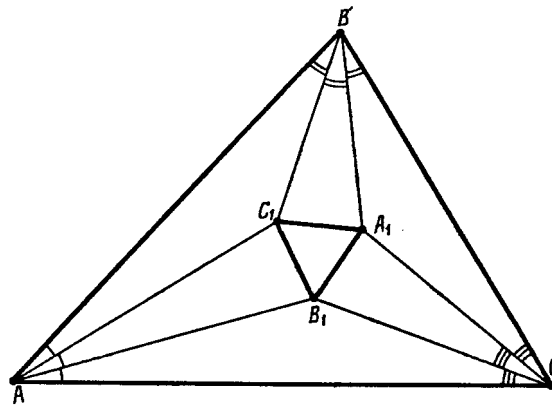


Figure 55 (5.56)

**5.57.** On the sides of an equilateral triangle $ABC$ as on bases, isosceles triangles $A_1BC$, $AB_1C$ and $ABC_1$ with angles $\alpha$, $\beta$ and $\gamma$ at the bases such that $\alpha + \beta + \gamma = 60°$ are constructed inwards. Lines $BC_1$ and $B_1C$ meet at point $A_2$, lines $AC_1$ and $A_1C$ meet at point $B_2$, and lines $AB_1$ and $A_1B$ meet at point $C_2$. Prove that the angles of triangle $A_2B_2C_2$ are equal to $3\alpha$, $3\beta$ and $3\gamma$.

## §7. Menelaus's theorem

Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be colinear vectors. Denote by $\frac{\overline{AB}}{\overline{CD}}$ the quantity $\pm\frac{AB}{CD}$, where the plus sign is taken if the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are codirected and the minus sign if the vectors are directed opposite to each other.

**5.58.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ (or on their extensions) points $A_1$, $B_1$ and $C_1$, respectively, are taken. Prove that points $A_1$, $B_1$ and $C_1$ lie on one line if and only if

$$\frac{\overline{BA_1}}{\overline{CA_1}} \cdot \frac{\overline{CB_1}}{\overline{AB_1}} \cdot \frac{\overline{AC_1}}{\overline{BC_1}} = 1. \qquad\qquad (\textit{Menelaus's theorem})$$

**5.59.** Prove Problem 5.85 a) with the help of Menelaus's theorem.

**5.60.** A circle $S$ is tangent to circles $S_1$ and $S_2$ at points $A_1$ and $A_2$, respectively. Prove that line $A_1A_2$ passes through the intersection point of either common outer or common inner tangents to circles $S_1$ and $S_2$.

**5.61.** a) The midperpendicular to the bisector $AD$ of triangle $ABC$ intersects line $BC$ at point $E$. Prove that $BE : CE = c^2 : b^2$.

b) Prove that the intersection point of the midperpendiculars to the bisectors of a triangle and the extensions of the corresponding sides lie on one line.

**5.62.** From vertex $C$ of the right angle of triangle $ABC$ height $CK$ is dropped and in triangle $ACK$ bisector $CE$ is drawn. Line that passes through point $B$ parallel to $CE$ meets $CK$ at point $F$. Prove that line $EF$ divides segment $AC$ in halves.

**5.63.** On lines $BC$, $CA$ and $AB$ points $A_1$, $B_1$ and $C_1$, respectively, are taken so that points $A_1$, $B_1$ and $C_1$ lie on one line. The lines symmetric to lines $AA_1$, $BB_1$ and $CC_1$ through the corresponding bisectors of triangle $ABC$ meet lines $BC$, $CA$ and $AB$ at points $A_2$, $B_2$ and $C_2$, respectively. Prove that points $A_2$, $B_2$ and $C_2$ lie on one line.

* * *

**5.64.** Lines $AA_1$, $BB_1$ and $CC_1$ meet at one point, $O$. Prove that the intersection points of lines $AB$ and $A_1B_1$, $BC$ and $B_1C_1$, $AC$ and $A_1C_1$ lie on one line. (*Desargues's theorem.*)

**5.65.** Points $A_1$, $B_1$ and $C_1$ are taken on one line and points $A_2$, $B_2$ and $C_2$ are taken on another line. The intersection pointa of lines $A_1B_2$ with $A_2B_1$, $B_1C_2$ with $B_2C_1$ and $C_1A_2$ with $C_2A_1$ are $C$, $A$ and $B$, respectively. Prove that points $A$, $B$ and $C$ lie on one line. (*Pappus' theorem.*)

**5.66.** On sides $AB$, $BC$ and $CD$ of quadrilateral $ABCD$ (or on their extensions) points $K$, $L$ and $M$ are taken. Lines $KL$ and $AC$ meet at point $P$, lines $LM$ and $BD$ meet at point $Q$. Prove that the intersection point of lines $KQ$ and $MP$ lies on line $AD$.

**5.67.** The extensions of sides $AB$ and $CD$ of quadrilateral $ABCD$ meet at point $P$ and the extensions of sides $BC$ and $AD$ meet at point $Q$. Through point $P$ a line is drawn that intersects sides $BC$ and $AD$ at points $E$ and $F$. Prove that the intersection points of the diagonals of quadrilaterals $ABCD$, $ABEF$ and $CDFE$ lie on the line that passes through point $Q$.

**5.68.** a) Through points $P$ and $Q$ triples of lines are drawn. Let us denote their intersection points as shown on Fig. 56. Prove that lines $KL$, $AC$ and $MN$ either meet at one point or are parallel.
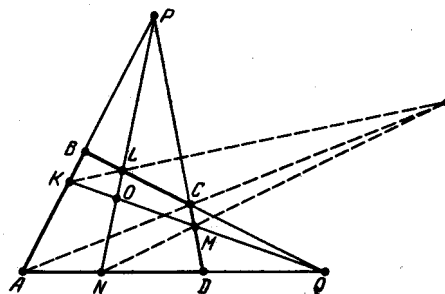


Figure 56 (5.68)

b) Prove further that if point $O$ lies on line $BD$, then the intersection point of lines $KL$, $AC$ and $MN$ lies on line $PQ$.

**5.69.** On lines $BC$, $CA$ and $AB$ points $A_1$, $B_1$ and $C_1$ are taken. Let $P_1$ be an arbitrary point of line $BC$, let $P_2$ be the intersection point of lines $P_1B_1$ and $AB$, let $P_3$ be the

intersection point of lines $P_2A_1$ and $CA$, let $P_4$ be the intersection point of $P_3C_1$ and $BC$, etc. Prove that points $P_7$ and $P_1$ coincide.

See also Problem 6.98.

## §8. Ceva's theorem

**5.70.** Triangle $ABC$ is given and on lines $AB$, $BC$ and $CA$ points $C_1$, $A_1$ and $B_1$, respectively, are taken so that $k$ of them lie on sides of the triangle and $3 - k$ on the extensions of the sides. Let
$$R = \frac{BA_1}{CA_1} \cdot \frac{CB_1}{AB_1} \cdot \frac{AC_1}{BC_1}.$$
Prove that

a) points $A_1$, $B_1$ and $C_1$ lie on one line if and only if $R = 1$ and $k$ is even. (*Menelaus's theorem.*)

b) lines $AA_1$, $BB_1$ and $CC_1$ either meet at one point or are parallel if and only if $R = 1$ and $k$ is odd. (*Ceva's theorem.*)

**5.71.** The inscribed (or an escribed) circle of triangle $ABC$ is tangent to lines $BC$, $CA$ and $AB$ at points $A_1$, $B_1$ and $C_1$, respectively. Prove that lines $AA_1$, $BB_1$ and $CC_1$ meet at one point.

**5.72.** Prove that the heights of an acute triangle intersect at one point.

**5.73.** Lines $AP, BP$ and $CP$ meet the sides of triangle $ABC$ (or their extensions) at points $A_1$, $B_1$ and $C_1$, respectively. Prove that:

a) lines that pass through the midpoints of sides $BC$, $CA$ and $AB$ parallel to lines $AP$, $BP$ and $CP$, respectively, meet at one point;

b) lines that connect the midpoints of sides $BC$, $CA$ and $AB$ with the midpoints of segments $AA_1$, $BB_1$, $CC_1$, respectively, meet at one point.

**5.74.** On sides $BC$, $CA$, and $AB$ of triangle $ABC$, points $A_1$, $B_1$ and $C_1$ are taken so that segments $AA_1$, $BB_1$ and $CC_1$ meet at one point. Lines $A_1B_1$ and $A_1C_1$ meet the line that passes through vertex $A$ parallel to side $BC$ at points $C_2$ and $B_2$, respectively. Prove that $AB_2 = AC_2$.

**5.75.** a) Let $\alpha$, $\beta$ and $\gamma$ be arbitrary angles such that the sum of any two of them is not less than $180°$. On sides of triangle $ABC$, triangles $A_1BC$, $AB_1C$ and $ABC_1$ with angles at vertices $A$, $B$, and $C$ equal to $\alpha$, $\beta$ and $\gamma$, respectively, are constructed outwards. Prove that lines $AA_1$, $BB_1$ and $CC_1$ meet at one point.

b) Prove a similar statement for triangles constructed on sides of triangle $ABC$ inwards.

**5.76.** Sides $BC$, $CA$ and $AB$ of triangle $ABC$ are tangent to a circle centered at $O$ at points $A_1$, $B_1$ and $C_1$. On rays $OA_1$, $OB_1$ and $OC_1$ equal segments $OA_2$, $OB_2$ and $OC_2$ are marked. Prove that lines $AA_2$, $BB_2$ and $CC_2$ meet at one point.

**5.77.** Lines $AB$, $BP$ and $CP$ meet lines $BC$, $CA$ and $AB$ at points $A_1$, $B_1$ and $C_1$, respectively. Points $A_2$, $B_2$ and $C_2$ are selected on lines $BC$, $CA$ and $AB$ so that
$$\overline{BA_2} : \overline{A_2C} = \overline{A_1C} : \overline{BA_1},$$
$$\overline{CB_2} : \overline{B_2A} = \overline{B_1A} : \overline{CB_1},$$
$$\overline{AC_2} : \overline{C_2B} = \overline{C_1B} : \overline{AC_1}.$$
Prove that lines $AA_2$, $BB_2$ and $CC_2$ also meet at one point, $Q$ (or are parallel).

Such points $P$ and $Q$ are called *isotomically conjugate with respect to triangle $ABC$*.

**5.78.** On sides $BC$, $CA$, $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ are taken so that lines $AA_1$, $BB_1$ and $CC_1$ intersect at one point, $P$. Prove that lines $AA_2, BB_2$ and $CC_2$ symmetric to these lines through the corresponding bisectors also intersect at one point, $Q$.

Such points $P$ and $Q$ are called *isogonally conjugate with respect to triangle $ABC$*.

**5.80.** The opposite sides of a convex hexagon are pairwise parallel. Prove that the lines that connect the midpoints of opposite sides intersect at one point.

**5.81.** From a point $P$ perpendiculars $PA_1$ and $PA_2$ are dropped to side $BC$ of triangle $ABC$ and to height $AA_3$. Points $B_1$, $B_2$ and $C_1$, $C_2$ are similarly defined. Prove that lines $A_1A_2$, $B_1B_2$ and $C_1C_2$ either meet at one point or are parallel.

**5.82.** Through points $A$ and $D$ lying on a circle tangents that intersect at point $S$ are drawn. On arc $\smile AD$ points $B$ and $C$ are taken. Lines $AC$ and $BD$ meet at point $P$, lines $AB$ and $CD$ meet at point $Q$. Prove that line $PQ$ passes through point $S$.

**5.83.** a) On sides $BC$, $CA$ and $AB$ of an isosceles triangle $ABC$ with base $AB$, points $A_1$, $B_1$ and $C_1$, respectively, are taken so that lines $AA_1$, $BB_1$ and $CC_1$ meet at one point. Prove that
$$\frac{AC_1}{C_1B} = \frac{\sin \angle ABB_1 \cdot \sin \angle CAA_1}{\sin \angle BAA_1 \cdot \sin \angle CBB_1}.$$

b) Inside an isosceles triangle $ABC$ with base $AB$ points $M$ and $N$ are taken so that $\angle CAM = \angle ABN$ and $\angle CBM = \angle BAN$. Prove that points $C$, $M$ and $N$ lie on one line.

**5.84.** In triangle $ABC$ bisectors $AA_1$, $BB_1$ and $CC_1$ are drawn. Bisectors $AA_1$ and $CC_1$ intersect segments $C_1B_1$ and $B_1A_1$ at points $M$ and $N$, respectively. Prove that $\angle MBB_1 = \angle NBB_1$.

See also Problems 10.56, 14.7, 14.38.

## §9. Simson's line

**5.85.** a) Prove that the bases of the perpendiculars dropped from a point $P$ of the circumscribed circle of a triangle to the sides of the triangle or to their extensions lie on one line.

This line is called *Simson's line* of point $P$ with respect to the triangle.

b) The bases of perpendiculars dropped from a point $P$ to the sides (or their extensions) of a triangle lie on one line. Prove that point $P$ lies on the circumscribed circle of the triangle.

**5.86.** Points $A$, $B$ and $C$ lie on one line, point $P$ lies outside this line. Prove that the centers of the circumscribed circles of triangles $ABP$, $BCP$, $ACP$ and point $P$ lie on one circle.

**5.87.** In triangle $ABC$ the bisector $AD$ is drawn and from point $D$ perpendiculars $DB'$ and $DC'$ are dropped to lines $AC$ and $AB$, respectively; point $M$ lies on line $B'C'$ and $DM \perp BC$. Prove that point $M$ lies on median $AA_1$.

**5.88.** a) From point $P$ of the circumscribed circle of triangle $ABC$ lines $PA_1$, $PB_1$ and $PC_1$ are drawn at a given (oriented) angle $\alpha$ to lines $BC$, $CA$ and $AB$, respectively, so that points $A_1$, $B_1$ and $C_1$ lie on lines $BC$, $CA$ and $AB$, respectively. Prove that points $A_1$, $B_1$ and $C_1$ lie on one line.

b) Prove that if in the definition of Simson's line we replace the angle $90°$ by an angle $\alpha$, i.e., replace the perpendiculars with the lines that form angles of $\alpha$, their intersection points with the sides lie on the line and the angle between this line and Simson's line becomes equal to $90° - \alpha$.

**5.89.** a) From a point $P$ of the circumscribed circle of triangle $ABC$ perpendiculars $PA_1$ and $PB_1$ are dropped to lines $BC$ and $AC$, respectively. Prove that $PA \cdot PA_1 = 2Rd$, where $R$ is the radius of the circumscribed circle, $d$ the distance from point $P$ to line $A_1B_1$.

b) Let $\alpha$ be the angle between lines $A_1B_1$ and $BC$. Prove that $\cos \alpha = \frac{PA}{2R}$.

**5.90.** Let $A_1$ and $B_1$ be the projections of point $P$ of the circumscribed circle of triangle $ABC$ to lines $BC$ and $AC$, respectively. Prove that the length of segment $A_1B_1$ is equal to the length of the projection of segment $AB$ to line $A_1B_1$.

**5.91.** Points $P$ and $C$ on a circle are fixed; points $A$ and $B$ move along the circle so that angle $\angle ACB$ remains fixed. Prove that Simson's lines of point $P$ with respect to triangle $ABC$ are tangent to a fixed circle.

**5.92.** Point $P$ moves along the circumscribed circle of triangle $ABC$. Prove that Simson's line of point $P$ with respect to triangle $ABC$ rotates accordingly through the angle equal to a half the angle value of the arc circumvent by $P$.

**5.93.** Prove that Simson's lines of two diametrically opposite points of the circumscribed circle of triangle $ABC$ are perpendicular and their intersection point lies on the circle of 9 points, cf. Problem 5.106.

**5.94.** Points $A$, $B$, $C$, $P$ and $Q$ lie on a circle centered at $O$ and the angles between vector $\overrightarrow{OP}$ and vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OQ}$ are equal to $\alpha$, $\beta$, $\gamma$ and $\frac{1}{2}(\alpha + \beta + \gamma)$, respectively. Prove that Simson's line of point $P$ with respect to triangle $ABC$ is parallel to $OQ$.

**5.95.** Chord $PQ$ of the circumscribed circle of triangle $ABC$ is perpendicular to side $BC$. Prove that Simson's line of point $P$ with respect to triangle $ABC$ is parallel to line $AQ$.

**5.96.** The heights of triangle $ABC$ intersect at point $H$; let $P$ be a point of its circumscribed circle. Prove that Simson's line of point $P$ with respect to triangle $ABC$ divides segment $PH$ in halves.

**5.97.** Quadrilateral $ABCD$ is inscribed in a circle; $l_a$ is Simson's line of point $A$ with respect to triangle $BCD$; let lines $l_b$, $l_c$ and $l_d$ be similarly defined. Prove that these lines intersect at one point.

**5.98.** a) Prove that the projection of point $P$ of the circumscribed circle of quadrilateral $ABCD$ onto Simson's lines of this point with respect to triangles $BCD$, $CDA$, $DAB$ and $BAC$ lie on one line. (*Simson's line of the inscribed quadrilateral.*)

b) Prove that by induction we can similarly define Simson's line of an inscribed $n$-gon as the line that contains the projections of a point $P$ on Simson's lines of all $(n-1)$-gons obtained by deleting one of the vertices of the $n$-gon.

See also Problems 5.10, 5.59.

## §10. The pedal triangle

Let $A_1$, $B_1$ and $C_1$ be the bases of the perpendiculars dropped from point $P$ to lines $BC$, $CA$ and $AB$, respectively. Triangle $A_1B_1C_1$ is called the *pedal* triangle of point $P$ with respect to triangle $ABC$.

**5.99.** Let $A_1B_1C_1$ be the pedal triangle of point $P$ with respect to triangle $ABC$. Prove that $B_1C_1 = \frac{BC \cdot AP}{2R}$, where $R$ is the radius of the circumscribed circle of triangle $ABC$.

**5.100.** Lines $AP, BP$ and $CP$ intersect the circumscribed circle of triangle $ABC$ at points $A_2, B_2$ and $C_2$; let $A_1B_1C_1$ be the pedal triangle of point $P$ with respect to triangle $ABC$. Prove that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

**5.101.** Inside an acute triangle $ABC$ a point $P$ is given. If we drop from it perpendiculars $PA_1$, $PB_1$ and $PC_1$ to the sides, we get $\triangle A_1B_1C_1$. Performing for $\triangle A_1B_1C_1$ the same operation we get $\triangle A_2B_2C_2$ and then we similarly get $\triangle A_3B_3C_3$. Prove that $\triangle A_3B_3C_3 \sim \triangle ABC$.

**5.102.** A triangle $ABC$ is inscribed in the circle of radius $R$ centered at $O$. Prove that the area of the pedal triangle of point $P$ with respect to triangle $ABC$ is equal to $\frac{1}{4}\left|1 - \frac{d^2}{R^2}\right| S_{ABC}$, where $d = |PO|$.

**5.103.** From point $P$ perpendiculars $PA_1$, $PB_1$ and $PC_1$ are dropped on sides of triangle $ABC$. Line $l_a$ connects the midpoints of segments $PA$ and $B_1C_1$. Lines $l_b$ and $l_c$ are similarly defined. Prove that $l_a$, $l_b$ and $l_c$ meet at one point.

**5.104.** a) Points $P_1$ and $P_2$ are isogonally conjugate with respect to triangle $ABC$, cf. Problem 5.79. Prove that their pedal triangles have a common circumscribed circle whose center is the midpoint of segment $P_1P_2$.

b) Prove that the above statement remains true if instead of perpendiculars we draw from points $P_1$ and $P_2$ lines forming a given (oriented) angle to the sides.

See also Problems 5.132, 5.133, 14.19 b).

## §11. Euler's line and the circle of nine points

**5.105.** Let $H$ be the point of intersection of heights of triangle $ABC$, $O$ the center of the circumscribed circle and $M$ the point of intersection of medians. Prove that point $M$ lies on segment $OH$ and $OM : MH = 1 : 2$.

The line that contains points $O$, $M$ and $H$ is called *Euler's line.*

**5.106.** Prove that the midpoints of sides of a triangle, the bases of heights and the midpoints of segments that connect the intersection point of heights with the vertices lie on one circle and the center of this circle is the midpoint of segment $OH$.

The circle defined above is called the *circle of nine points.*

**5.107.** The heights of triangle $ABC$ meet at point $H$.

a) Prove that triangles $ABC$, $HBC$, $AHC$ and $ABH$ have a common circle of 9 points.

b) Prove that Euler's lines of triangles $ABC$, $HBC$, $AHC$ and $ABH$ intersect at one point.

c) Prove that the centers of the circumscribed circles of triangles $ABC$, $HBC$, $AHC$ and $ABH$ constitute a quadrilateral symmetric to quadrilateral $HABC$.

**5.108.** What are the sides the Euler line intersects in an acute and an obtuse triangles?

**5.109.** a) Prove that the circumscribed circle of triangle $ABC$ is the circle of 9 points for the triangle whose vertices are the centers of escribed circles of triangle $ABC$.

b) Prove that the circumscribed circle divides the segment that connects the centers of the inscribed and an escribed circles in halves.

**5.110.** Prove that Euler's line of triangle $ABC$ is parallel to side $BC$ if and only if $\tan B \tan C = 3$.

**5.111.** On side $AB$ of acute triangle $ABC$ the circle of 9 points singles out a segment. Prove that the segment subtends an angle of $2|\angle A - \angle B|$ with the vertex at the center.

**5.112.** Prove that if Euler's line passes through the center of the inscribed circle of a triangle, then the triangle is an isosceles one.

**5.113.** The inscribed circle is tangent to the sides of triangle $ABC$ at points $A_1$, $B_1$ and $C_1$. Prove that Euler's line of triangle $A_1B_1C_1$ passes through the center of the circumscribed circle of triangle $ABC$.

**5.114.** In triangle $ABC$, heights $AA_1$, $BB_1$ and $CC_1$ are drawn. Let $A_1A_2$, $B_1B_2$ and $C_1C_2$ be diameters of the circle of nine points of triangle $ABC$. Prove that lines $AA_2$, $BB_2$ and $CC_2$ either meet at one point or are parallel.

See also Problems 3.65 a), 13.34 b).

## §12. Brokar's points

**5.115.** a) Prove that inside triangle $ABC$ there exists a point $P$ such that $\angle ABP = \angle CAP = \angle BCP$.

b) On sides of triangle $ABC$, triangles $CA_1B$, $CAB_1$ and $C_1AB$ similar to $ABC$ are constructed outwards (the angles at the first vertices of all the four triangles are equal, etc.). Prove that lines $AA_1$, $BB_1$ and $CC_1$ meet at one point and this point coincides with the point found in heading a).

This point $P$ is called *Brokar's point* of triangle $ABC$. The proof of the fact that there exists *another* Brokar's point $Q$ for which $\angle BAQ = \angle ACQ = \angle CBQ$ is similar to the proof of existence of $P$ given in what follows. We will refer to $P$ and $Q$ as the *first* and the *second* Brokar's points.

**5.116.** a) Through Brokar's point $P$ of triangle $ABC$ lines $AB$, $BP$ and $CP$ are drawn. They intersect the circumscribed circle at points $A_1$, $B_1$ and $C_1$, respectively. Prove that $\triangle ABC = \triangle B_1C_1A_1$.

b) Triangle $ABC$ is inscribed into circle $S$. Prove that the triangle formed by the intersection points of lines $PA$, $PB$ and $PC$ with circle $S$ can be equal to triangle $ABC$ for no more than 8 distinct points $P$. (We suppose that the intersection points of lines $PA$, $PB$ and $PC$ with the circle are distinct from points $A$, $B$ and $C$.)

**5.117.** a) Let $P$ be Brokar's point of triangle $ABC$. Let $\varphi = \angle ABP = \angle BCP = \angle CAP$. Prove that $\cot \varphi = \cot \alpha + \cot \beta + \cot \gamma$.

The angle $\varphi$ from Problem 5.117 is called *Brokar's angle* of triangle $ABC$.

b) Prove that Brokar's points of triangle $ABC$ are isogonally conjugate to each other (cf. Problem 5.79).

c) The tangent to the circumscribed circle of triangle $ABC$ at point $C$ and the line passing through point $B$ parallel to $AC$ intersect at point $A_1$. Prove that Brokar's angle of triangle $ABC$ is equal to angle $\angle A_1AC$.

**5.118.** a) Prove that Brokar's angle of any triangle does not exceed $30°$.

b) Inside triangle $ABC$, point $M$ is taken. Prove that one of the angles $\angle ABM$, $\angle BCM$ and $\angle CAM$ does not exceed $30°$.

**5.119.** Let $Q$ be the second Brokar's point of triangle $ABC$, let $O$ be the center of its circumscribed circle; $A_1$, $B_1$ and $C_1$ the centers of the circumscribed circles of triangles $CAQ$, $ABQ$ and $BCQ$, respectively. Prove that $\triangle A_1B_1C_1 \sim \triangle ABC$ and $O$ is the first Brokar's point of triangle $A_1B_1C_1$.

**5.120.** Let $P$ be Brokar's point of triangle $ABC$; let $R_1$, $R_2$ and $R_3$ be the radii of the circumscribed circles of triangles $ABP$, $BCP$ and $CAP$, respectively. Prove that $R_1R_2R_3 = R^3$, where $R$ is the radius of the circumscribed circle of triangle $ABC$.

**5.121.** Let $P$ and $Q$ be the first and the second Brokar's points of triangle $ABC$. Lines $CP$ and $BQ$, $AP$ and $CQ$, $BP$ and $AQ$ meet at points $A_1$, $B_1$ and $C_1$, respectively. Prove that the circumscribed circle of triangle $A_1B_1C_1$ passes through points $P$ and $Q$.

**5.122.** On sides $CA$, $AB$ and $BC$ of an acute triangle $ABC$ points $A_1$, $B_1$ and $C_1$, respectively, are taken so that $\angle AB_1A_1 = \angle BC_1B_1 = \angle CA_1C_1$. Prove that $\triangle A_1B_1C_1 \sim \triangle ABC$ and the center of the rotational homothety that sends one triangle into another coincides with the first Brokar's point of both triangles.

See also Problem 19.55.

## §13. Lemoine's point

Let $AM$ be a median of triangle $ABC$ and line $AS$ be symmetric to line $AM$ through the bisector of angle $A$ (point $S$ lies on segment $BC$). Then segment $AS$ is called a *simedian* of triangle $ABC$; sometimes the whole ray $AS$ is referred to as a simedian.

Simedians of a triangle meet at the point *isogonally conjugate* to the intersection point of medians (cf. Problem 5.79). The intersection point of simedians of a triangle is called *Lemoine's point.*

**5.123.** Let lines $AM$ and $AN$ be symmetric through the bisector of angle $\angle A$ of triangle $ABC$ (points $M$ and $N$ lie on line $BC$). Prove that $\frac{BM \cdot BN}{CM \cdot CN} = \frac{c^2}{b^2}$. In particular, if $AS$ is a simedian, then $\frac{BS}{CS} = \frac{c^2}{b^2}$.

**5.124.** Express the length of simedian $AS$ in terms of the lengths of sides of triangle $ABC$.

Segment $B_1C_1$, where points $B_1$ and $C_1$ lie on rays $AC$ and $AB$, respectively, is said to be *antiparallel* to side $BC$ if $\angle AB_1C_1 = \angle ABC$ and $\angle AC_1B_1 = \angle ACB$.

**5.125.** Prove that simedian $AS$ divides any segment $B_1C_1$ antiparallel to side $BC$ in halves.

**5.126.** The tangent at point $B$ to the circumscribed circle $S$ of triangle $ABC$ intersects line $AC$ at point $K$. From point $K$ another tangent $KD$ to circle $S$ is drawn. Prove that $BD$ is a simedian of triangle $ABC$.

**5.127.** Tangents to the circumscribed circle of triangle $ABC$ at points $B$ and $C$ meet at point $P$. Prove that line $AP$ contains simedian $AS$.

**5.128.** Circle $S_1$ passes through points $A$ and $B$ and is tangent to line $AC$, circle $S_2$ passes through points $A$ and $C$ and is tangent to line $AB$. Prove that the common chord of these circles is a simedian of triangle $ABC$.

**5.129.** Bisectors of the outer and inner angles at vertex $A$ of triangle $ABC$ intersect line $BC$ at points $D$ and $E$, respectively. The circle with diameter $DE$ intersects the circumscribed circle of triangle $ABC$ at points $A$ and $X$. Prove that $AX$ is a simedian of triangle $ABC$.

* * *

**5.130.** Prove that Lemoine's point of right triangle $ABC$ with right angle $\angle C$ is the midpoint of height $CH$.

**5.131.** Through a point $X$ inside triangle $ABC$ three segments antiparallel to its sides are drawn, cf. Problem 5.125?. Prove that these segments are equal if and only if $X$ is Lemoine's point.

**5.132.** Let $A_1$, $B_1$ and $C_1$ be the projections of Lemoine's point $K$ to the sides of triangle $ABC$. Prove that $K$ is the intersection point of medians of triangle $A_1B_1C_1$.

**5.133.** Let $A_1$, $B_1$ and $C_1$ be the projections of Lemoine's point $K$ of triangle $ABC$ on sides $BC$, $CA$ and $AB$, respectively. Prove that median $AM$ of triangle $ABC$ is perpendicular to line $B_1C_1$.

**5.134.** Lines $AK$, $BK$ and $CK$, where $K$ is Lemoine's point of triangle $ABC$, intersect the circumscribed circle at points $A_1$, $B_1$ and $C_1$, respectively. Prove that $K$ is Lemoine's point of triangle $A_1B_1C_1$.

**5.135.** Prove that lines that connect the midpoints of the sides of a triangle with the midpoints of the corresponding heights intersect at Lemoine's point.

See also Problems 11.22, 19.54, 19.55.

## Problems for independent study

**5.136.** Prove that the projection of the diameter of a circumscribed circle perpendicular to a side of the triangle to the line that contains the second side is equal to the third side.

**5.137.** Prove that the area of the triangle with vertices in the centers of the escribed circles of triangle $ABC$ is equal to $2pR$.

**5.138.** An isosceles triangle with base $a$ and the lateral side $b$, and an isosceles triangle with base $b$ and the lateral side $a$ are inscribed in a circle of radius $R$. Prove that if $a \neq b$, then $ab = \sqrt{5}R^2$.

**5.139.** The inscribed circle of right triangle $ABC$ is tangent to the hypothenuse $AB$ at point $P$; let $CH$ be a height of triangle $ABC$. Prove that the center of the inscribed circle of triangle $ACH$ lies on the perpendicular dropped from point $P$ to $AC$.

**5.140.** The inscribed circle of triangle $ABC$ is tangent to sides $CA$ and $AB$ at points $B_1$ and $C_1$, respectively, and an escribed circle is tangent to the extension of sides at points $B_2$ and $C_2$. Prove that the midpoint of side $BC$ is equidistant from lines $B_1C_1$ and $B_2C_2$.

**5.141.** In triangle $ABC$, bisector $AD$ is drawn. Let $O$, $O_1$ and $O_2$ be the centers of the circumscribed circles of triangles $ABC$, $ABD$ and $ACD$, respectively. Prove that $OO_1 = OO_2$.

**5.142.** The triangle constructed from a) medians, b) heights of triangle $ABC$ is similar to triangle $ABC$. What is the ratio of the lengths of the sides of triangle $ABC$?

**5.143.** Through the center $O$ of an equilateral triangle $ABC$ a line is drawn. It intersects lines $BC$, $CA$ and $AB$ at points $A_1$, $B_1$ and $C_1$, respectively. Prove that one of the numbers $\frac{1}{OA_1}$, $\frac{1}{OB_1}$ and $\frac{1}{OC_1}$ is equal to the sum of the other two numbers.

**5.144.** In triangle $ABC$ heights $BB_1$ and $CC_1$ are drawn. Prove that if $\angle A = 45°$, then $B_1C_1$ is a diameter of the circle of nine points of triangle $ABC$.

**5.145.** The angles of triangle $ABC$ satisfy the relation $\sin^2 \angle A + \sin^2 \angle B + \sin^2 \angle C = 1$. Prove that the circumscribed circle and the circle of nine points of triangle $ABC$ intersect at a right angle.

## Solutions

**5.1.** Let $AC_1 = AB_1 = x$, $BA_1 = BC_1 = y$ and $CA_1 = CB_1 = z$. Then

$$a = y + z, \quad b = z + x \quad \text{and} \quad c = x + y.$$

Subtracting the third equality from the sum of the first two ones we get $z = \frac{a+b-c}{2}$. Hence, if triangle $ABC$ is given, then the position of points $A_1$ and $B_1$ is uniquely determined. Similarly, the position of point $C_1$ is also uniquely determined. It remains to notice that the tangency points of the inscribed circle with the sides of the triangle satisfy the relations indicated in the hypothesis of the problem.

**5.2.** Rays $CO_a$ and $CO_b$ are the bisectors of the outer angles at vertex $C$, hence, $C$ lies on line $O_aO_b$ and $\angle O_aCB = \angle O_bCA$. Since $CO_c$ is the bisector of angle $\angle BCA$, it follows that $\angle BCO_c = \angle ACO_c$. Adding these equalities we get: $\angle O_aCO_c = \angle O_cCO_b$, i.e., $O_cC$ is a height of triangle $O_aO_bO_c$. We similarly prove that $O_aA$ and $O_bB$ are heights of this triangle.

**5.3.** Clearly,

$$\angle BOC = 180° - \angle CBO - \angle BCO = 180° - \frac{\angle B}{2} - \frac{\angle C}{2} = 90° + \frac{\angle A}{2}$$

and $\angle BO_aC = 180° - \angle BOC$, because $\angle OBO_a = \angle OCO_a = 90°$.

**5.4.** Let $AA_1$, $BB_1$ and $CC_1$ be the bisectors of triangle $ABC$ and $O$ the intersection point of these bisectors. Suppose that $x > 1$. Then $\angle PAB > \angle PAC$, i.e., point $P$ lies

inside triangle $AA_1C$. Similarly, point $P$ lies inside triangles $CC_1B$ and $BB_1A$. But the only common point of these three triangles is point $O$. Contradiction. The case $x < 1$ is similarly treated.

**5.5.** Let $d_a$, $d_b$ and $d_c$ be the distances from point $O$ to sides $BC$, $CA$ and $AB$. Then $ad_a + bd_b + cd_c = 2S$ and $ah_a = bh_b = ch_c = 2S$. If $h_a - d_a = h_b - d_b = h_c - d_c = x$, then

$$(a + b + c)x = a(h_a - d_a) = b(h_b - d_b) + c(h_c - d_c) = 6S - 2S = 4S.$$

Hence, $x = \frac{4S}{2p} = 2r$.

**5.6.** Let us prove that point $O$ is the center of the escribed circle of triangle $PBQ$ tangent to side $PQ$. Indeed, $\angle POQ = \angle A = 90° - \frac{1}{2}\angle B$. The angle of the same value with the vertex at the center of the escribed circle subtends segment $PQ$ (Problem 5.3). Moreover, point $O$ lies on the bisector of angle $B$. Hence, the semiperimeter of triangle $PBQ$ is equal to the length of the projection of segment $OB$ to line $CB$.

**5.7.** Let $P$ be the tangent point of the inscribed circle with side $BC$, let $PQ$ be a diameter of the inscribed circle, $R$ the intersection point of lines $AQ$ and $BC$. Since $CR = BP$ (cf. Problem 19.11 a)) and $M$ is the midpoint of side $BC$, we have: $RM = PM$. Moreover, $O$ is the midpoint of diameter $PQ$, hence, $MO \parallel QR$ and since $AH \parallel PQ$, we have $AE = OQ$.

**5.8.** The given circle can be the inscribed as well as the escribed circle of triangle $ABC$ cut off by the tangent from the angle. Making use of the result of Problem 3.2 we can verify that in either case

$$\frac{uv}{w^2} = \frac{(p - b)(p - c) \sin \angle B \sin \angle C}{h_a^2}.$$

It remains to notice that $h_a = b \sin \angle C = c \sin \angle B$ and $\frac{(p-b)(p-c)}{bc} = \sin^2 \frac{1}{2}\angle A$ (Problem 12.13).

**5.9.** Let $A_1$, $B_1$ and $C_1$ be points symmetric to point $H$ through sides $BC$, $CA$ and $AB$, respectively. Since $AB \perp CH$ and $BC \perp AH$, it follows that $\angle(AB, BC) = \angle(CH, HA)$ and since triangle $AC_1H$ is an isosceles one, $\angle(CH, HA) = \angle(AC_1, C_1C)$. Hence, $\angle(AB, BC) = \angle(AC_1, C_1C)$, i.e., point $C_1$ lies on the circumscribed circle of triangle $ABC$. We similarly prove that points $A_1$ and $B_1$ lie on this same circle.

**5.10.** Let $R$ be the radius of the circumscribed circle of triangle $ABC$. This circle is also the circumscribed circle of triangles $ABP$, $APC$ and $PBC$. Clearly, $\angle ABP = 180° - \angle ACP = \alpha$, $\angle BAP = \angle BCP = \beta$ and $\angle CAP = \angle CBP = \gamma$. Hence,

$$PX = PB \sin \gamma = 2R \sin \beta \sin \gamma, \ PY = 2R \sin \alpha \sin \gamma \text{ and } P = 2R \sin \alpha \sin \beta.$$

It is also clear that

$$BC = 2R \sin \angle BAC = 2R \sin(\beta + \gamma), \ AC = 2R \sin(\alpha - \gamma), \ AB = 2R \sin(\alpha + \beta).$$

It remains to verify the equality

$$\frac{\sin(\beta + \gamma)}{\sin \beta \sin \gamma} = \frac{\sin(\alpha - \gamma)}{\sin \alpha \sin \gamma} + \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}$$

which is subject to a direct calculation.

**5.11.** a) Let $M$ be the intersection point of line $AI$ with the circumscribed circle. Drawing the diameter through point $I$ we get

$$AI \cdot IM = (R + d)(R - d) = R^2 - d^2.$$

Since $IM = CM$ (by Problem 2.4 a)), it follows that $R^2 - d^2 = AI \cdot CM$. It remains to observe that $AI = \frac{r}{\sin \frac{1}{2}\angle A}$ and $CM = 2R \sin \frac{1}{2}\angle A$.

b) Let $M$ be the intersection point of line $AI_a$ with the circumscribed circle. Then $AI_a \cdot I_a M = d_a^2 - R^2$. Since $I_a M = CM$ (by Problem 2.4 a)), it follows that $d_a^2 - R^2 = AI_a \cdot CM$. It remains to notice that $AI_a = \frac{r_a}{\sin \frac{1}{2} \angle A}$ and $CM = 2R \sin \frac{1}{2} \angle A$.

**5.12.** a) Since $B_1$ is the center of the circumscribed circle of triangle $AMC$ (cf. Problem 2.4 a)), $AM = 2MB_1 \sin \angle ACM$. It is also clear that $MC = \frac{r}{\sin \angle ACM}$. Hence, $\frac{MA \cdot MC}{MB_1} = 2r$.

b) Since

$$\angle MBC_1 = \angle BMC_1 = 180° - \angle BMC \quad \text{and} \quad \angle BC_1 M = \angle A,$$

it follows that

$$\frac{MC_1}{BC} = \frac{BM}{BC} \cdot \frac{MC_1}{BM} = \frac{\sin \angle BCM}{\sin \angle BMC} \cdot \frac{\sin \angle MBC_1}{\sin \angle BC_1 M} = \frac{\sin \angle BCM}{\sin \angle A}.$$

Moreover, $MB = 2MA_1 \sin \angle BCM$. Therefore, $\frac{MC_1 \cdot MA_1}{MB} = \frac{BC}{2 \sin \angle A} = R$.

**5.13.** Let $M$ be the midpoint of side $AC$, and $N$ the tangent point of the inscribed circle with side $BC$. Then $BN = p - b$ (see Problem 3.2), hence, $BN = AM$ because $p = \frac{3}{2}b$ by assumption. Moreover, $\angle OBN = \angle B_1 AM$ and, therefore, $\triangle OBN = \triangle B_1 AM$, i.e., $OB = B_1 A$. But $B_1 A = B_1 O$ (see Problem 2.4 a)).

**5.14.** Let $O$ and $O_1$ be the centers of the inscribed and circumscribed circles of triangle $ABC$. Let us consider the circle of radius $d = OO_1$ centered at $O$. In this circle, let us draw chords $O_1 M$ and $O_1 N$ parallel to sides $AB$ and $AC$, respectively. Let $K$ be the tangent point of the inscribed circle with side $AB$ and $L$ the midpoint of side $AB$. Since $OK \perp AB$, $O_1 L \perp AB$ and $O_1 M \parallel AB$, it follows that

$$O_1 M = 2KL = 2BL - 2BK = c - (a + c - b) = b - a = AE.$$

Similarly, $O_1 N = AD$ and, therefore, $\triangle MO_1 N = \triangle EAD$. Consequently, the radius of the circumscribed circle of triangle $EAD$ is equal to $d$.

**5.15.** Let the inscribed circle be tangent to side $AC$ at point $K$ and the escribed circle be tangent to the extension of side $AC$ at point $L$. Then $r = CK$ and $r_c = CL$. It remains to make use of the result of Problem 3.2.

**5.16.** Since $\frac{1}{2}AB = AM = BM$, it follows that $CM = \frac{1}{2}AB$ if and only if point $C$ lies on the circle with diameter $AB$.

**5.17.** Let $M$ and $N$ be the midpoints of sides $AB$ and $CD$. Triangle $APB$ is a right one; hence, $PM = \frac{1}{2}AB$ and $\angle MPA = \angle PAM$ and, therefore, $PM \parallel AD$. Similar arguments show that points $P$, $M$ and $Q$ lie on one line and

$$PQ = PM + MN + NQ = \frac{AB + (BC + AD) + CD}{2}.$$

**5.18.** Let $F$ be the intersection point of lines $DE$ and $BC$; let $K$ be the midpoint of segment $EC$. Segment $CD$ is simultaneosly a bisector and a height of triangle $ECF$, hence, $ED = DF$ and, therefore, $DK \parallel FC$. Median $DK$ of right triangle $EDC$ is twice shorter its hypothenuse $EC$ (Problem 5.16), hence, $AD = DK = \frac{1}{2}EC$.

**5.19.** Let the sum of the angles at the base $AD$ of trapezoid $ABCD$ be equal to 90°. Denote the intersection point of lines $AB$ and $CD$ by $O$. Point $O$ lies on the line that passes through the midpoints of the bases. Let us draw through point $C$ line $CK$ parallel to this line and line $CE$ parallel to line $AB$ (points $K$ and $E$ lie on base $AD$). Then $CK$ is a median of right triangle $ECD$, hence, $CK = \frac{ED}{2} = \frac{AD - BC}{2}$ (cf. Problem 5.16).

**5.20.** It is clear that $\angle CEB = \angle A + \angle ACE = \angle BCK + \angle KCE = \angle BCE$.

**5.21.** Segments $CF$ and $DK$ are bisectors in similar triangles $ACB$ and $CDB$ and, therefore, $AB : FB = CB : KB$. Hence, $FK \parallel AC$. We similarly prove that $LF \parallel CB$.

Therefore, $CLFK$ is a rectangle whose diagonal $CF$ is the bisector of angle $LCK$, i.e., the rectangle is a square.

**5.22.** Since $\frac{\sin \angle ACQ}{AQ} = \frac{\sin \angle AQC}{AC}$, it follows that

$$\frac{\sin \alpha}{a} = \frac{\sin(180° - \alpha - 90° - \varphi)}{a \cos \varphi} = \frac{\cos(\alpha + \varphi)}{a \cos \varphi},$$

where $a$ is the (length of the) side of square $ABPQ$ and $\varphi = \angle CAB$. Hence, $\cot \alpha = 1 + \tan \varphi$. Similarly,

$$\cot \gamma = 1 + \tan(90° - \varphi) = 1 + \cot \varphi.$$

It follows that

$$\tan \alpha + \tan \gamma = \frac{1}{1 + \tan \varphi} + \frac{1}{1 + \cot \varphi} = 1$$

and, therefore,

$$\cos \alpha \cos \gamma = \cos \alpha \sin \gamma + \cos \gamma \sin \alpha = \sin(\alpha + \gamma) = \cos \beta.$$

**5.23.** By Pythagoras theorem

$$AP^2 + BQ^2 + CR^2 + (AM^2 - PM^2) + (BM^2 - QM^2) + (CM^2 - RM^2)$$

and

$$PB^2 + QC^2 + RA^2 = (BM^2 - PM^2) + (CM^2 - QM^2) + (AM^2 - RM^2).$$

These equations are equal.

Since

$$AP^2 + BQ^2 + CR^2 = (a - PB)^2 + (a - QC)^2 + (a - RA)^2 =$$
$$3a^2 - 2a(PB + QC + RA) + PB^2 + QC^2 + RA^2,$$

where $a = AB$, it follows that $PB + QC + RA = \frac{3}{2}a$.

**5.24.** Let point $F$ divide segment $BC$ in the ratio of $CF : FB = 1 : 2$; let $P$ and $Q$ be the intersection points of segment $AF$ with $BD$ and $CE$, respectively. It is clear that triangle $OPQ$ is an equilateral one. Making use of the result of Problem 1.3 it is easy to verify that $AP : PF = 3 : 4$ and $AQ : QF = 6 : 1$. Hence, $AP : PQ : QF = 3 : 3 : 1$ and, therefore, $AP = PQ = OP$. Hence, $\angle AOP = \frac{180° - \angle APO}{2} = 30°$ and $\angle AOC = \angle AOP + \angle POQ = 90°$.

**5.25.** Let $A$ and $B$, $C$ and $D$, $E$ and $F$ be the intersection points of the circle with sides $PQ$, $QR$, $RP$, respectively, of triangle $PQR$. Let us consider median $PS$. It connects the midpoints of parallel chords $FA$ and $DC$ and, therefore, is perpendicular to them. Hence, $PS$ is a height of triangle $PQR$ and, therefore, $PQ = PR$. Similarly, $PQ = QR$.

**5.26.** Let $H$ be the intersection point of heights $AA_1$, $BB_1$ and $CC_1$ of triangle $ABC$. By hypothesis, $A_1H \cdot BH = B_1H \cdot AH$. On the other hand, since points $A_1$ and $B_1$ lie on the circle with diameter $AB$, then $AH \cdot A_1H = BH \cdot B_1H$. It follows that $AH = BH$ and $A_1H = B_1H$ and, therefore, $AC = BC$. Similarly, $BC = AC$.

**5.27.** a) Suppose that triangle $ABC$ is not an equilateral one; for instance, $a \neq b$. Since $a + h_a = a + b \sin \gamma$ and $b + h_b = b + a \sin \gamma$, it follows that $(a - b)(1 - \sin \gamma) = 0$; hence, $\sin \gamma = 0$, i.e., $\gamma = 90°$. But then $a \neq c$ and similar arguments show that $\beta = 90°$. Contradiction.

b) Let us denote the (length of the) side of the square two vertices of which lie on side $BC$ by $x$. The similarity of triangles $ABC$ and $APQ$, where $P$ and $Q$ are the vertices of the square that lie on $AB$ and $AC$, respectively, yields $\frac{x}{a} = \frac{h_a - x}{h_a}$, i.e., $x = \frac{ah_a}{a + h_a} = \frac{2S}{a + h_a}$.

Similar arguments for the other squares show that $a + h_a = b + h_b = c + h_c$.

**5.28.** If $\alpha$, $\beta$ and $\gamma$ are the angles of triangle $ABC$, then the angles of triangle $A_1B_1C_1$ are equal to $\frac{\beta + \gamma}{2}$, $\frac{\gamma + \alpha}{2}$ and $\frac{\alpha + \beta}{2}$. Let, for definiteness, $\alpha \geq \beta \geq \gamma$. Then $\frac{\alpha + \beta}{2} \geq \frac{\alpha + \gamma}{2} \geq \frac{\beta + \gamma}{2}$. Hence, $\alpha = \frac{\alpha + \beta}{2}$ and $\gamma = \frac{\beta + \gamma}{2}$, i.e., $\alpha = \beta$ and $\beta = \gamma$.

**5.29.** In any triangle a height is longer than the diameter of the inscribed circle. Therefore, the lengths of heights are integers greater than 2, i.e., all of them are not less than 3. Let $S$ be the area of the triangle, $a$ the length of its longest side and $h$ the corresponding height.

Suppose that the triangle is not an equilateral one. Then its perimeter $P$ is shorter than $3a$. Therefore, $3a > P = \mathrm{Pr} = 2S = ha$, i.e., $h < 3$. Contradiction.

**5.30.** Since the outer angle at vertex $A$ of triangle $ABA_1$ is equal to $120°$ and $\angle A_1AB_1 = 60°$, it follows that $AB_1$ is the bisector of this outer angle. Moreover, $BB_1$ is the bisector of the outer angle at vertex $B$, hence, $A_1B_1$ is the bisector of angle $\angle AA_1C$. Similarly, $A_1C_1$ is the bisector of angle $\angle AA_1B$. Hence,

$$\angle B_1A_1C_1 = \frac{\angle AA_1C + \angle AA_1B}{2} = 90°.$$

**5.31.** Thanks to the solution of the preceding problem ray $A_1C_1$ is the bisector of angle $\angle AA_1B$. Let $K$ be the intersection point of the bisectors of triangle $A_1AB$. Then

$$\angle C_1KO = \angle A_1KB = 90° + \frac{\angle A}{2} = 120°.$$

Hence, $\angle C_1KO + \angle C_1AO = 180°$, i.e., quadrilateral $AOKC_1$ is an inscribed one. Hence, $\angle A_1C_1O = \angle KC_1O = \angle KAO = 30°$.

**5.32.** a) Let $S$ be the circumscribed circle of triangle $ABC$, let $S_1$ be the circle symmetric to $S$ through line $BC$. The orthocenter $H$ of triangle $ABC$ lies on circle $S_1$ (Problem 5.9) and, therefore, it suffices to verify that the center $O$ of circle $S$ also belongs to $S_1$ and the bisector of the outer angle $A$ passes through the center of circle $S_1$. Then $POAH$ is a rhombus, because $PO \parallel HA$.

Let $PQ$ be the diameter of circle $S$ perpendicular to line $BC$; let points $P$ and $A$ lie on one side of line $BC$. Then $AQ$ is the bisector of angle $A$ and $AP$ is the bisector of the outer angle $\angle A$. Since $\angle BPC = 120° = \angle BOC$, point $P$ is the center of circle $S_1$ and point $O$ belongs to circle $S_1$.

b) Let $S$ be the circumscribed circle of triangle $ABC$ and $Q$ the intersection point of the bisector of angle $\angle BAC$ with circle $S$. It is easy to verify that $Q$ is the center of circle $S_1$ symmetric to circle $S$ through line $BC$. Moreover, points $O$ and $H$ lie on circle $S_1$ and since $\angle BIC = 120°$ and $\angle BI_aC = 60°$ (cf. Problem 5.3), it follows that $II_a$ is a diameter of circle $S_1$. It is also clear that $\angle OQI = \angle QAH = \angle AQH$, because $OQ \parallel AH$ and $HA = QO = QH$. Hence, points $O$ and $H$ are symmetric through line $II_a$.

**5.33.** On side $AC$ of triangle $ABC$, construct outwards an equilateral triangle $AB_1C$. Since $\angle A = 120°$, point $A$ lies on segment $BB_1$. Therefore, $BB_1 = b + c$ and, moreover, $BC = a$ and $B_1C = b$, i.e., triangle $BB_1C$ is the desired one.

**5.34.** a) Let $M_1$ and $N_1$ be the midpoints of segments $BH$ and $CH$, respectively; let $BB_1$ and $CC_1$ be heights. Right triangles $ABB_1$ and $BHC_1$ have a common acute angle — the one at vertex $B$; hence, $\angle C_1HB = \angle A = 60°$. Since triangle $BMH$ is an isosceles one, $\angle BHM = \angle HBM = 30°$. Therefore, $\angle C_1HM = 60° - 30° = 30° = \angle BHM$, i.e., point $M$ lies on the bisector of angle $\angle C_1HB$. Similarly, point $N$ lies on the bisector of angle $\angle B_1HC$.

b) Let us make use of the notations of the preceding problem and, moreover, let $B'$ and $C'$ be the midpoints of sides $AC$ and $AB$. Since $AC_1 = AC \cos \angle A = \frac12 AC$, it follows that $C_1C' = \frac12|AB - AC|$. Similarly, $B_1B' = \frac12|AB - AC|$, i.e., $B_1B' = C_1C'$. It follows that the parallel lines $BB_1$ and $B'O$, $CC_1$ and $C'O$ form not just a parallelogram but a rhombus. Hence, its diagonal $HO$ is the bisector of the angle at vertex $H$.

**5.35.** Since

$$\angle BB_1C = \angle B_1BA + \angle B_1AB > \angle B_1BA = \angle B_1BC,$$

it follows that $BC > B_1C$. Hence, point $K$ symmetric to $B_1$ through bisector $CC_1$ lies on side $BC$ and not on its extension. Since $\angle CC_1B = 30°$, we have $\angle B_1C_1K = 60°$ and, therefore, triangle $B_1C_1K$ is an equilateral one. In triangles $BC_1B_1$ and $BKB_1$ side $BB_1$ is a common one and sides $C_1B_1$ and $KB_1$ are equal; the angles $C_1BB_1$ and $KBB_1$ are also equal but these angles are not the ones between equal sides. Therefore, the following two cases are possible:

1) $\angle BC_1B_1 = \angle BKB_1$. Then $\angle BB_1C_1 = \angle BB_1K = \frac{60°}{2} = 30°$. Therefore, if $O$ is the intersection point of bisectors $BB_1$ and $CC_1$, then

$$\angle BOC = \angle B_1OC_1 = 180° - \angle OC_1B_1 - \angle OB_1C_1 = 120°.$$

On the other hand, $\angle BOC = 90° + \frac{\angle A}{2}$ (cf. Problem 5.3), i.e., $\angle A = 60°$.

2) $\angle BC_1B_1 + \angle BKB_1 = 180°$. Then quadrilateral $BC_1B_1K$ is an inscribed one and since triangle $B_1C_1K$ is an equilateral one, $\angle B = 180° - \angle C_1B_1K = 120°$.

**5.36.** Let $BM$ be a median, $AK$ a bisector of triangle $ABC$ and $BM \perp AK$. Line $AK$ is a bisector and a height of triangle $ABM$, hence, $AM = AB$, i.e., $AC = 2AM = 2AB$. Therefore, $AB = 2$, $BC = 3$ and $AC = 4$.

**5.37.** Let $a$ and $b$ be legs and $c$ the hypothenuse of the given triangle. If numbers $a$ and $b$ are odd, then the remainder after division of $a^2 + b^2$ by 4 is equal to 2 and $a^2 + b^2$ cannot be a perfect square. Hence, one of the numbers $a$ and $b$ is even and another one is odd; let, for definiteness, $a = 2p$. The numbers $b$ and $c$ are odd, hence, $c + b = 2q$ and $c - b = 2r$ for some $q$ and $r$. Therefore, $4p^2 = a^2 = c^2 - b^2 = 4qr$. If $d$ is a common divisor of $q$ and $r$, then $a = 2\sqrt{qr}$, $b = q - r$ and $c = q + r$ are divisible by $d$. Therefore, $q$ and $r$ are relatively prime, ??? since $p^2 = qr$, it follows that $q = m^2$ and $r = n^2$. As a result we get $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$.

It is also easy to verify that if $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$, then $a^2 + b^2 = c^2$.

**5.38.** Let $p$ be the semiperimeter of the triangle and $a$, $b$, $c$ the lengths of the triangle's sides. By Heron's formula $S^2 = p(p - a)(p - b)(p - c)$. On the other hand, $S^2 = p^2r^2 = p^2$ since $r = 1$. Hence, $p = (p - a)(p - b)(p - c)$. Setting $x = p - a$, $y = p - b$, $z = p - c$ we rewrite our equation in the form

$$x + y + z = xyz.$$

Notice that $p$ is either integer or half integer (i.e., of the form $\frac{2n+1}{2}$, where $n$ is an integer) and, therefore, all the numbers $x$, $y$, $z$ are simultaneously either integers or half integers. But if they are half integers, then $x + y + z$ is a half integer and $xyz$ is of the form $\frac{m}{8}$, where $m$ is an odd number. Therefore, numbers $x$, $y$, $z$ are integers. Let, for definiteness, $x \le y \le z$. Then $xyz = x + y + z \le 3z$, i.e., $xy \le 3$. The following three cases are possible:

1) $x = 1$, $y = 1$. Then $2 + z = z$ which is impossible.

2) $x = 1$, $y = 2$. Then $3 + z = 2z$, i.e., $z = 3$.

3) $x = 1$, $y = 3$. Then $4 + z = 3z$, i.e., $z = 2 < y$ which is impossible.

Thus, $x = 1$, $y = 2$, $z = 3$. Therefore, $p = x + y + z = 6$ and $a = p - x = 5$, $b = 4$, $c = 3$.

**5.39.** Let $a_1$ and $b_1$, $a_2$ and $b_2$ be the legs of two distinct Pythagorean triangles, $c_1$ and $c_2$ their hypothenuses. Let us take two perpendicular lines and mark on them segments $OA = a_1a_2$, $OB = a_1b_2$, $OC = b_1b_2$ and $OD = a_2b_1$ (Fig. 57). Since $OA \cdot OC = OB \cdot OD$, quadrilateral $ABCD$ is an inscribed one. By Problem 2.71

$$4R^2 = OA^2 + OB^2 + OC^2 + OD^2 = (c_1c_2)^2,$$

i.e., $R = \frac{c_1 c_2}{2}$. Magnifying, if necessary, quadrilateral $ABCD$ twice, we get the quadrilateral to be found.
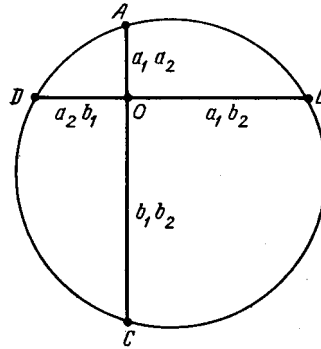


Figure 57 (Sol. 5.39)

**5.40.** a) The lengths of hypothenuses of right triangles with legs 5 and 12, 9 and 12 are equal to 13 and 15, respectively. Identifying the equal legs of these triangles we get a triangle whose area is equal to $\frac{12(5+9)}{2} = 84$.

b) First, suppose that the length of the shortest side of the given triangle is an even number, i.e., the lengths of the sides of the triangle are equal to $2n$, $2n + 1$, $2n + 2$. Then by Heron's formula

$$16S^2 = (6n + 3)(2n + 3)(2n + 1)(2n - 1) = 4(3n^2 + 6n + 2)(4n^2 - 1) + 4n^2 - 1.$$

We have obtained a contradiction since the number in the right-hand side is not divisible by 4. Consequently, the lengths of the sides of the triangle are equal to $2n - 1$, $2n$ and $2n + 1$, where $S^2 = 3n^2(n^2 - 1)$. Hence, $S = nk$, where $k$ is an integer and $k^2 = 3(n^2 - 1)$. It is also clear that $k$ is the length of the height dropped to the side of length $2n$. This height divides the initial triangle into two right triangles with a common leg of length $k$ and hypothenuses of length $2n + 1$ and $2n - 1$ the squares of the lengths of the other legs of these triangles are equal to

$$(2n \pm 1)^2 - k^2 = 4n^2 \pm 4n + 1 - 3n^2 + 3 = (n \pm 2)^2.$$

**5.41.** a) Since $AB^2 - AB_1^2 = BB_1^2 = BC^2 - (AC \pm AB_1)^2$, we see that $AB_1 = \pm \frac{AB^2 + AC^2 - BC^2}{2AC}$.

b) Let diagonals $AC$ and $BD$ meet at point $O$. Let us prove, for example, that the number $q = \frac{BO}{OD}$ is a rational one (then the number $OD = \frac{BD}{q+1}$ is also a rational one). In triangles $ABC$ and $ADC$ draw heights $BB_1$ and $DD_1$. By heading a) the numbers $AB_1$ and $CD_1$ — the lengths of the corresponding sides — are rational and, therefore, the number $B_1D_1$ is also rational.

Let $E$ be the intersection point of line $BB_1$ and the line that passes through point $D$ parallel to $AC$. In right triangle $BDE$, we have $ED = B_1D_1$ and the lengths of leg $ED$ and hypothenuse $BD$ are rational numbers; hence, $BE^2$ is also a rational number. From triangles $ABB_1$ and $CDD_1$ we derive that numbers $BB_1^2$ and $DD_1^2$ are rational. Since

$$BE^2 = (BB_1 + DD_1)^2 = BB_1^2 + DD_1^2 + 2BB_1 \cdot DD_1,$$

number $BB_1 \cdot DD_1$ is rational. It follows that the number

$$\frac{BO}{OD} = \frac{BB_1}{DD_1} = \frac{BB_1 \cdot DD_1}{DD_1^2}$$

is a rational one.

**5.42.** Triangles $ABC$ and $A_1B_1C_1$ cannot have two pairs of corresponding angles whose sum is equal to $180°$ since otherwise their sum would be equal to $360°$ and the third angles of these triangles should be equal to zero. Now, suppose that the angles of the first triangle are equal to $\alpha$, $\beta$ and $\gamma$ and the angles of the second one are equal to $180° - \alpha$, $\beta$ and $\gamma$. The sum of the angles of the two triangles is equal to $360°$, hence, $180° + 2\beta + 2\gamma = 360°$, i.e., $\beta + \gamma = 90°$. It follows that $\alpha = 90° = 180° - \alpha$.

**5.43.** Clearly, $\overrightarrow{A_1C} = \overrightarrow{BO}$ and $\overrightarrow{CB_1} = \overrightarrow{OA}$, hence, $\overrightarrow{A_1B_1} = \overrightarrow{BA}$. Similarly, $\overrightarrow{B_1C_1} = \overrightarrow{CB}$ and $\overrightarrow{C_1A_1} = \overrightarrow{AC}$, i.e., $\triangle ABC = \triangle A_1B_1C_1$. Moreover, $ABA_1B_1$ and $ACA_1C_1$ are parallelograms. It follows that segments $BB_1$ and $CC_1$ pass through the midpoint of segment $AA_1$.

**5.44.** Since $\angle MAO = \angle PAO = \angle AOM$, it follows that $AMOP$ is a rhombus. Similarly, $BNOQ$ is a rhombus. It follows that

$$MN = MO + ON = AM + BN \quad \text{and} \quad OP + PQ + QO = AP + PQ + QB = AB.$$

**5.45.** a) Through vertices of triangle $ABC$ let us draw lines parallel to the triangle's opposite sides. As a result we get triangle $A_1B_1C_1$; the midpoints of the sides of the new triangle are points $A$, $B$ and $C$. The heights of triangle $ABC$ are the midperpendiculars to the sides of triangle $A_1B_1C_1$ and, therefore, the center of the circumscribed circle of triangle $A_1B_1C_1$ is the intersection point of heights of triangle $ABC$.

b) Point $H$ is the center of the circumscribed circle of triangle $A_1B_1C_1$, hence,

$$4R^2 = B_1H^2 = B_1A^2 + AH^2 = BC^2 + AH^2.$$

Therefore,

$$AH^2 = 4R^2 - BC^2 = \left(\frac{1}{\sin^2\alpha} - 1\right)BC^2 = (BC\cot\alpha)^2.$$

**5.46.** Let $AD$ be the bisector of an equilateral triangle $ABC$ with base $AB$ and angle $36°$ at vertex $C$. Then triangle $ACD$ is an isosceles one and $\triangle ABC \sim \triangle BDA$. Therefore, $CD = AD = AB = 2xBC$ and $DB = 2xAB = 4x^2BC$; hence,

$$BC = CD + DB = (2x + 4x^2)BC.$$

**5.47.** Let $B_1$ and $B_2$ be the projections of point $A$ to bisectors of the inner and outer angles at vertex $B$; let $M$ the midpoint of side $AB$. Since the bisectors of the inner and outer angles are perpendicular, it follows that $AB_1BB_2$ is a rectangular and its diagonal $B_1B_2$ passes through point $M$. Moreover,

$$\angle B_1MB = 180° - 2\angle MBB_1 = 180° - \angle B.$$

Hence, $B_1B_2 \parallel BC$ and, therefore, line $B_1B_2$ coincides with line $l$ that connects the midpoints of sides $AB$ and $AC$.

We similarly prove that the projections of point $A$ to the bisectors of angles at vertex $C$ lie on line $l$.

**5.48.** Suppose that the bisectors of angles $A$ and $B$ are equal but $a > b$. Then $\cos\frac{1}{2}\angle A < \cos\frac{1}{2}\angle B$ and $\frac{1}{c} + \frac{1}{b} > \frac{1}{c} + \frac{1}{a}$, i.e., $\frac{bc}{b+c} < \frac{ac}{a+c}$. By multiplying these inequalities we get a contradiction, since $l_a = \frac{2bc\cos\frac{\angle A}{2}}{b+c}$ and $l_b = \frac{2ac\cos\frac{\angle B}{2}}{a+c}$ (cf. Problem 4.47).

**5.49.** a) By Problem 4.47 the length of the bisector of angle $\angle B$ of triangle $ABC$ is equal to $\frac{2ac\cos\frac{\angle B}{2}}{a+c}$ and, therefore, it suffices to verify that the system of equations

$$\frac{ac}{a+c} = p, \quad a^2 + c^2 - 2ac\cos\angle B = q$$

has (up to a transposition of $a$ with $c$) a unique positive solution. Let $a + c = u$. Then $ac = pu$ and $q = u^2 - 2pu(1 + \cos\beta)$. The product of the roots of this quadratic equation for $u$ is equal to $-q$ and, therefore, it has one positive root. Clearly, the system of equations

$$a + c = u, \quad ac = pu$$

has a unique solution.

b) In triangles $AA_1B$ and $CC_1B$, sides $AA_1$ and $CC_1$ are equal; the angles at vertex $B$ are equal, and the bisectors of the angles at vertex $B$ are also equal. Therefore, these triangles are equal and either $AB = BC$ or $AB = BC_1$. The second equality cannot take place.

**5.50.** Let points $M$ and $N$ lie on sides $AB$ and $AC$. If $r_1$ is the radius of the circle whose center lies on segment $MN$ and which is tangent to sides $AB$ and $AC$, then $S_{AMN} = qr_1$, where $q = \frac{AM + AN}{2}$. Line $MN$ passes through the center of the inscribed circle if and only if $r_1 = r$, i.e., $\frac{S_{AMN}}{q} = \frac{S_{ABC}}{p} = \frac{S_{BCNM}}{p - q}$.

**5.51.** a) On the extension of segment $AC$ beyond point $C$ take a point $B'$ such that $CB' = CB$. Triangle $BCB'$ is an isosceles one; hence, $\angle AEB = \angle ACB = 2\angle CBB'$ and, therefore, $E$ is the center of the circumscribed circle of triangle $ABB'$. It follows that point $F$ divides segment $AB'$ in halves; hence, line $C_1F$ divides the perimeter of triangle $ABC$ in halves.

b) It is easy to verify that the line drawn through point $C$ parallel to $BB'$ is the bisector of angle $ACB$. Since $C_1F \parallel BB'$, line $C_1F$ is the bisector of the angle of the triangle with vertices at the midpoints of triangle $ABC$. The bisectors of this new triangle meet at one point.

**5.52.** Let $X$ be the intersection point of lines $AD_2$ and $CD_1$; let $M$, $E_1$ and $E_2$ be the projections of points $X$, $D_1$ and $D_2$, respectively, to line $AC$. Then $CE_2 = CD_2 \sin\gamma = a\sin\gamma$ and $AE_1 = c\sin\alpha$. Since $a\sin\gamma = c\sin\alpha$, it follows that $CE_2 = AE_1 = q$. Hence,

$$\frac{XM}{AM} = \frac{D_2E_2}{AE_2} = \frac{a\cos\gamma}{b + q} \quad \text{and} \quad \frac{XM}{CM} = \frac{c\cos\alpha}{b + q}.$$

Therefore, $AM : CM = c\cos\alpha : a\cos\gamma$. Height $BH$ divides side $AC$ in the same ratio.

**5.53.** a) By the law of cosines

$$B_1C_1^2 = AC_1^2 + AB_1^2 - 2AC_1 \cdot AB_1 \cdot \cos(90° + \alpha),$$

i.e.,

$$a_1^2 = \frac{c^2}{2} + \frac{b^2}{2} + bc\sin\alpha = \frac{b^2 + c^2}{2} + 2S.$$

Writing similar equalities for $b_1^2$ and $c_1^2$ and taking their sum we get the statement desired.

b) For an acute triangle $ABC$, add to $S$ the areas of triangles $ABC_1$, $AB_1C$ and $A_1BC$; add to $S_1$ the areas of triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$. We get equal quantities (for a triangle with an obtuse angle $\angle A$ the area of triangle $AB_1C_1$ should be taken with a minus sign). Hence,

$$S_1 = S + \frac{a^2 + b^2 + c^2}{4} - \frac{ab\cos\gamma + ac\cos\beta + bc\cos\alpha}{4}.$$

It remains to notice that

$$ab\cos\gamma + bc\cos\alpha + ac\cos\beta = 2S(\cot\gamma + \cot\alpha + \cot\beta) = \frac{a^2 + b^2 + c^2}{2};$$

cf. Problem 12.44 a).

**5.54.** First, let us prove that point $B'$ lies on the circumscribed circle of triangle $AHC$, where $H$ is the intersection point of heights of triangle $ABC$. We have

$$\angle(AB', B'C) = \angle(AA_1, CC_1) =$$
$$\angle(AA_1, BC) + \angle(BC, AB) + \angle(AB, CC_1) = \angle(BC, AB).$$

But as follows from the solution of Problem 5.9 $\angle(BC, AB) = \angle(AH, HC)$ and, therefore, points $A$, $B'$, $H$ and $C$ lie on one circle and this circle is symmetric to the circumscribed circle of triangle $ABC$ through line $AC$. Hence, both these circles have the same radius, $R$, consequently,

$$B'H = 2R \sin B'AH = 2R \cos \alpha.$$

Similarly, $A'H = 2R \cos \alpha = C'H$. This completes solution of heading a); to solve heading b) it remains to notice that $\triangle A'B'C' \sim \triangle ABC$ since after triangle $A'B'C'$ is rotated through an angle of $\alpha$ its sides become parallel to the sides of triangle $ABC$.

**5.55.** Let $a_1 = BA_1$, $a_2 = A_1C$, $b_1 = CB_1$, $b_2 = B_1A$, $c_1 = AC_1$ and $c_2 = C_1B$. The products of the lengths of segments of intersecting lines that pass through one point are equal and, therefore, $a_1(a_1 + x) = c_2(c_2 - z)$, i.e.,

$$a_1 x + c_2 z = c_2^2 - a_1^2.$$

We similarly get two more equations for $x$, $y$ and $z$:

$$b_1 y + a_2 x = a_2^2 - b_1^2 \quad \text{and} \quad c_1 z + b_2 y = b_2^2 - c_1^2.$$

Let us multiply the first equation by $b^{2n}$; multiply the second and the third ones by $c^{2n}$ and $a^{2n}$, respectively, and add the equations obtained. Since, for instance, $c_2 b^n - c_1 a^n = 0$ by the hypothesis, we get zero in the right-hand side. The coefficient of, say, $x$ in the left-hand side is equal to

$$a_1 b^{2n} + a_2 c^{2n} = \frac{ac^n b^{2n} + ab^n c^{2n}}{b^n + c^n} = ab^n c^n.$$

Hence,

$$ab^n c^n x + ba^n c^n y + ca^n b^n z = 0.$$

Dividing both sides of this equation by $(abc)^n$ we get the statement desired.

**5.56.** Let in the initial triangle $\angle A = 3\alpha$, $\angle B = 3\beta$ and $\angle C = 3\gamma$. Let us take an equilateral triangle $A_2B_2C_2$ and construct on its sides as on bases isosceles triangles $A_2B_2R$, $B_2C_2P$ and $C_2A_2Q$ with angles at the bases equal to $60° - \gamma$, $60° - \alpha$, $60° - \beta$, respectively (Fig. 58).

Let us extend the lateral sides of these triangles beyond points $A_2$, $B_2$ and $C_2$; denote the intersection point of the extensions of sides $RB_2$ and $QC_2$ by $A_3$, that of $PC_2$ and $RA_2$ by $B_3$, that of $QA_2$ and $PB_2$ by $C_3$. Through point $B_2$ draw the line parallel to $A_2C_2$ and denote by $M$ and $N$ the its intersection points with lines $QA_3$ and $QC_3$, respectively. Clearly, $B_2$ is the midpoint of segment $MN$. Let us compute the angles of triangles $B_2C_3N$ and $B_2A_3M$:

$$\angle C_3B_2N = \angle PB_2M = \angle C_2B_2M = \angle C_2B_2P = \alpha;$$
$$\angle B_2NC_3 = 180° - \angle C_2A_2Q = 120° + \beta;$$

hence, $\angle B_2C_3N = 180° - \alpha - (120° + \beta) = \gamma$. Similarly, $\angle A_3B_2M = \gamma$ and $\angle B_2A_3M = \alpha$. Hence, $\triangle B_2C_3N \sim \triangle A_3B_2M$. It follows that $C_3B_2 : B_2A_3 = C_3N : B_2M$ and since $B_2M = B_2N$ and $\angle C_3B_2A_3 = \angle C_3NB_2$, it follows that $C_3B_2 : B_2A_3 = C_3N : NB_2$ and $\triangle C_3B_2A_3 \sim \triangle C_3NB_2$; hence, $\angle B_2C_3A_3 = \gamma$.
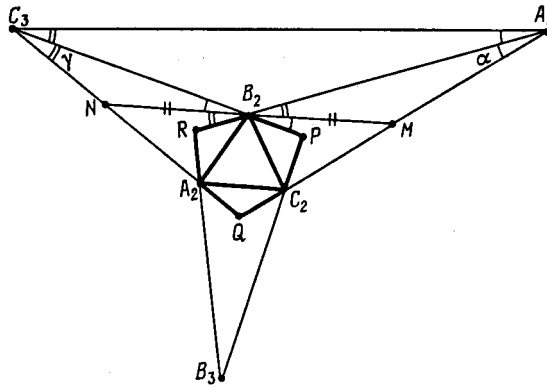
Figure 58 (Sol. 5.56)

Similarly, $\angle A_2 C_3 B_3 = \gamma$ and, therefore, $\angle A_3 C_3 B_3 = 3\gamma = \angle C$ and $C_3 B_3$, $C_3 A_2$ are the trisectors of angle $C_3$ of triangle $A_3 B_3 C_3$. Similar arguments for vertices $A_3$ and $B_3$ show that $\triangle ABC \sim \triangle A_3 B_3 C_3$ and the intersection points of the trisectors of triangle $A_3 B_3 C_3$ are vertices of an equilateral triangle $A_2 B_2 C_2$.

**5.57.** Point $A_1$ lies on the bisector of angle $\angle BAC$, hence, point $A$ lies on the extension of the bisector of angle $\angle B_2 A_1 C_2$. Moreover, $\angle B_2 A C_2 = \alpha = \frac{180° - \angle B_2 A_1 C_2}{2}$. Hence, $A$ is the center of an escribed circle of triangle $B_2 A_1 C_2$ (cf. Problem 5.3). Let $D$ be the intersection point of lines $AB$ and $CB_2$. Then

$$\angle AB_2 C_2 = \angle AB_2 D = 180° - \angle B_2 AD - \angle ADB_2 = 180° - \gamma - (60° + \alpha) = 60° + \beta.$$

Since

$$\angle AB_2 C = 180° - (\alpha + \beta) - (\beta + \gamma) = 120° - \beta,$$

it follows that

$$\angle CB_2 C_2 = \angle AB_2 C - \angle AB_2 C_2 = 60° - 2\beta.$$

Similarly, $\angle AB_2 A_2 = 60° - 2\beta$. Hence,

$$\angle A_2 B_2 C_2 = \angle AB_2 C - \angle AB_2 A_2 - \angle CB_2 C_2 = 3\beta.$$

Similarly, $\angle B_2 A_2 C_2 = 3\alpha$ and $\angle A_2 C_2 B_2 = 3\gamma$.

**5.58.** Let the projection to a line perpendicular to line $A_1 B_1$ send points $A$, $B$ and $C$ to $A'$, $B'$ and $C'$, respectively; point $C_1$ to $Q$ and points $A_1$ and $B_1$ into one point, $P$. Since

$$\frac{\overline{A_1 B}}{\overline{A_1 C}} = \frac{\overline{PB'}}{\overline{PC'}}, \quad \frac{\overline{B_1 C}}{\overline{B_1 A}} = \frac{\overline{PC'}}{\overline{PA'}} \quad \text{and} \quad \frac{\overline{C_1 A}}{\overline{C_1 B}} = \frac{\overline{QA'}}{\overline{QB'}},$$

it follows that

$$\frac{\overline{A_1 B}}{\overline{A_1 C}} \cdot \frac{\overline{B_1 C}}{\overline{B_1 A}} \cdot \frac{\overline{C_1 A}}{\overline{C_1 B}} = \frac{\overline{PB'}}{\overline{PC'}} \cdot \frac{\overline{PC'}}{\overline{PA'}} \cdot \frac{\overline{QA'}}{\overline{QB'}} = \frac{\overline{PB'}}{\overline{PA'}} \cdot \frac{\overline{QA'}}{\overline{QB'}} = \frac{b'}{a'} \cdot \frac{a' + x}{b' + x},$$

where $|x| = PQ$. The equality $\frac{b'}{a'} \cdot \frac{a' + x}{b' + x} = 1$ is equivalent to the fact that $x = 0$. (We have to take into account that $a' \neq b'$ since $A' \neq B'$.) But the equality $x = 0$ means that $P = Q$, i.e., point $C_1$ lies on line $A_1 B_1$.

**5.59.** Let point $P$ lie on arc $\smile BC$ of the circumscribed circle of triangle $ABC$. Then

$$\frac{\overline{BA_1}}{\overline{CA_1}} = -\frac{BP \cos \angle PBC}{CP \cos \angle PCB}, \quad \frac{\overline{CB_1}}{\overline{AB_1}} = -\frac{CP \cos \angle PCA}{AP \cos PAC}, \quad \frac{\overline{AC_1}}{\overline{BC_1}} = -\frac{AP \cos \angle PAB}{PB \cos \angle PBA}.$$

By multiplying these equalities and taking into account that

$$\angle PAC = \angle PBC, \quad \angle PAB = \angle PCB \quad \text{and} \quad \angle PAC + \angle PBA = 180°$$

we get

$$\frac{\overline{BA_1}}{\overline{CA_1}} \cdot \frac{\overline{CB_1}}{\overline{AB_1}} \cdot \frac{\overline{AC_1}}{\overline{BC_1}} = 1.$$

**5.60.** Let $O$, $O_1$ and $O_2$ be the centers of circles $S$, $S_1$ and $S_2$; let $X$ be the intersection point of lines $O_1O_2$ and $A_1A_2$. By applying Menelaus's theorem to triangle $OO_1O_2$ and points $A_1$, $A_2$ and $X$ we get

$$\frac{O_1X}{O_2X} \cdot \frac{O_2A_2}{OA_2} \cdot \frac{OA_1}{O_1A_1} = 1$$

and, therefore, $O_1X : O_2X = R_1 : R_2$, where $R_1$ and $R_2$ are the radii of circles $S_1$ and $S_2$, respectively. It follows that $X$ is the intersection point of the common outer or common inner tangents to circles $S_1$ and $S_2$.

**5.61.** a) Let, for definiteness, $\angle B < \angle C$. Then $\angle DAE = \angle ADE = \angle B + \frac{\angle A}{2}$; hence, $\angle CAE = \angle B$. Since

$$\frac{BE}{AB} = \frac{\sin \angle BAE}{\sin \angle AEB} \quad \text{and} \quad \frac{AC}{CE} = \frac{\sin \angle AEC}{\sin \angle CAE},$$

it follows that

$$\frac{BE}{CE} = \frac{c \sin \angle BAE}{b \sin \angle CAE} = \frac{c \sin(\angle A + \angle B)}{b \sin \angle B} = \frac{c \sin \angle C}{b \sin \angle B} = \frac{c^2}{b^2}.$$

b) In heading a) point $E$ lies on the extension of side $BC$ since $\angle ADC = \angle BAD + \angle B > \angle CAD$. Therefore, making use of the result of heading a) and Menelaus's theorem we get the statement desired.

**5.62.** Since $\angle BCE = 90° - \frac{\angle B}{2}$, we have: $\angle BCE = \angle BEC$ and, therefore, $BE = BC$. Hence,

$$CF : KF = BE : BK = BC : BK \quad \text{and} \quad AE : KE = CA : CK = BC : BK.$$

Let line $EF$ intersect $AC$ at point $D$. By Menelaus's theorem $\frac{AD}{CD} \cdot \frac{CF}{KF} \cdot \frac{KE}{AE} = 1$. Taking into account that $CF : KF = AE : KE$ we get the statement desired.

**5.63.** Proof is similar to that of Problem 5.79; we only have to consider the ratio of *oriented* segments and angles.

**5.64.** Let $A_2$, $B_2$ and $C_2$ be the intersection points of lines $BC$ with $B_1C_1$, $AC$ with $A_1C_1$, $AB$ with $A_1B_1$, respectively. Let us apply Menelaus's theorem to the following triangles and points on their sides: $OAB$ and $(A_1, B_1, C_2)$, $OBC$ and $(B_1, C_1, A_2)$, $OAC$ and $(A_1, C_1, B_2)$. Then

$$\frac{\overline{AA_1}}{\overline{OA_1}} \cdot \frac{\overline{OB_1}}{\overline{BB_1}} \cdot \frac{\overline{BC_2}}{\overline{AC_2}} = 1, \quad \frac{\overline{OC_1}}{\overline{CC_1}} \cdot \frac{\overline{BB_1}}{\overline{OB_1}} \cdot \frac{\overline{CA_2}}{\overline{BA_2}} = 1, \quad \frac{\overline{OA_1}}{\overline{AA_1}} \cdot \frac{\overline{CC_1}}{\overline{OC_1}} \cdot \frac{\overline{AB_2}}{\overline{CB_2}} = 1.$$

By multiplying these equalities we get

$$\frac{\overline{BC_2}}{\overline{AC_2}} \cdot \frac{\overline{AB_2}}{\overline{CB_2}} \cdot \frac{\overline{CA_2}}{\overline{BA_2}} = 1.$$

Menelaus's theorem implies that points $A_2$, $B_2$, $C_2$ lie on one line.

**5.65.** Let us consider triangle $A_0B_0C_0$ formed by lines $A_1B_2$, $B_1C_2$ and $C_1A_2$ (here $A_0$ is the intersection point of lines $A_1B_2$ and $A_2C_1$, etc), and apply Menelaus's theorem to this triangle and the following five triples of points:

$$(A, B_2, C_1), \quad (B, C_2, A_1), \quad (C, A_2, B_1), \quad (A_1, B_1, C_1) \quad \text{and} \quad (A_2, B_2, C_2).$$

As a result we get

$$\frac{\overline{B_0 A}}{\overline{C_0 A}} \cdot \frac{\overline{A_0 B_2}}{\overline{B_0 B_2}} \cdot \frac{\overline{C_0 C_1}}{\overline{A_0 C_1}} = 1, \qquad \frac{\overline{C_0 B}}{\overline{A_0 B}} \cdot \frac{\overline{B_0 C_2}}{\overline{C_0 C_2}} \cdot \frac{\overline{A_0 A_1}}{\overline{B_0 A_1}} = 1,$$

(2)
$$\frac{\overline{A_0 C}}{\overline{B_0 C}} \cdot \frac{\overline{C_0 A_2}}{\overline{A_0 A_2}} \cdot \frac{\overline{B_0 B_1}}{\overline{C_0 B_1}} = 1, \qquad \frac{\overline{B_0 A_1}}{\overline{A_0 A_1}} \cdot \frac{\overline{C_0 B_1}}{\overline{B_0 B_1}} \cdot \frac{\overline{A_0 C_1}}{\overline{C_0 C_1}} = 1,$$

(3)
$$\frac{\overline{A_0 A_2}}{\overline{C_0 A_2}} \cdot \frac{\overline{B_0 B_2}}{\overline{A_0 B_2}} \cdot \frac{\overline{C_0 C_2}}{\overline{B_0 C_2}} = 1.$$

By multiplying these equalities we get $\frac{\overline{B_0 A}}{\overline{C_0 A}} \cdot \frac{\overline{C_0 B}}{\overline{A_0 B}} \cdot \frac{\overline{A_0 C}}{\overline{B_0 C}} = 1$ and, therefore, points $A$, $B$ and $C$ lie on one line.

**5.66.** Let $N$ be the intersection point of lines $AD$ and $KQ$, $P'$ the intersection point of lines $KL$ and $MN$. By Desargue's theorem applied to triangles $KBL$ and $NDM$ we derive that $P'$, $A$ and $C$ lie on one line. Hence, $P' = P$.

**5.67.** It suffices to apply Desargues's theorem to triangles $AED$ and $BFC$ and Pappus' theorem to triples of points $(B, E, C)$ and $(A, F, D)$.

**5.68.** a) Let $R$ be the intersection point of lines $KL$ and $MN$. By applying Pappus' theorem to triples of points $(P, L, N)$ and $(Q, M, K)$, we deduce that points $A$, $C$ and $R$ lie on one line.

b) By applying Desargues's theorem to triangles $NDM$ and $LBK$ we see that the intersection points of lines $ND$ with $LB$, $DM$ with $BK$, and $NM$ with $LK$ lie on one line.

**5.69.** Let us make use of the result of Problem 5.68 a). For points $P$ and $Q$ take points $P_2$ and $P_4$, for points $A$ and $C$ take points $C_1$ and $P_1$ and for $K$, $L$, $M$ and $N$ take points $P_5$, $A_1$, $B_1$ and $P_3$, respectively. As a result we see that line $P_6 C_1$ passes through point $P_1$.

**5.70.** a) This problem is a reformulation of Problem 5.58 since the number $\overline{BA_1} : \overline{CA_1}$ is negative if point $A_1$ lies on segment $BC$ and positive otherwise.

b) First, suppose that lines $AA_1$, $BB_1$ and $CC_1$ meet at point $M$. Any three (nonzero) vectors in plane are linearly dependent, i.e., there exist numbers $\lambda$, $\mu$ and $\nu$ (not all equal to zero) such that $\lambda \overrightarrow{AM} + \mu \overrightarrow{BM} + \nu \overrightarrow{CM} = 0$. Let us consider the projection to line $BC$ parallel to line $AM$. This projection sends points $A$ and $M$ to $A_1$ and points $B$ and $C$ into themselves. Therefore, $\mu \overrightarrow{BA_1} + \nu \overrightarrow{CA_1} = 0$, i.e.,

$$\frac{\overline{BA_1}}{\overline{CA_1}} = -\frac{\nu}{\mu}.$$

Similarly,

$$\frac{\overline{CB_1}}{\overline{AB_1}} = -\frac{\lambda}{\nu} \quad \text{and} \quad \frac{\overline{AC_1}}{\overline{BC_1}} = -\frac{\mu}{\lambda}.$$

By multiplying these three equalities we get the statement desired.

If lines $AA_1$, $BB_1$ and $CC_1$ are parallel, in order to get the proof it suffices to notice that

$$\frac{\overline{BA_1}}{\overline{CA_1}} = \frac{\overline{BA}}{\overline{C_1 A}} \quad \text{and} \quad \frac{\overline{CB_1}}{\overline{AB_1}} = \frac{\overline{C_1 B}}{\overline{AB}}.$$

Now, suppose that the indicated relation holds and prove that then lines $AA_1$, $BB_1$ and $CC_1$ intersect at one point. Let $C_1^*$ be the intersection point of line $AB$ with the line that passes through point $C$ and the intersection point of lines $AA_1$ and $BB_1$. For point $C_1^*$ the same relation as for point $C_1$ holds. Therefore, $C_1^* A : C_1^* B = C_1 A : C_1 B$. Hence, $C_1^* = C_1$, i.e., lines $AA_1$, $BB_1$ and $CC_1$ meet at one point.

It is also possible to verify that if the indicated relation holds and two of the lines $AA_1$, $BB_1$ and $CC_1$ are parallel, then the third line is also parallel to them.

**5.71.** Clearly, $AB_1 = AC_1$, $BA_1 = BC_1$ and $CA_1 = CB_1$, and, in the case of the inscribed circle, on sides of triangle $ABC$, there are three points and in the case of an escribed circle there is just one point on sides of triangle $ABC$. It remains to make use of Ceva's theorem.

**5.72.** Let $AA_1$, $BB_1$ and $CC_1$ be heights of triangle $ABC$. Then

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{b\cos\angle A}{a\cos\angle B} \cdot \frac{c\cos\angle B}{b\cos\angle C} \cdot \frac{a\cos\angle C}{c\cos\angle A} = 1.$$

**5.73.** Let $A_2$, $B_2$ and $C_2$ be the midpoints of sides $BC$, $CA$ and $AB$. The considered lines pass through the vertices of triangle $A_2B_2C_2$ and in heading a) they divide its sides in the same ratios in which lines $AP$, $BP$ and $CP$ divide sides of triangle $ABC$ whereas in heading b) they divide them in the inverse ratios. It remains to make use of Ceva's theorem.

**5.74.** Since $\triangle AC_1B_2 \sim \triangle BC_1A_1$ and $\triangle AB_1C_2 \sim \triangle CB_1A_1$, it follows that $AB_2\cdot C_1B = AC_1 \cdot BA_1$ and $AC_2 \cdot CB_1 = A_1C \cdot B_1A$. Hence,

$$\frac{AB_2}{AC_2} = \frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

**5.75.** Let lines $AA_1$, $BB_1$ and $CC_1$ intersect lines $BC$, $CA$ and $AB$ at points $A_1$, $B_2$ and $C_2$.

a) If $\angle B + \beta < 180°$ and $\angle C + \gamma < 180°$, then

$$\frac{BA_2}{A_2C} = \frac{S_{ABA_1}}{S_{ACA_1}} = \frac{AB \cdot BA_1 \sin(\angle B + \beta)}{AC \cdot CA_1 \sin(\angle C + \gamma)} = \frac{AB}{AC} \cdot \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin(\angle B + \beta)}{\sin(\angle C + \gamma)}.$$

The latter expression is equal to $\overline{BA_2} : \overline{A_2C}$ in all the cases. Let us write similar expressions for $\overline{CB_2} : \overline{B_2A}$ and $\overline{AC_2} : \overline{C_2B}$ and multiply them. Now it remains to make use of Ceva's theorem.

b) Point $A_2$ lies outside segment $BC$ only if precisely one of the angles $\beta$ and $\gamma$ is greater than the corresponding angle $\angle B$ or $\angle C$. Hence,

$$\frac{\overline{BA_2}}{\overline{A_2C}} = \frac{AB}{AC} \cdot \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin(\angle B - \beta)}{\sin(\angle C - \gamma)}.$$

**5.76.** It is easy to verify that this problem is a particular case of Problem 5.75.

REMARK. A similar statement is also true for an escribed circle.

**5.77.** The solution of the problem obviously follows from Ceva's theorem.

**5.78.** By applying the sine theorem to triangles $ACC_1$ and $BCC_1$ we get

$$\frac{AC_1}{C_1C} = \frac{\sin\angle ACC_1}{\sin\angle A} \quad \text{and} \quad \frac{CC_1}{C_1B} = \frac{\sin\angle B}{\sin\angle C_1CB},$$

i.e.,

$$\frac{AC_1}{C_1B} = \frac{\sin\angle ACC_1}{\sin\angle C_1CB} \cdot \frac{\sin\angle B}{\sin\angle A}.$$

Similarly,

$$\frac{BA_1}{A_1C} = \frac{\sin\angle BAA_1}{\sin\angle A_1AC} \cdot \frac{\sin\angle C}{\sin\angle B} \quad \text{and} \quad \frac{CB_1}{B_1A} = \frac{\sin\angle CBB_1}{\sin\angle B_1BA} \cdot \frac{\sin\angle A}{\sin\angle C}.$$

To complete the proof it remains to multiply these equalities.

REMARK. A similar statement is true for the ratios of oriented segments and angles in the case when the points are taken on the extensions of sides.

**5.79.** We may assume that points $A_2$, $B_2$ and $C_2$ lie on the sides of triangle $ABC$. By Problem 5.78

$$\frac{AC_2}{C_2B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} = \frac{\sin \angle ACC_2}{\sin \angle C_2CB} \cdot \frac{\sin \angle BAA_2}{\sin \angle A_2AC} \cdot \frac{\sin \angle CBB_2}{\sin \angle B_2BA}.$$

Since lines $AA_2$, $BB_2$ and $CC_2$ are symmetric to lines $AA_1$, $BB_1$ and $CC_1$, respectively, through the bisectors, it follows that $\angle ACC_2 = \angle C_1CB$, $\angle C_2CB = \angle ACC_1$ etc., hence,

$$\frac{\sin \angle ACC_2}{\sin \angle C_2CB} \cdot \frac{\sin \angle BAA_2}{\sin \angle A_2AC} \cdot \frac{\sin \angle CBB_2}{\sin \angle B_2BA} = \frac{\sin \angle C_1CB}{\sin \angle ACC_1} \cdot \frac{\sin \angle A_1AC}{\sin \angle BAA_1} \cdot \frac{\sin \angle B_1BA}{\sin \angle CBB_1} =$$
$$\frac{C_1B}{AC_1} \cdot \frac{A_1C}{BA_1} \cdot \frac{B_1A}{CB_1} = 1.$$

Therefore,

$$\frac{AC_2}{C_2B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} = 1,$$

i.e., lines $AA_2$, $BB_2$ and $CC_2$ meet at one point.

REMARK. The statement holds also in the case when points $A_1$, $B_1$ and $C_1$ are taken on the extensions of sides if only point $P$ does not lie on the circumscribed circle $S$ of triangle $ABC$; if $P$ does lie on $S$, then lines $AA_2$, $BB_2$ and $CC_2$ are parallel (cf. Problem 2.90).

**5.80.** Let diagonals $AD$ and $BE$ of the given hexagon $ABCDEF$ meet at point $P$; let $K$ and $L$ be the midpoints of sides $AB$ and $ED$, respectively. Since $ABDE$ is a trapezoid, segment $KL$ passes through point $P$ (by Problem 19.2). By the law of sines

$$\sin \angle APK : \sin \angle AKP = AK : AP \quad \text{and} \quad \sin \angle BPK : \sin \angle BKP = BK : BP.$$

Since $\sin \angle AKP = \sin \angle BKP$ and $AK = BK$, we have

$$\sin \angle APK : \sin \angle BPK = BP : AP = BE : AD.$$

Similar relations can be also written for the segments that connect the midpoints of the other two pairs of the opposite sides. By multiplying these relations and applying the result of Problem 5.78 to the triangle formed by lines $AD$, $BE$ and $CF$, we get the statement desired.

**5.81.** Let us consider the homothety with center $P$ and coefficient 2. Since $PA_1A_3A_2$ is a rectangle, this homothety sends line $A_1A_2$ into line $l_a$ that passes through point $A_3$; lines $l_a$ and $A_3P$ are symmetric through line $A_3A$. Line $A_3A$ divides the angle $B_3A_3C_3$ in halves (Problem 1.56 a)).

We similarly prove that lines $l_b$ and $l_c$ are symmetric to lines $B_3P$ and $C_3P$, respectively, through bisectors of triangle $A_3B_3C_3$. Therefore, lines $l_a$, $l_b$ and $l_c$ either meet at one point or are parallel (Problem 1.79) and, therefore, lines $A_1A_2$, $B_1B_2$ and $C_1C_2$ meet at one point.

**5.82.** By Problems 5.78 and 5.70 b)) we have

$$\frac{\sin \angle ASP}{\sin \angle PSD} \cdot \frac{\sin \angle DAP}{\sin \angle PAS} \cdot \frac{\sin \angle SDP}{\sin \angle PDA} = 1 = \frac{\sin \angle ASQ}{\sin \angle QSD} \cdot \frac{\sin \angle DAQ}{\sin \angle QAS} \cdot \frac{\sin \angle SDQ}{\sin \angle QDA}.$$

But

$$\angle DAP = \angle SDQ, \; \angle SDP = \angle DAQ, \; \angle PAS = \angle QDA \; \text{and} \; \angle PDA = \angle QAS.$$

Hence,

$$\frac{\sin \angle ASP}{\sin \angle PSD} = \frac{\sin \angle ASQ}{\sin \angle QSD}.$$

This implies that points $S$, $P$ and $Q$ lie on one line, since the function $\frac{\sin(\alpha-x)}{\sin x}$ is monotonous with respect to $x$: indeed,

$$\frac{d}{dx}\left(\frac{\sin(\alpha-x)}{\sin x}\right) = -\frac{\sin\alpha}{\sin^2 x}.$$

**5.83.** a) By Ceva's theorem

$$\frac{AC_1}{C_1B} = \frac{CA_1}{A_1B} \cdot \frac{AB_1}{B_1C}$$

and by the law of sines

$$\begin{array}{llll} CA_1 &=& \frac{CA\sin\angle CAA_1}{\sin\angle AA_1B}, & A_1B &=& \frac{AB\sin\angle BAA_1}{\sin\angle AA_1B}, \\ AB_1 &=& \frac{AB\sin\angle ABB_1}{\sin\angle AB_1B}, & B_1C &=& \frac{BC\sin\angle CBB_1}{\sin\angle AB_1B}. \end{array}$$

Substituting the last four identities in the first identity and taking into account that $AC = BC$, we get the statement desired.

b) Let us denote the intersection points of lines $CM$ and $CN$ with base $AB$ by $M_1$ and $N_1$, respectively. We have to prove that $M_1 = N_1$. From heading a) it follows that $AM_1 : M_1B = AN_1 : N_1B$, i.e., $M_1 = N_1$.

**5.84.** Let segments $BM$ and $BN$ meet side $AC$ at points $P$ and $Q$, respectively. Then

$$\frac{\sin\angle PBB_1}{\sin PBA} = \frac{\sin\angle PBB_1}{\sin\angle BPB_1} \cdot \frac{\sin\angle APB}{\sin\angle PBA} = \frac{PB}{BB_1} \cdot \frac{AB}{PA}.$$

If $O$ is the intersection point of bisectors of triangle $ABC$, then $\frac{AP}{PB_1} \cdot \frac{B_1O}{OB} \cdot \frac{BC_1}{C_1A} = 1$ and, therefore,

$$\frac{\sin\angle PBB_1}{\sin\angle PBA} = \frac{AB}{BB_1} \cdot \frac{B_1O}{OB} \cdot \frac{BC_1}{C_1A}.$$

Observe that $BC_1 : C_1A = BC : CA$ and perform similar calculations for $\sin\angle QBB_1 : \sin\angle QBC$; we deduce that

$$\frac{\sin\angle PBB_1}{\sin\angle PBA} = \frac{\sin\angle QBB_1}{\sin\angle QBC}.$$

Since $\angle ABB_1 = \angle CBB_1$, we have: $\angle PBB_1 = \angle QBB_1$.

**5.85.** a) Let point $P$ lie on arc $\smile AC$ of the circumscribed circle of triangle $ABC$; let $A_1$, $B_1$ and $C_1$ be the bases of perpendiculars dropped from point $P$ to lines $BC$, $CA$ and $AB$. The sum of angles at vertices $A_1$ and $C_1$ of quadrilateral $A_1BC_1P$ is equal to $180°$, hence, $\angle A_1PC_1 = 180° - \angle B = \angle APC$. Therefore, $\angle APC_1 = \angle A_1PC$, where one of points $A_1$ and $C_1$ (say, $A_1$) lies on a side of the triangle and the other point lies on the extension of a side. Quadrilaterals $AB_1PC_1$ and $A_1B_1PC$ are inscribed ones, hence,

$$\angle AB_1C_1 = \angle APC_1 = \angle A_1PC = \angle A_1B_1C$$

and, therefore, point $B_1$ lies on segment $A_1C_1$.

b) By the same arguments as in heading a) we get

$$\angle(AP, PC_1) = \angle(AB_1, B_1C) = \angle(CB_1, B_1A_1) = \angle(CP, PA_1).$$

Add $\angle(PC_1, PC)$ to $\angle(AP, PC_1)$; we get

$$\angle(AP, PC) = \angle(PC_1, PA_1) = \angle(BC_1, BA_1) = \angle(AB, BC),$$

i.e., point $P$ lies on the circumscribed circle of triangle $ABC$.

**5.86.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of segments $PA$, $PB$ and $PC$, respectively; let $O_a$, $O_b$ and $O_c$ be the centers of the circumscribed circles of triangles $BCP$, $ACP$ and $ABP$, respectively. Points $A_1$, $B_1$ and $C_1$ are the bases of perpendiculars dropped from point

$P$ to sides of triangle $O_aO_bO_c$ (or their extensions). Points $A_1$, $B_1$ and $C_1$ lie on one line, hence, point $P$ lies on the circumscribed circle of triangle $O_aO_cO_c$ (cf. Problem 5.85, b).

**5.87.** Let the extension of the bisector $AD$ intersect the circumscribed circle of triangle $ABC$ at point $P$. Let us drop from point $P$ perpendiculars $PA_1$, $PB_1$ and $PC_1$ to lines $BC$, $CA$ and $AB$, respectively; clearly, $A_1$ is the midpoint of segment $BC$. The homothety centered at $A$ that sends $P$ to $D$ sends points $B_1$ and $C_1$ to $B'$ and $C'$ and, therefore, it sends point $A_1$ to $M$, because $M(???)$ lies on line $B_1C_1$ and $PA_1 \parallel DM$.

**5.88.** a) The solution of Problem 5.85 can be adapted without changes to this case.

b) Let $A_1$ and $B_1$ be the bases of perpendiculars dropped from point $P$ to lines $BC$ and $CA$, respectively, and let points $A_2$ and $B_2$ from lines $BC$ and $AC$, respectively, be such that $\angle(PA_2, BC) = \alpha = \angle(PB_2, AC)$. Then $\triangle PA_1A_2 \sim \triangle PB_1B_2$ hence, points $A_1$ and $B_1$ turn under a rotational homothety centered at $P$ into $A_2$ and $B_2$ and $\angle A_1PA_2 = 90° - \alpha$ is the angle of the rotation.

**5.89.** a) Let the angle between lines $PC$ and $AC$ be equal to $\varphi$. Then $PA = 2R\sin\varphi$. Since points $A_1$ and $B_1$ lie on the circle with diameter $PC$, the angle between lines $PA_1$ and $A_1B_1$ is also equal to $\varphi$. Hence, $PA_1 = \frac{d}{\sin\varphi}$ and, therefore, $PA \cdot PA_1 = 2Rd$.

b) Since $PA_1 \perp BC$, it follows that $\cos\alpha = \sin\varphi = \frac{d}{PA_1}$. It remains to notice that $PA_1 = \frac{2Rd}{PA}$.

**5.90.** Points $A_1$ and $B_1$ lie on the circle with diameter $PC$, hence, $A_1B_1 = PC\sin\angle A_1CB_1 = PC\sin\angle C$. Let the angle between lines $AB$ and $A_1B_1$ be equal to $\gamma$ and $C_1$ be the projection of point $P$ to line $A_1B_1$. Lines $A_1B_1$ and $B_1C_1$ coincide, hence, $\cos\gamma = \frac{PC}{2R}$ (cf. Problem 5.89). Therefore, the length of the projection of segment $AB$ to line $A_1B_1$ is equal to

$$AB\cos\gamma = \frac{(2R\sin\angle C)PC}{2R} = PC\sin\angle C.$$

**5.91.** Let $A_1$ and $B_1$ be the bases of perpendiculars dropped from point $P$ to lines $BC$ and $AC$. Points $A_1$ and $B_1$ lie on the circle with diameter $PC$. Since $\sin\angle A_1CB_1 = \sin\angle ACB$, the chords $A_1B_1$ of this circle are of the same length. Therefore, lines $A_1B_1$ are tangent to a fixed circle.

**5.92.** Let $A_1$ and $B_1$ be the bases of perpendiculars dropped from point $P$ to lines $BC$ and $CA$. Then

$$\angle(A_1B_1, PB_1) = \angle(A_1C, PC) = \frac{\smile BP}{2}.$$

It is also clear that for all points $P$ lines $PB_1$ have the same direction.

**5.93.** Let $P_1$ and $P_2$ be diametrically opposite points of the circumscribed circle of triangle $ABC$; let $A_i$ and $B_i$ be the bases of perpendiculars dropped from point $P_i$ to lines $BC$ and $AC$, respectively; let $M$ and $N$ be the midpoints of sides $AC$ and $BC$, respectively; let $X$ be the intersection point of lines $A_1B_1$ and $A_2B_2$, respectively. By Problem 5.92 $A_1B_1 \perp A_2B_2$. It remains to verify that $\angle(MX, XN) = \angle(BC, AC)$. Since $AB_2 = B_1C$, it follows that $XM$ is a median of right triangle $B_1XB_2$. Hence, $\angle(XM, XB_2) = \angle(XB_2, B_2M)$.

Similarly, $\angle(XA_1, XN) = \angle(A_1N, XA_1)$. Therefore,

$$\angle(MX, XN) = \angle(XM, XB_2) + \angle(XB_2, XA_1) + \angle(XA_1, XN) =$$
$$\angle(XB_2, B_2M) + \angle(A_1N, XA_1) + 90°.$$

Since
$$\angle(XB_2, B_2M) + \angle(AC, CB) + \angle(NA_1, A_1X) + 90° = 0°,$$
we have: $\angle(MN, XN) + \angle(AC, CB) = 0°$.

**5.94.** If point $R$ on the given circle is such that $\angle(\overrightarrow{OP}, \overrightarrow{OR}) = \frac{1}{2}(\beta + \gamma)$, then $OR \perp BC$. It remains to verify that $\angle(OR, OQ) = \angle(PA_1, A_1B_1)$. But $\angle(OR, OQ) = \frac{1}{2}\alpha$ and

$$\angle(PA_1, A_1B_1) = \angle(PB, BC_1) = \frac{\angle(\overrightarrow{OP}, \overrightarrow{OA})}{2} = \frac{\alpha}{2}.$$

**5.95.** Let lines $AC$ and $PQ$ meet at point $M$. In triangle $MPC$ draw heights $PB_1$ and $CA_1$. Then $A_1B_1$ is Simson's line of point $P$ with respect to triangle $ABC$. Moreover, by Problem 1.52 $\angle(MB_1, B_1A_1) = \angle(CP, PM)$. It is also clear that $\angle(CP, PM) = \angle(CA, AQ) = \angle(MB_1, AQ)$. Hence, $A_1B_1 \parallel AQ$.

**5.96.** Let us draw chord $PQ$ perpendicular to $BC$. Let points $H'$ and $P'$ be symmetric to points $H$ and $P$, respectively, through line $BC$; point $H'$ lies on the circumscribed circle of triangle $ABC$ (Problem 5.9). First, let us prove that $AQ \parallel P'H$. Indeed, $\angle(AH', AQ) = \angle(PH', PQ) = \angle(AH', P'H)$. Simson's line of point $P$ is parallel to $AQ$ (Problem 5.95), i.e., it passes through the midpoint of side $PP'$ of triangle $PP'H$ and is parallel to side $P'H$; hence, it passes through the midpoint of side $PH$.

**5.97.** Let $H_a$, $H_b$, $H_c$ and $H_d$ be the orthocenters of triangles $BCD$, $CDA$, $DAB$ and $ABC$, respectively. Lines $l_a$, $l_b$, $l_c$ and $l_d$ pass through the midpoints of segments $AH_a$, $BH_b$, $CH_c$ and $DH_d$, respectively (cf. Problem 5.96). The midpoints of these segments coincide with point $H$ such that $2\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$, where $O$ is the center of the circle (cf. Problem 13.33).

**5.98.** a) Let $B_1$, $C_1$ and $D_1$ be the projections of point $P$ to lines $AB$, $AC$ and $AD$, respectively. Points $B_1$, $C_1$ and $D_1$ lie on the circle with diameter $AP$. Lines $B_1C_1$, $C_1D_1$ and $D_1B_1$ are Simson's lines of point $P$ with respect to triangles $ABC$, $ACD$ and $ADB$, respectively. Therefore, projections of point $P$ to Simson's lines of these triangles lie on one line — Simson's line of triangle $B_1C_1D_1$.

We similarly prove that any triple of considered points lies on one line.

b) Let $P$ be a point of the circumscribed circle of $n$-gon $A_1 \ldots A_n$; let $B_2$, $B_3$, $\ldots$, $B_n$ be the projections of point $P$ to lines $A_1A_2$, $\ldots$, $A_1A_n$, respectively. Points $B_2$, $\ldots$, $B_n$ lie on the circle with diameter $A_1P$.

Let us prove by induction that Simson's line of point $P$ with respect to $n$-gon $A_1 \ldots A_n$ coincides with Simson's line of point $P$ with respect to $(n-1)$-gon $B_2 \ldots B_n$ (for $n = 4$ this had been proved in heading a)). By the inductive hypothesis Simson's line of the $(n-1)$-gon $A_1A_3 \ldots A_n$ coincides with Simson's line of $(n-2)$-gon $B_3 \ldots B_n$. Hence, the projections of point $P$ to Simson's line of $(n-1)$-gons whose vertices are obtained by consecutive deleting points $A_2, \ldots, A_n$ from the collection $A_1, \ldots, A_n$ ????? lie on Simson's line of the $(n-1)$-gon $B_2 \ldots B_n$. The projection of point $P$ to Simson's line of the $(n-1)$-gon $A_2 \ldots A_n$ lies on the same line, because our arguments show that any $n-1$ of the considered $n$ points of projections lie on one line.

**5.99.** Points $B_1$ and $C_1$ lie on the circle with diameter $AP$. Hence, $B_1C_1 = AP \sin \angle B_1AC_1 = AP\left(\frac{BC}{2R}\right)$.

**5.100.** This problem is a particular case of Problem 2.43.

**5.101.** Clearly,

$$\angle C_1AP = \angle C_1B_1P = \angle A_2B_1P = \angle A_2C_2P = \angle B_3C_2P = \angle B_3A_3P.$$

(The first, third and fifth equalities are obtained from the fact that the corresponding quadrilaterals are inscribed ones; the remaining equalities are obvious.) Similarly, $\angle B_1AP = \angle C_3A_3P$. Hence,

$$\angle B_3A_3C_3 = \angle B_3A_3P + \angle C_3A_3P = \angle C_1AP + \angle BAP = \angle BAC.$$

Similarly, the equalities of the remaining angles of triangles $ABC$ and $A_3B_3C_3$ are similarly obtained.

**5.102.** Let $A_1$, $B_1$ and $C_1$ be the bases of perpendiculars dropped from point $P$ to lines $BC$, $CA$ and $AB$, respectively; let $A_2$, $B_2$ and $C_2$ be the intersection points of lines $PA$, $PB$ and $PC$, respectively, with the circumscribed circle of triangle $ABC$. Further, let $S$, $S_1$ and $S_2$ be areas of triangles $ABC$, $A_1B_1C_1$ and $A_2B_2C_2$, respectively. It is easy to verify that $a_1 = \frac{a \cdot AP}{2R}$ (Problem 5.99) and $a_2 = \frac{a \cdot B_2P}{CP}$. Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar (Problem 5.100); hence, $\frac{S_1}{S_2} = k^2$, where $k = \frac{a_1}{a_2} = \frac{AP \cdot CP}{2R \cdot B_2P}$. Since $B_2P \cdot BP = |d^2 - R^2|$, we have:

$$\frac{S_1}{S_2} = \frac{(AP \cdot BP \cdot CP)^2}{4R^2(d^2 - R^2)^2}.$$

Triangles $A_2B_2C_2$ and $ABC$ are inscribed in one circle, hence, $\frac{S_2}{S} = \frac{a_2 b_2 c_2}{abc}$ (cf. Problem 12.1). It is also clear that, for instance,

$$\frac{a_2}{a} = \frac{B_2P}{CP} = \frac{|d^2 - R^2|}{BP \cdot CP}.$$

Therefore,

$$S_2 : S = |d^2 - R^2|^3 : (AP \cdot BP \cdot CP)^2.$$

Hence,

$$\frac{S_1}{S} = \frac{S_1}{S_2} \cdot \frac{S_2}{S} = \frac{|d^2 - R^2|}{4R^2}.$$

**5.103.** Points $B_1$ and $C_1$ lie on the circle with diameter $PA$ and, therefore, the midpoint of segment $PA$ is the center of the circumscribed circle of triangle $AB_1C_1$. Consequenly, $l_a$ is the midperpendicular to segment $B_1C_1$. Hence, lines $l_a$, $l_b$ and $l_c$ pass through the center of the circumscribed circle of triangle $A_1B_1C_1$.

**5.104.** a) Let us drop from points $P_1$ and $P_2$ perpendiculars $P_1B_1$ and $P_2B_2$, respectively, to $AC$ and perpendiculars $P_1C_1$ and $P_2C_2$ to $AB$. Let us prove that points $B_1$, $B_2$, $C_1$ and $C_2$ lie on one circle. Indeed,

$$\angle P_1B_1C_1 = \angle P_1AC_1 = \angle P_2AB_2 = \angle P_2C_2B_2;$$

and, since $\angle P_1B_1A = \angle P_2C_2A$, it follows that $\angle C_1B_1A = \angle B_2C_2A$. The center of the circle on which the indicated points lie is the intersection point of the midperpendiculars to segments $B_1B_2$ and $C_1C_2$; observe that both these perpendiculars pass through the midpoint $O$ of segment $P_1P_2$, i.e., $O$ is the center of this circle. In particular, points $B_1$ and $C_1$ are equidistant from point $O$. Similarly, points $A_1$ and $B_1$ are equidistant from point $O$, i.e., $O$ is the center of the circumscribed circle of triangle $A_1B_1C_1$. Moreover, $OB_1 = OB_2$.

b) The preceding proof passes virtually without changes in this case as well.

**5.105.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$. Triangles $A_1B_1C_1$ and $ABC$ are similar and the similarity coefficient is equal to 2. The heights of triangle $A_1B_1C_1$ intersect at point $O$; hence, $OA_1 : HA = 1 : 2$. Let $M'$ be the intersection point of segments $OH$ and $AA_1$. Then $OM' : M'H = OA_1 : HA = 1 : 2$ and $AM' : M'A_1 = OA_1 : HA = 1 : 2$, i.e., $M' = M$.

**5.106.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$, respectively; let $A_2$, $B_2$ and $C_2$ the bases of heights; $A_3$, $B_3$ and $C_3$ the midpoints of segments that connect the intersection point of heights with vertices. Since $A_2C_1 = C_1A = A_1B_1$ and $A_1A_2 \parallel B_1C_1$, point $A_2$ lies on the circumscribed circle of triangle $A_1B_1C_1$. Similarly, points $B_2$ and $C_2$ lie on the circumscribed circle of triangle $A_1B_1C_1$.

Now, consider circle $S$ with diameter $A_1A_3$. Since $A_1B_3 \parallel CC_2$ and $A_3B_3 \parallel AB$, it follows that $\angle A_1B_3A_3 = 90°$ and, therefore, point $B_3$ lies on $S$. We similarly prove that points $C_1$,

$B_1$ and $C_3$ lie on $S$. Circle $S$ passes through the vertices of triangle $A_1B_1C_1$; hence, it is its circumscribed circle.

The homothety with center $H$ and coefficient $\frac{1}{2}$ sends the circumscribed circle of triangle $ABC$ into the circumscribed circle of triangle $A_3B_3C_3$, i.e., into the circle of 9 points. Therefore, this homothety sends point $O$ into the center of the circle of nine points.

**5.107.** a) Let us prove that, for example, triangles $ABC$ and $HBC$ share the same circle of nine points. Indeed, the circles of nine points of these triangles pass through the midpoint of side $BC$ and the midpoints of segments $BH$ and $CH$.

b) Euler's line passes through the center of the circle of 9 points and these triangles share one circle of nine points.

c) The center of symmetry is the center of the circle of 9 points of these triangles.

**5.108.** Let $AB > BC > CA$. It is easy to verify that for an acute and an obtuse triangles the intersection point $H$ of heights and the center $O$ of the circumscribed circle are positioned precisely as on Fig. 59 (i.e., for an acute triangle point $O$ lies inside triangle $BHC_1$ and for an acute triangle points $O$ and $B$ lie on one side of line $CH$).
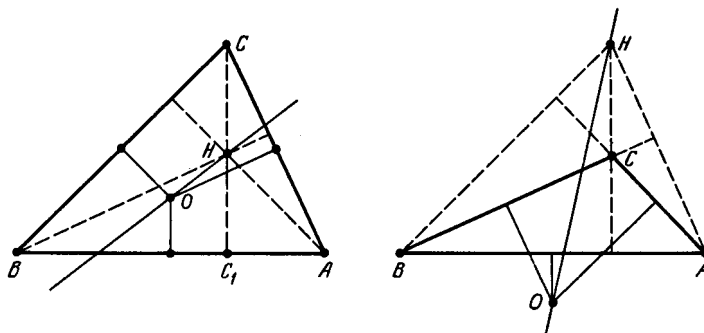


Figure 59 (Sol. 5.108)

Therefore, in an acute triangle Euler's line intersects the longest side $AB$ and the shortest side $AC$, whereas in an acute triangle it intersects the longest side $AB$, and side $BC$ of intermediate length.

**5.109.** a) Let $O_a$, $O_b$ and $O_c$ be the centers of the escribed circles of triangle $ABC$. The vertices of triangle $ABC$ are the bases of the heights of triangle $O_aO_bO_c$ (Problem 5.2) and, therefore, the circle of 9 points of triangle $O_aO_bO_c$ passes through point $A$, $B$ and $C$.

b) Let $O$ be the intersection point of heights of triangle $O_aO_bO_c$, i.e., the intersection point of the bisectors of triangle $ABC$. The circle of 9 points of triangle $O_aO_bO_c$ divides segment $OO_a$ in halves.

**5.110.** Let $AA_1$ be an height, $H$ the intersection point of heights. By Problem 5.45 b) $AH = 2R|\cos \angle A|$. The medians are divided by their intersection point in the ratio of 1:2, hence, Euler's line is parallel to $BC$ if and only if $AH : AA_1 = 2 : 3$ and vectors $\overrightarrow{AH}$ and $\overrightarrow{AA_1}$ are codirected, i.e.,

$$2R \cos \angle A : 2R \sin \angle B \sin \angle C = 2 : 3.$$

Taking into account that

$$\cos \angle A = -\cos(\angle B + \angle C) = \sin \angle B \sin \angle C - \cos \angle B \cos \angle C$$

we get

$$\sin \angle B \sin \angle C = 3 \cos \angle B \cos \angle C.$$

**5.111.** Let $CD$ be a height, $O$ the center of the circumscribed circle, $N$ the midpoint of side $AB$ and let point $E$ divide the segment that connects $C$ with the intersection point of the

heights in halves. Then $CENO$ is a parallelogram, hence, $\angle NED = \angle OCH = |\angle A - \angle B|$ (cf. Problem 2.88). Points $N$, $E$ and $D$ lie on the circle of 9 points, hence, segment $ND$ is seen from its center under an angle of $2\angle NED = 2|\angle A - \angle B|$.

**5.112.** Let $O$ and $I$ be the centers of the circumscribed and inscribed circles, respectively, of triangle $ABC$, let $H$ be the intersection point of the heights; lines $AI$ and $BI$ intersect the circumscribed circle at points $A_1$ and $B_1$. Suppose that triangle $ABC$ is not an isosceles one. Then $OI : IH = OA_1 : AH$ and $OI : IH = OB_1 : BH$. Since $OB_1 = OA_1$, we see that $AH = BH$ and, therefore, $AC = BC$. Contradiction.

**5.113.** Let $O$ and $I$ be the centers of the circumscribed and inscribed circles, respectively, of triangle $ABC$, $H$ the orthocenter of triangle $A_1B_1C_1$. In triangle $A_1B_1C_1$, draw heights $A_1A_2$, $B_1B_2$ and $C_1C_2$. Triangle $A_1B_1C_1$ is an acute one (e.g., $\angle B_1A_1C_1 = \frac{\angle B + \angle C}{2} < 90°$), hence, $H$ is the center of the inscribed circle of triangle $A_2B_2C_2$ (cf. Problem 1.56, a). The corresponding sides of triangles $ABC$ and $A_2B_2C_2$ are parallel (cf. Problem 1.54 a) and, therefore, there exists a homothety that sends triangle $ABC$ to triangle $A_2B_2C_2$. This homothety sends point $O$ to point $I$ and point $I$ to point $H$; hence, line $IH$ passes through point $O$.

**5.114.** Let $H$ be be the intersection point of the heights of triangle $ABC$, let $E$ and $M$ be the midpoints of segments $CH$ and $AB$, see Fig. 60. Then $C_1MC_2E$ is a rectangle.
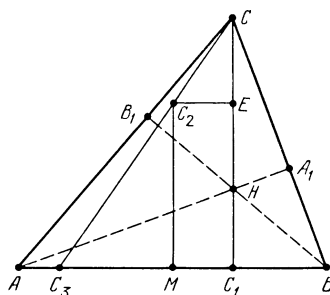


Figure 60 (Sol. 5.114)

Let line $CC_2$ meet line $AB$ at point $C_3$. Let us prove that $\overline{AC_3} : \overline{C_3B} = \tan 2\alpha : \tan 2\beta$. It is easy to verify that

$$\overline{C_3M} : \overline{C_2E} = \overline{MC_2} : \overline{EC}, \quad \overline{EC} = R\cos\gamma,$$
$$\overline{MC_2} = \overline{C_1E} = 2R\sin\alpha\sin\beta - R\cos\gamma$$
$$\text{and} \quad \overline{C_2E} = \overline{MC_1} = R\sin(\beta - \alpha)$$

Hence,

$$\overline{C_3M} = \frac{R\sin(\beta - \alpha)(2\sin\beta\sin\alpha - \cos\gamma)}{\cos\gamma} = \frac{R\sin(\beta - \alpha)\cos(\beta - \alpha)}{\cos\gamma}.$$

Therefore,

$$\frac{\overline{AC_3}}{\overline{C_3B}} = \frac{\overline{AM} + \overline{MC_3}}{\overline{C_3M} + \overline{MB}} = \frac{\sin 2\gamma + \sin 2(\alpha - \beta)}{\sin 2\gamma - \sin 2(\alpha - \beta)} = \frac{\tan 2\alpha}{\tan 2\beta}.$$

Similar arguments show that

$$\frac{\overline{AC_3}}{\overline{C_3B}} \cdot \frac{\overline{BA_3}}{\overline{A_3C}} \cdot \frac{\overline{CB_3}}{\overline{B_3A}} = \frac{\tan 2\alpha}{\tan 2\beta} \cdot \frac{\tan 2\beta}{\tan 2\gamma} \cdot \frac{\tan 2\gamma}{\tan 2\alpha} = 1.$$

**5.115.** Let us solve a more general heading b). First, let us prove that lines $AA_1$, $BB_1$ and $CC_1$ meet at one point. Let the circumscribed circles of triangles $A_1BC$ and $AB_1C$

intersect at point $O$. Then

$$\angle(BO, OA) \;\; = \angle(BO, OC) + \angle(OC, OA) = \angle(BA_1, A_1C) + \angle(CB_1, B_1A) =$$
$$= \angle(BA, AC_1) + \angle(C_1B, BA) = \angle(C_1B, AC_1),$$

i.e., the circumscribed circle of triangle $ABC_1$ also passes through point $O$. Hence,

$$\angle(AO, OA_1) = \angle(AO, OB) + \angle(BO, OA_1) = \angle(AC_1, C_1B + \angle(BC, CA_1) = 0°,$$

i.e., line $AA_1$ passes through point $O$. We similarly prove that lines $BB_1$ and $CC_1$ pass through point $O$.

Now, let us prove that point $O$ coincides with point $P$ we are looking for. Since $\angle BAP = \angle A - \angle CAP$, the equality $\angle ABP = \angle CAP$ is equivalent to the equality $\angle BAP + \angle ABP = \angle A$, i.e., $\angle APB = \angle B + \angle C$. For point $O$ the latter equality is obvious since it lies on the circumscribed circle of triangle $ABC_1$.

**5.116.** a) Let us prove that $\smile AB =\smile B_1C_1$, i.e., $AB = B_1C_1$. Indeed, $\smile AB =\smile AC_1 + \smile C_1B$ and $\smile C_1B =\smile AB_1$; hence, $\smile AB =\smile AC_1 + \smile AB_1 =\smile B_1C_1$.

b) Let us assume that triangles $ABC$ and $A_1B_1C_1$ are inscribed in one circle, where triangle $ABC$ is fixed and triangle $A_1B_1C_1$ rotates. Lines $AA_1$, $BB_1$ and $CC_1$ meet at one point for not more than one position of triangle $A_1B_1C_1$, see Problem 7.20 b). We can obtain 12 distinct families of triangles $A_1B_1C_1$: triangles $ABC$ and $A_1B_1C_1$ can be identified after a rotation or an axial symmetry; moreover, there are 6 distinct ways to associate symbols $A_1$, $B_1$ and $C_1$ to the vertices of the triangle.

From these 12 families of triangles 4 families can never produce the desired point $P$. For similarly oriented triangles the cases

$$\triangle ABC = \triangle A_1C_1B_1, \quad \triangle ABC = \triangle C_1B_1A_1, \quad \triangle ABC = \triangle B_1A_1C_1$$

are excluded: for example, if $\triangle ABC = \triangle A_1C_1B_1$, then point $P$ is the intersection point of line $BC = B_1C_1$ with the tangent to the circle at point $A = A_1$; in this case triangles $ABC$ and $A_1B_1C_1$ coincide.

For differently oriented triangles the case $\triangle ABC = \triangle A_1B_1C_1$ is excluded: in this case $AA_1 \parallel BB_1 \parallel CC_1$.

REMARK. Brokar's points correspond to differently oriented triangles; for the first Brokar's point $\triangle ABC = \triangle B_1C_1A_1$ and for the second Brokar's point we have $\triangle ABC = \triangle C_1A_1B_1$.

**5.117.** a) Since $PC = \frac{AC \sin \angle CAP}{\sin \angle APC}$ and $PC = \frac{BC \sin \angle CBP}{\sin \angle BPC}$, it follows that

$$\frac{\sin \varphi \sin \beta}{\sin \gamma} = \frac{\sin(\beta - \varphi) \sin \alpha}{\sin \beta}.$$

Taking into account that

$$\sin(\beta - \gamma) = \sin \beta \cos \varphi - \cos \beta \sin \varphi$$

we get $\cot \varphi = \cot \beta + \frac{\sin \beta}{\sin \alpha \sin \gamma}$. It remains to notice that

$$\sin \beta = \sin(\alpha + \gamma) = \sin \alpha \cos \gamma + \sin \gamma \cos \alpha.$$

b) For the second Brokar's angle we get precisely the same expression as in heading a). It is also clear that both Brokar's angles are acute ones.

c) Since $\angle A_1BC = \angle BCA$ and $\angle BCA_1 = \angle CAB$, it follows that $\triangle CA_1B \sim \triangle ABC$. Therefore, Brokar's point $P$ lies on segment $AA_1$ (cf. Problem 5.115 b)).

**5.118.** a) By Problem 10.38 a)

$$\cot \varphi = \cot \alpha + \cot \beta + \cot \gamma \geq \sqrt{3} = \cot 30°;$$

hence, $\varphi \leq 30°$.

b) Let $P$ be the first Brokar's point of triangle $ABC$. Point $M$ lies inside (or on the boundary of) one of the triangles $ABP$, $BCP$ and $CAP$. If, for example, point $M$ lies inside triangle $ABP$, then $\angle ABM \leq \angle ABP \leq 30°$.

**5.119.** Lines $A_1B_1$, $B_1C_1$ and $C_1A_1$ are the midperpendiculars to segments $AQ$, $BQ$ and $CQ$, respectively. Therefore, we have, for instance, $\angle B_1A_1C_1 = 180° - \angle AQC = \angle A$. For the other angles the proof is similar.

Moreover, lines $A_1O$, $B_1O$ and $C_1O$ are the midperpendiculars to segments $CA$, $AB$ and $BC$, respectively. Hence, acute angles $\angle OA_1C_1$ and $\angle ACQ$, for example, have pairwise perpendicular sides and, consecutively, they are equal. Similar arguments show that $\angle OA_1C_1 = \angle OB_1A_1 = \angle OC_1B_1 = \varphi$, where $\varphi$ is the Brokar's angle of triangle $ABC$.

**5.120.** By the law of sines

$$R_1 = \frac{AB}{2\sin\angle APB}, \quad R_2 = \frac{BC}{2\sin\angle BPC} \quad \text{and} \quad R_3 = \frac{CA}{2\sin\angle CPA}.$$

It is also clear that

$$\sin\angle APB = \sin\angle A, \quad \sin\angle BPC = \sin\angle B \quad \text{and} \quad \sin\angle CPA = \sin\angle C.$$

**5.121.** Triangle $ABC_1$ is an isosceles one and the angle at its base $AB$ is equal to Brokar's angle $\varphi$. Hence, $\angle(PC_1, C_1Q) = \angle(BC_1, C_1A) = 2\varphi$. Similarly

$$\angle(PA_1, A_1Q) = \angle(PB_1, B_1Q) = \angle(PC_1, C_1Q) = 2\varphi.$$

**5.122.** Since $\angle CA_1B_1 = \angle A + \angle AB_1A_1$ and $\angle AB_1A_1 = \angle CA_1C_1$, we have $\angle B_1A_1C_1 = \angle A$. We similarly prove that the remaining angles of triangles $ABC$ and $A_1B_1C_1$ are equal.

The circumscribed circles of triangles $AA_1B_1$, $BB_1C_1$ and $CC_1A_1$ meet at one point $O$. (Problem 2.80 a). Clearly, $\angle AOA_1 = \angle AB_1A_1 = \varphi$. Similarly, $\angle BOB_1 = \angle COC_1 = \varphi$. Hence, $\angle AOB = \angle A_1OB_1 = 180° - \angle A$. Similarly, $\angle BOC = 180° - \angle B$ and $\angle COA = 180° - \angle C$, i.e., $O$ is the first Brokar's point of both triangles. Hence, the rotational homothety by angle $\varphi$ with center $O$ and coefficient $\frac{AO}{A_1O}$ sends triangle $A_1B_1C_1$ to triangle $ABC$.

**5.123.** By the law of sines $\frac{AB}{BM} = \frac{\sin\angle AMB}{\sin\angle BAM}$ and $\frac{AB}{BN} = \frac{\sin\angle ANB}{\sin\angle BAN}$. Hence,

$$\frac{AB^2}{BM \cdot BN} = \frac{\sin\angle AMB \sin\angle ANB}{\sin\angle BAM \sin\angle BAN} = \frac{\sin\angle AMC \sin\angle ANC}{\sin\angle CAN \sin\angle CAM} = \frac{AC^2}{CM \cdot CN}.$$

**5.124.** Since $\angle BAS = \angle CAM$, we have

$$\frac{BS}{CM} = \frac{S_{BAS}}{S_{CAM}} = \frac{AB \cdot AS}{AC \cdot AM},$$

i.e., $\frac{AS}{AM} = \frac{2b \cdot BS}{ac}$. It remains to observe that, as follows from Problems 5.123 and 12.11 a), $BS = \frac{ac^2}{b^2+c^2}$ and $2AM = \sqrt{2b^2 + 2c^2 - a^2}$.

**5.125.** The symmetry through the bisector of angle $A$ sends segment $B_1C_1$ into a segment parallel to side $BC$, it sends line $AS$ to line $AM$, where $M$ is the midpoint of side $BC$.

**5.126.** On segments $BC$ and $BA$, take points $A_1$ and $C_1$, respectively, so that $A_1C_1 \parallel BK$. Since $\angle BAC = \angle CBK = \angle BA_1C_1$, segment $A_1C_1$ is antiparallel to side $AC$. On the other hand, by Problem 3.31 b) line $BD$ divides segment $A_1C_1$ in halves.

**5.127.** It suffices to make use of the result of Problem 3.30.

**5.128.** Let $AP$ be the common chord of the considered circles, $Q$ the intersection point of lines $AP$ and $BC$. Then

$$\frac{BQ}{AB} = \frac{\sin\angle BAQ}{\sin\angle AQB} \quad \text{and} \quad \frac{AC}{CQ} = \frac{\sin\angle AQC}{\sin\angle CAQ}.$$

Hence, $\frac{BQ}{CQ} = \frac{AB \sin \angle BAP}{AC \sin \angle CAP}$. Since $AC$ and $AB$ are tangents to circles $S_1$ and $S_2$, it follows that $\angle CAP = \angle ABP$ and $\angle BAP = \angle ACP$ and, therefore, $\angle APB = \angle APC$. Hence,

$$\frac{AB}{AC} = \frac{AB}{AP} \cdot \frac{AP}{AC} = \frac{\sin \angle APB}{\sin \angle ABP} \cdot \frac{\sin \angle ACP}{\sin \angle APC} = \frac{\sin \angle ACP}{\sin \angle ABP} = \frac{\sin \angle BAP}{\sin \angle CAP}.$$

It follows that $\frac{BQ}{CQ} = \frac{AB^2}{AC^2}$.

**5.129.** Let $S$ be the intersection point of lines $AX$ and $BC$. Then $\frac{AS}{AB} = \frac{CS}{CX}$ and $\frac{AS}{AC} = BSBX$ and, therefore,

$$\frac{CS}{BS} = \frac{AC}{AB} \cdot \frac{XC}{XB}.$$

It remains to observe that $\frac{XC}{XB} = \frac{AC}{AB}$ (see the solution of Problem 7.16 a)).

**5.130.** Let $L$, $M$ and $N$ be the midpoints of segments $CA, CB$ and $CH$. Since $\triangle BAC \sim \triangle CAH$, it follows that $\triangle BAM \sim \triangle CAN$ and, therefore, $\angle BAM = \angle CAN$. Similarly, $\angle ABL = \angle CBN$.

**5.131.** Let $B_1C_1$, $C_2A_2$ and $A_3B_3$ be given segments. Then triangles $A_2XA_3$, $B_1XB_3$ and $C_1XC_2$ are isosceles ones; let the lengths of their lateral sides be equal to $a$, $b$ and $c$. Line $AX$ divides segment $B_1C_1$ in halves if and only if this line contains a simedian. Hence, if $X$ is Lemoin's point, then $a = b$, $b = c$ and $c = a$. And if $B_1C_1 = C_2A_2 = A_3B_3$, then $b + c = c + a = a + b$ and, therefore, $a = b = c$.

**5.132.** Let $M$ be the intersection point of medians of triangle $ABC$; let $a_1$, $b_2$, $c_1$ and $a_2$, $b_2$, $c_2$ be the distances from points $K$ and $M$, respectively, to the sides of the triangle. Since points $K$ and $M$ are isogonally conjugate, $a_1a_2 = b_1b_2 = c_1c_2$. Moreover, $aa_2 = bb_2 = cc_2$ (cf. Problem 4.1). Therefore, $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$. Making use of this equality and taking into account that areas of triangles $A_1B_1K$, $B_1C_1K$ and $C_1A_1K$ are equal to $\frac{a_1b_1c}{4R}$, $\frac{b_1c_1a}{4R}$ and $\frac{c_1a_1b}{4R}$, respectively, where $R$ is the radius of the circumscribed circle of triangle $ABC$, we deduce that the areas of these triangles are equal. Moreover, point $K$ lies inside triangle $A_1B_1C_1$. Therefore, $K$ is the intersection point of medians of triangle $A_1B_1C_1$ (cf. Problem 4.2).

**5.133.** Medians of triangle $A_1B_1C_1$ intersect at point $K$ (Problem 5.132); hence, the sides of triangle $ABC$ are perpendicular to the medians of triangle $A_1B_1C_1$. After a rotation through an angle of $90°$ the sides of triangle $ABC$ become pairwise parallel to the medians of triangle $A_1B_1C_1$ and, therefore, the medians of triangle $ABC$ become parallel to the corresponding sides of triangle $A_1B_1C_1$ (cf. Problem 13.2). Hence, the medians of triangle $ABC$ are perpendicular to the corresponding sides of triangle $A_1B_1C_1$.

**5.134.** Let $A_2$, $B_2$ and $C_2$ be the projections of point $K$ to lines $BC$, $CA$ and $AB$, respectively. Then $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$ (Problem 5.100) and $K$ is the intersection point of medians of triangle $A_2B_2C_2$ (Problem 5.132). Hence, the similarity transformation that sends triangle $A_2B_2C_2$ to triangle $A_1B_1C_1$ sends point $K$ to the intersection point $M$ of medians of triangle $A_1B_1C_1$. Moreover, $\angle KA_2C_2 = \angle KBC_2 = \angle B_1A_1K$, i.e., points $K$ and $M$ are isogonally conjugate with respect to triangle $A_1B_1C_1$ and, therefore, $K$ is Lemoin's point of triangle $A_1B_1C_1$.

**5.135.** Let $K$ be Lemoin's point of triangle $ABC$; let $A_1$, $B_1$ and $C_1$ be the projections of point $K$ on the sides of triangle $ABC$; let $L$ be the midpoint of segment $B_1C_1$ and $N$ the intersection point of line $KL$ and median $AM$; let $O$ be the midpoint of segment $AK$ (Fig. 61). Points $B_1$ and $C_1$ lie on the circle with diameter $AK$, hence, by Problem 5.132 $OL \perp B_1C_1$. Moreover, $AN \perp B_1C_1$ (Problem 5.133) and $O$ is the midpoint of segment $AK$, consequently, $OL$ is the midline of triangle $AKN$ and $KL = LN$. Therefore, $K$ is the midpoint of segment $A_1N$. It remains to notice that the homothety with center $M$ that sends $N$ to $A$ sends segment $NA_1$ to height $AH$.
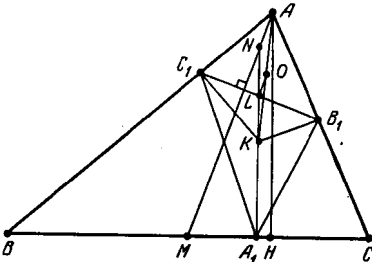
Figure 61 (Sol. 5.135)

# Chapter 6. POLYGONS

## Background

1) A polygon is called *a convex* one if it lies on one side of any line that connects two of its neighbouring vertices.

2) A convex polygon is called *a circumscribed* one if all its sides are tangent to a circle. A convex quadrilateral is a circumscribed one if and only if $AB + CD = BC + AD$.

A convex polygon is called *an inscribed* one if all its vertices lie on one circle. *A convex quadrilateral is an inscribed one if and only if*

$$\angle ABC + \angle CDA = \angle DAB + \angle BCD.$$

3) A convex polygon is called *a regular* one if all its sides are equal and all its angles are also equal.

A convex $n$-gon is a regular one if and only if under a rotation by the angle of $\frac{2\pi}{n}$ with center at point $O$ it turns into itself. This point $O$ is called *the center* of the regular polygon.

## Introductory problems

**1.** Prove that a convex quadrilateral $ABCD$ can be inscribed into a circle if and only if $\angle ABC + \angle CDA = 180°$.

**2.** Prove that a circle can be inscribed in a convex quadrilateral $ABCD$ if and only if $AB + CD = BC + AD$.

**3.** a) Prove that the axes of symmetry of a regular polygon meet at one point.

b) Prove that a regular $2n$-gon has a center of symmetry.

**4.** a) Prove that the sum of the angles at the vertices of a convex $n$-gon is equal to $(n - 2) \cdot 180°$.

b) A convex $n$-gon is divided by nonintersecting diagonals into triangles. Prove that the number of these triangles is equal to $n - 2$.

## §1. The inscribed and circumscribed quadrilaterals

**6.1.** Prove that if the center of the circle inscribed in a quadrilateral coincides with the intersection point of the quadrilateral's diagonals, then this quadrilateral is a rhombus.

**6.2.** Quadrilateral $ABCD$ is circumscribed about a circle centered at $O$. Prove that $\angle AOB + \angle COD = 180°$.

**6.3.** Prove that if there exists a circle tangent to all the sides of a convex quadrilateral $ABCD$ and a circle tangent to the extensions of all its sides then the diagonals of such a quadrilateral are perpendicular.

**6.4.** A circle singles out equal chords on all the four sides of a quadrilateral. Prove that a circle can be inscribed into this quadrilateral.

**6.5.** Prove that if a circle can be inscribed into a quadrilateral, then the center of this circle lies on one line with the centers of the diagonals.

**6.6.** Quadrilateral $ABCD$ is circumscribed about a circle centered at $O$. In triangle $AOB$ heights $AA_1$ and $BB_1$ are drawn. In triangle $COD$ heights $CC_1$ and $DD_1$ are drawn. Prove that points $A_1$, $B_1$, $C_1$ and $D_1$ lie on one line.

**6.7.** The angles at base $AD$ of trapezoid $ABCD$ are equal to $2\alpha$ and $2\beta$. Prove that the trapezoid is a circumscribed one if and only if $\frac{BC}{AD} = \tan\alpha \tan\beta$.

**6.8.** In triangle $ABC$, segments $PQ$ and $RS$ parallel to side $AC$ and a segment $BM$ are drawn as plotted on Fig. 62. Trapezoids $RPKL$ and $MLSC$ are circumscribed ones. Prove that trapezoid $APQC$ is also a circumscribed one.
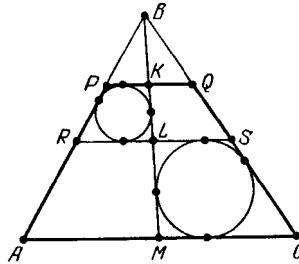

Figure 62 (6.8)

**6.9.** Given convex quadrilateral $ABCD$ such that rays $AB$ and $CD$ intersects at a point $P$ and rays $BC$ and $AD$ intersect at a point $Q$. Prove that quadrilateral $ABCD$ is a circumscribed one if and only if one of the following conditions hold:

$$AB + CD = BC + AD, \quad AP + CQ = AQ + CP \quad BP + BQ = DP + DQ.$$

**6.10.** Through the intersection points of the extension of sides of convex quadrilateral $ABCD$ two lines are drawn that divide it into four quadrilaterals. Prove that if the quadrilaterals adjacent to vertices $B$ and $D$ are circumscribed ones, then quadrilateral $ABCD$ is also a circumscribed one.

**6.11.** Prove that the intersection point of the diagonals of a circumscribed quadrilateral coincides with the intersection point of the diagonals of the quadrilateral whose vertices are the tangent points of the sides of the initial quadrilateral with the inscribed circle.

* * *

**6.12.** Quadrilateral $ABCD$ is an inscribed one; $H_c$ and $H_d$ are the orthocenters of triangles $ABD$ and $ABC$ respectively. Prove that $CDH_cH_d$ is a parallelogram.

**6.13.** Quadrilateral $ABCD$ is an inscribed one. Prove that the centers of the inscribed circles of triangles $ABC$, $BCD$, $CDA$ and $DAB$ are the vertices of a rectangle.

**6.14.** The extensions of the sides of quadrilateral $ABCD$ inscribed in a circle centered at $O$ intersect at points $P$ and $Q$ and its diagonals intersect at point $S$.

a) The distances from points $P$, $Q$ and $S$ to point $O$ are equal to $p$, $q$ and $s$, respectively, and the radius of the circumscribed circle is equal to $R$. Find the lengths of the sides of triangle $PQS$.

b) Prove that the heights of triangle $PQS$ intersect at point $O$.

* * *

**6.15.** Diagonal $AC$ divides quadrilateral $ABCD$ into two triangles whose inscribed circles are tangent to diagonal $AC$ at one point. Prove that the inscribed circles of triangle $ABD$ and $BCD$ are also tangent to diagonal $BD$ at one point and their tangent points with the sides of the quadrilateral lie on one circle.

**6.16.** Prove that the projections of the intersection point of the diagonals of the inscribed quadrilateral to its sides are vertices of a circumscribed quadrilateral only if the projections do not lie on the extensions of the sides.

**6.17.** Prove that if the diagonals of a quadrilateral are perpendicular, then the projections of the intersection points of the diagonals on its sides are vertices of an inscribed quadrilateral.

See also Problem 13.33, 13.34, 16.4.

## §2. Quadrilaterals

**6.18.** The angle between sides $AB$ and $CD$ of quadrilateral $ABCD$ is equal to $\varphi$. Prove that

$$AD^2 = AB^2 + BC^2 + CD^2 - 2(AB \cdot BC \cos B + BC \cdot CD \cos C + CD \cdot AB \cos \varphi).$$

**6.19.** In quadrilateral $ABCD$, sides $AB$ and $CD$ are equal and rays $AB$ and $DC$ intersect at point $O$. Prove that the line that connects the midpoints of the diagonals is perpendicular to the bisector of angle $AOD$.

**6.20.** On sides $BC$ and $AD$ of quadrilateral $ABCD$, points $M$ and $N$, respectively, are taken so that $BM : MC = AN : ND = AB : CD$. Rays $AB$ and $DC$ intersect at point $O$. Prove that line $MN$ is parallel to the bisector of angle $AOD$.

**6.21.** Prove that the bisectors of the angles of a convex quadrilateral form an inscribed quadrilateral.

**6.22.** Two distinct parallelograms $ABCD$ and $A_1B_1C_1D_1$ with corresponding parallel sides are inscribed into quadrilateral $PQRS$ (points $A$ and $A_1$ lie on side $PQ$, points $B$ and $B_1$ lie on side $QR$, etc.). Prove that the diagonals of the quadrilateral are parallel to the corresponding sides of the parallelograms.

**6.23.** The midpoints $M$ and $N$ of diagonals $AC$ and $BD$ of convex quadrilateral $ABCD$ do not coincide. Line $MN$ intersects sides $AB$ and $CD$ at points $M_1$ and $N_1$. Prove that if $MM_1 = NN_1$, then $AD \parallel BC$.

**6.24.** Prove that two quadrilaterals are similar if and only if four of their corresponding angles are equal and the corresponding angles between the diagonals are also equal.

**6.25.** Quadrilateral $ABCD$ is a convex one; points $A_1$, $B_1$, $C_1$ and $D_1$ are such that $AB \parallel C_1D_1$ and $AC \parallel B_1D_1$, etc. for all pairs of vertices. Prove that quadrilateral $A_1B_1C_1D_1$ is also a convex one and $\angle A + \angle C_1 = 180°$.

**6.26.** From the vertices of a convex quadrilateral perpendiculars are dropped on the diagonals. Prove that the quadrilateral with vertices at the basis of the perpendiculars is similar to the initial quadrilateral.

**6.27.** A convex quadrilateral is divided by the diagonals into four triangles. Prove that the line that connects the intersection points of the medians of two opposite triangles is perpendicular to the line that connects the intersection points of the heights of the other two triangles.

**6.28.** The diagonals of the circumscribed trapezoid $ABCD$ with bases $AD$ and $BC$ intersect at point $O$. The radii of the inscribed circles of triangles $AOD$, $AOB$, $BOC$ and $COD$ are equal to $r_1$, $r_2$, $r_3$ and $r_4$, respectively. Prove that $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$.

**6.29.** A circle of radius $r_1$ is tangent to sides $DA$, $AB$ and $BC$ of a convex quadrilateral $ABCD$; a circle of radius $r_2$ is tangent to sides $AB$, $BC$ and $CD$; the radii $r_3$ and $r_4$ are similarly defined. Prove that $\frac{AB}{r_1} + \frac{CD}{r_3} = \frac{BC}{r_2} + \frac{AD}{r_4}$.

**6.30.** A quadrilateral $ABCD$ is convex and the radii of the circles inscribed in triangles $ABC$, $BCD$, $CDA$ and $DAB$ are equal. Prove that $ABCD$ is a rectangle.

**6.31.** Given a convex quadrilateral $ABCD$ and the centers $A_1$, $B_1$, $C_1$ and $D_1$ of the circumscribed circles of triangles $BCD$, $CDA$, $DAB$ and $ABC$, respectively. For quadrilateral $A_1B_1C_1D_1$ points $A_2$, $B_2$, $C_2$ and $D_2$ are similarly defined. Prove that quadrilaterals $ABCD$ and $A_2B_2C_2D_2$ are similar and their similarity coefficient is equal to

$$\frac{1}{4}\left|(\cot A + \cot C)(\cot B + \cot D)\right|.$$

**6.32.** Circles whose diameters are sides $AB$ and $CD$ of a convex quadrilateral $ABCD$ are tangent to sides $CD$ and $AB$, respectively. Prove that $BC \parallel AD$.

**6.33.** Four lines determine four triangles. Prove that the orthocenters of these triangles lie on one line.

## §3. Ptolemy's theorem

**6.34.** Quadrilateral $ABCD$ is an inscribed one. Prove that

$$AB \cdot CD + AD \cdot BC = AC \cdot BD \quad \text{(Ptolemy's theorem)}.$$

**6.35.** Quadrilateral $ABCD$ is an inscribed one. Prove that

$$\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{BA \cdot BC + DA \cdot DC}.$$

**6.36.** Let $\alpha = \frac{\pi}{7}$. Prove that

$$\frac{1}{\sin \alpha} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 3\alpha}.$$

**6.37.** The distances from the center of the circumscribed circle of an acute triangle to its sides are equal to $d_a$, $d_b$ and $d_c$. Prove that $d_a + d_b + d_c = R + r$.

**6.38.** The bisector of angle $\angle A$ of triangle $ABC$ intersects the circumscribed circle at point $D$. Prove that $AB + AC \leq 2AD$.

**6.39.** On arc $\smile CD$ of the circumscribed circle of square $ABCD$ point $P$ is taken. Prove that $PA + PC = \sqrt{2}PB$.

**6.40.** Parallelogram $ABCD$ is given. A circle passing through point $A$ intersects segments $AB$, $AC$ and $AD$ at points $P$, $Q$ and $R$, respectively. Prove that

$$AP \cdot AB + AR \cdot AD = AQ \cdot AC.$$

**6.41.** On arc $\smile A_1A_{2n+1}$ of the circumscribed circle $S$ of a regular $(2n+1)$-gon $A_1 \ldots A_{2n+1}$ a point $A$ is taken. Prove that:

a) $d_1 + d_3 + \cdots + d_{2n+1} = d_2 + d_4 + \cdots + d_{2n}$, where $d_i = AA_i$;

b) $l_1 + \cdots + l_{2n+1} = l_2 + \cdots + l_{2n}$, where $l_i$ is the length of the tangent drawn from point $A$ to the circle of radius $r$ tangent to $S$ at point $A_i$ (all the tangent points are simultaneously either inner or outer ones).

**6.42.** Circles of radii $x$ and $y$ are tangent to a circle of radius $R$ and the distance between the tangent points is equal to $a$. Calculate the length of the following common tangent to the first two circles:

a) the outer one if both tangents are simultaneously either outer or inner ones;

b) the inner one if one tangent is an inner one and the other one is an outer one.

**6.43.** Circles $\alpha$, $\beta$, $\gamma$ and $\delta$ are tangent to a given circle at vertices $A$, $B$, $C$ and $D$, respectively, of convex quadrilateral $ABCD$. Let $t_{\alpha\beta}$ be the length of the common tangent to circles $\alpha$ and $\beta$ (the outer one if both tangent are simultaneously either inner or outer

ones and the inner one if one tangent is an inner one and the other one is an outer one); $t_{\beta\gamma}$, $t_{\gamma\delta}$, etc. are similarly determined. Prove that

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta} \qquad \text{(The \textit{generalized Ptolemy's theorem})}$$

See also Problem 9.67.

## §4. Pentagons

**6.44.** In an equilateral (non-regular) pentagon $ABCDE$ we have angle $\angle ABC = 2\angle DBE$. Find the value of angle $\angle ABC$.

**6.45.** a) Diagonals $AC$ and $BE$ of a regular pentagon $ABCDE$ intersect at point $K$. Prove that the inscribed circle of triangle $CKE$ is tangent to line $BC$.

b) Let $a$ be the length of the side of a regular pentagon, $d$ the length of its diagonal. Prove that $d^2 = a^2 + ad$.

**6.46.** Prove that a square can be inscribed in a regular pentagon so that the vertices of the square would lie on four sides of the pentagon.
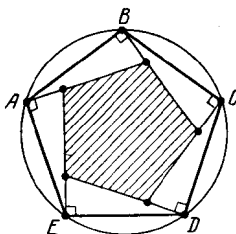


Figure 63 (6.46)

**6.47.** Regular pentagon $ABCDE$ with side $a$ is inscribed in circle $S$. The lines that pass through the pentagon's vertices perpendicularly to the sides form a regular pentagon with side $b$ (Fig. 63). A side of a regular pentagon circumsribed about circle $S$ is equal to $c$. Prove that $a^2 + b^2 = c^2$.

See also Problems 2.59, 4.9, 9.23, 9.44, 10.63, 10.67, 13.10, 13.56, 20.11.

## §5. Hexagons

**6.48.** The opposite sides of a convex hexagon $ABCDEF$ are pairwise parallel. Prove that:

a) the area of triangle $ACE$ constitutes not less than a half area of the hexagon.

b) the areas of triangles $ACE$ and $BDF$ are equal.

**6.49.** All the angles of a convex hexagon $ABCDEF$ are equal. Prove that

$$|BC - EF| = |DE - AB| = |AF - CD|.$$

**6.50.** The sums of the angles at vertices $A$, $C$, $E$ and $B$, $D$, $F$ of a convex hexagon $ABCDEF$ with equal sides are equal. Prove that the opposite sides of this hexagon are parallel.

**6.51.** Prove that if in a convex hexagon each of the three diagonals that connect the opposite vertices divides the area in halves then these diagonals intersect at one point.

**6.52.** Prove that if in a convex hexagon each of the three segments that connect the midpoints of the opposite sides divides the area in halves then these segments intersect at one point.

See also problems 2.11, 2.20, 2.46, 3.66, 4.6, 4.28, 4.31, 5.80, 9.45 a), 9.76–9.78, 13.3, 14.6, 18.22, 18.23.

## §6. Regular polygons

**6.53.** The number of sides of a polygon $A_1 \ldots A_n$ is odd. Prove that:

a) if this polygon is an inscribed one and all its angles are equal, then it is a regular polygon;

b) if this polygon is a circumscribed one and all its sides are equal, then it is a regular polygon.

**6.54.** All the angles of a convex polygon $A_1 \ldots A_n$ are equal; an inner point $O$ of the polygon is the vertex of equal angles that subtend all the polygon's sides. Prove that the polygon is a regular one.

**6.55.** A paper band of constant width is tied in a simple knot and then tightened in order to make the knot flat, cf. Fig. 64. Prove that the knot is of the form of a regular pentagon.
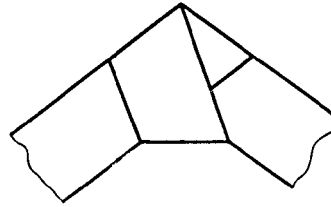


Figure 64 (6.55)

**6.56.** On sides $AB$, $BC$, $CD$ and $DA$ of square $ABCD$ equilateral triangles $ABK$, $BCL$, $CDM$ and $DAN$ are constructed inwards. Prove that the midpoints of sides of these triangles (which are not the sides of a square) and the midpoints of segments $KL$, $LM$, $MN$ and $NK$ form a regular 12-gon.

$$* \, * \, *$$

**6.57.** Does there exist a regular polygon the length of one of whose diagonal is equal to the sum of lengths of some other two diagonals?

**6.58.** A regular $(4k+2)$-gon is inscribed in a circle of radius $R$ centered at $O$. Prove that the sum of the lengths of segments singled out by the legs of angle $\angle A_k O A_{k+1}$ on lines $A_1 A_{2k}$, $A_2 A_{2k-1}, \ldots, A_k A_{k+1}$ is equal to $R$.

**6.59.** In regular 18-gon $A_1 \ldots A_{18}$, diagonals $A_a A_d$, $A_b A_e$ and $A_c A_f$ are drawn. Let $k = a - b$, $p = b - c$, $m = c - d$, $q = d - e$, $n = e - f$ and $r = f - a$. Prove that the indicated diagonals intersect at one point in any of the following cases and only in these cases:

a) $\overrightarrow{k, m, n} = \overrightarrow{p, q, r}$;

b) $\overrightarrow{k, m, n} = \overrightarrow{1, 2, 7}$ and $\overrightarrow{p, q, r} = \overrightarrow{1, 3, 4}$;

c) $\overrightarrow{k, m, n} = \overrightarrow{1, 2, 8}$ and $\overrightarrow{p, q, r} = \overrightarrow{2, 2, 3}$.

REMARK. The equality $\overrightarrow{k, m, n} = \overrightarrow{x, y, z}$ means that the indicated tuples of numbers coincide; the order in which they are written in not taken into account.

**6.60.** In a regular 30-gon three diagonals are drawn. For them define tuples $\overrightarrow{k, m, n}$ and $\overrightarrow{p, q, r}$ as in the preceding problem. Prove that if $\overrightarrow{k, m, n} = \overrightarrow{1, 3, 14}$ and $\overrightarrow{p, q, r} = \overrightarrow{2, 2, 8}$, then the diagonals intersect at one point.

**6.61.** In a regular $n$-gon ($n \geq 3$) the midpoints of all its sides and the diagonals are marked. What is the greatest number of marked points that lie on one circle?

**6.62.** The vertices of a regular $n$-gon are painted several colours so that the points of one colour are the vertices of a regular polygon. Prove that among these polygons there are two equal ones.

**6.63.** Prove that for $n \geq 6$ a regular $(n-1)$-gon is impossible to inscribe in a regular $n$-gon so that on every side of the $n$-gon except one there lies exactly one vertex of the $(n-1)$-gon.

* * *

**6.64.** Let $O$ be the center of a regular $n$-gon $A_1 \ldots A_n$ and $X$ an arbitrary point. Prove that

$$\overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n} = \vec{0} \quad \text{and} \quad \overrightarrow{XA_1} + \cdots + \overrightarrow{XA_n} = n\overrightarrow{XO}.$$

**6.65.** Prove that it is possible to place real numbers $x_1, \ldots, x_n$ all distinct from zero in the vertices of a regular $n$-gon so that for any regular $k$-gon all vertices of which are vertices of the initial $n$-gon the sum of the numbers at the vertices of the $k$-gon is equal to zero.

**6.66.** Point $A$ lies inside regular 10-gon $X_1 \ldots X_{10}$ and point $B$ outside it. Let $\mathbf{a} = \overrightarrow{AX_1} + \ldots \overrightarrow{AX_{10}}$ and $\mathbf{b} = \overrightarrow{BX_1} + \ldots \overrightarrow{BX_{10}}$. Is it possible that $|\mathbf{a}| > |\mathbf{b}|$?

**6.67.** A regular polygon $A_1 \ldots A_n$ is inscribed in the circle of radius $R$ centered at $O$; let $X$ be an arbitrary point. Prove that

$$A_1 X^2 + \cdots + A_n X^2 = n(R^2 + d^2), \quad \text{where} \quad d = OX.$$

**6.68.** Find the sum of squares of the lengths of all the sides and diagonals of a regular $n$-gon inscribed in a circle of radius $R$.

**6.69.** Prove that the sum of distances from an arbitrary $X$ to the vertices of a regular $n$-gon is the least if $X$ is the center of the $n$-gon.

**6.70.** A regular $n$-gon $A_1 \ldots A_n$ is inscribed in the circle of radius $R$ centered at $O$; let $\mathbf{e}_i = \overrightarrow{OA_i}$ and $\mathbf{x} = \overrightarrow{OX}$ be an arbitrary vector. Prove that

$$\sum (\mathbf{e}_i, \mathbf{x})^2 = \frac{nR^2 \cdot OX^2}{2}.$$

**6.71.** Find the sum of the squared distances from the vertices of a regular $n$-gon inscribed in a circle of radius $R$ to an arbitrary line that passes through the center of the $n$-gon.

**6.72.** The distance from point $X$ to the center of a regular $n$-gon is equal to $d$ and $r$ is the radius of the inscribed circle of the $n$-gon. Prove that the sum of squared distances from point $X$ to the lines that contain the sides of the $n$-gon is equal to $n\left(r^2 + \frac{d^2}{2}\right)$.

**6.73.** Prove that the sum of squared lengths of the projections of the sides of a regular $n$-gon to any line is equal to $\frac{1}{2}na^2$, where $a$ is the length of the side of the $n$-gon.

**6.74.** A regular $n$-gon $A_1 \ldots A_n$ is inscribed in a circle of radius $R$; let $X$ be a point on this circle. Prove that

$$XA_1^4 + \cdots + XA_n^4 = 6nR^4.$$

**6.75.** a) A regular $n$-gon $A_1 \ldots A_n$ is inscribed in the circle of radius 1 centered at 0, let $\mathbf{e}_i = \overrightarrow{OA_i}$ and $\mathbf{u}$ an arbitrary vector. Prove that $\sum (\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i = \frac{1}{2}n\mathbf{u}$.

b) From an arbitrary point $X$ perpendiculars $XA_1, \ldots, XA_n$ are dropped to the sides (or their extensions) of a regular $n$-gon. Prove that $\sum \overrightarrow{XA_i} = \frac{1}{2}n\overrightarrow{XO}$, where $O$ is the center of the $n$-gon.

**6.76.** Prove that if the number $n$ is not a power of a prime, then there exists a convex $n$-gon with sides of length 1, 2, $\ldots, n$, all the angles of which are equal.

See also Problems 2.9, 4.59, 4.62, 6.36, 6.41, 6.45–6.47, 9.83, 9.84, 11.46, 11.48, 17.31, 18.30, 19.47, 23.8, 24.2.

## §7. The inscribed and circumscribed polygons

**6.77.** On the sides of a triangle three squares are constructed outwards. What should be the values of the angles of the triangle in order for the six vertices of these squares distinct from the vertices of the triangle belong to one circle?

**6.78.** A $2n$-gon $A_1 \ldots A_{2n}$ is inscribed in a circle. Let $p_1$, $\ldots$, $p_{2n}$ be the distances from an arbitrary point $M$ on the circle to sides $A_1A_2$, $A_2A_3$, $\ldots$, $A_{2n}A_1$. Prove that $p_1 p_3 \ldots p_{2n-1} = p_2 p_4 \ldots p_{2n}$.

**6.79.** An inscribed polygon is divided by nonintersecting diagonals into triangles. Prove that the sum of radii of all the circles inscribed in these triangles does not depend on the partition.

**6.80.** Two $n$-gons are inscribed in one circle and the collections of the length of their sides are equal but the corresponding sides are not necessarily equal. Prove that the areas of these polygons are equal.

**6.81.** Positive numbers $a_1$, $\ldots$, $a_n$ are such that $2a_i < a_1 + \cdots + a_n$ for all $i = 1, \ldots,$ $n$. Prove that there exists an inscribed $n$-gon the lengths of whose sides are equal to $a_1, \ldots,$ $a_n$.

* * *

**6.82.** A point inside a circumscribed $n$-gon is connected by segments with all the vertices and tangent points. The triangles formed in this way are alternately painted red and blue. Prove that the product of areas of red triangles is equal to the product of areas of blue triangles.

**6.83.** In a $2n$-gon ($n$ is odd) $A_1 \ldots A_{2n}$ circumscribed about a circle centered at $O$ the diagonals $A_1A_{n+1}$, $A_2A_{n+2}$, $\ldots$, $A_{n-1}A_{2n-1}$ pass through point $O$. Prove that the diagonals $A_nA_{2n}$ also passes through point $O$.

**6.84.** A circle of radius $r$ is tangent to the sides of a polygon at points $A_1$, $\ldots$, $A_n$ and the length of the side on which point $A_i$ lies is equal to $a_i$. The distance from point $X$ to the center of the circle is equal to $d$. Prove that

$$a_1 X A_1^2 + \cdots + a_n X A_n^2 = P(r^2 + d^2),$$

where $P$ is the perimeter of the polygon.

**6.85.** An $n$-gon $A_1 \ldots A_n$ is circumscribed about a circle; $l$ is an arbitrary tangent to the circle that does not pass through any vertex of the $n$-gon. Let $a_i$ be the distance from vertex $A_i$ to line $l$ and $b_i$ the distance from the tangent point of side $A_iA_{i+1}$ with the circle to line $l$. Prove that:

a) the value $\frac{b_1 \ldots b_n}{a_1 \ldots a_n}$ does not depend on the choice of line $l$;

b) the value $\frac{a_1 a_3 \ldots a_{2m-1}}{a_2 a_4 \ldots a_{2m}}$ does not depend on the choice of line $l$ if $n = 2m$.

**6.86.** Certain sides of a convex polygon are red; the other ones are blue. The sum of the lengths of the red sides is smaller than the semiperimeter and there is no pair of neighbouring blue sides. Prove that it is impossible to inscribe this polygon in a circle.

See also Problems 2.12, 4.39, 19.6.

## §8. Arbitrary convex polygons

**6.87.** What is the greatest number of acute angles that a convex polygon can have?

**6.88.** How many sides whose length is equal to the length of the longest diagonal can a convex polygon have?

**6.89.** For which $n$ there exists a convex $n$-gon one side of which is of length 1 and the lengths of the diagonals are integers?

**6.90.** Can a convex non-regular pentagon have exactly four sides of equal length and exactly four diagonals of equal lengths? Can the fifth side of such a pentagon have a common point with the fifth diagonal?

**6.91.** Point $O$ that lies inside a convex polygon forms, together with each two of its vertices, an isosceles triangle. Prove that point $O$ is equidistant from the vertices of this polygon.

See also Problems 4.49, 4.50, 9.82, 9.85, 9.86, 11.35, 13.14, 14.26, 16.8, 17.33, 17.34, 19.9, 23.13, 23.15.

## §9. Pascal's theorem

**6.92.** Prove that the intersection points of the opposite sides (if these sides are not parallel) of an inscribed hexagon lie on one line. (*Pascal's theorem.*)

**6.93.** Point $M$ lies on the circumscribed cirlce of triangle $ABC$; let $R$ be an arbitrary point. Lines $AR$, $BR$ and $CR$ intersect the circumscribed circle at points $A_1$, $B_1$ and $C_1$, respectively. Prove that the intersection points of lines $MA_1$ and $BC$, $MB_1$ and $CA$, $MC_1$ and $AB$ lie on one line and this line passes through point $R$.

**6.94.** In triangle $ABC$, heights $AA_1$ and $BB_1$ and bisectors $AA_2$ and $BB_2$ are drawn; the inscribed circle is tangent to sides $BC$ and $AC$ at points $A_3$ and $B_3$, respectively. Prove that lines $A_1B_1$, $A_2B_2$ and $A_3B_3$ either intersect at one point or are parallel.

**6.95.** Quadrilateral $ABCD$ is inscribed in circle $S$; let $X$ be an arbitrary point, $M$ and $N$ be the other intersection points of lines $XA$ and $XD$ with circle $S$. Lines $DC$ and $AX$, $AB$ and $DX$ intersect at points $E$ and $F$, respectively. Prove that the intersection point of lines $MN$ and $EF$ lies on line $BC$.

**6.96.** Points $A$ and $A_1$ that lie inside a circle centered at $O$ are symmetric through point $O$. Rays $AP$ and $A_1P_1$ are codirected, rays $AQ$ and $A_1Q_1$ are also codirected. Prove that the intersection point of lines $P_1Q$ and $PQ_1$ lies on line $AA_1$. (Points $P$, $P_1$, $Q$ and $Q_1$ lie on the circle.)

**6.97.** On a circle, five points are given. With the help of a ruler only construct a sixth point on this circle.

**6.98.** Points $A_1$, ..., $A_6$ lie on one circle and points $K$, $L$, $M$ and $N$ lie on lines $A_1A_2$, $A_3A_4$, $A_1A_6$ and $A_4A_5$, respectively, so that $KL \parallel A_2A_3$, $LM \parallel A_3A_6$ and $MN \parallel A_6A_5$. Prove that $NK \parallel A_5A_2$.

## Problems for independent study

**6.99.** Prove that if $ABCD$ is a rectangle and $P$ is an arbitrary point, then $AP^2 + CP^2 = DP^2 + BP^2$.

**6.100.** The diagonals of convex quadrilateral $ABCD$ are perpendicular. On the sides of the quadrilateral, squares centered at $P$, $Q$, $R$ and $S$ are constructed outwards. Prove that segment $PR$ passes through the intersection point of diagonals $AC$ and $BD$ so that $PR = \frac{1}{2}(AC + BD)$.

**6.101.** On the longest side $AC$ of triangle $ABC$, points $A_1$ and $C_1$ are taken so that $AC_1 = AB$ and $CA_1 = CB$ and on sides $AB$ and $BC$ points $A_2$ and $C_2$ are taken so that $AA_1 = AA_2$ and $CC_1 = CC_2$. Prove that quadrilateral $A_1A_2C_2C_1$ is an inscribed one.

**6.102.** A convex 7-gon is inscribed in a circle. Prove that if certain three of its angles are equal to 120° each, then some two of its sides are equal.

**6.103.** In plane, there are given a regular $n$-gon $A_1 \ldots A_n$ and point $P$. Prove that from segments $A_1P$, ..., $A_nP$ a closed broken line can be constructed.

**6.104.** Quadrilateral $ABCD$ is inscribed in circle $S_1$ and circumscribed about circle $S_2$; let $K$, $L$, $M$ and $N$ be tangent points of its sides with circle $S_2$. Prove that $KM \perp LN$.

**6.105.** Pentagon $ABCDE$ the lengths of whose sides are integers and $AB = CD = 1$ is circumscribed about a circle. Find the length of segment $BK$, where $K$ is the tangent point of side $BC$ with the circle.

**6.106.** Prove that in a regular $2n$-gon $A_1 \ldots A_{2n}$ the diagonals $A_1A_{n+2}$, $A_{2n-1}A_3$ and $A_{2n}A_5$ meet at one point.

**6.107.** Prove that in a regular 24-gon $A_1 \ldots A_{24}$ diagonals $A_1A_7$, $A_3A_{11}$ and $A_5A_{21}$ intersect at a point that lies on diameter $A_4A_{16}$.

## Solutions

**6.1.** Let $O$ be the center of the inscribed circle and the intersection point of the diagonals of quadrilateral $ABCD$. Then $\angle ACB = \angle ACD$ and $\angle BAC = \angle CAD$. Hence, triangles $ABC$ and $ADC$ are equal, since they have a common side $AC$. Therefore, $AB = DA$. Similarly, $AB = BC = CD = DA$.

**6.2.** Clearly,

$$\angle AOB = 180° - \angle BAO - \angle ABO = 180° - \frac{\angle A + \angle B}{2}$$

and $\angle COD = 180° - \frac{\angle C + \angle D}{2}$. Hence,

$$\angle AOB + \angle COD = 360° - \frac{\angle A + \angle B + \angle C + \angle D}{2} = 180°.$$

**6.3.** Let us consider two circles tangent to the sides of the given quadrilateral and their extensions. The lines that contain the sides of the quadrilateral are the common inner and outer tangents to these circles. The line that connects the midpoints of the circles contains a diagonal of the quadrilateral and besides it is an axis of symmetry of the quadrilateral. Hence, the other diagonal is perpendicular to this line.

**6.4.** Let $O$ be the center of the given circle, $R$ its radius, $a$ the length of chords singled out by the circle on the sides of the quadrilateral. Then the distances from point $O$ to the sides of the quadrilateral are equal to $\sqrt{R^2 - \frac{a^2}{4}}$, i.e., point $O$ is equidistant from the sides of the quadrilateral and is the center of the inscribed circle.

**6.5.** For a parallelogram the statement of the problem is obvious therefore, we can assume that lines $AB$ and $CD$ intersect. Let $O$ be the center of the inscribed circle of quadrilateral $ABCD$; let $M$ and $N$ be the midpoints of diagonals $AC$ and $BD$. Then

$$S_{ANB} + S_{CND} = S_{AMB} + S_{CMD} = S_{AOB} + S_{COD} = \frac{S_{ABCD}}{2}.$$

It remains to make use of the result of Problem 7.2.

**6.6.** Let the inscribed circle be tangent to sides $DA$, $AB$ and $BC$ at points $M$, $H$ and $N$, respectively. Then $OH$ is a height of triangle $AOB$ and the symmetries through lines $AO$ and $BO$ sends point $H$ into points $M$ and $N$, respectively. Hence, by Problem 1.57 points $A_1$ and $B_1$ lie on line $MN$. Similarly, points $C_1$ and $D_1$ lie on line $MN$.

**6.7.** Let $r$ be the distance from the intersection point of bisectors of angles $A$ and $D$ to the base $AD$, let $r'$ be the distance from the intersection point of bisectors of angles $B$ and

$C$ to base $BC$. Then $AD = r(\cot \alpha + \cot \beta)$ and $BC = r'(\tan \alpha + \tan \beta)$. Hence, $r = r'$ if and only if

$$\frac{BC}{AD} = \frac{\tan \alpha + \tan \beta}{\cot \alpha \cot \beta} = \tan \alpha \cdot \tan \beta.$$

**6.8.** Let $\angle A = 2\alpha$, $\angle C = 2\beta$ and $\angle BMA = 2\varphi$. By Problem 6.7, $\frac{PK}{RL} = \frac{\tan \alpha}{\tan \varphi}$ and $\frac{LS}{MC} = \cot \varphi \tan \beta$. Since $\frac{PQ}{RS} = \frac{PK}{RL}$ and $\frac{RS}{AC} = \frac{LS}{MC}$, it follows that

$$\frac{PQ}{AC} = \frac{PK}{RL} \frac{LS}{MC} = \tan \alpha \tan \beta.$$

Hence, trapezoid $APQC$ is a circumscribed one.

**6.9.** First, let us prove that if quadrilateral $ABCD$ is a circumscribed one, then all the conditions take place. Let $K$, $L$, $M$ and $N$ be the tangent points of the inscribed circle with sides $AB$, $BC$, $CD$ and $DA$. Then

$$\begin{aligned}
AB + CD &= AK + BK + CM + DM = AN + BL + CL + DN = BC + AD, \\
AP + CQ &= AK + PK + QL - CL = AN + PM + QN - CM = AQ + CP, \\
BP + BQ &= AP - AB + BC + CQ = (AP + CQ) + (BC - AB) = \\
&\quad AQ + CP + CD - AD = DP + DQ.
\end{aligned}$$

Now, let us prove, for instance, that if $BP + BQ = DP + DQ$, then quadrilateral $ABCD$ is a circumscribed one. For this let us consider the circle tangent to side $BC$ and rays $BA$ and $CD$. Assume that line $AD$ is not tangent to this circle; let us shift this line in order for it to touch the circle (Fig. 65).
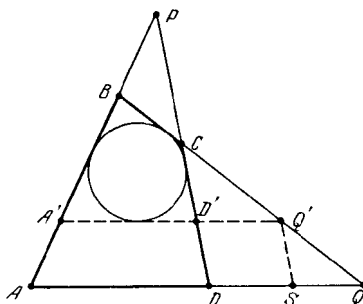

Figure 65 (Sol. 6.9)

Let $S$ be a point on line $AQ$ such that $Q'S \parallel DD'$. Since $BP + BQ = DP + DQ$ and $BP + BQ' = D'P + D'Q'$, it follows that $QS + SQ' = QQ'$. Contradiction.

In the other two cases the proof is similar.

**6.10.** Let rays $AB$ and $DC$ intersect at point $P$, let rays $BC$ and $AD$ intersect at point $Q$; let the given lines passing through points $P$ and $Q$ intersect at point $O$. By Problem 6.9 we have $BP + BQ = OP + OQ$ and $OP + OQ = DP + DQ$. Hence, $BP + BQ = DP + DQ$ and, therefore, quadrilateral $ABCD$ is a circumscribed one.

**6.11.** Let sides $AB$, $BC$, $CD$ and $DA$ of quadrilateral $ABCD$ be tangent of the inscribed circle at points $E$, $F$, $G$ and $H$, respectively. First, let us show that lines $FH, EG$ and $AC$ intersect at one point. Denote the points at which lines $FH$ and $EG$ intersect line $AC$ by $M$ and $M'$, respectively. Since $\angle AMH = \angle BFM$ as angles between the tangents and chord $HF$, it follows that $\sin \angle AHM = \sin \angle CFM$. Hence,

$$\frac{AM \cdot MH}{FM \cdot MC} = \frac{S_{AMH}}{S_{FMC}} = \frac{AH \cdot MH}{FC \cdot FM},$$

i.e., $\frac{AM}{MC} = \frac{AH}{FC}$. Similarly,

$$\frac{AM'}{M'C} = \frac{AE}{CG} = \frac{AH}{FC} = \frac{AM}{MC};$$

hence, $M = M'$, i.e., lines $FH$, $EG$ and $AC$ intersect at one point.

Similar arguments show that lines $FH$, $EG$ and $BD$ intersect at one point and therefore, lines $AC$, $BD$, $FH$ and $EG$ intersect at one point.

**6.12.** Segments $CH_d$ and $DH_c$ are parallel because they are perpendicular to line $BC$. Moreover, since $\angle BCA = \angle BDA = \varphi$, the lengths of these segments are equal to $AB|\cot\varphi|$, cf. Problem 5.45 b).

**6.13.** Let $O_a$, $O_b$, $O_c$ and $O_d$ be the centers of the inscribed circles of triangles $BCD$, $ACD$, $ABD$ and $ABC$, respectively. Since $\angle ADB = \angle ACB$, it follows that

$$\angle AO_cB = 90° + \frac{\angle ADB}{2} = 90° + \frac{\angle ACB}{2} = \angle AO_dB,$$

cf. Problem 5.3. Therefore, quadrilateral $ABO_dO_c$ is an inscribed one, i.e.,

$$\angle O_cO_dB = 180° - \angle O_cAB = 180° - \frac{\angle A}{2}.$$

Similarly, $\angle O_aO_dB = 180° - \frac{\angle C}{2}$. Since $\angle A + \angle C = 180°$, it follows that $\angle O_cO_dB + \angle O_aO_bB = 270°$ and, therefore, $\angle O_aO_dO_c = 90°$. We similarly prove that the remaining angles of quadrilateral $O_aO_bO_cO_d$ are equal to $90°$.

**6.14.** a) Let rays $AB$ and $DC$ intersect at point $P$ and rays $BC$ and $AD$ intersect at point $Q$. Let us prove that point $M$ at which the circumscribed circles of triangles $CBP$ and $CDQ$ intersect lies on segment $PQ$. Indeed,

$$\angle CMP + \angle CMQ = \angle ABC + \angle ADC = 180°.$$

Hence, $PM + QM = PQ$ and since

$$PM \cdot PQ = PD \cdot PC = p^2 - R^2 \quad \text{and } QM \cdot PQ = QD \cdot QA = q^2 - R^2,$$

it follows that $PQ^2 = PM \cdot PQ + QM \cdot PQ = p^2 + q^2 - 2R^2$. Let $N$ be the intersection point of the circumscribed circles of triangles $ACP$ and $ABS$. Let us prove that point $S$ lies on segment $PN$. Indeed,

$$\angle ANP = \angle ACP = 180° - \angle ACD = 180° - \angle ABD = \angle ANS.$$

Hence, $PN - SN = PS$ and since

$$PN \cdot PS = PA \cdot PB = p^2 - R^2 \quad \text{and } SN \cdot PS = SA \cdot SC = R^2 - s^2,$$

it follows that

$$PS^2 = PN \cdot PS - SN \cdot PS = p^2 + s^2 - 2R^2.$$

Similarly, $QS^2 = q^2 + s^2 - 2R^2$.

b) By heading a)

$$PQ^2 - PS^2 = q^2 - s^2 = OQ^2 - OS^2.$$

Hence, $OP \perp QS$, cf. Problem 7.6. We similarly prove that $OQ \perp PS$ and $OS \perp PQ$.

**6.15.** Let the inscribed circles of triangles $ABC$ and $ACD$ be tangent to diagonal $AC$ at points $M$ and $N$, respectively. Then

$$AM = \frac{AC + AB - BC}{2} \quad \text{and} quad AN = \frac{AC + AD - CD}{2},$$

cf. Problem 3.2. Points $M$ and $N$ coincide if and only if $AM = AN$, i.e., $AB + CD = BC + AD$. Thus, if points $M$ and $N$ coincide, then quadrilateral $ABCD$ is a circumscribed

one and similar arguments show that the tangent points of the inscribed circles of triangles $ABD$ and $BCD$ with the diagonal $BD$ coincide.

Let the inscribed circle of triangle $ABC$ be tangent to sides $AB$, $BC$ and $CA$ at points $P$, $Q$ and $M$, respectively and the inscribed circle of triangle $ACD$ be tangent to sides $AC$, $CD$ and $DA$ at points $M$, $R$ and $S$, respectively. Since $AP = AM = AS$ and $CQ = CM = CR$, it follows that triangles $APS$, $BPQ$, $CQR$ and $DRS$ are isosceles ones; let $\alpha$, $\beta$, $\gamma$ and $\delta$ be the angles at the bases of these isosceles triangles. The sum of the angles of these triangles is equal to

$$2(\alpha + \beta + \gamma + \delta) + \angle A + \angle B + \angle C + \angle D;$$

hence, $\alpha + \beta + \gamma + \delta = 180°$. Therefore,

$$\angle SPQ + \angle SRQ = 360° - (\alpha + \beta + \gamma + \delta) = 180°,$$

i.e., quadrilateral $PQRS$ is an inscribed one.

**6.16.** Let $O$ be the intersection point of diagonals $AC$ and $BD$; let $A_1$, $B_1$, $C_1$ and $D_1$ be the projections of $O$ to sides $AB$, $BC$, $CD$ and $DA$, respectively. Points $A_1$ and $D_1$ lie on the circle with diameter $AO$, hence, $\angle OA_1D_1 = \angle OAD_1$. Similarly, $\angle OA_1B_1 = \angle OBB_1$. Since $\angle CAD = \angle CBD$, we have: $\angle OA_1D_1 = \angle OA_1B_1$.

We similarly prove that $B_1O$, $C_1O$ and $D_1O$ are the bisectors of the angles of quadrilateral $A_1B_1C_1D_1$, i.e., $O$ is the center of its inscribed circle.
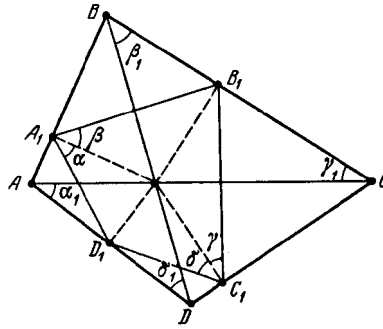


Figure 66 (Sol. 6.17)

**6.17.** Let us make use of the notations on Fig. 66. The condition that quadrilateral $A_1B_1C_1D_1$ is an inscribed one is equivalent to the fact that $(\alpha+\beta)+(\gamma+\delta) = 180°$ and the the fact that $AC$ and $BD$ are perpendicular is equivalent to the fact that $(\alpha_1 + \delta_1) + (\beta_1 + \gamma_1) = 180°$. It is also clear that $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \gamma_1$ and $\delta = \delta_1$.

**6.18.** By the law of cosines

$$AD^2 = AC^2 + CD^2 - 2AC \cdot CD \cos ACD, \quad AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos B.$$

Since the length of the projection of segment $AC$ to line $l$ perpendicular to $CD$ is equal to the sum of the lengths of projections of segments $AB$ and $BC$ to line $l$,

$$AC \cos ACD = AB \cos \varphi + BC \cos C.$$

**6.19.** Let $\angle AOD = 2\alpha$; then the distances from point $O$ to the projections of the midpoints of diagonals $AC$ and $BD$ to the bisector of angle $\angle AOD$ are equal to $\frac{OA+OC}{2} \cos \alpha$ and $\frac{OB+OD}{2} \cos \alpha$, respectively. Since

$$OA + OC = AB + OB + OC = CD + OB + OC = OB + OD,$$

these projections coincide.

**6.20.** Let us complement triangles $ABM$ and $DCM$ to parallelograms $ABMM_1$ and $DCMM_2$. Since $AM_1 : DM_2 = BM : MC = AN : DN$, it follows that $\triangle ANM_1 \sim \triangle DNM_2$. Hence, point $N$ lies on segment $M_1M_2$ and

$$MM_1 : MM_2 = AB : CD = AN : ND = M_1N : M_2N,$$

i.e., $MN$ is the bisector of angle $M_1MM_2$.

**6.21.** Let $a$, $b$, $c$ and $d$ be (the lengths of) the bisectors of the angles at vertices $A$, $B$, $C$ and $D$. We have to verify that $\angle(a,b) + \angle(c,d) = 0°$. Clearly,

$$\angle(a,b) = \angle(a,AB) + \angle(AB,b) \quad \text{and} \quad \angle(c,d) = \angle(c,CD) + \angle(CD,d).$$

Since quadrilateral $ABCD$ is a convex one and

$$\angle(a,AB) = \frac{\angle(AD,AB)}{2}, \quad \angle(AB,b) = \frac{\angle(AB,BC)}{2},$$

$$\angle(c,CD) = \frac{\angle(CB,CD)}{2}, \quad \angle(CD,d) = \frac{\angle(CD,DA)}{2},$$

it follows that

$$\angle(a,b) + \angle(c,d) = \frac{\angle(AD,AB) + \angle(AB,BC) + \angle(CB,CD) + \angle(CD,DA)}{2} =$$

$$\frac{360°}{2} = 0°$$

(see Background to Chapter 2).

**6.22.** Let, for definiteness, $AB > A_1B_1$. The parallel translation by vector $\overrightarrow{CB}$ sends triangle $SD_1C_1$ to $S'D_1'C_1'$ and segment $CD$ to $BA$. Since $QA_1 : QA = A_1B_1 : AB = S'D_1' : S'A$, we see that $QS' \parallel A_1D_1'$. Hence, $QS \parallel AD$. Similarly, $PR \parallel AB$.

**6.23.** Suppose that lines $AD$ and $BC$ are not parallel. Let $M_2$, $K$, $N_2$ be the midpoints of sides $AB$, $BC$, $CD$, respectively. If $MN \parallel BC$, then $BC \parallel AD$, because $AM = MC$ and $BN = ND$. Therefore, let us assume that lines $MN$ and $BC$ are not parallel, i.e., $M_1 \neq M_2$ and $N_1 \neq N_2$. Clearly, $\overrightarrow{M_2M} = \frac{1}{2}\overrightarrow{BC} = \overrightarrow{NN_2}$ and $\overrightarrow{M_1M} = \overrightarrow{NN_1}$. Hence, $M_1M_2 \parallel N_1N_2$. Therefore, $KM \parallel AB \parallel CD \parallel KN$, i.e., $M = N$. Contradiction.

**6.24.** By a similarity transformation we can identify one pair of the corresponding sides of quadrilaterals, therefore, it suffices to consider quadrilaterals $ABCD$ and $ABC_1D_1$ whose points $C_1$ and $D_1$ lie on rays $BC$ and $AD$ and such that $CD \parallel C_1D_1$. Denote the intersection points of diagonals of quadrilaterals $ABCD$ and $ABC_1D_1$ by $O$ and $O_1$, respectively.

Suppose that points $C$ and $D$ lie closer to points $B$ and $A$, then points $C_1$ and $D_1$, respectively. Let us prove then that $\angle AOB > \angle AO_1B$. Indeed, $\angle C_1BA > \angle CAB$ and $\angle D_1BA > \angle DBA$, hence,

$$\angle AO_1B = 180° - \angle C_1AB - \angle D_1BA < 180° - \angle CAB - \angle DBA = \angle AOB.$$

We have obtained a contradiction and, therefore, $C_1 = C$, $D_1 = D$.

**6.25.** Any quadrilateral is determined up to similarity by the directions of its sides and diagonals. Therefore, it suffices to construct one example of a quadrilateral $A_1B_1C_1D_1$ with the required directions of sides and diagonals. Let $O$ be the intersection point of diagonals $AC$ and $BD$. On ray $OA$, take an arbitrary point $D_1$ and draw $D_1A_1 \parallel BC$, $A_1B_1 \parallel CD$ and $B_1C_1 \parallel DA$ (Fig. 67).
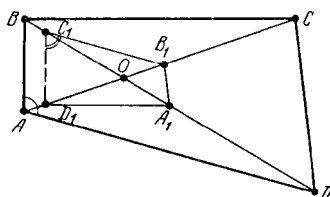
Figure 67 (Sol. 6.25)

Since
$$OC_1 : OB_1 = OD : OA, \quad OB_1 : OA_1 = OC : OD \quad \text{and} \quad OA_1 : OD_1 = OB : OC,$$
it follows that $OC_1 : OD_1 = OB : OA$, consequently, $C_1D_1 \parallel AB$. The obtained plot shows that $\angle A + \angle C_1 = 180°$.

**6.26.** Let $O$ be the intersection point of the diagonals of quadrilateral $ABCD$. Without loss of generality we may assume that $\alpha = \angle AOB < 90°$. Let us drop perpendiculars $AA_1$, $BB_1$, $CC_1$, $DD_1$ to the diagonals of quadrilateral $ABCD$. Since
$$OA_1 = OA\cos\alpha, \quad OB_1 = OB\cos\alpha, \quad OC_1 = OC\cos\alpha, \quad OD_1 = OD\cos\alpha,$$
it follows that the symmetry through the bisector of angle $AOB$ sends quadrilateral $ABCD$ into a quadrilateral homothetic to quadrilateral $A_1B_1C_1D_1$ with coefficient $\overrightarrow{BC}1\cos\alpha$.

**6.27.** Let the diagonals of quadrilateral $ABCD$ intersect at point $O$; let $H_a$ and $H_b$ be the orthocentres of triangles $AOB$ and $COD$; let $K_a$ and $K_b$ be the midpoints of sides $BC$ and $AD$; let $P$ be the midpoint of diagonal $AC$. The intersection point of medians of triangles $AOD$ and $BOC$ divide segments $K_aO$ and $K_bO$ in the ratio of $1 : 2$ and, therefore, we have to prove that $H_aH_b \perp K_aK_b$.

Since $OH_a = AB|\cot\varphi|$ and $OH_b = CD|\cot\varphi|$, where $\varphi = \angle AOB$, cf. Problem 5.45 b), then $OH_a : OH_b = PK_a : PK_b$. The correspondiong legs of angles $\angle H_aOH_b$ and $\angle K_aPK_b$ are perpendicular; moreover, vectors $\overrightarrow{OH_a}$ and $\overrightarrow{OH_b}$ are directed *towards* lines $AB$ and $CD$ for $\varphi < 90°$ and *away from* these lines for $\varphi > 90°$. Hence, $\angle H_aOH_b = \angle K_aPK_b$ and $\triangle H_aOH_b \sim \triangle K_aPK_b$. It follows that $H_aH_b \perp K_aK_b$.

**6.28.** Let $S = S_{AOD}$, $x = AO$, $y = DO$, $a = AB$, $b = BC$, $c = CD$, $d = DA$ and $k$ the similarity coefficient of triangles $BOC$ and $AOD$. Then
$$2\left(\frac{1}{r_1} + \frac{1}{r_3}\right) = \frac{d + x + y}{S} + \frac{kd + kx + ky}{k^2 S},$$
$$2\left(\frac{1}{r_2} + \frac{1}{r_4}\right) = \frac{a + x + ky}{kS} + \frac{c + kx + y}{kS}$$
because $S_{BOC} = k^2 S$ and $S_{AOB} = S_{COD} = kS$. Since
$$\frac{x + y}{S} + \frac{x + y}{k^2 S} = \frac{x + ky}{kS} + \frac{kx + y}{kS},$$
it remains to notice that $a + c = b + d = kd + d$.

**6.29.** It is easy to verify that
$$AB = r_1\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right) \quad \text{and} \quad CD = r_3\left(\cot\frac{C}{2} + \cot\frac{D}{2}\right).$$
Hence,
$$\frac{AB}{r_1} + \frac{CD}{r_3} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} + \cot\frac{D}{2} = \frac{BC}{r_2} + \frac{AD}{r_4}.$$

**6.30.** Let us complete triangles $ABD$ and $DBC$ to parallelograms $ABDA_1$ and $DBCC_1$. The segments that connect point $D$ with the vertices of parallelogram $ACC_1A_1$ divide it into four triangles equal to triangles $DAB$, $CDA$, $BCD$ and $ABC$ and, therefore, the radii of the inscribed circles of these triangles are equal.

Let us prove that point $D$ coincides with the intersection point $O$ of the diagonals of the parallelogram. If $D \neq 0$, then we may assume that point $D$ lies inside triangle $AOC$. Then $r_{ADC} < r_{AOC} = r_{A_1OC_1} < r_{A_1DC_1} = r_{ABC}$, cf. Problem 10.86. We have obtained a contradiction, hence, $D = O$.

Since $p = \overrightarrow{BC}Sr$ and the areas and radii of the inscribed circles of triangles into which the diagonals divide the parallelogram $ACC_1A_1$ are equal, the triangles' perimeters are equal. Hence, $ACC_1A_1$ is a rhombus and $ABCD$ is a rectangular.

**6.31.** Points $C_1$ and $D_1$ lie on the midperpendicular to segment $AB$, hence, $AB \perp C_1D_1$. Similarly, $C_1D_1 \perp A_2B_2$ and, therefore, $AB \parallel A_2B_2$. We similarly prove that the remaining corresponding sides and the diagonals of quadrilaterals $ABCD$ and $A_2B_2C_2D_2$ are parallel. Therefore, these quadrilaterals are similar.

Let $M$ be the midpoint of segment $AC$. Then $B_1M = |AM \cot D|$ and $D_1M = |AM \cot B|$, where $B_1D_1 = |\cot B + \cot D| \cdot \frac{1}{2}AC$. Let us rotate quadrilateral $A_1B_1C_1D_1$ by $90°$. Then making use of the result of Problem 6.25 we see that this quadrilateral is a convex one and $\cot A = -\cot C_1$, etc. Therefore,

$$A_2C_2 = |\cot A + \cot C| \cdot \frac{1}{2}B_1D_1 = \frac{1}{4}|(\cot A + \cot C)(\cot B + \cot D)| \cdot AC.$$

**6.32.** Let $M$ and $N$ be the midpoints of sides $AB$ and $CD$, respectively. Let us drop from point $D$ perpendicular $DP$ to line $MN$ and from point $M$ perpendicular $MQ$ to line $CD$. Then $Q$ is the tangent point of line $CD$ and a circle with diameter $AB$. Right triangles $PDN$ and $OMN$ are similar, hence,

$$DP = \frac{ND \cdot MQ}{MN} = \frac{ND \cdot MA}{MN}.$$

Similarly, the distance from point $A$ to line $MN$ is equal to $\overrightarrow{BC}ND \cdot MAMN$. Therefore, $AD \parallel MN$. Similarly, $BC \parallel MN$.

**6.33.** It suffices to verify that the orthocentres of any three of the four given triangles lie on one line. Let a certain line intersect lines $B_1C_1$, $C_1A_1$ and $A_1B_1$ at points $A$, $B$ and $C$, respectively; let $A_2$, $B_2$ and $C_2$ be the orthocentres of triangles $A_1BC$, $AB_1C$ and $ABC_1$, respectively. Lines $AB_2$ and $A_2B$ are perpendicular to line $A_1B_1$ and, therefore, they are parallel. Similarly, $BC_2 \parallel B_2C$ and $CA_2 \parallel C_2A$. Points $A$, $B$ and $C$ lie on one line and, therefore, points $A_2$, $B_2$ and $C_2$ also lie on one line, cf. Problem 1.12 b).

**6.34.** On diagonal $BD$, take point $M$ so that $\angle MCD = \angle BCA$. Then $\triangle ABC \sim \triangle DMC$, because angles $\angle BAC$ and $\angle BDC$ subtend the same arc. Hence, $AB \cdot CD = AC \cdot MD$. Since $\angle MCD = \angle BCA$, then $\angle BCM = \angle ACD$ and $\triangle BCM \sim \triangle ACD$ because angles $\angle CBD$ and $\angle CAD$ subtend one arc. Hence, $BC \cdot AD = AC \cdot BM$. It follows that

$$AB \cdot CD + AD \cdot BC = AC \cdot MD + AC \cdot BM = AC \cdot BD.$$

**6.35.** Let $S$ be the area of quadrilateral $ABCD$, let $R$ be the radius of its circumscribed circle. Then

$$S = S_{ABC} + S_{ADC} = \frac{AC(AB \cdot BC + AD \cdot DC)}{4R},$$

cf. Problem 12.1. Similarly,

$$S = \frac{BD(AB \cdot AD + BC \cdot CD)}{4R}.$$

By equating these equations for $S$ we get the desired statement.

**6.36.** Let regular hexagon $A_1 \ldots A_7$ be inscribed in a circle. By applying Ptolemey's theorem to qudrilateral $A_1 A_3 A_4 A_5$ we get

$$A_1 A_3 \cdot A_5 A_4 + A_3 A_4 \cdot A_1 A_5 = A_1 A_4 \cdot A_3 A_5,$$

i.e.,

$$\sin 2\alpha \sin \alpha + \sin \alpha \sin 3\alpha = \sin 3\alpha \sin 2\alpha.$$

**6.37.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$, respectively. By Ptolemy's theorem

$$AC_1 \cdot OB_1 + AB_1 \cdot OC_1 = AO \cdot B_1 C_1,$$

where $O$ is the center of the circumscribed circle. Hence, $cd_b + bd_c = aR$. Similarly, $ad_c + cd_a = bR$ and $ad_b + bd_a = cR$. Moreover, $ad_a + bd_b + cd_c = 2S = (a + b + c)r$. By adding all these equalities and dividing by $a + b + c$ we get the desired statement.

**6.38.** By Ptolemy's theorem

$$AB \cdot CD + AC \cdot BD = AD \cdot BC.$$

Taking into account that $CD = BD \geq \frac{1}{2} BC$ we get the desired statement.

**6.39.** By applying Ptolemy's theorem to quadrilateral $ABCP$ and dividing by the lengths of the square's side we get the desired statement.

**6.40.** By applying Ptolemy's theorem to quadrilateral $APQR$ we get

$$AP \cdot RQ + AR \cdot QP = AQ \cdot PR.$$

Since $\angle ACB = \angle RAQ = \angle RPQ$ and $\angle RQP = 180° - \angle PAR = \angle ABC$, it follows that $\triangle RQP \sim \triangle ABC$ and, therefore, $RQ : QP : PR = AB : BC : CA$. It remains to notice that $BC = AD$.

**6.41.** a) Let us express Ptolemy's theorem for all quadrilaterals with vertices at point $A$ and three consecutive vertices of the given polygon; then let us group in the obtained equalities the factors in which $d_i$ with even indices enter in the right-hand side. By adding these equalities we get

$$(2a + b)(d_1 + \cdots + d_{2n+1}) = (2a + b)(d_2 + \cdots + d_{2n}),$$

where $a$ is the side of the given polygon and $b$ is its shortest diagonal.

b) Let $R$ be the radius of circle $S$. Then $l_i = d_i \sqrt{\frac{R \pm r}{R}}$, cf. Problem 3.20. It remains to make use of the result of heading a).
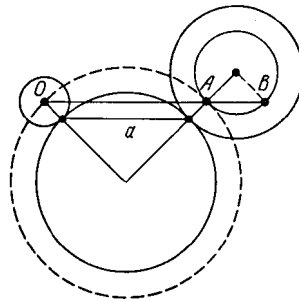


Figure 68 (Sol. 6.42)

**6.42.** Let both tangent be exterior ones and $x \leq y$. The line that passes through the center $O$ of the circle of radius $x$ parallel to the segment that connects the tangent points intersects the circle of radius $y - x$ (centered in the center of the circle of radius $y$) at points $A$ and $B$ (Fig. 68).

Then $OA = \frac{a(R+x)}{R}$ and

$$OB = OA + \frac{a(y-x)}{R} = \frac{a(R+y)}{R}.$$

The square of the length to be found of the common outer tangent is equal to

$$OA \cdot OB = \left(\frac{a}{R}\right)^2 (R+x)(R+y).$$

Similar arguments show that if both tangent are inner ones, then the square of the lengths of the outer tangent is equal to $\left(\frac{a}{R}\right)^2 (R-x)(R-y)$ and if the circle of radius $x$ is tangent from the outside and the circle of radius $y$ from the inside, then the square of the length of the inner tangent is equal to $\left(\frac{a}{R}\right)^2 (R-y)(R+x)$.

REMARK. In the case of an inner tangency of the circles we assume that $R > x$ and $R > y$.

**6.43.** Let $R$ be the radius of the circumscribed circle of quadrilateral $ABCD$; let $r_a$, $r_b$, $r_c$ and $r_d$ be the radii of circles $\alpha$, $\beta$, $\gamma$ and $\delta$, respectively. Further, let $a = \sqrt{R \pm r_a}$, where the plus sign is taken if the tangent is an outer one and the minus sign if it is an inner one; numbers $b$, $c$ and $d$ are similarly defined. Then $t_{\alpha\beta} = \frac{ab \cdot AB}{R}$, cf. Problem 6.42, etc. Therefore, by multiplying the equality

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$

by $\frac{abcd}{R}$ we get the desired statement.

**6.44.** Since $\angle EBD = \angle ABE + \angle CBD$, it is possible to take a point $P$ on side $ED$ so that $\angle EBP = \angle ABE = \angle AEB$, i.e., $BP \parallel AE$. Then $\angle PBD = \angle EBD - \angle EBP = \angle CBD = \angle BDC$, i.e., $BP \parallel CD$. Therefore, $AE \parallel CD$ and since $AE = CD$, $CDEA$ is a parallelogram. Hence, $AC = ED$, i.e., triangle $ABC$ is an equilateral one and $\angle ABC = 60°$.

**6.45.** a) Let $O$ be the center of the circumscribed circle of triangle $CKE$. It suffices to verify that $\angle COK = 2\angle KCB$. It is easy to calculate both these angles:

$$\angle COK = 180° - 2\angle OKC = 180° - \angle EKC = 180° - \angle EDC = 72°$$

and $\angle KCB = \frac{180° - \angle ABC}{2} = 36°$.

b) Since $BC$ is a tangent to the circumscribed circle of triangle $CKE$, then $BE \cdot BK = BC^2$, i.e., $d(d-a) = a^2$.

**6.46.** Let the perpendiculars erected to line $AB$ at points $A$ and $B$ intersect sides $DE$ and $CD$ at points $P$ and $Q$, respectively. Any point of segment $CQ$ is a vertex of a rectangle inscribed in pentagon $ABCDE$ (the respective sides of this pentagon are parallel to $AB$ and $AP$); as this point moves from $Q$ to $C$ the ratio of the lengths of the sides of the rectangles varies from $\frac{AP}{AB}$ to 0. Since angle $\angle AEP$ is an obtuse one, $AP > AE = AB$. Therefore, for a certain point of segment $QC$ the ratio of the lengths of the sides of the rectangle is equal to 1.

**6.47.** Let points $A_1$, ..., $E_1$ be symmetric to points $A$, ..., $E$ through the center of circle $S$; let $P$, $Q$ and $R$ be the intersection points of lines $BC_1$ and $AB_1$, $AE_1$ and $BA_1$, $BA_1$ and $CB_1$, see Fig. 69.

Then $PQ = AB = a$ and $QR = b$. Since $PQ \parallel AB$ and $\angle ABA_1 = 90°$, it follows that $PR^2 = PQ^2 + QR^2 = a^2 + b^2$. Line $PR$ passes through the center of circle $S$ and $\angle AB_1C = 4 \cdot 18° = 72°$, hence, $PR$ is a side of a regular pentagon circumscribed about the circle with center $B_1$ whose radius $B_1O$ is equal to the radius of circle $S$.
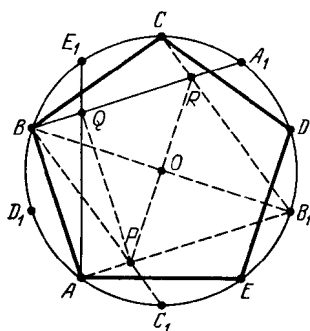
Figure 69 (Sol. 6.47)
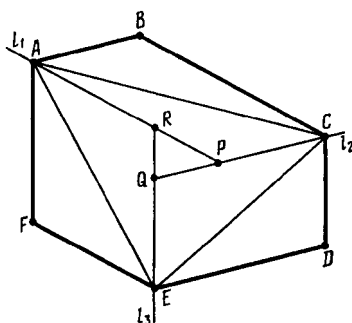


Figure 70 (Sol. 6.48)

**6.48.** Through points $A$, $C$ and $E$ draw lines $l_1$, $l_2$ and $l_3$ parallel to lines $BC$, $DE$ and $FA$, respectively. Denote the intersection points of lines $l_1$ and $l_2$, $l_2$ and $l_3$, $l_3$ and $l_1$ by $P$, $Q$, $R$, see Fig. 70. Then

$$S_{ACE} = \frac{S_{ABCDEF} - S_{PQR}}{2} + S_{PQR} = \frac{S_{ABCDEF} + S_{PQR}}{2} \geq \frac{S_{ABCDEF}}{2}.$$

Similarly, $S_{BDF} = \frac{1}{2}(S_{ABCDEF} + S_{P'Q'R'})$. Clearly,

$$PQ = |AB - DE|, \quad QR = |CD - AF|, \quad PR = |EF - BC|,$$

hence, triangles $PQR$ and $P'Q'R'$ are equal. Therefore, $S_{ACE} = S_{BDF}$.

**6.49.** Let us construct triangle $PQR$ as in the preceding problem. This triangle is an equilateral one and

$$PQ = |AB - DE|, \quad QR = |CD - AF|, \quad RP = |EF - BC|.$$

Hence, $|AB - DE| = |CD - AF| = |EF - BC|$.

**6.50.** The sum of the angles at vertices $A$, $C$ and $E$ is equal to $360°$, hence, from isosceles triangles $ABF$, $CBD$ and $EDF$ we can construct a triangle by juxtaposing $AB$ to $CB$, $ED$ to $CD$ and $EF$ to $AF$. The sides of the obtained triangle are equal to the respective sides of triangle $BDF$. Therefore, the symmetry through lines $FB$, $BD$ and $DF$ sends points $A$, $C$ and $E$, respectively, into the center $O$ of the circumscribed circle of triangle $BDF$, and, therefore, $AB \parallel OF \parallel DE$.

**6.51.** Let us suppose that the diagonals of the hexagon form triangle $PQR$. Denote the vertices of the hexagon as follows: vertex $A$ lies on ray $QP$, vertex $B$ on $RP$, vertex $C$ on $RQ$, etc. Since lines $AD$ and $BE$ divide the area of the hexagon in halves, then

$$S_{APEF} + S_{PED} = S_{PDCB} + S_{ABP} \quad \text{and} \quad S_{APEF} + S_{ABP} = S_{PDCB} + S_{PED}.$$

Hence, $S_{ABP} = S_{PED}$, i.e.,

$$AP \cdot BP = EP \cdot DP = (ER + RP)(DQ + QP) > ER \cdot DQ.$$

Similarly, $CQ \cdot DQ > AP \cdot FR$ and $FR \cdot ER > BP \cdot CQ$. By multiplying these inequalities we get

$$AP \cdot BP \cdot CQ \cdot DQ \cdot FR \cdot ER > ER \cdot DQ \cdot AP \cdot FR \cdot BP \cdot CQ$$

which is impossible. Hence, the diagonals of the hexagon intersect at one point.
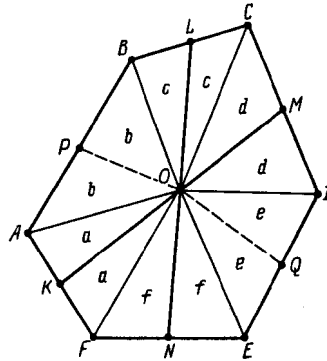


Figure 71 (Sol. 6.52)

**6.52.** Denote the midpoints of the sides of convex hexagon $ABCDEF$ as plotted on Fig. 71. Let $O$ be the intersection point of segments $KM$ and $LN$. Let us denote the areas of triangles into which the segments that connect point $O$ with the vertices and the midpoints of the sides divide the hexagon as indicated on the same figure. It is easy to verify that $S_{KONF} = S_{LOMC}$, i.e., $a + f = c + d$. Therefore, the broken line $POQ$ divides the hexagon into two parts of equal area; hence, segment $PQ$ passes through point $O$.

**6.53.** a) Let $O$ be the center of the circumscribed circle. Since

$$\angle A_k O A_{k+2} = 360° - 2\angle A_k A_{k+1} A_{k+2} = \varphi$$

is a constant, the rotation through an angle of $\varphi$ with center $O$ sends point $A_k$ into $A_{k+2}$. For $n$ odd this implies that all the sides of polygon $A_1 \ldots A_n$ are equal.

b) Let $a$ be the length of the side of the given polygon. If one of its sides is divided by the tangent point with the inscribed circle into segments of length $x$ and $a - x$, then its neighbouring sides are also divided into segments of length $x$ and $a - x$ (the neighbouring segments of neighbouring sides are equal), etc. For $n$ odd this implies that all the sides of polygon $A_1 \ldots A_n$ are divided by the tangent points with the inscribed circle in halves; therefore, all the angles of the polygon are equal.

**6.54.** The sides of polygon $A_1 \ldots A_n$ are parallel to respective sides of a regular $n$-gon. On rays $OA_1, \ldots, OA_n$ mark equal segments $OB_1, \ldots, OB_n$. Then polygon $B_1 \ldots B_n$ is a regular one and the sides of polygon $A_1 \ldots A_n$ form equal angles with the respective sides of polygon $B_1 \ldots B_n$. Therefore,

$$OA_1 : OA_2 = OA_2 : OA_3 = \cdots = OA_n : OA_1 = k,$$

i.e.,

$$OA_1 = kOA_2 = k^2 OA_3 = \cdots = k^n OA_1;$$

thus, $k = 1$.

**6.55.** Denote the vertices of the pentagon as indicated on Fig. 72. Notice that if in a triangle two heights are equal, then the sides on which these heights are dropped are also equal.
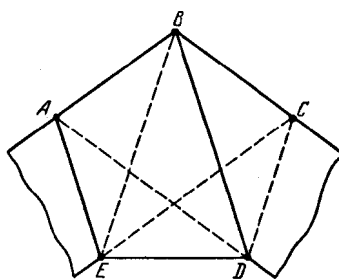
Figure 72 (Sol. 6.55)

From consideration of triangles $EAB$, $ABC$ and $BCD$ we deduce that $EA = AB$, $AB = BC$ and $BC = CD$. Therefore, trapezoids $EABC$ and $ABCD$ are isosceles ones, i.e., $\angle A = \angle B = \angle C$. By considering triangles $ABD$ and $BCE$ we get $AD = BD$ and $BE = CE$. Since triangles $EAB$, $ABC$, $BCD$ are equal, it follows that $BE = AC = BD$. Hence, $AD = BE$ and $BD = CE$, i.e., trapezoids $ABDE$ and $CDEB$ are isosceles ones. Therefore, $ED = AB = BC = CD = AE$ and $\angle E = \angle A = \angle B = \angle C = \angle D$, i.e., $ABCDE$ is a regular pentagon.

**6.56.** Triangles $BAM$ and $BCN$ are isosceles ones with angle $15°$ at the base, cf. Problem 2.26, and, therefore, triangle $BMN$ is an equilateral one. Let $O$ be the centre of the square, $P$ and $Q$ the midpoints of segments $MN$ and $BK$ (Fig. 73). Since $OQ$ is the midline of triangle $MBK$, it follows that $OQ = \frac{1}{2}BM = MP = OP$ and $\angle QON = \angle MBA = 15°$. Therefore, $\angle POQ = \angle PON - \angle QON = 30°$.

The remaining part of the proof is carried out similarly.
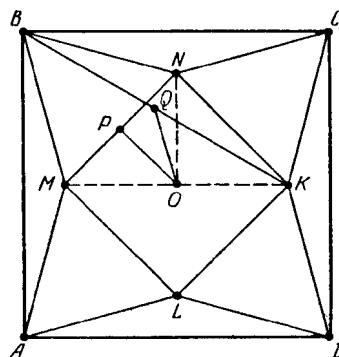


Figure 73 (Sol. 6.56)

**6.57.** Let us consider a regular 12-gon $A_1 \ldots A_{12}$ inscribed in a circle of radius $R$. Clearly, $A_1A_7 = 2R$, $A_1A_3 = A_1A_{11} = R$. Hence, $A_1A_7 = A_1A_3 + A_1A_{11}$.

**6.58.** For $k = 3$ the solution of the problem is clear from Fig. 74. Indeed, $A_3A_4 = OQ$, $KL = QP$ and $MN = PA_{14}$ and, therefore,

$$A_3A_4 + KL + MN = OQ + QP + PA_{14} = OA_{14} = R.$$

Proof is carried out in a similar way for any $k$.

**6.59.** In the proof if suffices to apply the result of Problems 5.78 and 5.70 b) to triangle $A_aA_cA_e$ and lines $A_aA_d$, $A_cA_f$ and $A_eA_b$. Solvling heading b) we have to notice additionally that

$$\sin 20° \sin 70° = \sin 20° \cos 20° = \frac{\sin 40°}{2} = \sin 30° \sin 40°$$

and in the solution of heading c) that $\sin 10° \sin 80° = \sin 30° \sin 20°$.
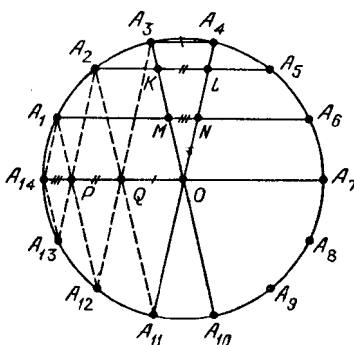
Figure 74 (Sol. 6.58)

**6.60.** As in the preceding problem we have to verify the equality

$$\sin 2\alpha \sin 2\alpha \sin 8\alpha = \sin \alpha \sin 3\alpha \sin 14\alpha, \text{ where } \alpha = \frac{180°}{30} = 6°.$$

Clearly, $\sin 14\alpha = \cos \alpha$, hence, $2 \sin \alpha \sin 3\alpha \sin 14\alpha = \sin 2\alpha \sin 3\alpha$. It remains to verify that

$$\sin 3\alpha = 2 \sin 2\alpha \sin 8\alpha = \cos 6\alpha - \cos 10\alpha = 1 - 2\sin^2 3\alpha - \frac{1}{2},$$

i.e., $4 \sin^2 18° + 2 \sin 18° = 1$, cf. Problem 5.46.

**6.61.** First, let $n = 2m$. The diagonals and sides of a regular $2m$-gon have $m$ distinct lengths. Therefore, the marked points lie on $m - 1$ concentric circles (having $n$ points each) or in the common center of these circles. Since distinct circles have not more than two common points, the circle that does not belong to this family of concentric circles contains not more than $1 + 2(m - 1) = 2m - 1 = n - 1$ of marked points.

Now, let $n = 2m + 1$. There are $m$ distinct lengths among the lengths of the diagonals and sides of a regular $(2m + 1)$-gon. Hence, the marked points lie on $m$ concentric circles ($n$ points on each). A circle that does not belong to this family of concentric circles contains not more than $2m = n - 1$ marked points.

In either case the greatest number of marked points that lie on one circle is equal to $n$.

**6.62.** Denote the center of the polygon by $O$ and the vertices of the polygon by $A_1, \ldots, A_n$. Suppose that there are no equal polygons among the polygons of the same colour, i.e., they have $m = m_1 < m_2 < m_3 < \cdots < m_k$ sides, respectively. Let us consider a transformation $f$ defined on the set of vertices of the $n$-gon as the one that sends vertex $A_k$ to vertex $A_{mk}$: $f(A_k) = A_{mk}$ (we assume that $A_{p+qn} = A_p$). This transformation sends the vertices of a regular $m$-gon into one point, $B$, hence, the sum of vectors $\overrightarrow{Of(A_i)}$, where $A_i$ are the vertices of an $m$-gon, is equal to $m\overrightarrow{OB} \neq \overrightarrow{0}$.

Since $\angle A_{mi}OA_{mj} = m\angle A_iOA_j$, the vertices of any regular polygon with the number of sides greater than $m$ pass under the considered transformation into the vertices of a regular polygon. Therefore, the sum of vectors $\overrightarrow{Of(A_i)}$ over all vertices of an $n$-gon and similar sums over the vertices of $m_2$-, $m_3$-, $\ldots, m_k$-gons are equal to zero. We have obtained a contradiction with the fact that the sum of vectors $\overrightarrow{Of(A_i)}$ over the vertices of an $m$-gon is not equal to zero.

Therefore, among the polygons of one color there are two equal ones.

**6.63.** Let a regular $(n - 1)$-gon $B_1 \ldots B_{n-1}$ be inscribed into a regular $n$-gon $A_1 \ldots A_n$. We may assume that $A_1$ and $B_1$ are the least distant from each other vertices of these polygons and points $B_2, B_3, B_4$ and $B_5$ lie on sides $A_2A_3, A_3A_4, A_4A_5$ and $A_5A_6$. Let $\alpha_i = \angle A_{i+1}B_iB_{i+1}$ and $\beta_i = \angle B_iB_{i+1}A_{i+1}$, where $i = 1, 2, 3, 4$. By the sine theorem

$A_2B_2 : B_1B_2 = \sin\alpha_1 : \sin\varphi$ and $B_2A_3 : B_2B_3 = \sin\beta_2 : \sin\varphi$, where $\varphi$ is the angle at a vertex of a regular $n$-gon. Therefore, $\sin\alpha_1 + \sin\beta_2 = \frac{a_n \sin\varphi}{a_{n-1}}$, where $a_n$ and $a_{n-1}$ are the (lengths of the) sides of the given polygons.

Similar arguments show that

$$\sin\alpha_1 + \sin\beta_2 = \sin\alpha_2 + \sin\beta_3 = \sin\alpha_3 + \sin\beta_4.$$

Now, observe that

$$\sin\alpha_i + \sin\beta_{i+1} = 2\sin\frac{\alpha_i + \beta_{i+1}}{2}\cos\frac{\alpha_i - \beta_{i+1}}{2}$$

and compute $\alpha_i + \beta_{i+1}$ and $\alpha_i - \beta_{i+1}$. Since $\alpha_i + \beta_i = \frac{2\pi}{n}$ and $\alpha_{i+1} + \beta_i = \frac{2\pi}{n-1}$, it follows that $\alpha_{i+1} = \alpha_i + \frac{2\pi}{n(n-1)}$ and $\beta_{i+1} = \beta_i - \frac{2\pi}{n(n-1)}$; therefore,

$$\alpha_i + \beta_{i+1} = \frac{2\pi}{n} - \frac{2\pi}{n(n-1)}$$

is a constant and

$$\alpha_i - \beta_{i+1} = \alpha_{i-1} - \beta_i + \frac{4\pi}{n(n-1)}.$$

Hence,

$$\cos\theta = \cos\left(\theta + \frac{2\pi}{n(n-1)}\right) = \cos\left(\theta + \frac{4\pi}{(n-1)n}\right) \quad \text{for} \quad \theta = \frac{\alpha_1 - \beta_2}{2}.$$

We have obtained a contradiction because on an interval shorter than $2\pi$ the cosine cannot attain the same value at three distinct points.

REMARK. A square *can* be inscribed in a regular pentagon, cf. Problem 6.46.

**6.64.** Let $\mathbf{a} = \overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n}$. A rotation about point $O$ by $\frac{360°}{n}$ sends point $A_i$ to $A_{i+1}$ and, therefore, sends vector $\mathbf{a}$ into itself, i.e., $\mathbf{a} = \mathbf{0}$.

Since $\overrightarrow{XA_i} = \overrightarrow{XO} + \overrightarrow{OA_i}$ and $\overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$, it follows that $\overrightarrow{XA_1} + \cdots + \overrightarrow{XA_n} = n\overrightarrow{XO}$.

**6.65.** Through the center of a regular polygon $A_1 \ldots A_n$, draw line $l$ that does not pass through the vertices of the polygon. Let $x_i$ be equal to the length of the projection of vector $\overrightarrow{OA_i}$ to a line perpendicular to $l$. Then all the $x_i$ are nonzero and the sum of numbers $x_i$ assigned to the vertices of a regular $k$-gon is equal to zero since the corresponding sum of vectors $\overrightarrow{OA_i}$ vanishes, cf. Problem 6.64.

**6.66.** By Problem 6.64 $\mathbf{a} = 10\overrightarrow{AO}$ and $\mathbf{b} = 10\overrightarrow{BO}$, where $O$ is the center of polygon $X_1 \ldots X_{10}$. Clearly, if point $A$ is situated rather close to a vertex of the polygon and point $B$ rather close to the midpoint of a side, then $AO > BO$.

**6.67.** Since

$$A_iX^2 = |\overrightarrow{A_iO} + \overrightarrow{OX}|^2 = A_iO^2 + OX^2 + 2(\overrightarrow{A_iO}, \overrightarrow{OX}) =$$

$$R^2 + d^2 + 2(\overrightarrow{A_iO}, \overrightarrow{OX}),$$

it follows that

$$\sum A_iX^2 = n(R^2 + d^2) + 2(\sum \overrightarrow{A_iO}, \ \overrightarrow{OX}) = n(R^2 + d^2),$$

cf. Problem 6.64.

**6.68.** Denote by $S_k$ the sum of squared distances from vertex $A_k$ to all the other vertices. Then

$$S_k = A_k A_1^2 + A_k A_2^2 + \cdots + A_k A_n^2 = A_k O^2 + 2(\overrightarrow{A_k O}, \overrightarrow{OA_1}) + A_1 O^2 + \ldots$$
$$+ A_k O^2 + 2(\overrightarrow{A_k O}, \overrightarrow{OA_n}) + A_n O^2 = 2nR^2$$

because $\sum_{i=1}^n \overrightarrow{OA_i} = \overrightarrow{0}$. Hence, $\sum_{i=1}^n S_k = 2n^2 R^2$. Since each squared side and diagonal enters this sum twice, the sum to be found is equal to $n^2 R^2$.

**6.69.** Consider the rotation of the given $n$-gon about the $n$-gon's center $O$ that sends $A_k$ to $A_1$. Let $X_k$ be the image of point $X$ under the rotation. This rotation sends segment $A_k X$ to $A_1 X_k$. Therefore,

$$A_1 X + \cdots + A_n X = A_1 X_1 + \cdots + A_1 X_n.$$

Since $n$-gon $X_1 \ldots X_n$ is a regular one,

$$\overrightarrow{A_1 X_1} + \cdots + \overrightarrow{A_1 X_n} = n\overrightarrow{A_1 O},$$

cf. Problem 6.64. Therefore, $A_1 X_1 + \cdots + A_n X_n \geq n\overrightarrow{A_1 O}$.

**6.70.** Let $B_i$ be the projection of point $X$ to line $OA_i$. Then

$$(\mathbf{e}_i, \mathbf{x}) = (\overrightarrow{OA_i}, \ \overrightarrow{OB_i} + \overrightarrow{B_i X}) = (\overrightarrow{OA_i}, \ \overrightarrow{OB_i}) = \pm R \cdot OB_i.$$

Points $B_1, \ldots, B_n$ lie on the circle with diameter $OX$ and are vertices of a regular $n$-gon for $n$ odd and vertices of an $\frac{n}{2}$-gon counted twice for $n$ even, cf. Problem 2.9. Therefore, $\sum OB_i^2 = \frac{1}{2} n \cdot OX^2$, cf. Problem 6.67.

**6.71.** Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the vectors that go from the center of the given $n$-gon into its vertices; $\mathbf{x}$ a unit vector perpendicular to line $l$. The sum to be found is equal to $\sum(\mathbf{e}_i, \mathbf{x})^2 = \frac{1}{2} n \cdot R^2$, cf. Problem 6.70.

**6.72.** Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the unit vectors directed from the center $O$ of a regular $n$-gon into the midpoints of its sides; $\mathbf{x} = \overrightarrow{OX}$. Then the distance from point $X$ to the $i$-th side is equal to $|(\mathbf{x}, \mathbf{e}_i) - r|$. Hence, the sum to be found is equal to

$$\sum((\mathbf{x}, \mathbf{e}_i)^2 - 2r(\mathbf{x}, \mathbf{e}_i) + r^2) = \sum(\mathbf{x}, \mathbf{e}_i)^2 + nr^2.$$

By Problem 6.70 $\sum(\mathbf{x}, \mathbf{e}_i)^2 = \frac{1}{2} nd^2$.

**6.73.** Let $\mathbf{x}$ be the unit vector parallel to line $l$ and $\mathbf{e}_i = \overrightarrow{A_i A_{i+1}}$. Then the squared length of the projection of side $A_i A_{i+1}$ to line $l$ is equal to $(\mathbf{x}, \mathbf{e}_i)^2$. By Problem 6.70 $\sum(\mathbf{x}, \mathbf{e}_i)^2 = \frac{1}{2} na^2$.

**6.74.** Let $\mathbf{a} = \overrightarrow{OX}, \mathbf{e}_i = \overrightarrow{OA_i}$. Then

$$XA_i^4 = |\mathbf{a} + \mathbf{e}_i|^4 = (|\mathbf{a}|^2 + 2(\mathbf{a}, \mathbf{e}_i) + |\mathbf{e}_i|^2)^2 =$$
$$4(R^2 + (\mathbf{a}, \mathbf{e}_i))^2 = 4(R^4 + 2R^2(\mathbf{a}, \mathbf{e}_i) + (\mathbf{a}, \mathbf{e}_i)^2).$$

Clearly, $\sum(\mathbf{a}, \mathbf{e}_i) = (\mathbf{a}, \sum \mathbf{e}_i) = \mathbf{0}$. By Problem 6.70 $\sum(\mathbf{a}, \mathbf{e}_i)^2 = \frac{1}{2} nR^4$; hence, the sum to be found is equal to $4\left(nR^4 + \frac{nR^4}{2}\right) = 6nR^4$.

**6.75.** a) First, let us prove the required relation for $\mathbf{u} = \mathbf{e}_1$. Let $\mathbf{e}_i = (\sin \varphi_i, \cos \varphi_i)$, where $\cos \varphi_1 = 1$. Then

$$\sum (\mathbf{e}_1, \mathbf{e}_i)\mathbf{e}_i = \sum \cos \varphi_i \mathbf{e}_i = \sum (\sin \varphi_i \cos \varphi_i, \cos^2 \varphi_i) =$$
$$\sum \left( \frac{\sin 2\varphi_i}{2}, \frac{1 + \cos 2\varphi_i}{2} \right) = \left( 0, \frac{n}{2} \right) = \frac{n\mathbf{e}_1}{2}.$$

For $\mathbf{u} = \mathbf{e}_2$ the proof is similar.

It remains to notice that any vector $\mathbf{u}$ can be represented in the form $\mathbf{u} = \lambda \mathbf{e}_1 + \mu \mathbf{e}_2$.

b) Let $B_1, \ldots, B_n$ be the midpoints of sides of the given polygon, $\mathbf{e}_i = \frac{\overrightarrow{OB_i}}{OB_i}$, $\mathbf{u} = \overrightarrow{XO}$. Then $\overrightarrow{XA_i} = \overrightarrow{OB_i} + (\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i$. Since $\sum \overrightarrow{OB_i} = \overrightarrow{0}$, it follows that

$$\sum \overrightarrow{XA_i} = \sum (\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i = \frac{n\mathbf{u}}{2} = \frac{n\overrightarrow{XO}}{2}.$$

**6.76.** Let $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ be the vectors of sides of a regular $n$-gon. It suffices to prove that by reordering these vectors we can get a set of vectors $\overrightarrow{\mathbf{a}_1}, \ldots, \overrightarrow{\mathbf{a}_n}$ such that $\sum_{k=1}^{n} k\mathbf{a}_k = \mathbf{0}$. A number $n$ which is not a power of a prime can be represented in the form $n = pq$, where $p$ and $q$ are relatively prime. Now, let us prove that the collection

$$\mathbf{e}_0, \mathbf{e}_p, \ldots, \mathbf{e}_{(q-1)p}; \mathbf{e}_q, \mathbf{e}_{q+p}, \ldots, \mathbf{e}_{q+(q-1)p};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{e}_{(p-1)q}, \mathbf{e}_{(p-1)q+p}, \ldots, \mathbf{e}_{(p-1)q+(q-1)p}$$

is the one to be found. First, notice that if

$$x_1 q + y_1 p \equiv x_2 q + y_2 p \ (\mod pq),$$

then $x_1 \equiv x_2 \ (\mod p)$ and $y_1 \equiv y_2 \ (\mod q)$; therefore, in the considered collection each of the vectors $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$ is encountered exactly once.

The endpoints of vectors $\mathbf{e}_q, \mathbf{e}_{q+p}, \ldots, \mathbf{e}_{q+(q-1)p}$ with a common beginning point distinguish a regular $q$-gon and, therefore, their sum is equal to zero. Moreover, vectors $\mathbf{e}_0, \mathbf{e}_p, \ldots, \mathbf{e}_{(q-1)p}$ turn into $\mathbf{e}_q, \mathbf{e}_{q+p}, \ldots, \mathbf{e}_{q+(p-1)q}$ under the rotation by an angle of $\varphi = \frac{2\pi}{p}$. Hence, if $\mathbf{e}_0 + 2\mathbf{e}_p + \cdots + q\mathbf{e}_{(q-1)p} = \mathbf{b}$, then

$$(q+1)\mathbf{e}_q + (q+2)\mathbf{e}_{q+p} + \cdots + 2q\mathbf{e}_{q+(q-1)p} =$$
$$q(\mathbf{e}_q + \cdots + \mathbf{e}_{q+(q-1)p}) + \mathbf{e}_q + 2\mathbf{e}_{q+p} + \cdots + q\mathbf{e}_{q+(q-1)p} = R^\varphi \mathbf{b},$$

where $R^\varphi \mathbf{b}$ is the vector obtained from $\mathbf{b}$ after the rotation by $\varphi = \frac{2\pi}{p}$. Similar arguments show that for the considered set of vectors we have

$$\sum_{k=1}^{n} k\mathbf{a}_k = \mathbf{b} + R^\varphi \mathbf{b} + \cdots + R^{(p-1)\varphi}\mathbf{b} = \mathbf{0}.$$

**6.77.** Suppose that on the sides of triangle $ABC$ squares $ABB_1A_1$, $BCC_2B_2$, $ACC_3A_3$ are constructed outwards and vertices $A_1$, $B_1$, $B_2$, $C_2$, $C_3$, $A_3$ lie on one circle $S$. The mid-perpendiculars to segments $A_1B_1$, $B_2C_2$, $A_3C_3$ pass through the center of circle $S$. It is clear

that the midperpendiculars to segments $A_1B_1$, $B_2C_2$, $A_3C_3$ coincide with the midperpendiculars to sides of triangle $ABC$ and therefore, the center of circle $S$ coincides with the center of the circumscribed circle of the triangle.

Denote the center of the circumscribed circle of triangle $ABC$ by $O$. The distance from $O$ to line $B_2C_2$ is equal to $R\cos\angle A + 2R\sin\angle A$, where $R$ is the radius of the circumscribed circle of triangle $ABC$. Hence,

$$OB_2^2 = (R\sin\angle A)^2 + (R\cos\angle A + 2R\sin\angle A)^2 =$$
$$R^2(3 + 2(\sin\angle 2A - \cos 2\angle A)) = R^2(3 - 2\sqrt{2}\cos(45° + 2\angle A)).$$

Clearly, in order for the triangle to possess the desired property, it is necessary and sufficient that $OB_2^2 = OC_3^2 = OA_1^2$, i.e.,

$$\cos(45° + 2\angle A) = \cos(45° + 2\angle B) = \cos(45° + 2\angle C).$$

This equality holds for $\angle A = \angle B = \angle C = 60°$. If, contrarywise, $\angle A \neq \angle B$, then $(45° + 2\angle A) + (45° + 2\angle B) = 360°$, i.e., $\angle A + \angle B = 135°$. Hence, $\angle C = 45°$ and $\angle A = \angle C = 45°$, $\angle B = 90°$ (or $\angle B = 45°$, $\angle A = 90°$). We see that the triangle should be either an equilateral or an isosceles one.

**6.78.** In any triangle we have $h_c = \frac{ab}{2R}$ (Problem 12.33); hence, $p_k = \frac{MA_k \cdot MA_{k+1}}{2R}$. Therefore,

$$p_1p_3\ldots p_{2n-1} = \frac{MA_1 \cdot MA_2 \ldots MA_{2n}}{(2R)^n} = p_2p_4\ldots p_{2n}.$$

**6.79.** Let $ABC$ be a triangle inscribed in circle $S$. Denote the distances from the center $O$ of $S$ to sides $BC$, $CA$ and $AB$ by $a$, $b$ and $c$, respectively. Then $R + r = a + b + c$ if point $O$ lies inside triangle $ABC$ and $R + r = -a + b + c$ if points $A$ and $O$ lie on various sides of line $BC$, cf. Problem 12.38.

Each of the diagonals of the partition belongs to two triangles of the partition. For one of these triangles point $O$ and the remaining vertex lie on one side of the diagonal, for the other one the points lie on different sides.

A partition of an $n$-gon by nonintersecting diagonals into triangles consists of $n - 2$ triangles. Therefore, the sum $(n-2)R + r_1 + \cdots + r_{n-2}$ is equal to the sum of distances from point $O$ to the sides of an $n$-gon (the distances to the sides are taken with the corresponding signs). This implies that the sum $r_1 + \cdots + r_{n-2}$ does not depend on the partition.

**6.80.** Let polygon $A_1 \ldots A_n$ be inscribed in a circle. Let us consider point $A_2'$ symmetric to point $A_2$ through the midperpendicular to segment $A_1A_3$. Then polygon $A_1A_2'A_3\ldots A_n$ is an inscribed one and its area is equal to the area of polygon $A_1\ldots A_n$. Therefore, we can transpose any two sides. Therefore, we can make any side, call it $X$, a neighbouring side of any given side, $Y$; next, make any of the remaining sides a neighbour of $X$, etc. Therefore, the area of an $n$-gon inscribed into the given circle only depends on the set of lengths of the sides but not on their order.

**6.81.** Without loss of generality we may assume that $a_n$ is the greatest of the numbers $a_1, \ldots, a_n$. Let $n$-gon $A_1 \ldots A_n$ be inscribed into a circle centered at $O$. Then

$$A_iA_{i+1} : A_1A_n = \sin\frac{\angle A_iOA_{i+1}}{2} : \sin\frac{\angle A_1OA_n}{2}.$$

Therefore, let us proceed as follows. From the relation $\sin\frac{\varphi_i}{2} : \sin\frac{\varphi}{2} = a_i : a_n$ the angle $\varphi_i$ is uniquely determined in terms of $\varphi$ if $\varphi_i < \pi$. On a circle of radius 1, fix a point $A_n$ and

consider variable points $A_1, \ldots, A_{n-1}, A'_n$ such that

$$\smile A_n A_1 = \varphi, \ \smile A_1 A_2 = \varphi_1, \ \ldots, \smile A_{n-2} A_{n-1} = \varphi_{n-2} \text{ and } \smile A_{n-1} A'_n = \varphi_{n-1}.$$

Denote these points in two distinct ways as plotted on Fig. 75. (The first way — Fig. 75 a) — corresponds to an $n$-gon that contains the center of the circle, and the second way — Fig. 75 b) — corresponds to an $n$-gon that does not contain the center of the circle). It remains to prove that as $\varphi$ varies from 0 to $\pi$, then in one of these cases point $A'_n$ coincides with $A_n$ (indeed, then up to a similarity we get the required $n$-gon). Suppose that in the first case points $A'_n$ and $A_n$ never coincide for $0 \leq \varphi \leq \pi$, i.e., for $\varphi = \pi$ we have $\varphi_1 + \cdots + \varphi_{n-1} < \pi$.
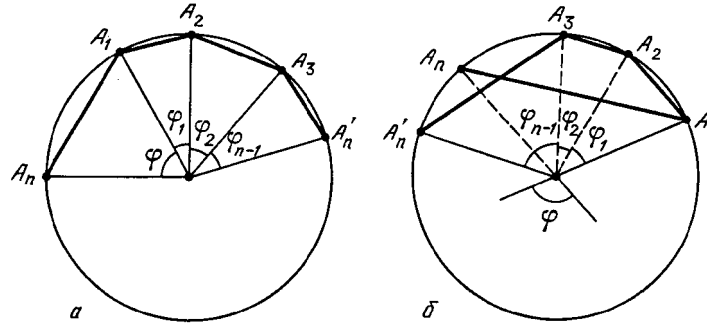


Figure 75 (Sol. 6.81)

Fig. 75 b) requires certain comments: $\sin \alpha \approx \alpha$ for small values of $\alpha$; hence, the conditions of the problem imply that for small angles point $A_n$ does indeed lie on arc $\smile A_1 A'_n$ because $\varphi_1 + \cdots + \varphi_{n-1} > \varphi$. Thus, for small angles $\varphi_1 + \cdots + \varphi_{n-1} > \varphi$ and if $\varphi = \pi$, then by the hypothesis $\varphi_1 + \cdots + \varphi_{n-1} < \pi = \varphi$. Hence, at certain moment $\varphi = \varphi_1 + \cdots + \varphi_{n-1}$, i.e., points $A_n$ and $A'_n$ coincide.

**6.82.** Let $h_1, \ldots, h_n$ be the distances from the given point to the corresponding sides; let $a_1, \ldots, a_n$ be the distances from the vertices of the polygon to tangent points. Then the product of areas of red as well as blue triangles is equal to $\frac{a_1 \ldots a_n h_1 \ldots h_n}{2^n}$.

**6.83.** Let $OH_i$ be a height of triangle $OA_i A_{i+1}$. Then $\angle H_{i-1} OA_i = \angle H_i OA_i = \varphi_i$. The conditions of the problem imply that

$$\varphi_1 + \varphi_2 = \varphi_{n+1} + \varphi_{n+2},$$
$$\varphi_{n+2} + \varphi_{n+3} = \varphi_2 + \varphi_3,$$
$$\varphi_3 + \varphi_4 = \varphi_{n+3} + \varphi_{n+4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\varphi_{n-2} + \varphi_{n-1} = \varphi_{2n-2} + \varphi_{2n-1}$$

(expressing the last equality we have taken into account that $n$ is odd) and

$$\varphi_{n-1} + 2\varphi_n + \varphi_{n+1} = \varphi_{2n-1} + 2\varphi_{2n} + \varphi_1.$$

Adding all these equalities we get

$$\varphi_{n-1} + \varphi_n = \varphi_{2n-1} + \varphi_{2n},$$

as required.

**6.84.** Let $O$ be the center of the given circle. Then $\overrightarrow{XA_i} = \overrightarrow{XO} + \overrightarrow{OA_i}$ and, therefore,

$$XA_i^2 = XO^2 + OA_i^2 + 2(\overrightarrow{OX}, \overrightarrow{OA_i}) = d^2 + r^2 + 2(\overrightarrow{XO}, \overrightarrow{OA_i}).$$

Since $a_1 \overrightarrow{OA_1} + \cdots + A_n \overrightarrow{OA_n} = \overrightarrow{0}$ (cf. Problem 13.4), it follows that

$$a_1 XA_1^2 + \cdots + a_n XA_n^2 = (a_1 + \cdots + a_n)(d^2 + r^2).$$

**6.85.** By Problem 5.8 $\frac{b_{i-1}b_i}{a_i^2} = \sin^2 \frac{\angle A_i}{2}$. To solve heading a) it suffices to multiply all these equalities and to solve heading b) we have to divide the product of all equalities with even index $i$ by the product of all equalities with odd index $i$.

**6.86.** Let $BC$ be a blue side, $AB$ and $CD$ be the sides neighbouring with $BC$. By the hypothesis sides $AB$ and $CD$ are red ones. Suppose that the polygon is a circumscribed one; let $P$, $Q$, $R$ be the tangent points of sides $AB$, $BC$, $CD$, respectively, with the inscribed circle. Clearly, $BP = BQ$, $CR = CQ$ and segments $BP$, $CR$ only neighbour one blue segment. Therefore, the sum of the lengths of the red sides is not smaller than the sum of the lengths of the blue sides. We have obtained a contradiction with the fact that the sum of the lengths of red sides is smaller than the semiperimeter. Therefore, a circle cannot be inscribed into the polygon.

**6.87.** Let the given $n$-gon have $k$ acute angles. Then the sum of its angles is smaller than $k \cdot 90° + (n-k) \cdot 180°$. On the other hand, the sum of the angles of the $n$-gon is equal to $(n-2) \cdot 180°$. Hence,

$$(n-2) \cdot 180° < k \cdot 90° + (n-k) \cdot 180°, \quad \text{i.e., } k < 4.$$

Since $k$ is an integer, $k \leq 3$.
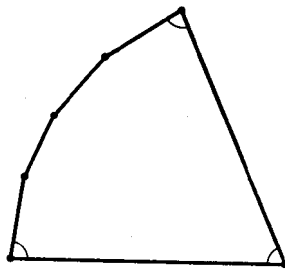


Figure 76 (Sol. 6.87)

For any $n \geq 3$ there exists a convex $n$-gon with three acute angles (Fig. 76).

**6.88.** Suppose that the lengths of nonadjacent sides $AB$ and $CD$ are equal to the length of the greatest diagonal. Then $AB + CD \geq AC + BD$. But by Problem 9.14 $AB + CD < AC + BD$. We have obtained a contradiction and therefore, the sides whose length is equal to the length of the longest diagonal should be adjacent ones, i.e., there are not more than two of such sides.
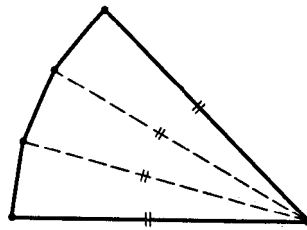


Figure 77 (Sol. 6.88)

An example of a polygon with two sides whose lengths are equal to the length of the longest diagonal is given on Fig. 77. Clearly, such an $n$-gon exists for any $n > 3$.

**6.89.** Let us prove that $n \leq 5$. Let $AB = 1$ and $C$ the vertex not adjacent to either $A$ or $B$. Then $|AC - BC| < AB = 1$. Hence, $AC = BC$, i.e., point $C$ lies on the midperpendicular

to side $AB$. Therefore, in addition to vertices $A$, $B$, $C$ the polygon can have only two more vertices.



ED=1
BE=BD=3
AD=EC=2
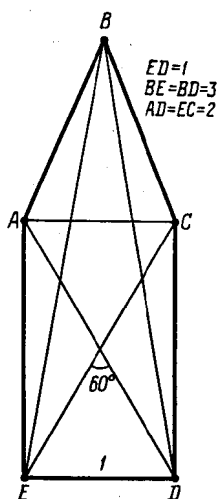
Figure 78 (Sol. 6.89)

An example of a pentagon with the required property is given on Fig. 78. Let us elucidate its construction. Clearly, $ACDE$ is a rectangle, $AC = ED = 1$ and $\angle CAD = 60°$. Point $B$ is determined from the condition $BE = BD = 3$.

An example of a quadrilateral with the desired property is rectangle $ACDE$ on the same figure.

**6.90.** An example of a pentagon satisfying the conditions of the problem is plotted on Fig. 79. Let us clarify its construction. Take an equilateral right triangle $EAB$ and draw midperpendiculars to sides $EA$, $AB$; on them construct points $C$ and $D$, respectively, so that $ED = BC = AB$ (i.e., lines $BC$ and $ED$ form angles of 30° with the corresponding midperpendiculars). Clearly,

$$DE = BC = AB = EA < EB < DC \quad \text{and} \quad DB = DA = CA = CE > EB.$$

Now, let us prove that the fifth side and the fifth diagonal cannot have a common point. Suppose that the fifth side $AB$ has a common point $A$ with the fifth diagonal. Then the fifth diagonal is either $AC$ or $AD$. Let us consider these two cases.
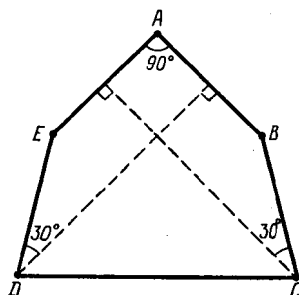


Figure 79 (Sol. 6.90)

In the first case $\triangle AED = \triangle CDE$; hence, under the symmetry through the midperpendicular to segment $ED$ point $A$ turns into point $C$. This symmetry preserves point $B$ because $BE = BD$. Therefore, segment $AB$ turns into $CB$, i.e., $AB = CB$. Contradiction.

In the second case $\triangle ACE = \triangle EBD$; hence, under the symmetry through the bisector of angle $\angle AED$ segment $AB$ turns into $DC$, i.e., $AB = CD$. Contradiction.

**6.91.** Let us consider two neighbouring vertices $A_1$ and $A_2$. If $\angle A_1OA_2 \geq 90°$, then $OA_1 = OA_2$ because neither right nor acute angle can be adjacent to the base of an isosceles triangle.
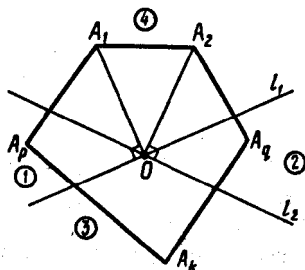


Figure 80 (Sol. 6.91)

Now, let $\angle A_1OA_2 < 90°$. Let us draw through point $O$ lines $l_1$ and $l_2$ perpendicular to lines $OA_1$ and $OA_2$, respectively. Denote the regions into which these lines divide the plane as indicated on Fig. 80. If in region 3 there is a vertex, $A_k$, then $A_1O = A_kO = A_2O$ because $\angle A_1OA_k \geq 90°$ and $\angle A_2OA_k \geq 90°$. If region 3 has no vertices of the polygon, then in region 1 there is a vertex $A_p$ and in region 2 there is a vertex $A_q$ (if neither of the regions 1 or 2 would have contained vertices of the polygon, then point $O$ would have been outside the polygon). Since $\angle A_1OA_q \geq 90°$, $\angle A_2OA_p \geq 90°$ and $\angle A_pOA_q \geq 90°$, it follows that $A_1O = A_qO = A_pO = A_2O$.

It remains to notice that if the distances from point $O$ to any pair of the neighbouring vertices of the polygon are equal, then all the distances from point $O$ to the vertices of the polygon are equal.

**6.92.** Let us prove that if $A$, $B$, $C$, $D$, $E$, $F$ are points on the circle placed in an arbitrary order; lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$, intersect at points $G$, $H$, $K$, respectively. Then points $G$, $H$ and $K$ lie on one line.

Let $a$, $b$, ..., $f$ be oriented angles between a fixed line and lines $OA$, $OB$, ..., $OF$, respectively, where $O$ is the center of the circumscribed circle of the hexagon. Then

$$\angle(AB, DE) = \frac{a + b - d - e}{2}, \quad \angle(CD, FA) = \frac{c + d - f - a}{2},$$

$$\angle(EF, BC) = \frac{e + f - b - c}{2}$$

and, therefore, the sum of these angles is equal to 0.

Let $Z$ be the intersection point of circumscribed circles of triangles $BDG$ and $DFK$. Let us prove that point $B$, $F$, $Z$ and $H$ lie on one circle. For this we have to verify that $\angle(BZ, ZF) = \angle(BH, HF)$. Clearly,

$$\angle(BZ, ZF) = \angle(BZ, ZD) + \angle(DZ, ZF),$$
$$\angle(BZ, ZD) = \angle(BG, GD) = \angle(AB, DE),$$
$$\angle(DZ, ZF) = \angle(DK, KF) = \angle(CD, FA)$$

and, as we have just proved,

$$\angle(AB, DE) + \angle(CD, FA) = -\angle(EF, BC) = \angle(BC, EF) = \angle(BH, HF).$$

Now, let us prove that points $H$, $Z$ and $G$ lie on one line. For this it suffices to verify that $\angle(GZ, ZB) = \angle(HZ, ZB)$. Clearly,

$$\angle(GZ, ZB) = \angle(GD, DB) = \angle(ED, DB), \quad \angle(HZ, ZB) = \angle(HF, FB) = \angle(ED, DB).$$

We similarly prove that points $K$, $Z$ and $G$ lie on one line:

$$\angle(DZ, ZG) = \angle(DB, BG) = \angle(DB, BA);$$
$$\angle(DZ, ZK) = \angle(DF, FK) = \angle(DB, BA)$$

We have deduced that points $H$ and $K$ lie on line $GZ$, consequently, points $G$, $H$ and $K$ lie on one line.

**6.93.** Let $A_2$, $B_2$ and $C_2$ be the indicated intersection points of lines. By applying Pascal's theorem to points $M$, $A_1$, $A$, $C$, $B$, $B_1$ we deduce that $A_2$, $B_2$ and $R$ lie on one line. Similarly, points $A_2$, $C_2$ and $R$ lie on one line. Hence, points $A_2$, $B_2$, $C_2$ and $R$ lie on one line.

**6.94.** Points $A_1$ and $B_1$ lie on circle $S$ of diameter $AB$. Let $A_4$ and $B_4$ be the intersection points of lines $AA_2$ and $BB_2$ with line $A_3B_3$. By Problem 2.41 a) these points lie on circle $S$. Lines $A_1B$ and $A_4A$ intersect at point $A_2$ and lines $BB_4$ and $AB_1$ at point $B_2$. Therefore, applying Pascal's theorem to points $B_1$, $A_1$, $B$, $B_4$, $A_4$, $A$ we see that the intersection point of lines $B_1A_1$ and $B_4A_4$ (the latter line coincides with $A_3B_3$) lies on line $A_2B_2$.

**6.95.** Let $K$ be the intersection point of lines $BC$ and $MN$. Apply Pascal's theorem to points $A$, $M$, $N$, $D$, $C$, $B$. We see that points $E$, $K$, $F$ lie on one line and, therefore, $K$ is the intersection point of lines $MN$ and $EF$.

**6.96.** Let rays $PA$ and $QA$ intersect the circle at points $P_2$ and $Q_2$, i.e., $P_1P_2$ and $Q_1Q_2$ are diameters of the given circle. Let us apply Pascal's theorem to hexagon $PP_2P_1QQ_2Q_1$. Lines $PP_2$ and $QQ_2$ intersect at point $A$ and lines $P_1P_2$ and $Q_1Q_2$ intersect at point $O$, hence, the intersection point of lines $P_1Q$ and $Q_1P$ lies on line $AO$.

**6.97.** Let given points $A$, $B$, $C$, $D$, $E$ lie on one line. Suppose that we have constructed point $F$ of the same circle. Denote by $K$, $L$, $M$ the intersection points of lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$, respectively. Then by Pascal's theorem points $K$, $L$, $M$ lie on one line.

The above implies the following construction. Let us draw through point $E$ an arbitrary line $a$ and denote its intersection point with line $BC$ by $L$. Then construct the intersection point $K$ of lines $AB$ and $DE$ and the intersection point $M$ of lines $KL$ and $CD$. Finally, let $F$ be the intersection point of lines $AM$ and $a$. Let us prove that $F$ lies on our circle. Let $F_1$ be the intersection point of the circle and line $a$. From Pascal's theorem it follows that $F_1$ lies on line $AM$, i.e., $F_1$ is the intersection point of $a$ and $AM$. Hence, $F_1 = F$.

**6.98.** Let $P$ and $Q$ be the intersection points of line $A_3A_4$ with $A_1A_2$ and $A_1A_6$, respectively, and $R$ and $S$ be the intersection points of line $A_4A_5$ with $A_1A_6$ and $A_1A_2$, respectively. Then

$$A_2K : A_3L = A_2P : A_3P, \quad A_3L : A_6M = A_3Q : A_6Q, \quad A_6M : A_5N = A_6R : A_5R.$$

Therefore, the desired relation $A_2K : A_5N = A_2S : A_5S$ takes the form

$$\frac{A_2P}{A_3P} \cdot \frac{A_3Q}{A_6Q} \cdot \frac{A_6R}{A_5R} \cdot \frac{A_5S}{A_2S} = 1.$$

Let $T$ be the intersection point of lines $A_2A_3$ and $A_5A_6$; by Pascal's theorem points $S$, $Q$ and $T$ lie on one line. By applying Menelau's theorem (cf. Problem 5.58) to triangle $PQS$ and points $T$, $A_2$, $A_3$ and also to triangle $RQS$ and points $T$, $A_5$, $A_6$ we get

$$\frac{A_2P}{A_2S} \cdot \frac{A_3Q}{A_3P} \cdot \frac{TS}{TQ} = 1 \quad \text{and} \quad \frac{TQ}{TS} \cdot \frac{A_5S}{A_5R} \cdot \frac{A_6R}{A_6Q} = 1.$$

By multiplying these equalities we get the statement desired. (The ratio of segments should be considered oriented ones.)

# Chapter 7. LOCI

## Background

1) A *locus* is a figure consisting of all points having a cirtain property.

2) A solution of a problem where a locus is to be found should contain the proof of the following facts:

a) the points with a required property belong to figure $\Phi$ which is the answer to the problem;

b) All points of $\Phi$ have the required property.

3) A locus possessing two properties is the intersection of two figures: (1) the locus of points possessing the first property and (2) the locus of points possessing the other property.

4) Three most important loci:

a) *The locus of points equidistant from points $A$ and $B$ is the midperpendicular to segment $AB$;*

b) *The locus of points whose distance from a given point $O$ is equal to $R$ is the circle of radius $R$ centered at $O$;*

c) *The locus of vertices of a given angle that subtend given segment $AB$ is the union of two arcs of circles symmetric through line $AB$ (points $A$ and $B$ do not belong to the locus).*

## Introductory problems

**1.** a) Find the locus of points equidistant from two parallel lines.

b) Find the locus of points equidistant from two intersecting lines.

**2.** Find the locus of the midpoints of segments with the endpoints on two given parallel lines.

**3.** Given triangle $ABC$, find the locus of points $X$ satisfying inequalities $AX \leq BX \leq$ CX.

**4.** Find the locus of points $X$ such that the tangents drawn from $X$ to the given circle have a given length.

**5.** A point $A$ on a circle is fixed. Find the locus of points $X$ that divide chords with $A$ as an endpoint in the ratio of $1 : 2$ counting from point $A$.

## §1. The locus is a line or a segment of a line

**7.1.** Two wheels of radii $r_1$ and $r_2$ roll along line $l$. Find the set of intersection points $M$ of their common inner tangents.

**7.2.** Sides $AB$ and $CD$ of quadrilateral $ABCD$ of area $S$ are not parallel. Inside the quadrilateral find the locus of points $X$ for which $S_{ABX} + S_{CDX} = \frac{1}{2}S$.

**7.3.** Given two lines that meet at point $O$. Find the locus of points $X$ for which the sum of the lengths of projections of segments $OX$ to these lines is a constant.

**7.4.** Given rectangle $ABCD$, find the locus of points $X$ for which $AX + BX = CX + DX$.

**7.5.** Find the locus of points $M$ that lie inside rhombus $ABCD$ and with the property that $\angle AMD + \angle BMC = 180°$.

* * *

**7.6.** Given points $A$ and $B$ in plane, find the locus of points $M$ for which the difference of the squared lengths of segments $AM$ and $PM$ is a constant.

**7.7.** A circle $S$ and a point $M$ outside it are given. Through point $M$ all possible circles $S_1$ that intersect $S$ are drawn; $X$ is the intersection point of the tangent at $M$ to $S_1$ with the extension of the common chord of circles $S$ and $S_1$. Find the locus of points $X$.

**7.8.** Given two nonintersecting circles, find the locus of the centers of circles that divide the given circles in halves (i.e., that intersect the given circles in diametrically opposite points).

**7.9.** A point $A$ inside a circle is taken. Find the locus of the intersection points of tangents to circles drawn through the endpoints of possible chords that contain point $A$.

**7.10.** a) Parallelogram $ABCD$ is given. Prove that the quantity

$$AX^2 + CX^2 - BX^2 - DX^2$$

does not depend on the choice of point $X$.

b) Quadrilateral $ABCD$ is not a parallelogram. Prove that all points $X$ that satisfy the relation $AX^2 + CX^2 = BX^2 + DX^2$ lie on one line perpendicular to the segment that connects the midpoints of the diagonals.

See also Problems 6.14, 15.14.

## §2. The locus is a circle or an arc of a circle

**7.11.** A segment moves along the plane so that its endpoints slide along the legs of a right angle $\angle ABC$. What is the trajectory traversed by the midpoint of this segment? (We naturally assume that the length of the segment does not vary while it moves.)

**7.12.** Find the locus of the midpoints of the chords of a given circle, provided the chords pass through a given point.

**7.13.** Given two points, $A$ and $B$ and two circles that are tangent to line $AB$: one circle is tangent at $A$ and the other one at $B$, and the circles are tangent to each other at point $M$. Find the locus of points $M$.

* * *

**7.14.** Two points, $A$ and $B$ in plane are given. Find the locus of points $M$ for which $AM : BM = k$. (*Apollonius's circle.*)

**7.15.** Let $S$ be Apollonius's circle for points $A$ and $B$ where point $A$ lies outside circle $S$. From point $A$ tangents $AP$ and $AQ$ to circle $S$ are drawn. Prove that $B$ is the midpoint of segment $PQ$.

**7.16.** Let $AD$ and $AE$ be the bisectors of the inner and outer angles of triangle $ABC$ and $S_a$ be the circle with diameter $DE$; circles $S_b$ and $S_c$ are similarly defined. Prove that:

a) circles $S_a$, $S_b$ and $S_c$ have two common points, $M$ and $N$, such that line $MN$ passes through the center of the circumscribed circle of triangle $ABC$;

b) The projections of point $M$ (and $N$) to the sides of triangle $ABC$ distinguish an equilateral triangle.

**7.17.** Triangle $ABC$ is an equilateral one, $M$ is a point. Prove that if the lengths of segments $AM$, $BM$ and $CM$ form a geometric progression, then the quotient of this progression is smaller than 2.

See also Problems 14.19 a), 18.14.

## §3. The inscribed angle

**7.18.** Points $A$ and $B$ on a circle are fixed and a point $C$ runs along the circle. Find the set of the intersection points of a) heights; b) bisectors of triangles $ABC$.

**7.19.** Point $P$ runs along the circumscribed circle of square $ABCD$. Lines $AP$ and $BD$ intersect at point $Q$ and the line that passes through point $Q$ parallel to $AC$ intersects line $BP$ at point $X$. Find the locus of points $X$.

**7.20.** a) Points $A$ and $B$ on a circle are fixed and points $A_1$ and $B_1$ run along the same circle so that the value of arc $\smile A_1B_1$ remains a constant; let $M$ be the intersection point of lines $AA_1$ and $BB_1$. Find the locus of points $M$.

b) Triangles $ABC$ and $A_1B_1C_1$ are inscribed in a circle; triangle $ABC$ is fixed and triangle $A_1B_1C_1$ rotates. Prove that lines $AA_1$, $BB_1$ and $CC_1$ intersect at one point for not more than one position of triangle $A_1B_1C_1$.

**7.21.** Four points in the plane are given. Find the locus of the centers of rectangles formed by four lines that pass through the given points.

**7.22.** Find the locus of points $X$ that lie inside equilateral triangle $ABC$ and such that $\angle XAB + \angle XBC + \angle XCA = 90°$.

See also Problems 2.5, 2.37.

## §4. Auxiliary equal triangles

**7.23.** A semicircle centered at $O$ is given. From every point $X$ on the extension of the diameter of the semicircle a ray tangent to the semicircle is drawn. On the ray segment $XM$ equal to segment $XO$ is marked. Find the locus of points $M$ obtained in this way.

**7.24.** Let $A$ and $B$ be fixed points in plane. Find the locus of points $C$ with the following property: height $h_b$ of triangle $ABC$ is equal to $b$.

**7.25.** A circle and a point $P$ inside it are given. Through every point $Q$ on the circle the tangent is drawn. The perpendicular dropped from the center of the circle to line $PQ$ and the tangent intersect at a point $M$. Find the locus of points $M$.

## §5. The homothety

**7.26.** Points $A$ and $B$ on a circle are fixed. Point $C$ runs along the circle. Find the set of the intersection points of the medians of triangles $ABC$.

**7.27.** Triangle $ABC$ is given. Find the locus of the centers of rectangles $PQRS$ whose vertices $Q$ and $P$ lie on side $AC$ and vertices $R$ and $S$ lie on sides $AB$ and $BC$, respectively.

**7.28.** Two circles intersect at points $A$ and $B$. Through point $A$ a line passes. It intersects the circles for the second time at points $P$ and $Q$. What is the line plotted by the midpoint of segment $PQ$ while the intersecting line rotates about point $A$.

**7.29.** Points $A$, $B$ and $C$ lie on one line; $B$ is between $A$ and $C$. Find the locus of points $M$ such that the radii of the circumscribed circles of triangles $AMB$ and $CMB$ are equal.

See also Problems 19.10, 19.21, 19.38.

## §6. A method of loci

**7.30.** Points $P$ and $Q$ move with the same constant speed $v$ along two lines that intersect at point $O$. Prove that there exists a fixed point $A$ in plane such that the distances from $A$ to $P$ and $Q$ are equal at all times.

**7.31.** Through the midpoint of each diagonal of a convex quadrilateral a line is drawn parallel to the other diagonal. These lines meet at point $O$. Prove that segments that connect

$O$ with the midpoints of the sides of the quadrilateral divide the area of the quadrilateral into equal parts.

**7.32.** Let $D$ and $E$ be the midpoints of sides $AB$ and $BC$ of an acute triangle $ABC$ and point $M$ lies on side $AC$. Prove that if $MD < AD$, then $ME > EC$.

**7.33.** Inside a convex polygon points $P$ and $Q$ are taken. Prove that there exists a vertex of the polygon whose distance from $Q$ is smaller than that from $P$.

**7.34.** Points $A$, $B$ and $C$ are such that for any fourth point $M$ either $MA \leq MB$ or $MA \leq MC$. Prove that point $A$ lies on segment $BC$.

**7.35.** Quadrilateral $ABCD$ is given; in it $AB < BC$ and $AD < DC$. Point $M$ lies on diagonal $BD$. Prove that $AM < MC$.

## §7. The locus with a nonzero area

**7.36.** Let $O$ be the center of rectangle $ABCD$. Find the locus of points $M$ for which $AM \geq OM$, $BM \geq OM$, $CM \geq OM$ and $DM \geq OM$.

**7.37.** Find the locus of points $X$ from which tangents to a given arc $AB$ of a circle can be drawn.

**7.38.** Let $O$ be the center of an equilateral triangle $ABC$. Find the locus of points $M$ satisfying the following condition: any line drawn through $M$ intersects either segment $AB$ or segment $CO$.

**7.39.** In plane, two nonintersecting disks are given. Does there necessarily exist a point $M$ outside these disks that satisfies the following condition: each line that passes through $M$ intersects at least one of these disks?

Find the locus of points $M$ with this property.

See also Problem 18.11.

## §8. Carnot's theorem

**7.40.** Prove that the perpendiculars dropped from points $A_1$, $B_1$ and $C_1$ to sides $BC$, $CA$, $AB$ of triangle $ABC$ intersect at one point if and only if

$$A_1B^2 + C_1A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1B^2. \qquad (Carnot's\ formula)$$

**7.41.** Prove that the heights of a triangle meet at one point.

**7.42.** Points $A_1$, $B_1$ and $C_1$ are such that $AB_1 = AC_1$, $BC = BA_1$ and $CA_1 = CB_1$. Prove that the perpendiculars dropped from points $A_1$, $B_1$ and $C_1$ to lines $BC$, $CA$ and $AB$ meet at one point.

**7.43.** a) The perpendiculars dropped from the vertices of triangle $ABC$ to the corresponding sides of triangle $A_1B_1C_1$ meet at one point. Prove that the perpendiculars dropped from the vertices of triangle $A_1B_1C_1$ to the corresponding sides of triangle $ABC$ also meet at one point.

b) Lines drawn through vertices of triangle $ABC$ parallelly to the corresponding sides of triangle $A_1B_1C_1$ intersect at one point. Prove that the lines drawn through the vertices of triangle $A_1B_1C_1$ parallelly to the corresponding sides of triangle $ABC$ also intersect at one point.

**7.44.** On line $l$ points $A_1$, $B_1$ and $C_1$ are taken and from the vertices of triangle $ABC$ perpendiculars $AA_2$, $BB_2$ and $CC_2$ are dropped to this line. Prove that the perpendiculars dropped from points $A_1$, $B_1$ and $C_1$ to lines $BC$, $CA$ and $AB$, respectively, intersect at one point if and only if

$$\overline{A_1B_1} : \overline{B_1C_1} = \overline{A_2B_2} : \overline{B_2C_2}.$$

The ratios of segments are oriented ones.

**7.45.** Triangle $ABC$ is an equilateral one, $P$ an arbitrary point. Prove that the perpendiculars dropped from the centers of the inscribed circles of triangles $PAB$, $PBC$ and $PCA$ to lines $AB$, $BC$ and $CA$, respectively, meet at one point.

**7.46.** Prove that if perpendiculars raised at the bases of bisectors of a triangle meet at one point, then the triangle is an isosceles one.

## §9. Fermat-Apollonius's circle

**7.47.** Prove that the set of points $X$ such that

$$k_1 A_1 X^2 + \cdots + k_n A_n X^2 = c$$

is either

a) a circle or the empty set if $k_1 + \cdots + k_n \neq 0$;

b) a line, a plane or the empty set if $k_1 + \cdots + k_n = 0$.

**7.48.** Line $l$ intersects two circles at four points. Prove that the quadrilateral formed by the tangents at these points is a circumscribed one and the center of its circumscribed circle lies on the line that connects the centers of the given circles.

**7.49.** Points $M$ and $N$ are such that $AM : BM : CM = AN : BN : CN$. Prove that line $MN$ passes through the center $O$ of the circumscribed circle of triangle $ABC$.

See also Problems 7.6, 7.14, 8.59–8.63.

## Problems for independent study

**7.50.** On sides $AB$ and $BC$ of triangle $ABC$, points $D$ and $E$ are taken. Find the locus of the midpoints of segments $DE$.

**7.51.** Two circles are tangent to a given line at two given points $A$ and $B$; the circles are also tangent to each other. Let $C$ and $D$ be the tangent points of these circles with another outer tangent. Both tangent lines to the circles are outer ones. Find the locus of the midpoints of segments $CD$.

**7.52.** The bisector of one of the angles of a triangle has inside the triangle a common point with the perpendicular erected from the midpoint of the side opposite the angle. Prove that the triangle is an isosceles one.

**7.53.** Triangle $ABC$ is given. Find the locus of points $M$ of this triangle for which the condition $AM \geq BM \geq CM$ holds. When the obtained locus is a) a pentagon; b) a triangle?

**7.54.** Square $ABCD$ is given. Find the locus of the midpoints of the sides of the squares inscribed in the given square.

**7.55.** An equilateral triangle $ABC$ is given. Find the locus of points $M$ such that triangles $AMB$ and $BCM$ are isosceles ones.

**7.56.** Find the locus of the midpoints of segments of length $\frac{2}{\sqrt{3}}$ whose endpoints lie on the sides of a unit square.

**7.57.** On sides $AB$, $BC$ and $CA$ of a given triangle $ABC$ points $P$, $Q$ and $R$, respectively, are taken, so that $PQ \parallel AC$ and $PR \parallel BC$. Find the locus of the midpoints of segments $QR$.

**7.58.** Given a semicircle with diameter $AB$. For any point $X$ on this semicircle, point $Y$ on ray $XA$ is taken so that $XY = XB$. Find the locus of points $Y$.

**7.59.** Triangle $ABC$ is given. On its sides $AB$, $BC$ and $CA$ points $C_1$, $A_1$ and $B_1$, respectively, are selected. Find the locus of the intersection points of the circumscribed circles of triangles $AB_1C_1$, $A_1BC_1$ and $A_1B_1C$.

## Solutions

**7.1.** Let $O_1$ and $O_2$ be the centers of the wheels of radii $r_1$ and $r_2$, respectively. If $M$ is the intersection point of the inner tangents, then $OM : O_2M = r_1 : r_2$. It is easy to derive from this condition that the distance from point $M$ to line $l$ is equal to $\frac{2r_1r_2}{r_1+r_2}$. Hence, all the intersection points of the common inner tangents lie on the line parallel to $l$ and whose distance from $l$ is equal to $\frac{2r_1r_2}{r_1+r_2}$.

**7.2.** Let $O$ be the intersection point of lines $AB$ and $CD$. On rays $OA$ and $OD$, mark segments $OK$ and $OL$ equal to $AB$ and $CD$, respectively. Then

$$S_{ABX} + S_{CDX} = S_{KOX} + S_{LOX} = S_{KOL} \pm S_{KXL}.$$

Therefore, the area of triangles $KXL$ is a constant, i.e., point $X$ lies on a line parallel to $KL$.

**7.3.** Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors parallel to the given lines; $\mathbf{x} = \overrightarrow{OX}$. The sum of the lengths of the projections of vector $\mathbf{x}$ to the given lines is equal to $|(\mathbf{a}, \mathbf{x})| + |(\mathbf{b}, \mathbf{x})| = |(\mathbf{a} \pm \mathbf{b}, \mathbf{x})|$, where the change of sign occurs on the perpendiculars to the given lines erected at point $O$. Therefore, the locus to be found is a rectangle whose sides are parallel to the bisectors of the angles between the given lines and the vertices lie on the indicated perpendiculars.

**7.4.** Let $l$ be the line that passes through the midpoints of sides $BC$ and $AD$. Suppose that point $X$ does not lie on $l$; for instance, points $A$ and $X$ lie on one side of $l$. Then $AX < DX$ and $BX < CX$ and, therefore, $AX + BX < CX + DX$. Hence, $l$ is the locus to be found.

**7.5.** Let $N$ be a point such that $\overrightarrow{MN} = \overrightarrow{DA}$. Then $\angle NAM = \angle DMA$ and $\angle NBM = \angle BMC$ and, therefore, quadrilateral $AMBN$ is an inscribed one. The diagonals of the inscribed quadrilateral $AMBN$ are equal, hence, either $AM \parallel BN$ or $BM \parallel AN$. In the first case $\angle AMD = \angle MAN = \angle AMB$ and in the second case $\angle BMC = \angle MBN = \angle BMA$. If $\angle AMB = \angle AMD$, then $\angle AMB + \angle BMC = 180°$ and point $M$ lies on diagonal $AC$ and if $\angle BMA = \angle BMC$, then point $M$ lies on diagonal $BD$. It is also clear that if point $M$ lies on one of the diagonals, then $\angle AMD + \angle BMC = 180°$.

**7.6.** Introduce a coordinate system selecting point $A$ as the origin and directing $Ox$-axis along ray $AB$. Let $(x, y)$ be the coordinates of $M$. Then $AM^2 = x^2 + y^2$ and $BM^2 = (x - a)^2 + y^2$, where $a = AB$. Hence, $AM^2 - BM^2 = 2ax - a^2$. This quantity is equal to $k$ for points $M$ whose coordinates are $(\frac{a^2+k}{2a}, y)$. All such points lie on a line perpendicular to $AB$.

**7.7.** Let $A$ and $B$ be the intersection points of circles $S$ and $S_1$. Then $XM^2 = XA \cdot XB = XO^2 - R^2$, where $O$ and $R$ are the center and the radius, respectively, of circle $S$. Hence, $XO^2 - XM^2 = R^2$ and, therefore, points $X$ lie on the perpendicular to line $OM$ (cf. Problem 7.6).

**7.8.** Let $O_1$ and $O_2$ be the centers of the given circles, $R_1$ and $R_2$ their respective radii. The circle of radius $r$ centered at $X$ intersects the first circle in the diametrically opposite points if and only if $r^2 = XO_1^2 + R_1^2$; hence, the locus to be found consists of points $X$ such that $XO_1^2 + R_1^2 = XO_2^2 + R_2^2$. All such points $X$ lie on a line perpendicular to $O_1O_2$, cf. Problem 7.6.

**7.9.** Let $O$ be the center of the circle, $R$ its radius, $M$ the intersection point of the tangents drawn through the endpoints of the chord that contains point $A$, and $P$ the midpoint of this chord. Then $OP \cdot OM = R^2$ and $OP = OA \cos\varphi$, where $\varphi = \angle AOP$. Hence,

$$AM^2 = OM^2 + OA^2 - 2OM \cdot OA \cos\varphi = OM^2 + OA^2 - 2R^2,$$

and, therefore, the quantity

$$OM^2 - AM^2 = 2R^2 - OA^2$$

is a constant. It follows that all points $M$ lie on a line perpendicular to $OA$, cf. Problem 7.6.

**7.10.** Let $P$ and $Q$ be the midpoints of diagonals $AC$ and $BD$. Then

$$AX^2 + CX^2 = 2PX^2 + \frac{AC^2}{2} \quad \text{and} \quad BX^2 + DX^2 = 2QX^2 + \frac{BD^2}{2}$$

(cf. Problem 12.11 a)) and, therefore, in heading b) the locus to be found consists of points $X$ such that $PX^2 - QX^2 = \frac{1}{4}(BD^2 - AC^2)$ and in heading a) $P = Q$ and, therefore, the considered quantity is equal to $\frac{1}{2}(BD^2 - AC^2)$.

**7.11.** Let $M$ and $N$ be the midpoints of the given segment, $O$ its midpoint. Point $B$ lies on the circle with diameter $MN$, hence, $OB = \frac{1}{2}MN$. The trajectory of point $O$ is the part of the circle of radius $\frac{1}{2}MN$ centered at $B$ confined inside angle $\angle ABC$.

**7.12.** Let $M$ be the given point, $O$ the center of the given circle. If $X$ is the midpoint of chord $AB$, then $XO \perp AB$. Therefore, the locus to be found is the circle with diameter $MO$.

**7.13.** Let us draw through point $M$ a common tangent to the circles. Let $O$ be the intersection point of this tangent with line $AB$. Then $AO = MO = BO$, i.e., $O$ is the midpoint of segment $AB$. Point $M$ lies on the circle with center $O$ and radius $\frac{1}{2}AB$. The locus of points $M$ is the circle with diameter $AB$ (points $A$ and $B$ excluded).

**7.14.** For $k = 1$ we get the midperpendicular to segment $AB$. In what follows we will assume that $k \neq 1$.

Let us introduce a coordinate system in plane so that the coordinates of $A$ and $B$ are $(-a, 0)$ and $(a, 0)$, respectively. If the coordinates of point $M$ are $(x, y)$, then

$$\frac{AM^2}{BM^2} = (x + a)^2 + \frac{y^2}{(x - a)^2} + y^2.$$

The equation $\frac{AM^2}{BM^2} = k^2$ takes the form

$$x + 1 - \frac{k^2}{1 - k^2 a} + y^2 = \frac{2ka}{1 - k^2}^2.$$

This is an equation of the circle with center $(-1 + \frac{k^2}{1 - k^2 a}, 0)$ and radius $\frac{2ka}{|1 - k^2|}$.

**7.15.** Let line $AB$ intersect circle $S$ at points $E$ and $F$ so that point $E$ lies on segment $AB$. Then $PE$ is the bisector of triangle $APB$, hence, $\angle EPB = \angle EPA = \angle EFP$. Since $\angle EPF = 90°$, it follows that $PB \perp EF$.

**7.16.** a) The considered circles are Apollonius's circles for the pairs of vertices of triangle $ABC$ and, therefore, if $X$ is a common point of circles $S_a$ and $S_b$, then $XB : XC = AB : AC$ and $XC : XA = BC : BA$, i.e., $XB : XA = CB : CA$ and, therefore, point $X$ belongs to circle $S_c$. It is also clear that if $AB > BC$, then point $D$ lies inside circle $S_b$ and point $A$ outside it. It follows that circles $S_a$ and $S_b$ intersect at two distinct points.

To complete the proof, it remains to make use of the result of Problem 7.49.

b) According to heading a) $MA = \frac{\lambda}{a}$, $MB = \frac{\lambda}{b}$ and $MC = \frac{\lambda}{c}$. Let $B_1$ and $C_1$ be the projections of point $M$ on lines $AC$ and $AB$, respectively. Points $B_1$ and $C_1$ lie on the circle with diameter $MA$, hence,

$$B_1 C_1 = MA \sin \angle B_1 A C_1 = \frac{\lambda}{a} \frac{a}{2R} = \frac{\lambda}{2R},$$

where $R$ is the radius of the circumscribed circle of triangle $ABC$. Similarly, $A_1 C_1 = A_1 B_1 = \frac{\lambda}{2R}$.

**7.17.** Let $O_1$ and $O_2$ be points such that $\overrightarrow{BO_1} = \frac{4}{3}\overrightarrow{BA}$ and $\overrightarrow{CO_2} = \frac{4}{3}\overrightarrow{CB}$. It is easy to verify that if $BM > 2AM$, then point $M$ lies inside circle $S_1$ of radius $\frac{2}{3}AB$ with center $O_1$ (cf. Problem 7.14) and if $CM > 2BM$, then point $M$ lies inside circle $S_2$ of radius $\frac{2}{3}AB$ centered at $O_2$. Since $O_1O_2 > BO_1 = \frac{4}{3}AB$ and the sum of the radii of circles $S_1$ and $S_2$ is equal to $\frac{4}{3}AB$, it follows that these circles do not intersect. Therefore, if $BM = qAM$ and $CM = qBM$, then $q < 2$.

**7.18.** a) Let $O$ be the intersection point of heights $AA_1$ and $BB_1$. The points $A_1$ and $B_1$ lie on the circle with diameter $CO$. Therefore, $\angle AOB = 180° - \angle C$. Hence, the locus to be found is the circle symmetric to the given one through line $AB$ (points $A$ and $B$ should be excluded).

b) If $O$ is the intersection point of the bisectors of triangle $ABC$, then $\angle AOB = 90° + \frac{1}{2}\angle C$. On each of the two arcs $\smile AB$ the angles $C$ are constant and, therefore, the desired locus of the vertices of angles of $90° + \frac{1}{2}\angle C$ that subtend segment $AB$ is the union of two arcs (points $A$ and $B$ should be excluded).

**7.19.** Points $P$ and $Q$ lie on the circle with diameter $DX$, hence,

$$\angle(QD, DX) = \angle(QP, PX) = \angle(AP, PB) = 45°,$$

i.e., point $X$ lies on line $CD$.

**7.20.** a) If point $A_1$ traverses along the circle an arc of value $2\varphi$, then point $B_1$ also traverses an arc of value $2\varphi$, consequently, lines $AA_1$ and $BB_1$ turn through an angle of $\varphi$ and the angle between them will not change.

Hence, point $M$ moves along a circle that contains points $A$ and $B$.

b) Let at some moment lines $AA_1$, $BB_1$ and $CC_1$ meet at point $P$. Then, for instance, the intersection point of lines $AA_1$ and $BB_1$ moves along the circumscribed circle of triangle $ABP$. It is also clear that the circumscribed circles of triangles $ABP$, $BCP$ and $CAP$ have a unique common point, $P$.

**7.21.** Suppose that points $A$ and $C$ lie on opposite sides of a rectangle. Let $M$ and $N$ be the midpoints of segments $AC$ and $BD$, respectively. Let us draw through point $M$ line $l_1$ parallel to the sides of the rectangle on which points $A$ and $C$ lie and through point $N$ line $l_2$ parallel to the sides of the rectangle on which points $B$ and $D$ lie. Let $O$ be the intersection point of lines $l_1$ and $l_2$. Clearly, point $O$ lies on circle $S$ constructed on segment $MN$ as on a diameter.

On the other hand, point $O$ is the center of the rectangle. Clearly, the rectangle can be constructed for any point $O$ that lies on circle $S$.

It remains to notice that on the opposite sides of the rectangle points $A$ and $B$ or $A$ and $D$ can also lie. Hence, the locus to be found is the union of three circles.

**7.22.** It is easy to verify that the points of heights of triangle $ABC$ possess the required property. Suppose that a point $X$ not belonging to any of the heights of triangle $ABC$ possesses the required property. Then line $BX$ intersects heights $AA_1$ and $CC_1$ at points $X_1$ and $X_2$. Since

$$\angle XAB + \angle XBC + \angle XCA = 90° = \angle X_1AB + \angle X_1BC + \angle X_1CA,$$

it follows that

$$\angle XAB - \angle X_1AB = \angle X_1CA - \angle XCA,$$

i.e., $\angle(XA, AX_1) = \angle(X_1C, CX)$. Therefore, point $X$ lies on the circumscribed circle of triangle $AXC'$, where point $C'$ is symmetric to $C$ through line $BX$. We similarly prove that point $X_2$ lies on the circle and, therefore, line $BX$ intersects this circle at three distinct points. Contradiction.

**7.23.** Let $K$ be the tangent point of line $MX$ with the given semicircle and $P$ the projection of point $M$ to the diameter. In right triangles $MPX$ and $OKX$, the hypothenuses are equal and $\angle PXM = \angle OXK$; hence, these triangles are equal. In particular, $MP = KO = R$, where $R$ is the radius of the given semicircle. It follows that point $M$ lies on line $l$ parallel to the diameter of the semicircle and tangent to the semicircle. Let $AB$ be the segment of line $l$ whose projection is the diameter of the semicircle. From a point on $l$ that does not belong to segment $AB$ a tangent to the given semicircle cannot be drawn because the tangent drawn to the circle should be tangent to the other semicircle as well.

The locus to be found is punctured segment $AB$: without points $A$, $B$, and the midpoint.

**7.24.** Let $H$ be the base of height $h_b$ of triangle $ABC$ and $h_b = b$. Denote by $B'$ the intersection point of the perpendicular to line $AB$ drawn through point $A$ and the perpendicular to line $AH$ drawn through point $C$. Right triangles $AB'C$ and $BAH$ are equal, becuase $\angle AB'C = \angle BAH$ and $\angle AC = BH$. Therefore, $AB' = AB$, i.e., point $C$ lies on the circle with diameter $AB'$.
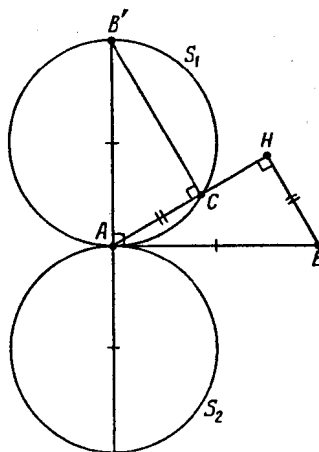


Figure 81 (Sol. 7.24)

Let $S_1$ and $S_2$ be the images of circle $S$ with diameter $AB$ under the rotations through angles of $\pm 90°$ with center at $A$ (Fig. 81). We have proved that point $C \neq A$ belongs to the union of circles $S_1$ and $S_2$.

Conversely, let a point $C$, $C \neq A$, belong to either of the circles $S_1$ or $S_2$; let $AB'$ be a diameter of the corresponding circle. Then $\angle AB'C = \angle HAB$ and $A'B = AB$; hence, $AC = HB$.

**7.25.** Let $O$ be the center of the circle, $N$ the intersection point of lines $OM$ and $QP$. Let us drop from point $M$ perpendicular $MS$ to line $OP$. Since $\triangle ONQ \sim \triangle OQM$ and $\triangle OPN \sim \triangle OMS$, we derive that

$$ON : OQ = OQ : OM \quad \text{and} \quad OP : ON = OM : OS.$$

By multiplying these equalities we get $OP : OQ = OQ : OS$. Hence, $OS = OQ^2 : OP$ is a constant. Since point $S$ lies on line $OP$, its position does not depend on the choice of point $Q$. The locus to be found is the line perpendicular to line $OP$ and passing through point $S$.

**7.26.** Let $O$ be the midpoint of segment $AB$, and $M$ the intersection point of the medians of triangle $ABC$. The homothety with center $O$ and coefficient $\frac{1}{3}$ sends point $C$ to point $M$. Therefore, the intersection point of the medians of triangle $ABC$ lies on circle $S$ which is the image of the initial circle under the homothety with center $O$ and coefficient $\frac{1}{3}$. To get the desired locus we have to delete from $S$ the images of points $A$ and $B$.

**7.27.** Let $O$ be the midpoint of height $BH$; let $M$, $D$ and $E$ be the midpoints of segment $AC$, and sides $RQ$ and $PS$, respectively (Fig. 82).
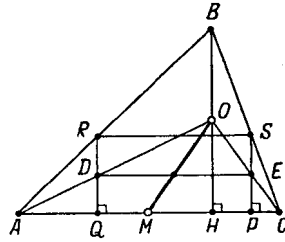


Figure 82 (Sol. 7.27)

Points $D$ and $E$ lie on lines $AO$ and $CO$, respectively. The midpoint of segment $DE$ is the center of rectangle $PQRS$. Clearly, this midpoint lies on segment $OM$. The locus in question is segment $OM$ without its endpoints.

**7.28.** Let $O_1$ and $O_2$ be the centers of the given circles (point $P$ lies on the circle centered at $O_1$); $O$ the midpoint of segment $O_1O_2$; $P'$, $Q'$ and $O'$ the projections of points $O_1$, $O_2$ and $O$ to line $PQ$. As line $PQ$ rotates, point $O'$ runs the circle $S$ with diameter $AO$. Clearly, the homothety with center $A$ and coefficient 2 sends segment $P'Q'$ to segment $PQ$, i.e., point $O'$ turns into the midpoint of segment $PQ$. Hence, the locus in question is the image of circle $S$ under this homothety.

**7.29.** Let $P$ and $Q$ be the centers of the circumscribed circles of triangles $AMB$ and $CMB$. Point $M$ belongs to the locus to be found if $BPMQ$ is a rhombus, i.e., point $M$ is the image of the midpoint of segment $PQ$ under the homothety with center $B$ and coefficient 2. Since the projections of points $P$ and $Q$ to line $AC$ are the midpoints of segments $AB$ and $BC$, respectively, the midpoints of all segments $PQ$ lie on one line. (The locus to be found is the above-obtained line without the intersection point with line $AC$.)

**7.30.** Point $P$ passes through point $O$ at time $t_1$, it passes point $Q$ at time $t_2$. At time $\frac{1}{2}(t_1 + t_2)$ the distances from $O$ to points $P$ and $Q$ are equal to $\frac{1}{2}|t_1 + t_2|v$. At this moment erect the perpendiculars to the lines at points $P$ and $Q$. It is easy to verify that the intersection point of these perpendiculars is the required one.

**7.31.** Denote the midpoints of diagonals $AC$ and $BD$ of quadrilateral $ABCD$ by $M$ and $N$, respectively. Clearly, $S_{AMB} = S_{BMC}$ and $S_{AMD} = S_{DMC}$, i.e., $S_{DABM} = S_{BCDM}$. Since the areas of quadrilaterals $DABM$ and $BCDM$ do not vary as point $M$ moves parallelly to $BD$, it follows that $S_{DABO} = S_{CDAO}$. Similar arguments for point $N$ show that $S_{ABCO} = S_{CDAO}$. Hence,

$$S_{ADO} + S_{ABO} = S_{BCO} + S_{CDO} \text{ and } S_{ABO} + S_{BCO} = S_{CDO} + S_{ADO}$$

and, therefore,

$$S_{ADO} = S_{BCO} = S_1 \text{ and } S_{ABO} = S_{CDO} = S_2,$$

i.e., the area of each of the four parts into which the segments that connect point $O$ with the midpoints of sides of the quadrilateral divide it is equal to $\frac{1}{2}(S_1 + S_2)$.

**7.32.** Let us drop height $BB_1$ from point $B$. Then $AD = B_1D$ and $CE = B_1E$. Clearly, if $MD < AD$, then point $M$ lies on segment $AB_1$, i.e., outside segment $B_1C$. Therefore, $ME > EC$.

**7.33.** Suppose that the distance from any vertex of the polygon to point $Q$ is not shorter than to point $P$. Then all the vertices of the polygon lie in the same half plane determined by the perpendicular to segment $PQ$ at point $P$; point $Q$ lies in the other half plane. Therefore, point $Q$ lies outside the polygon. This contradicts the hypothesis.

**7.34.** Let us find the locus of points $M$ for which $MA > MB$ and $MA > MC$. Let us draw midperpendiculars $l_1$ and $l_2$ to segments $AB$ and $AC$. We have $MA > MB$ for the points that lie inside the half-plane bounded by line $l_1$ and the one without point $A$. Therefore, the locus in question is the intersection of half-planes (without boundaries) bounded by lines $l_1$ and $l_2$ and not containing point $A$.

If points $A$, $B$ and $C$ do not lie on one line, then this locus is always nonempty. If $A$, $B$, $C$ lie on one line but $A$ does not lie on segment $BC$, then this locus is also nonempty. If point $A$ lies on segment $BC$, then this locus is empty, i.e., for any point $M$ either $MA \leq MB$ or $MA \leq MC$.

**7.35.** Let $O$ be the midpoint of diagonal $AC$. The projections of points $B$ and $D$ to line $AC$ lie on segment $AO$, hence, the projection of point $M$ also lies on segment $AO$.

**7.36.** Let us draw the midperpendicular $l$ to segment $AO$. Clearly, $AM \geq OM$ if and only if point $M$ lies on the same side of line $l$ as $O$ (or lies on line $l$ itself). Therefore, the locus in question is the rhombus formed by the midperpendiculars to segments $OA$, $OB$, $OC$ and $OD$.

**7.37.** The locus to be found is shaded on Fig. 83 (the boundary belongs to the locus).
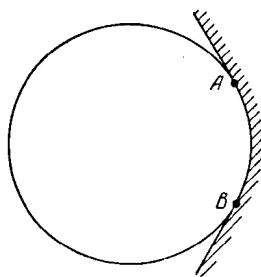


Figure 83 (Sol. 7.37)

**7.38.** Let $A_1$ and $B_1$ be the midpoints of sides $CB$ and $AC$, respectively. The locus to be found is the interior of quadrilateral $OA_1CB_1$.
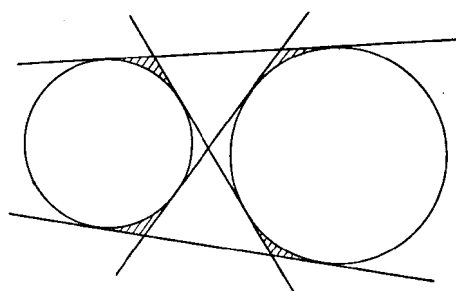


Figure 84 (Sol. 7.39)

**7.39.** Let us draw the common tangents to given disks (Fig. 84). It is easy to verify that the points that belong to the shaded domains (but not to their boundaries) satisfy the required condition and the points that do not belong to these domains do not satisfy this condition.

**7.40.** Let the perpendiculars dropped from points $A_1$, $B_1$, $C_1$ to lines $BC$, $CA$, $AB$, respectively, intersect at point $M$. Since points $B_1$ and $M$ lie on one perpendicular to line $AC$, we have

$$B_1A^2 - B_1C^2 = MA^2 - MC^2.$$

Similarly,

$$C_1B^2 - C_1A^2 = MB^2 - MA^2 \ \text{ and } \ A_1C^2 - A_1B^2 = MC^2 - MB^2.$$

Adding these equalities we get

$(*)$ $$A_1B^2 + C_1A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1B^2.$$

Conversely, let $(*)$ hold. Denote the intersection point of the perpendiculars dropped from points $A_1$ and $B_1$ to lines $BC$ and $AC$, respectively, by $M$. Let us draw through point $M$ line $l$ perpendicular to line $AB$. If $C_1'$ is a point on line $l$, then by the above

$$A_1B^2 + C_1'A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1'B^2.$$

Hence, $C_1'A^2 - C_1'B^2 = C_1A^2 - C_1B^2$. By Problem 7.6 the locus of points $X$ for which $XA^2 - XB^2 = k$ is a line perpendicular to segment $AB$. Therefore, the perpendicular dropped from point $C_1$ to line $AB$ passes through point $M$, as required.

**7.41.** Set $A_1 = A$, $B_1 = B$ and $C_1 = C$. From the obvious identity

$$AB^2 + CA^2 + BC^2 = BA^2 + AC^2 + CB^2$$

we derive that the heights dropped from points $A$, $B$ and $C$ to sides $BC$, $CA$ and $AB$, respectively, intersect at one point.

**7.42.** It suffices to make use of the result of Problem 7.40.

**7.43.** a) This problem is an obvious corollary of Problem 7.40.

b) Let the rotation by $90°$ about a point send triangle $A_1B_1C_1$ to triangle $A_2B_2C_2$. The perpendiculars to sides of triangle $A_2B_2C_2$ are parallel to the corresponding sides of triangle $A_1B_1C_1$ and, therefore, the perpendiculars dropped from the vertices of triangle $ABC$ to the corresponding sides of triangle $A_2B_2C_2$ intersect at one point. It follows that the perpendiculars dropped from the vertices of triangle $A_2B_2C_2$ to the corresponding sides of triangle $ABC$ intersect at one point. It remains to notice that the rotation by $90°$ that sends triangle $A_2B_2C_2$ to triangle $A_1B_1C_1$ sends these perpendiculars to the lines that pass through the sides of triangle $A_1B_1C_1$ parallelly the corresponding sides of triangle $ABC$.

**7.44.** We have to find out when the identity

$$AB_1^2 + BC_1^2 + CA_1^2 = BA_1^2 + CB_1^2 + AC_1^2$$

holds. By subtracting $AA_2^2 + BB_2^2 + CC_2^2$ from both sides of this identity we get

$$A_2B_1^2 + B_2C_1^2 + C_2A_1^2 = B_2A_1^2 + C_2B_1^2 + A_2C_1^2,$$

i.e.,

$$(b_1 - a_2)^2 + (c_1 - b_2)^2 + (a_1 - c_2)^2 = (a_1 - b_2)^2 + (b_1 - c_2)^2 + (c_1 - a_2)^2,$$

where $a_i$, $b_i$ and $c_i$ are the coordinates of points $A_i$, $B_i$ and $C_i$ on line $l$. After simplification we get

$$a_2b_1 + b_2c_1 + c_2a_1 = a_1b_2 + b_1c_2 + c_1a_2$$

and, therefore,

$$(b_2 - a_2)(c_1 - b_1) = (b_1 - a_1)(c_2 - b_2), \quad \text{i.e., } A_2B_2 : B_2C_2 = A_1B_1 : B_1C_1.$$

**7.45.** We may assume that the length of a side of the given equilateral triangle is equal to 2. Let $PA = 2a$, $PB = 2b$ and $PC = 2c$; let $A_1$, $B_1$ and $C_1$ be the projections of the

centers of the inscribed circles of triangles $PBC$, $PCA$ and $PAB$ to lines $BC$, $CA$ and $AB$, respectively. By Problem 3.2 we have

$$AB_1^2 + BC_1^2 + CA_1^2 = (1 + a - c)^2 + (1 + b - a)^2 + (1 + c - b)^2 =$$
$$3 + (a - c)^2 + (b - a)^2 + (c - b)^2 = BA_1^2 + CB_1^2 + AC_1^2.$$

**7.46.** The segments into which the bisectors divide the sides of the triangle are easy to calculate. As a result we see that if the perpendiculars raised from the bases of the bisectors intersect, then

$$(\frac{ac}{b} + c)^2 + (\frac{ab}{a} + c)^2 + (\frac{bc}{a} + b)^2 = (\frac{ab}{b} + c)^2 + (\frac{bc}{a} + c)^2 + (\frac{ac}{a} + b)^2,$$

i.e.,

$$0 = a^2 \frac{c - b}{b + c} + b^2 \frac{a - c}{a + c} + c^2 \frac{b - a}{a + b} = -(b - a)(a - c) \frac{a^2 + b^2 + c^2}{(a + b)(a + c)(b + c)}.$$

**7.47.** Let $(a_i, b_i)$ be the coordinates of point $A_i$ and $(x, y)$ the coordinates of point $X$. Then the equation satisfied by point $X$ takes the form

$$c = \sum k_i((x - a_i)^2 + (x - b_i)^2) =$$
$$(\sum k_i)(x^2 + y^2) - (2 \sum k_i a_i)x - (2 \sum k_i b_i)y + \sum k_i(a_i^2 + b_i^2).$$

If the coefficient of $x^2 + y^2$ is nonzero, then this equation determines either a circle or the empty set and if it is zero, then the equation determines either a line, or a plane, or the empty set.

REMARK. If in case a) points $A_1, \ldots, A_n$ lie on one line $l$, then this line can be taken for $Ox$-axis. Then $b_i = 0$ and, therefore, the coefficient of $y$ is equal to zero, i.e., the center of the circle lies on $l$.

**7.48.** Let line $l$ cut on the given circles arcs $\smile A_1 B_1$ and $\smile A_2 B_2$ whose values are $2\alpha_1$ and $2\alpha_2$, respectively; let $O_1$ and $O_2$ be the centers of the circles, $R_1$ and $R_2$ their respective radii. Let $K$ be the intersection point of the tangents at points $A_1$ and $A_2$. By the law of sines $KA_1 : KA_2 = \sin \alpha_2 : \sin \alpha_1$, i.e., $KA_1 \sin \alpha_1 = KA_2 \sin \alpha_2$. Since

$$KO_1^2 = KA_1^2 + R_1^2 \quad \text{and} \quad KO_2^2 = KA_2^2 + R_2^2,$$

it follows that

$$(\sin^2 \alpha_2)KO_1^2 - (\sin^2 \alpha_2)KO_2^2 = (R_1 \sin \alpha_1)^2 - (R_2 \sin \alpha_2)^2 = q.$$

We similarly prove that the other intersection points of the tangents belong to the locus of points $X$ such that

$$(\sin^2 \alpha_1)XO_1^2 - (\sin^2 \alpha_2)XO_2^2 = q.$$

This locus is a circle whose center lies on line $O_1 O_2$ (cf. Remark to Problem 7.47).

**7.49.** Let $AM : BM : CM = p : q : r$. All the points $X$ that satisfy

$$(q^2 - r^2)AX^2 + (r^2 - p^2)BX^2 + (p^2 - q^2)CX^2 = 0$$

lie on one line (cf. Problem 7.47) and points $M$, $N$ and $O$ satisfy this relation.

# Chapter 8. CONSTRUCTIONS

## §1. The method of loci

**8.1.** Construct triangle $ABC$ given $a$, $h_a$ and $R$.

**8.2.** Inside triangle $ABC$ construct point $M$ so that $S_{ABM} : S_{BCM} : S_{ACM} = 1 : 2 : 3$.

**8.3.** Through given point $P$ inside a given circle draw a chord so that the difference of the lengths of the segments into which $P$ divides the chord would be equal to the given value $a$.

**8.4.** Given a line and a circle without common points, construct a circle of a given radius $r$ tangent to them.

**8.5.** Given point $A$ and circle $S$ draw a line through point $A$ so that the chord cut by circle $S$ on this line would be of given length $d$.

**8.6.** Quadrilateral $ABCD$ is given. Inscribe in it a parallelogram with given directions of sides.

## §2. The inscribed angle

**8.7.** Given $a$, $m_c$ and angle $\angle A$, construct triangle $ABC$.

**8.8.** A circle and two points $A$ and $B$ inside it are given. Inscribe a right triangle in the circle so that the legs would pass through the given points.

**8.9.** The extensions of sides $AB$ and $CD$ of rectangle $ABCD$ intersect a line at points $M$ and $N$, respectively, and the extensions of sides $AD$ and $BC$ intersect the same line at points $P$ and $Q$, respectively. Construct rectangle $ABCD$ given points $M$, $N$, $P$, $Q$ and the length $a$ of side $AB$.

**8.10.** Construct a triangle given its bisector, median and height drawn from one vertex.

**8.11.** Construct triangle $ABC$ given side $a$, angle $Construct at triangle A$ and the radius $r$ of the inscribed circle.

## §3. Similar triangles and a homothety

**8.12.** Construct a triangle given two angles $\angle A$, $\angle B$ and the perimeter $P$.

**8.13.** Construct triangle $ABC$ given $m_a$, $m_b$ and $m_c$.

**8.14.** Construct triangle $ABC$ given $h_a$, $h_b$ and $h_c$.

**8.15.** In a given acute triangle $ABC$ inscribe square $KLMN$ so that vertices $K$ and $N$ lie on sides $AB$ and $AC$ and vertices $L$ and $M$ lie on side $BC$.

**8.16.** Construct triangle $ABC$ given $h_a$, $b - c$ and $r$.

Cf. also Problems 19.15-19.20, 19.39, 19.40.

## §4. Construction of triangles from various elements

In the problems of this section it is necessary to construct triangle $ABC$ given the elements indicated below.

**8.17.** $c$, $m_a$ and $m_b$.

**8.18.** $a$, $b$ and $h_a$.

**8.19.** $h_b$, $h_c$ and $m_a$.

**8.20.** $\angle A$, $h_b$ and $h_c$.

**8.21.** $a$, $h_b$ and $m_b$.

**8.22.** $h_a$, $m_a$ and $h_b$.

**8.23.** $a$, $b$ and $m_c$.

**8.24.** $h_a$, $m_a$ and $\angle A$.

**8.25.** $a$, $b$ and $l_c$.

**8.26.** $\angle A$, $h_a$ and $p$.

See also Problems 17.6-17.8.

## §5. Construction of triangles given various points

**8.27.** Construct triangle $ABC$ given (1) line $l$ containing side $AB$ and (2) bases $A_1$ and $B_1$ of heights dropped on sides $BC$ and $AC$, respectively.

**8.28.** Construct an equilateral triangle given the bases of its bisectors.

**8.29.** a) Construct triangle $ABC$ given three points $A'$, $B'$, $C'$ at which the bisectors of the angles of triangle $ABC$ intersect the circumscribed circle (both triangles are supposed to be acute ones).

b) Construct triangle $ABC$ given three points $A'$, $B'$, $C'$ at which the heights of the triangle intersect the circumscribed circle (both triangles are supposed to be acute ones).

**8.30.** Construct triangle $ABC$ given three points $A'$, $B'$, $C'$ symmetric to the center $O$ of the circumscribed circle of this triangle through sides $BC$, $CA$, $AB$, respectively.

**8.31.** Construct triangle $ABC$ given three points $A'$, $B'$, $C'$ symmetric to the intersection point of the heights of the triangle through sides $BC$, $CA$, $AB$, respectively (both triangles are supposed to be acute ones).

**8.32.** Construct triangle $ABC$ given three points $P$, $Q$, $R$ at which the height, the bisector and the median drawn from vertex $C$, respectively, intersect the circumscribed circle.

**8.33.** Construct triangle $ABC$ given the position of points $A_1$, $B_1$, $C_1$ that are the centeres of the escribed circles of triangle $ABC$.

**8.34.** Construct triangle $ABC$ given the center of the circumscribed circle $O$, the intersection point of medians, $M$, and the base $H$ of height $CH$.

**8.35.** Construct triangle $ABC$ given the centers of the inscribed, the circumscribed, and one of the escribed circles.

## §6. Triangles

**8.36.** Construct points $X$ and $Y$ on sides $AB$ and $BC$, respectively, of triangle $ABC$ so that $AX = BY$ and $XY \parallel AC$.

**8.37.** Construct a triangle from sides $a$ and $b$ if it is known that the angle opposite one of the sides is three times the angle opposite the other side.

**8.38.** In given triangle $ABC$ inscribe rectangle $PRQS$ (vertices $R$ and $Q$ lie on sides $AB$ and $BC$ and vertices $P$ and $S$ lie on side $AC$) so that its diagonal would be of a given length.

**8.39.** Through given point $M$ draw a line so that it would cut from the given angle with vertex $A$ a triangle $ABC$ of a given perimeter $2p$.

**8.40.** Construct triangle $ABC$ given its median $m_c$ and bisector $l_c$ if $\angle C = 90°$.

**8.41.** Given triangle $ABC$ such that $AB < BC$, construct on side $AC$ point $D$ so that the perimeter of triangle $ABD$ would be equal to the length of side $BC$.

**8.42.** Construct triangle $ABC$ from the radius of its circumscribed circle and the bisector of angle $\angle A$ if it is known that $\angle B - \angle C = 90°$.

**8.43.** On side $AB$ of triangle $ABC$ point $P$ is given. Draw a line (distinct from $AB$) through point $P$ that cuts rays $CA$ and $CB$ at points $M$ and $N$, respectively, such that $AM = BN$.

**8.44.** Construct triangle $ABC$ from the radius of the inscribed circle $r$ and (nonzero) lengths of segments $AO$ and $AH$, where $O$ is the center of the inscribed circle and $H$ the orthocenter.

See also Problems 15.12 b), 17.12-17.15, 18.10, 18.29.

## §7. Quadrilaterals

**8.45.** Construct a rhombus two sides of which lie on two given parallel lines and two other sides pass through two given points.

**8.46.** Construct quadrilateral $ABCD$ given the lengths of the four sides and the angle between $AB$ and $CD$.

**8.47.** Through vertex $A$ of convex quadrilateral $ABCD$ draw a line that divides $ABCD$ into two parts of equal area.

**8.48.** In a convex quadrilateral three sides are equal. Given the midpoints of the equal sides construct the quadrilateral.

**8.49.** A quadrilateral is both inscribed and circumscribed. Given three of its vertices, construct its fourth vertex.

**8.50.** Given vertices $A$ and $C$ of an isosceles circumscribed trapezoid $ABCD$ ($AD \parallel BC$) and the directions of its bases, construct vertices $B$ and $D$.

**8.51.** On the plane trapezoid $ABCD$ is drawn ($AD \parallel BC$) and perpendicular $OK$ from the intersection point $O$ is dropped on base $AD$; the midpoint $EF$ is drawn. Then the trapezoid itself was erased. How to recover the plot of the trapezoid from the remaining segments $OK$ and $EF$?

**8.52.** Construct a convex quadrilateral given the lengths of all its sides and one of the midlines.

**8.53.** (*Brachmagupta.*) Construct an inscribed quadrilateral given its four sides.

See also Problems 15.10, 15.13, 16.17, 17.4, 17.5.

## §8. Circles

**8.54.** Inside an angle two points $A$ and $B$ are given. Construct a circle that passes through these points and intercepts equal segments on the sides of the angle.

**8.55.** Given circle $S$, point $A$ on it and line $l$. Construct a circle tangent to the given circle at point $A$ and tangent to the given line.

**8.56.** a) Two points, $A$, $B$ and line $l$ are given. Construct a circle that passes through point $A$, $B$ and is tangent to $l$.

b) Two points $A$, $B$ and circle $S$ are given. Construct a circle that passes through points $A$ and $B$ and is tangent to $S$.

**8.57.** Three points $A$, $B$ and $C$ are given. Construct three circles that are pairwise tangent at these points.

**8.58.** Construct a circle the tangents to which drawn from three given points $A$, $B$ and $C$ have given lengths $a$, $b$ and $c$, respectively.

See also Problems 15.8, 15.9, 15.11, 15.12 a), 16.13, 16.14, 16.18–16.20, 18.24.

## §9. Apollonius' circle

**8.59.** Construct triangle $ABC$ given $a$, $h_a$ and $\frac{b}{c}$.

**8.60.** Construct triangle $ABC$ given the length of bisector $CD$ and the lengths of segments $AD$ and $BD$ into which the bisector divides side $AB$.

**8.61.** On a line four points $A$, $B$, $C$, $D$ are given in the indicated order. Construct point $M$ — the vertex of equal angles that subtend segments $AB$, $BC$, $CD$.

**8.62.** Two segments $AB$ and $A'B'$ are given in plane. Construct point $O$ so that triangles $AOB$ and $A'OB'$ would be similar (equal letters stand for the corresponding vertices of similar triangles).

**8.63.** Points $A$ and $B$ lie on a diameter of a given circle. Through $A$ and $B$ draw two equal chords with a common endpoint.

## §10. Miscellaneous problems

**8.64.** a) On parallel lines $a$ and $b$, points $A$ and $B$ are given. Through a given point $C$ draw line $l$ that intersects lines $a$ and $b$ at points $A_1$ and $B_1$, respectively, and such that $AA_1 = BB_1$.

b) Through point $C$ draw a line equidistant from given points $A$ and $B$.

**8.65.** Construct a regular decagon.

**8.66.** Construct a rectangle with the given ratio of sides knowing one point on each of its sides.

**8.67.** Given diameter $AB$ of a circle and point $C$ on the diameter. On this circle, construct points $X$ and $Y$ symmetric through line $AB$ and such that lines $AX$ and $YC$ are perpendicular.

See also Problems 15.7, 16.15, 16.16, 16.21, 17.9–17.11, 17.27–17.29, 18.41.

## §11. Unusual constructions

**8.68.** With the help of a ruler and a compass divide the angle of $19°$ into 19 equal parts.

**8.69.** Prove that an angle of value $n°$, where $n$ is an integer not divisible by 3, can be divided into $n$ equal parts with the help of a compass and ruler.

**8.70.** On a piece of paper two lines are drawn. They form an angle whose vertex lies outside this piece of paper. With the help of a ruler and a compass draw the part of the bisector of the angle that lies on this piece of paper.

**8.71.** With the help of a two-sided ruler construct the center of the given circle whose diameter is greater than the width of the ruler.

**8.72.** Given points $A$ and $B$; the distance between them is greater than 1 m. The length of a ruler is 10 cm. With the help of the ruler only construct segment $AB$. (Recall that with the help of a ruler one can only draw straight lines.)

**8.73.** On a circle of radius $a$ a point is given. With the help of a coin of radius $a$ construct the point diametrically opposite to the given one.

## §12. Construction with a ruler only

In the problems of this section we have to perform certain constructions with the help of a ruler only, without a compass or anything else. With the help of one ruler it is almost impossible to construct anything. For example, it is even impossible to construct the midpoint of a segment (Problem 30.59).

But if certain additional lines are drawn on the plane, it is possible to perform certain constructions. In particular, if an additional circle is drawn on the plane and its center is

marked, then with the help of a ruler one can perform all the constructions that can be performed with the help of a ruler and a compass. One has, however, to convene that a circle is "constructed" whenever its center and one of its points are marked.

REMARK. If a circle is drawn on the plane but its center is not marked then to construct its center with the help of a ruler only is impossible (Problem 30.60).

**8.74.** Given two parallel lines and a segment that lies on one of the given lines. Divide the segment in halves.

**8.75.** Given two parallel lines and a segment that lies on one of the given lines. Double the segment.

**8.76.** Given two parallel lines and a segment that lies on one of the given lines. Divide the segment into $n$ equal parts.

**8.77.** Given two parallel lines and point $P$, draw a line through $P$ parallel to the given lines.

**8.78.** A circle, its diameter $AB$ and point $P$ are given. Through point $P$ draw the perpendicular to line $AB$.

**8.79.** In plane circle $S$ and its center $O$ are given. Then with the help of a ruler only one can:

a) additionally given a line, draw a line through any point parallel to the given line and drop the perpendicular to the given line from this point;

b) additionally given a line a point on it and a length of a segment, on the given line, mark a segment of length equal to the given one and with one of the endpoints in the given point;

c) additionally given lengths of $a$, $b$, $c$ of segments, construct a segment of length $\frac{ab}{c}$;

d) additionally given line $l$, point $A$ and the length $r$ of a segment, construct the intersection points of line $l$ with the circle whose center is point $A$ and the radius is equal to $r$;

e) additionally given two points and two segments, construct the intersection points of the two circles whose centers are the given points and the radii are the given segments.

See also Problem 6.97.

## §13. Constructions with the help of a two-sided ruler

In problems of this section we have to perform constructions with the help of a ruler with two parallel sides (without a compass or anything else). *With the help of a two-sided ruler one can perform all the constructions that are possible to perform with the help of a compass and a ruler.*

Let $a$ be the width of a two-sided ruler. By *definition* of the two-sided ruler with the help of it one can perform the following elementary constructions:

1) draw the line through two given points;

2) draw the line parallel to a given one and with the distance between the lines equal to $a$;

3) through two given points $A$ and $B$, where $AB \geq a$, draw a pair of parallel lines the distance between which is equal to $a$ (there are two pairs of such lines).

**8.80.** a) Construct the bisector of given angle $\angle AOB$.

b) Given acute angle $\angle AOB$, construct angle $\angle BOC$ whose bisector is ray $OA$.

**8.81.** Erect perpendicular to given line $l$ at given point $A$.

**8.82.** a) Given a line and a point not on the line. Through the given point draw a line parallel to the given line.

b) Construct the midpoint of a given segment.

**8.83.** Given angle $\angle AOB$, line $l$ and point $P$ on it, draw through $P$ lines that form together with $l$ an angle equal to angle $\angle AOB$.

**8.84.** Given segment $AB$, a non-parallel to it line $l$ and point $M$ on it, construct the intersection points of line $l$ with the circle of radius $AB$ centered at $M$.

**8.85.** Given line $l$ and segment $OA$, parallel to $l$, construct the intersection points of $l$ with the circle of radius $OA$ centered at $O$.

**8.86.** Given segments $O_1A_1$ and $O_2A_2$, construct the radical axis of circles of radii $O_1A_1$ and $O_2A_2$ centered at $O_1$ and $O_2$, respectively.

## §14. Constructions using a right angle

In problems of this section we have to perform the constructions indicated using a *right angle*. A right angle enables one to perform the following elementary constructions:

a) given a line and a point not on it, place the right angle so that one of its legs lies on the given line and the other leg runs through the given point;

b) given a line and two points not on it, place the right angle so that its vertex lies on the given line and thelegs pass through two given points (if, certainly, for the given line and points such a position of the right angle exists).

Placing the right angle in one of the indicated ways we can draw rays corresponding to its sides.

**8.87.** Given line $l$ and point $A$ not on it, draw a line parallel to $l$.

**8.88.** Given segment $AB$, construct

a) the midpoint of $AB$;

b) segment $AC$ whose midpoint is point $B$.

**8.89.** Given angle $\angle AOB$, construct

a) an angle of value $2\angle AOB$;

b) an angle of value $\frac{1}{2}\angle AOB$.

**8.90.** Given angle $\angle AOB$ and line $l$, draw line $l_1$ so that the angle between lines $l$ and $l_1$ is equal to $\angle AOB$.

**8.91.** Given segment $AB$, line $l$ and point $O$ on it, construct on $l$ point $X$ such that $OX = AB$.

**8.92.** Given segment $OA$ parallel to line $l$, construct the locus of points in which the disc segment of radius $OA$ centered at $O$ intersects $l$.

### Problems for independent study

**8.93.** Construct a line tangent to two given circles (consider all the possible cases).

**8.94.** Construct a triangle given (the lengths of) the segments into which a height divides the base and a median drawn to a lateral side.

**8.95.** Construct parallelogram $ABCD$ given vertex $A$ and the midpoints of sides $BC$ and $CD$.

**8.96.** Given 3 lines, a line segment and a point. Construct a trapezoid whose lateral sides lie on the given lines, the diagonals intersect at the given point and one of the bases is of the given length.

**8.97.** Two circles are given. Draw a line so that it would be tangent to one of the circles and the other circle would intersept on it a chord of a given length.

**8.98.** Through vertex $C$ of triangle $ABC$ draw line $l$ so that the areas of triangles $AA_1C$ and $BB_1C$, where $A_1$ and $B_1$ are projections of points $A$ and $B$ on line $l$, are equal.

**8.99.** Construct triangle $ABC$ given sides $AB$ and $AC$ if it is given that bisector $AD$, median $BM$, and height $CH$ meet at one point.

**8.100.** Points $A_1$, $B_1$ and $C_1$ that divide sides $BC$, $CA$ and $AB$, respectively, of triangle $ABC$ in the ratio of $1:2$ are given. Recover triangle $ABC$ from this data.

## Solutions

**8.1.** Let us construct segment $BC$ of length $a$. The center $O$ of the circumscribed circle of triangle $ABC$ is the intersection point of two circles of radius $R$ centered at $B$ and $C$. Select one of these intersection points and construct the circumscribed circle $S$ of triangle $ABC$. Point $A$ is the intersection point of circle $S$ and a line parallel line $BC$ and whose distance from $BC$ is equal to $h_a$ (there are two such lines).

**8.2.** Let us construct points $A_1$ and $B_1$ on sides $BC$ and $AC$, respectively, so that $BA_1 : A_1C = 1 : 3$ and $AB_1 : B_1C = 1 : 2$. Let point $X$ lie inside triangle $ABC$. Clearly, $S_{ABX} : S_{BCX} = 1 : 2$ if and only if point $X$ lies on segment $BB_1$ and $S_{ABX} : S_{ACX} = 1 : 3$ if and only if point $X$ lies on segment $AA_1$. Therefore, the point $M$ to be constucted is the intersection point of segments $AA_1$ and $BB_1$.

**8.3.** Let $O$ be the center of the given circle, $AB$ a chord that passes through point $P$ and $M$ the midpoint of $AB$. Then $|AP - BP| = 2PM$. Since $\angle PMO = 90°$, point $M$ lies on circle $S$ with diameter $OP$. Let us construct chord $PM$ of circle $S$ so that $PM = \frac{1}{2}a$ (there are two such chords). The chord to be constructed is determined by line $PM$.

**8.4.** Let $R$ be the radius of the given circle, $O$ its center. The center of the circle to be constructed lies on circle $S$ of radius $R + r$ centered at $O$. On the other hand, the center to be constructed lies on line $l$ passing parallelly to the given line at distance $r$ (there are two such lines). Any intersection point of $S$ with $l$ can serve as the center of the circle to be constructed.

**8.5.** Let $R$ be the radius of circle $S$ and $O$ its center. If circle $S$ intersepts on the line that passes through point $A$ chord $PQ$ and $M$ is the midpoint of $PQ$, then

$$OM^2 + OQ^2 - NQ^2 = R^2 - \frac{d^2}{4}.$$

Therefore, the line to be constructed is tangent to the circle of radius $\sqrt{R^2 - \frac{d^2}{4}}$ centered at $O$.

**8.6.** On lines $AB$ and $CD$ take points $E$ and $F$ so that lines $BF$ and $CE$ would have had prescribed directions. Let us considered all possible parallelograms $PQRS$ with prescribed directions of sides whose vertices $P$ and $R$ lie on rays $BA$ and $CD$ and vertex $Q$ lies on side $BC$ (Fig. 85).
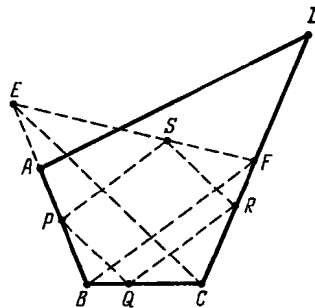


Figure 85 (8.6)

Let us prove that the locus of vertices $S$ is segment $EF$. Indeed, $\frac{SR}{EC} = \frac{PQ}{EC} = \frac{BQ}{BC} = \frac{FR}{FC}$, i.e., point $S$ lies on segment $EF$. Conversely if point $S'$ lies on segment $EF$ then let us draw lines $S'P'$, $P'Q'$ and $Q'R'$ so that $S'P' \parallel BF$, $P'Q' \parallel EC$ and $Q'R' \parallel BF$, where $P'$, $Q'$ and $R'$ are some points on lines $AB$, $BC$, $CD$, respectively. Then $\frac{S'P'}{BF} = \frac{P'E}{BE} = \frac{Q'C}{BC} = \frac{Q'R'}{BF}$, i.e., $S'P' = Q'R'$ and $P'Q'R'S'$ is a parallelogram.

This implies the following construction. First, construct points $E$ and $F$. Vertex $S$ is the intersection point of segments $AD$ and $EF$. The continuation of construction is obvious.

**8.7.** Suppose that triangle $ABC$ is constructed. Let $A_1$ and $C_1$ be the midpoints of sides $CD$ and $AB$, respectively. Since $C_1A_1 \parallel AC$, it follows that $\angle A_1C_1B = \angle A$. This implies the following construction.

First, let us construct segment $CD$ of length $a$ and its midpoint, $A_1$. Point $C_1$ is the intersection point of the circle of radius $m_c$ centered at $C$ and the arcs of the circles whose points are vertices of the angles equal to $\angle A$ that segment $A_1B$ subtends. Construct point $C_1$, then mark on ray $BC_1$ segment $BA = 2BC_1$. Then $A$ is the vertex of the triangle to be constructed.

**8.8.** Suppose that the desired triangle is constructed and $C$ is the vertex of its right angle. Since $\angle ACB = 90°$, point $C$ lies on circle $S$ with diameter $AB$. Hence, point $C$ is the intersection point of circle $S$ and the given circle. Constructing point $C$ and drawing lines $CA$ and $AB$, we find the remaining vertices of the triangle to be constructed.

**8.9.** Suppose that rectangle $ABCD$ is constructed. Let us drop perpendicular $PR$ from point $P$ to line $BC$. Point $R$ can be constructed because it lies on the circle with diameter $PQ$ and $PR = AB = a$. Constructing point $R$, let us construct lines $BC$ and $AD$ and drop on them perpendiculars from points $M$ and $N$, respectively.

**8.10.** Suppose that triangle $ABC$ is constructed, $AH$ is its height, $AD$ its bisector, $AM$ its median. By Problem 2.67 point $D$ lies between $M$ and $H$. Point $E$, the intersection point of line $AD$ with the perpendicular drawn from point $M$ to side $BC$, lies on the circumscribed circle of triangle $ABC$. Hence, the center $O$ of the circumscribed circle lies on the intersection of the midperpendicular to segment $AE$ and the perpendicular to side $BC$ drawn through point $M$.

The sequence of constructions is as follows: on an arbitrary line (which in what follows turns out to be line $BC$) construct point $H$, then consecutively construct points $A$, $D$, $M$, $E$, $O$. The desired vertices $B$ and $C$ of triangle $ABC$ are intersection points of the initial line with the circle of radius $OA$ centered at $O$.

**8.11.** Suppose that triangle $ABC$ is constructed and $O$ is the center of its inscribed circle. Then $\angle BOC = 90° + \frac{1}{2}\angle A$ (Problem 5.3). Point $O$ is the vertex of an angle of $90° + \frac{1}{2}\angle A$ that subtends segment $BC$; the distance from $O$ to line $BC$ is equal to $r$, hence, $BC(??)$ can be constructed. Further, let us construct the inscribed circle and draw the tangents to it from points $B$ and $C$.

**8.12.** Let us construct any triangle with angles $\angle A$ and $\angle B$ and find its perimeter $P_1$. The triangle to be found is similar to the constructed triangle with coefficient $\frac{P}{P_1}$.

**8.13.** Suppose that triangle $ABC$ is constructed. Let $AA_1$, $BB_1$ and $CC_1$ be its medians, $M$ their intersection point, $M'$ the point symmetric to $M$ through point $A_1$. Then $MM' = \frac{2}{3}m_a$, $MC = \frac{2}{3}m_c$ and $M'C = \frac{2}{3}m_b$; hence, triangle $MM'C$ can be constructed. Point $A$ is symmetric to $M'$ through point $M$ and point $B$ is symmetric to $C$ through the midpoint of segment $MM'$.

**8.14.** Clearly,

$$BC : AC : AB = \frac{S}{h_a} : \frac{S}{h_b} : \frac{S}{h_c} = \frac{1}{h_a} : \frac{1}{h_b} : \frac{1}{h_c}.$$

Let us take an arbitrary segment $B'C'$ and construct triangle $A'B'C'$ so that $B'C' : A'C' = h_b : h_a$ and $B'C' : A'B' = h_c : h_a$. Let $h'_a$ be the height of triangle $A'B'C'$ dropped from vertex $A'$. The triangle to be found is similar to triangle $A'B'C'$ with coefficient $\frac{h_a}{h'_a}$.

**8.15.** On side $AB$, take an arbitrary point $K'$ and drop from it perpendicular $K'L'$ to side $BC$; then construct square $K'L'M'N'$ that lies inside angle $\angle ABC$. Let line $BN'$ intersect side $AC$ at point $N$. Clearly, the square to be constructed is the image of square $K'L'M'N'$ under the homothety with center $B$ and coefficient $BN : BN'$.

**8.16.** Suppose that the desired triangle $ABC$ is constructed. Let $Q$ be the tangent point of the inscribed circle with side $BC$; let $PQ$ be a diameter of the circle, $R$ the tangent point of an escribed circle with side $BC$. Clearly,

$$BR = \frac{a+b+c}{2} - c = \frac{a+b-c}{2} \quad \text{and} \quad BQ = \frac{a+c-b}{2}.$$

Hence, $RQ = |BR - BQ| = |b - c|$. The inscribed circle of triangle $ABC$ and the escribed circle tangent to side $BC$ are homothetic with $A$ being the center of homothety. Hence, point $A$ lies on line $PR$ (Fig. 86).
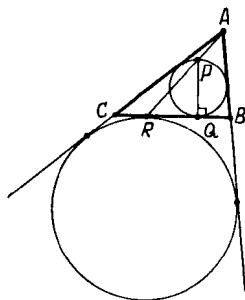


Figure 86 (Sol. 8.16)

This implies the following construction. Let us construct right triangle $PQR$ from the known legs $PQ = 2r$ and $RQ = |b - c|$. Then draw two lines parallel to line $RQ$ and whose distances from $RQ$ are equal to $h_a$. Vertex $A$ is the intersection point of one of these lines with ray $RP$. Since the length of diameter $PQ$ of the inscribed circle is known, it can be constructed. The intersection points of the tangents to this circle drawn from point $A$ with line $RQ$ are vertices $B$ and $C$ of the triangle.

**8.17.** Suppose that triangle $ABC$ is constructed. Let $M$ be the intersection point of medians $AA_1$ and $BB_1$. Then $AM = \frac{2}{3}m_a$ and $BM = \frac{2}{3}m_b$. Triangle $ABM$ can be constructed from the lengths of sides $AB = c$, $AM$ and $BM$. Then on rays $AM$ and $BM$ segments $AA_1 = m_a$ and $BB_1 = m_b$ should be marked. Vertex $C$ is the intersection point of lines $AB_1$ and $A_1B$.

**8.18.** Suppose triangle $ABC$ is constructed. Let $H$ be the base of the height dropped from vertex $A$. Right triangle $ACH$ can be constructed from its hypothenuse $AC = b$ and leg $AH = h_a$. Then on line $CH$ construct point $B$ so that $CB = a$.

**8.19.** Suppose that triangle $ABC$ is constructed. Let us draw from the midpoint $A_1$ of side $BC$ perpendiculars $A_1B'$ and $A_1C'$ to lines $AC$ and $AB$, respectively. Clearly, $AA_1 = m_a$, $A_1B' = \frac{1}{2}h_b$ and $A_1C' = \frac{1}{2}h_c$. This implies the following construction.

First, let us construct segment $AA_1$ of length $m_a$. Then construct right triangles $AA_1B'$ and $AA_1C'$ from the known legs and hypothenuse so that they would lie on distinct sides of line $AA_1$. It remains to construct points $B$ and $C$ on sides $AC'$ and $AB'$ of angle $C'AB'$ so that segment $BC$ would be divided by points $A_1$ in halves. For this let us mark on ray $AA_1$ segment $AD = 2AA_1$ and then draw through point $D$ the lines parallel to the legs of
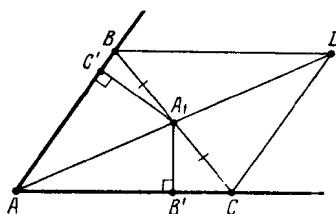
Figure 87 (Sol. 8.19)

angle $\angle C'AB'$. The intersection points of these lines with the legs of angle $\angle C'AB'$ are the vertices of the triangle to be constructed (Fig. 87).

**8.20.** Let us construct angle $\angle B'AC'$ equal to $\angle A$. Point $B$ is constructed as the intersection of ray $AB'$ with a line parallel to ray $AC'$ and passsing at distance $h_b$ from it. Point $C$ is similarly constructed.

**8.21.** Suppose that triangle $ABC$ is constructed. Let us drop height $BH$ from point $B$ and draw median $BB_1$. In right triangles $CBH$ and $B_1BH$, leg $BH$ and hypothenuses $CB$ and $B_1B$ are known; hence, these segments can be constructed. Then on ray $CB_1$ we mark segment $CA = 2CB_1$. The problem has two solutions because we can construct triangles $CBH$ and $B_1BH$ either on one or on distinct sides of line $BH$.

**8.22.** Suppose that triangle $ABC$ is constructed. Let $M$ be the midpoint of segment $BC$. From point $A$ drop height $AH$ and from point $M$ drop perpendicular $MD$ to side $AC$. Clearly, $MD = \frac{1}{2}h_b$. Hence, triangles $AMD$ and $AMH$ can be constructed.

Vertex $C$ is the intersection point of lines $AD$ and $MH$. On ray $CM$, mark segment $CB = 2CM$. The problem has two solutions because triangles $AMD$ and $AMH$ can be constructed either on one or on distinct sides of line $AM$.

**8.23.** Suppose that triangle $ABC$ is constructed. Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$, respectively. In triangle $CC_1B_1$ all the sides are known: $CC_1 = m_c$, $C_1B_1 = \frac{1}{2}a$ and $CB_1 = \frac{1}{2}b$; hence, it can be constructed. Point $A$ is symmetric to $C$ through point $B_1$ and point $B$ is symmetric to $A$ through $C_1$.

**8.24.** Suppose that triangle $ABC$ is constructed, $AM$ is its median, $AH$ its height. Let point $A'$ be symmetric to $A$ through point $M$.

Let us construct segment $AA' = 2m_a$. Let $M$ be the midpoint of $AA'$. Let us construct right triangle $AMH$ with hypothenuse $AM$ and leg $AH = h_a$. Point $C$ lies on an arc of the circle whose points are the vertices of the angles that subtend segment $AA'$; the values of these angles are equal to $180° - \angle A$ because $\angle ACA' = 180° - \angle CAB$. Hence, point $C$ is the intersection point of this arc and line $MH$. Point $B$ is symmetric to $C$ through point $M$.

**8.25.** Suppose triangle $ABC$ is constructed. Let $CD$ be its bisector. Let us draw line $MD$ parallel to side $BC$ (point $M$ lies on side $AC$). Triangle $CMD$ is an isosceles one because $\angle MCD = \angle DCB = \angle MDC$. Since

$$MC : AM = DB : AD = CB : AC = a : b \quad \text{and} \quad AM + MC = b,$$

it follows that $MC = \frac{ab}{a+b}$. Let us construct an isosceles triangle $CMD$ from its base $CD = l_c$ and lateral sides $MD = MC = \frac{ab}{a+b}$. Further, on ray $CM$, mark segment $CA = b$ and on the ray symmetric to ray $CM$ through line $CD$ mark segment $CB = a$.

**8.26.** Suppose that triangle $ABC$ is constructed. Let $S_1$ be the escribed circle tangent to side $BC$. Denote the tangent points of circle $S_1$ with the extensions of sides $AB$ and $AC$ by $K$ and $L$, respectively, and the tangent point of $S_1$ with side $BC$ by $M$. Since

$$AK = AL, AL = AC + CM \quad \text{and} \quad AK = AB + BM,$$

it follows that $AK = AL = p$. Let $S_2$ be the circle of radius $h_a$ centered at $A$. Line $BC$ is a common inner tangent to circles $S_1$ and $S_2$.

This implies the following construction. Let us construct angle $\angle KAL$ whose value is equal to that of $A$ so that $KA = LA = p$. Next, construct circle $S_1$ tangent to the sides of angle $\angle KAL$ at points $K$ and $L$ and circle $S_2$ of radius $h_a$ centered at $A$. Then let us draw a common inner tangent to circles $S_1$ and $S_2$. The intersection points of this tangent with the legs of angle $\angle KAL$ are vertices $B$ and $C$ of the triangle to be constructed.

**8.27.** Points $A_1$ and $B_1$ lie on the circle $S$ with diameter $AB$. The center $O$ of this circle lies on the midperpendicular to chord $A_1B_1$. This implies the following construction. First, let us construct point $O$ which is the intersection point of the midperpendicular to segment $A_1B_1$ with line $l$. Next, construct the circle of radius $OA_1 = OB_1$ centered at $O$. The vertices $A$ and $B$ are the intersection points of circle $S$ with line $l$. Vertex $C$ is the intersection point of lines $AB_1$ and $BA_1$.

**8.28.** Let $AB = BC$ and $A_1$, $B_1$, $C_1$ the bases of the bisectors of triangle $ABC$. Then $\angle A_1C_1C = \angle C_1CA = \angle C_1CA_1$, i.e., triangle $CA_1C_1$ is an isosceles one and $A_1C = A_1C_1$.

This implies the following construction.

Let us draw through point $B_1$ line $l$ parallel to $A_1C_1$. On $l$, construct point $C$ such that $CA_1 = C_1A_1$ and $\angle C_1A_1C > 90°$. Point $A$ is symmetric to point $C$ through point $B_1$ and vertex $B$ is the intersection point of lines $AC_1$ and $A_1C$.

**8.29.** a) By Problem 2.19 a) points $A$, $B$ and $C$ are the intersection points of the extensions of heights of triangle $A'B'C'$ with its circumscribed circle.

b) By Problem 2.19 b) points $A$, $B$ and $C$ are the intersection points of the extensions of bisectors of the angles of triangle $A'B'C'$ with its circumscribed circle.

**8.30.** Denote the midpoints of sides $BC$, $CA$, $AB$ of the triangle by $A_1$, $B_1$, $C_1$, respectively. Since $BC \parallel B_1C_1 \parallel B'C'$ and $OA_1 \perp BC$, it follows that $OA' \perp B'C'$. Similarly, $OB' \perp A'C'$ and $OC' \perp A'B'$, i.e., $O$ is the intersection point of the heights of triangle $A'B'C'$. Constructing point $O$, let us draw the midperpendiculars to segments $OA'$, $OB'$, $OC'$. These lines form triangle $ABC$.

**8.31.** Thanks to Problem 5.9 our problem coincides with Problem 8.29 b).

**8.32.** Let $O$ be the center of the circumscribed circle, $M$ the midpoint of side $AB$ and $H$ the base of the height dropped from point $C$. Point $Q$ is the midpoint of arc $\smile AB$, therefore, $OQ \perp AB$. This implies the following construction. First, the three given points determine the circumscribed circle $S$ of triangle $PQR$. Point $C$ is the intersection point of circle $S$ and the line drawn parallelly to $OQ$ through point $P$. Point $M$ is the intersection point of line $OQ$ and line $RC$. Line $AB$ passes through point $M$ and is perpendicular to $OQ$.

**8.33.** By Problem 5.2, points $A$, $B$ and $C$ are the bases of the heights of triangle $A_1B_1C_1$.

**8.34.** Let $H_1$ be the intersection point of heights of triangle $ABC$. By Problem 5.105, $OM : MH_1 = 1 : 2$ and point $M$ lies on segment $OH_1$. Therefore, we can construct point $H_1$. Then let us draw line $H_1H$ and erect at point $H$ of this line perpendicular $l$. Dropping perpendicular from point $O$ to line $l$ we get point $C_1$ (the midpoint of segment $AB$). On ray $C_1M$, construct point $C$ so that $CC_1 : MC_1 = 3 : 1$. Points $A$ and $B$ are the intersection points of line $l$ with the circle of radius $CO$ centered at $O$.

**8.35.** Let $O$ and $I$ be the centers of the circumscribed and inscribed circles, $I_c$ the center of the escribed circle tangent to side $AB$. The circumscribed circle of triangle $ABC$ divides segment $II_c$ (see Problem 5.109 b)) in halves and segment $II_c$ divides arc $\smile AB$ in halves. It is also clear that points $A$ and $B$ lie on the circle with diameter $II_c$. This implies the following construction.

Let us construct circle $S$ with diameter $II_c$ and circle $S_1$ with center $O$ and radius $OD$, where $D$ is the midpoint of segment $II_c$. Circles $S$ and $S_1$ intersect at points $A$ and $B$. Now, we can construct the inscribed circle of triangle $ABC$ and draw tangents to it at points $A$ and $B$.

**8.36.** Suppose that we have constructed points $X$ and $Y$ on sides $AB$ and $BC$, respectively, of triangle $ABC$ so that $AX = BY$ and $XY \parallel AC$. Let us draw $YY_1$ parallel to $AB$ and $Y_1C_1$ parallel to $BC$ (points $Y_1$ and $C_1$ lie on sides $AC$ and $AB$, respectively). Then $Y_1Y = AX = BY$, i.e., $BYY_1C$ is a rhombus and $BY_1$ is the bisector of angle $\angle B$.

This implies the following construction. Let us draw bisector $BY_1$, then line $Y_1Y$ parallel to side $AB$ (we assume that $Y$ lies on $BC$). Now, it is obvious how to construct point $X$.

**8.37.** Let, for definiteness, $a < b$. Suppose that triangle $ABC$ is constructed. On side $AC$, take point $D$ such that $\angle ABD = \angle BAC$. Then $\angle BDC = 2\angle BAC$ and

$$\angle CBD = 3\angle BAC - \angle BAC = 2\angle BAC,$$

i.e., $CD = CB = a$. In triangle $BCD$ all the sides are known: $CD = CB = a$ and $DB = AD = b - a$. Constructing triangle $BCD$, draw ray $BA$ that does not intersect side $CD$ so that $\angle DBA = \frac{1}{2}\angle DBC$. Vertex $A$ to be constructed is the intersection point of line $CD$ and this ray.

**8.38.** Let point $B'$ lie on line $l$ that passes through point $B$ parallelly to $AC$. Sides of triangles $ABC$ and $AB'C$ intersept equal segments on $l$. Hence, rectangles $P'R'Q'S'$ and $PRQS$ inscribed in triangles $ABC$ and $AB'C$, respectively, are equal if points $R$, $Q$, $R'$ and $Q'$ lie on one line.

On line $l$, take point $B'$ so that $\angle B'AC = 90°$. It is obvious how to inscribe rectangle $P'R'Q'S'$ with given diagonal $P'Q'$ in triangle $AB'C$ (we assume that $P' = A$). Draw line $R'Q'$; we thus find vertices $R$ and $Q$ of the rectangle to be found.

**8.39.** Suppose that triangle $ABC$ is constructed. Let $K$ and $L$ be points at which the escribed circle tangent to side $BC$ is tangent to the extensions of sides $AB$ and $AC$, respectively. Since $AK = AL = p$, this escribed circle can be constructed; it remains to draw the tangent through the given point $M$ to the constructed circle.

**8.40.** Let the extension of the bisector $CD$ intersect the circumscribed circle of triangle $ABC$ (with right angle $\angle C$) at point $P$, let $PQ$ be a diameter of the circumscribed circle and $O$ its center. Then $PD : PO = PQ : PC$, i.e., $PD \cdot PC = 2R^2 = 2m_c^2$. Therefore, drawing a tangent of length $\sqrt{2}m_c$ to the circle with diameter $CD$, it is easy to construct a segment of length $PC$. Now, the lengths of all the sides of triangle $OPC$ are known.

**8.41.** Let us construct point $K$ on side $AC$ so that $AK = BC - AB$. Let point $D$ lie on segment $AC$. The equality $AD + BD + AB = BC$ is equivalent to the equality $AD + BD = AK$. For point $D$ that lies on segment $AK$ the latter equality takes the form $AD + BD = AD + DK$ and for point $D$ outside segment $AK$ it takes the form $AD + BD = AD - DK$. In the first case $BD = DK$ and the second case is impossible. Hence, point $D$ is the intersection point of the midperpendicular to segment $BK$ and segment $AC$.

**8.42.** Suppose that triangle $ABC$ is constructed. Let us draw diameter $CD$ of the circumscribed circle. Let $O$ be the center of the circumscribed circle, $L$ the intersection point of the extension of the bisector $AK$ with the circumscribed circle (Fig. 88). Since $\angle ABC - \angle ACB = 90°$, it follows that $\angle ABD = \angle ACB$; hence, $\smile DA = \smile AB$. It is also clear that $\smile BL = \smile LC$. Therefore, $\angle AOL = 90°$.

This implies the following construction. Let us construct circle $S$ with center $O$ and a given radius. On circle $S$ select an arbitrary point $A$. Let us construct a point $L$ on circle $S$ so that $\angle AOL = 90°$. On segment $AL$, construct segment $AK$ whose length is equal to
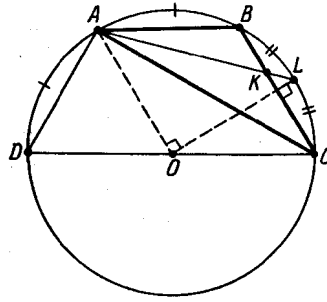
Figure 88 (Sol. 8.42)

that of the given bisector. Let us draw through point $K$ line $l$ perpendicular to $OL$. The intersection points of $l$ with circle $S$ are vertices $B$ and $C$ of triangle $ABC$ to be constructed.

**8.43.** On sides $BC$ and $AC$, take points $A_1$ and $B_1$ such that $PA_1 \parallel AC$ and $PB_1 \parallel BC$. Next, on rays $A_1B$ and $B_1A$ mark segments $A_1B_2 = AB_1$ and $B_1A_2 = BA_1$. Let us prove that line $A_2B_2$ is the one to be found. Indeed, let $k = \frac{AP}{AB}$. Then

$$\frac{BA_2}{BP} = \frac{(1-k)a}{ka} = \frac{(1-k)a + (1-k)b}{ka + kb} = \frac{CA_2}{CB_2},$$

i.e., $\triangle A_2B_1P \sim \triangle A_2CB_2$ and line $A_2B_2$ passes through point $P$. Moreover, $AA_2 = |(1-k)a - kb| = BB_2$.

**8.44.** Suppose that triangle $ABC$ is constructed. Let $B_1$ be the tangent point of the inscribed circle with side $AC$. In right triangle $AOB_1$ leg $OB_1 = r$ and hypothenuse $AO$ are known, therefore, we can construct angle $\angle OAB_1$, hence, angle $\angle BAC$. Let $O_1$ be the center of the circumscribed circle of triangle $ABC$, let $M$ be the midpoint of side $BC$. In right triangle $BO_1M$ leg $O_1M = \frac{1}{2}AH$ is known (see solution to Problem 5.105) and angle $\angle BO_1M$ is known (it is equal to either $\angle A$ or $180° - \angle A$); hence, it can be constructed. Next, we can determine the length of segment $OO_1 = \sqrt{R(R - 2r)}$, cf. Problem 5.11 a). Thus, we can construct segments of length $R$ and $OO_1 = d$.

After this take segment $AO$ and construct point $O_1$ for which $AO_1 = R$ and $OO_1 = d$ (there could be two such points). Let us draw from point $A$ tangents to the circle of radius $r$ centered at $O$. Points $B$ and $C$ to be found lie on these tangents and their distance from point $O_1$ is equal to $R$; obviously, points $B$ and $C$ are distinct from point $A$.

**8.45.** Let the distance between the given parallel lines be equal to $a$. We have to draw parallel lines through points $A$ and $B$ so that the distance between the lines is equal to $a$. To this end, let us construct the circle with segment $AB$ as its diameter and find the intersection points $C_1$ and $C_2$ of this circle with the circle of radius $a$ centered at $B$. A side of the rhombus to be constructed lies on line $AC_1$ (another solution: it lies on $AC_2$). Next, let us draw through point $B$ the line parallel to $AC_1$ (resp. $AC_2$).

**8.46.** Suppose that quadrilateral $ABCD$ is constructed. Let us denote the midpoints of sides $AB$, $BC$, $CD$ and $DA$ by $P$, $Q$, $R$ and $S$, respectively, and the midpoints of diagonals $AC$ and $BD$ by $K$ and $L$, respectively. In triangle $KSL$ we know $KS = \frac{1}{2}CD$, $LS = \frac{1}{2}AB$ and angle $\angle KSL$ equal to the angle between the sides $AB$ and $CD$.

Having constructed triangle $KSL$, we can construct triangle $KRL$ because the lengths of all its sides are known. After this we complement triangles $KSL$ and $KRL$ to parallelograms $KSLQ$ and $KRLP$, respectively. Points $A$, $B$, $C$, $D$ are vertices of parallelograms $PLSA$, $QKPB$, $RLQC$, $SKRD$ (Fig. 89).

**8.47.** Let us drop perpendiculars $BB_1$ and $DD_1$ from vertices $B$ and $D$, respectively, to diagonal $AC$. Let, for definiteness, $DD_1 > BB_1$. Let us construct a segment of length
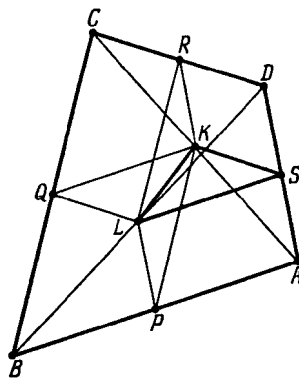
Figure 89 (Sol. 8.46)

$a = DD_1 - BB_1$; draw a line parallel to line $AC$ and such that the distance between this line and $AC$ is equal to $a$ and which intersects side $CD$ at a point, $E$. Clearly,

$$S_{AED} = \frac{ED}{CD}S_{ACD} = \frac{BB_1}{DD_1}S_{ACD} = S_{ABC}.$$

Therefore, the median of triangle $AEC$ lies on the line to be constructed.

**8.48.** Let $P$, $Q$, $R$ be the midpoints of equal sides $AB$, $BC$, $CD$ of quadrilateral $ABCD$. Let us draw the midperpendiculars $l_1$ and $l_2$ to segments $PQ$ and $QR$. Since $AB = BC = CD$, it is clear that points $B$ and $C$ lie on lines $l_1$ and $l_2$ and $BQ = QC$.

This implies the following construction. Let us draw the midperpendiculars $l_1$ and $l_2$ to segments $PQ$ and $QR$, respectively. Then through point $Q$ we draw a segment with endpoints on lines $l_1$ and $l_2$ so that $Q$ were its midpoint, cf. Problem 16.15.

**8.49.** Let vertices $A$, $B$ and $C$ of quadrilateral $ABCD$ which is both inscribed and circumscribed be given and $AB \geq BC$. Then $AD - CD = AB - BC \geq 0$. Hence, on side $AD$ we can mark segment $DC_1$ equal to $DC$. In triangle $AC_1C$ the lengths of sides $AC$ and $AC_1 = AB - BC$ are known and $\angle AC_1C = 90° + \frac{1}{2}\angle D = 180° - \frac{1}{2}\angle B$. Since angle $\angle AC_1C$ is an obtuse one, triangle $AC_1C$ is uniquely recoverable from these elements. The remaining part of the construction is obvious.

**8.50.** Let $ABCD$ be a circumscribed equilateral trapezoid with bases $AD$ and $BC$ such that $AD > BC$; let $C_1$ be the projection of point $C$ to line $AD$. Let us prove that $AB = AC_1$. Indeed, if $P$ and $Q$ are the tangent points of sides $AB$ and $AD$ with the inscribed circle, then $AB = AP + PB = AQ + \frac{1}{2}BC = AQ + QC_1 = AC_1$.

This implies the following construction. Let $C_1$ be the projection of point $C$ to base $AD$. Then $B$ is the intersection point of line $BC$ and the circle of radius $AC_1$ centered at $A$. A trapezoid with $AD < BC$ is similarly constructed.

**8.51.** Let us denote the midpoints of bases $AD$ and $BC$ by $L$ and $N$ and the midpoint of segment $EF$ by $M$. Points $L$, $O$, $N$ lie on one line (by Problem 19.2). Clearly, point $M$ also lies on this line. This implies the following construction.

Let us draw through point $K$ line $l$ perpendicular to line $OK$. Base $AD$ lies on $l$. Point $L$ is the intersection point of $l$ and line $OM$. Point $N$ is symmetric to point $L$ through point $M$. Let us draw lines through point $O$ parallel to lines $EN$ and $FN$. The intersection points of the lines we have just drawn are vertices $A$ and $D$ of the trapezoid. Vertices $B$ and $C$ are symmetric to vertices $A$ and $D$ through points $E$ and $F$, respectively.

**8.52.** Suppose that we have constructed quadrilateral $ABCD$ with given lengths of sides and a given midline $KP$ (here $K$ and $P$ are the midpoints of sides $AB$ and $CD$, respectively). Let $A_1$ and $B_1$ be the points symmetric to points $A$ and $B$, respectively, through point $P$. Triangle $A_1BC$ can be constructed because its sides $BC$, $CA_1 = AD$ and $BA_1 = 2KP$ are

known. Let us complement triangle $A_1BC$ to parallelogram $A_1EBC$. Now we can construct point $D$ because $CD$ and $ED = BA$ are known. Making use of the fact that $\overrightarrow{DA} = \overrightarrow{A_1C}$ we construct point $A$.

**8.53.** Making use of the formulas of Problems 6.34 and 6.35 it is easy to express the lengths of the diagonals of the inscribed triangle in terms of the lengths of its sides. The obtained formulas can be applied for the construction of the diagonals (for convenience it is advisable to introduce an arbitrary segment $e$ as the measure of unit length and construct segments of length $pq$, $\frac{p}{q}$ and $\sqrt{p}$ as $\frac{pq}{e}$, $\frac{pe}{q}$ and $\sqrt{pe}$).

**8.54.** A circle intercepts equal segments on the legs of an angle if and only if the center of the circle lies on the bisector of the angle. Therefore, the center of the circle to be found is the intersection point of the midperpendicular to segment $AB$ and the bisector of the given angle.

**8.55.** Let us suppose that we have constructed circle $S'$ tangent to the given circle $S$ at point $A$ and the given line $l$ at a point, $B$. Let $O$ and $O'$ be the centers of circles $S$ and $S'$, respectively (Fig. 90). Clearly, points $O$, $O'$ and $A$ lie on one line and $O'B = O'A$. Hence, we have to construct point $O'$ on line $OA$ so that $O'A = O'B$, where $B$ is the base of the perpendicular dropped from point $O'$ to line $l$.
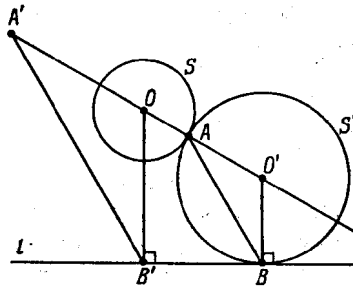


Figure 90 (Sol. 8.55)

To this end let us drop perpendiculr $OB'$ on line $l$. Next, on line $AO$ mark segment $OA'$ of length $OB'$. Let us draw through point $A$ line $AB$ parallel to $A'B'$ (point $B$ lies on line $l$). Point $O'$ is the intersection point of line $OA$ and the perpendicular to $l$ drawn through point $B$.

**8.56.** a) Let $l_1$ be the midperpendicular to segment $AB$, let $C$ be the interception point of lines $l_1$ and $l$; let $l'$ be the line symmetric to $l$ through line $l_1$. The problem reduces to the necessity to construct a circle that passes through point $A$ and is tangent to lines $l$ and $l'$, cf. Problem 19.15.

b) We may assume that the center of circle $S$ does not lie on the midperpendicular to segment $AB$ (otherwise the construction is obvious). Let us take an arbitrary point $C$ on circle $S$ and construct the circumscribed circle of triangle $ABC$; this circle intersects $S$ at a point $D$. Let $M$ be the intersection point of lines $AB$ and $CD$. Let us draw tangents $MP$ and $MQ$ to circle $S$. Then the circumscribed circles of triangles $ABP$ and $ABQ$ are the ones to be found since $MP^2 = MQ^2 = MA \cdot MB$.

**8.57.** Suppose we have constructed circles $S_1$, $S_2$ and $S_3$ tangent to each other pairwise at given points: $S_1$ and $S_2$ are tangent at point $C$; circles $S_1$ and $S_3$ are tangent at point $B$; circles $S_2$ and $S_3$ are tangent at point $A$. Let $O_1$, $O_2$ and $O_3$ be the centers of circles $S_1$, $S_2$ and $S_3$, respectively. Then points $A$, $B$ and $C$ lie on the sides of triangle $O_1O_2O_3$ and $O_1B = O_1C$, $O_2C = O_2A$ and $O_3A = O_3B$. Hence, points $A$, $B$ and $C$ are the tangent points of the inscribed circle of triangle $O_1O_2O_3$ with its sides.

This implies the following construction. First, let us construct the circumscribed circle of triangle $ABC$ and draw tangents to it at points $A$, $B$ and $C$. The intersection points of these tangents are the centers of circles to be found.

**8.58.** Suppose that we have constructed circle $S$ whose tangents $AA_1$, $BB_1$ and $CC_1$, where $A_1$, $B_1$ and $C_1$ are the tangent points, are of length $a$, $b$ and $c$, respectively. Let us construct circles $S_a$, $S_b$ and $S_c$ with the centers $A$, $B$ and $C$ and radii $a$, $b$ and $c$, respectively (Fig. 91). If $O$ is the center of circle $S$, then segments $OA_1$, $OB_1$ and $OC_1$ are radii of circle $S$ and tangents to circles $S_a$, $S_b$ and $S_c$ as well. Hence, point $O$ is the radical center (cf. §3.10) of circles $S_a$, $S_b$ and $S_c$.
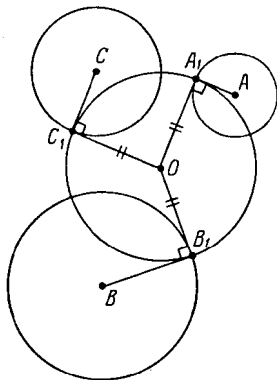


Figure 91 (Sol. 8.58)

This implies the following construction. First, construct circles $S_a$, $S_b$ and $S_c$. Then let us construct their radical center $O$. The circle to be found is the circle with center $O$ and the radius whose length is equal to that of the tangent drawn from point $O$ to circle $S_a$.

**8.59.** First, let us construct segment $BC$ of length $a$. Next, let us construct the locus of points $X$ for which $CX : BX = b : c$, cf. Problem 7.14. For vertex $A$ we can take any of the intersection points of this locus with a line whose distance from line $BC$ is equal to $h_a$.

**8.60.** Given the lengths of segments $AD'$ and $BD$, we can construct segment $AB$ and point $D$ on this segment. Point $C$ is the intersection point of the circle of radius $CD$ centered at $D$ and the locus of points $X$ for which $AX : BX = AD : BD$.

**8.61.** Let $X$ be a point that does not lie on line $AB$. Clearly, $\angle AXB = \angle BCX$ if and only if $AX : CX = AB : CB$. Hence, point $M$ is the intersection point of the locus of points $X$ for which $AX : CX = AB : CB$ and the locus of points $Y$ for which $BY : DY = BC : DC$ (it is possible for these loci not to intersect).

**8.62.** We have to construct a point $O$ for which $AO : A'O = AB : A'B'$ and $BO : B'O = AB : A'B'$. Point $O$ is the intersection point of the locus of points $X$ for which $AX : A'X = AB : A'B'$ and the locus of points $Y$ for which $BY : B'Y = AB : A'B'$.

**8.63.** Let $O$ be the center of the given circle. Chords $XP$ and $XQ$ that pass through points $A$ and $B$ are equal if and only if $XO$ is the bisector of angle $PXO$, i.e., $AX : BX = AO : BO$. The point $X$ to be found is the intersection point of the corresponding Apollonius's circle with the given circle.

**8.64.** a) If line $l$ does not intersect segment $AB$, then $ABB_1A_1$ is a parallelogram and $l \parallel AB$. If line $l$ intersects segment $AB$, then $AA_1BB_1$ is a parallelogram and $l$ passes through the midpoint of segment $AB$.

b) One of the lines to be found is parallel to line $AB$ and another one passes through the midpoint of segment $AB$.

**8.65.** Let us construct a circle of radius 1 and in it draw two perpendicular diameters, $AB$ and $CD$. Let $O$ be the center of the circle, $M$ the midpoint of segment $OC$, $P$ the intersection

point of line $AM$ and the circle with diameter $OC$ (Fig. 92). Then $AM^2 = 1 + \frac{1}{4} = \frac{5}{4}$ and, therefore, $AP = AM - PM = \frac{\sqrt{5}-1}{2} = 2\sin 18°$ (cf. Problem 5.46), i.e., $AP$ is the length of a side of a regular decagon inscribed in the given circle.
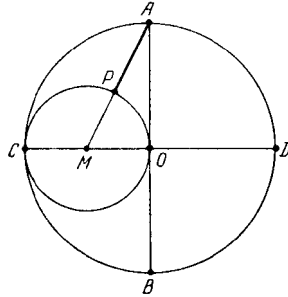


Figure 92 (Sol. 8.65)

**8.66.** Suppose we have constructed rectangle $PQRS$ so that the given points $A$, $B$, $C$, $D$ lie on sides $PQ$, $QR$, $RS$, $SP$, respectively, and $PQ : QR = a$, where a is the given ratio of sides. Let $F$ be the intersection point of the line drawn through point $D$ perpendicularly to line $AC$ and line $QR$. Then $DF : AC = a$.

This implies the following construction. From point $D$ draw a ray that intersects segment $AC$ at a right angle and on this ray construct a point $F$ so that $DF = a \cdot AC$. Side $QR$ lies on line $BF$. The continuation of the construction is obvious.

**8.67.** Suppose that points $X$ and $Y$ with the required properties are constructed. Denote the intersection point of lines $AX$ and $YC$ by $M$, that of lines $AB$ and $XY$ by $K$. Right triangles $AXK$ and $YXM$ have a common acute angle $\angle X$, hence, $\angle XAK = \angle XYM$. Angles $\angle XAB$ and $\angle XYB$ subtend the same arc, hence, $\angle XAB = \angle XYB$. Therefore, $\angle XYM = \angle XYB$. Since $XY \perp AB$, it follows that $K$ is the midpoint of segment $CB$.

Conversely, if $K$ is the midpoint of segment $CB$, then $\angle MYX = \angle BYX = \angle XAB$. Triangles $AXK$ and $YXM$ have a common angle $\angle X$ and $\angle XAK = \angle XYM$; hence, $\angle YMX = \angle AKX = 90°$.

This implies the following construction. Through the midpoint $K$ of segment $CB$ draw line $l$ perpendicular to line $AB$. Points $X$ and $Y$ are the intersection points of line $l$ with the given circle.

**8.68.** If we have an angle of value $\alpha$, then we can construct angles of value $2\alpha$, $3\alpha$, etc. Since $19 \cdot 19° = 361°$, we can construct an angle of $361°$ that coincides with the angle of $1°$.

**8.69.** First, let us construct an angle of $36°$, cf. Problem 8.65. Then we can construct the angle of $\frac{36° - 30°}{2} = 3°$. If $n$ is not divisible by 3, then having at our disposal angles of $n°$ and $3°$ we can construct an angle of $1°$. Indeed, if $n = 3k + 1$, then $1° = n° - k \cdot 3°$ and if $n = 3k + 2$, then $1° = 2n° - (2k + 1) \cdot 3°$.

**8.70.** The sequence of constructions is as follows. On the piece of paper take an arbitrary point $O$ and perform the homothety with center $O$ and sufficiently small coefficient $k$ so that this homothety sends the image of the intersection point of the given lines on the piece of paper. Then we can construct the bisector of the angle between the images of the lines. Next, let us perform the homothety with the same center and coefficient $\frac{1}{k}$ which yields the desired segment of the bisector.

**8.71.** Let us construct with the help of a two-sided ruler two parallel chords $AB$ and $CD$. Let $P$ and $Q$ be the intersection points of lines $AC$ with $BD$ and $AD$ with $BC$, respectively. Then line $PQ$ passes through the center of the given circle. Constructing similarly one more such line we find the center of the circle.

**8.72.** Let us draw through point $A$ two rays $p$ and $q$ that form a small angle inside which point $B$ lies (the rays can be constructed by replacing the ruler). Let us draw through point $B$ segments $PQ_1$ and $P_1Q$ (Fig. 93). If $PQ < 10$ cm and $P_1Q_1 < 10$ cm, then we can construct point $O$ at which lines $PQ$ and $P_1Q_1$ intersect.
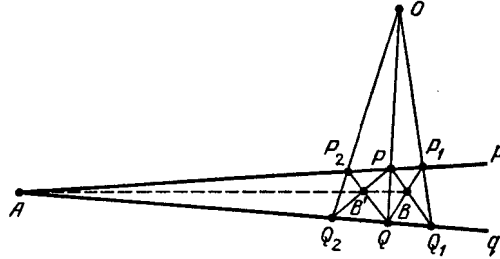


Figure 93 (Sol. 8.72)

Through point $O$ draw line $P_2Q_2$. If $PQ_2 < 10$ cm and $P_2Q < 10$ cm; then we can construct point $B'$ at which lines $PQ_2$ and $P_2Q$ intersect. If $BB' < 10$ cm, then by Problem 5.67 we can construct line $BB'$; this line passes through point $A$.

**8.73.** The construction is based on the fact that if $A$ and $B$ are the intersection points of equal circles centered at $P$ and $Q$, then $\overrightarrow{PA} = \overrightarrow{BQ}$. Let $S_1$ be the initial circle, $A_1$ the given point. Let us draw circle $S_2$ through point $A_1$ and circle $S_3$ through the intersection point $A_2$ of circles $S_1$ and $S_2$; circle $S_4$ through the intersection point $A_3$ of circles $S_2$ and $S_3$ and, finally, circle $S_5$ through the intersection points $B_1$ and $A_4$ of circles $S_1$ and $S_3$, respectively, with circle $S_4$. Let us prove that the intersection point $B_2$ of circles $S_5$ and $S_1$ is the one to be found.

Let $O_i$ be the center of circle $S_i$. Then

$$\overrightarrow{A_1O_1} = \overrightarrow{O_2A_2} = \overrightarrow{A_3O_3} = \overrightarrow{O_4A_4} = \overrightarrow{B_1O_5} = \overrightarrow{O_1B_2}.$$

REMARK. There are two intersection points of circles $S_1$ and $S_4$; for point $B_1$ we can take any of them.

**8.74.** Let $AB$ be the given segment, $P$ an arbitrary point not on the given lines. Let us construct the intersection points $C$ and $D$ of the second of the given lines with lines $PA$ and $PB$, respectively, and the intersection point $Q$ of lines $AD$ and $BC$. By Problem 19.2 line $PQ$ passes through the midpoint of segment $AB$.

**8.75.** Let $AB$ be the given segment; let $C$ and $D$ be arbitrary points on the second of given lines. By the preceding problem we can construct the midpoint, $M$, of segment $CD$. Let $P$ be the intersection point of lines $AM$ and $BD$; let $E$ be the intersection point of lines $PC$ and $AB$. Let us prove that $EB$ is the segment to be found.

Since $\triangle PMC \sim \triangle PAE$ and $\triangle PMD \sim \triangle PAB$, it follows that

$$\frac{AB}{AE} = \frac{AB}{AP} : \frac{AE}{AP} = \frac{MD}{MP} : \frac{MC}{MP} = \frac{MD}{MC} = 1.$$

**8.76.** Let $AB$ be the given segment; let $C$ and $D$ be arbitrary points on the second of the given lines. By the preceding problem we can construct points $D_1 = D, D_2, \ldots, D_n$ such that all the segments $D_iD_{i+1}$ are equal to segment $CD$. Let $P$ be the intersection point of lines $AC$ and $BD_n$ and let $B_1, \ldots, B_{n-1}$ be the intersection points of line $AB$ with lines $PD_1, \ldots, PD_{n-1}$, respectively. Clearly, points $B_1, \ldots, B_{n-1}$ divide segment $AB$ in $n$ equal parts.

**8.77.** On one of the given lines take segment $AB$ and construct its midpoint, $M$ (cf. Problem 8.74). Let $A_1$ and $M_1$ be the intersection points of lines $PA$ and $PM$ with the second of the given lines, $Q$ the intersection point of lines $BM_1$ and $MA_1$. It is easy to verify that line $PQ$ is parallel to the given lines.

**8.78.** In the case when point $P$ does not lie on line $AB$, we can make use of the solution of Problem 3.36. If point $P$ lies on line $AB$, then we can first drop perpendiculars $l_1$ and $l_2$ from some other points and then in accordance with Problem 8.77 draw through point $P$ the line parallel to lines $l_1$ and $l_2$.

**8.79.** a) Let $A$ be the given point, $l$ the given line. First, let us consider the case when point $O$ does not lie on line $l$. Let us draw through point $O$ two arbitrary lines that intersect line $l$ at points $B$ and $C$. By Problem 8.78, in triangle $OBC$, heights to sides $OB$ and $OC$ can be dropped. Let $H$ be their intersection point. Then we can draw line $OH$ perpendicular to $l$. By Problem 8.78 we can drop the perpendicular from point $A$ to $OH$. This is the line to be constructed that passes through $A$ and is parallel to $l$. In order to drop the perpendicular from $A$ to $l$ we have to erect perpendicular $l'$ to $OH$ at point $O$ and then drop the perpendicular from $A$ to $l'$.

If point $O$ lies on line $l$, then by Problem 8.78 we can immediately drop the perpendicular $l'$ from point $A$ to line $l$ and then erect the perpendicular to line $l'$ from the same point $A$.

b) Let $l$ be the given line, $A$ the given point on it and $BC$ the given segment. Let us draw through point $O$ lines $OD$ and $OE$ parallel to lines $l$ and $BC$, respectively ($D$ and $E$ are the intersection points of these lines with circle $S$). Let us draw through point $C$ the line parallel to $OB$ to its intersection with line $OE$ at point $F$ and through point $F$ the line parallel to $ED$ to its intersection with $OD$ at point $G$ and, finally, through point $G$ the line parallel to $OA$ to its intersection with $l$ at point $H$. Then $AH = OG = OF = BC$, i.e., $AH$ is the segment to be constructed.

c) Let us take two arbitrary lines that intersect at point $P$. Let us mark on one of them segment $PA = a$ and on the other one segments $PB = b$ and $PC = c$. Let $D$ be the intersection point of line $PA$ with the line that passes through $B$ and is parallel to $AC$. Clearly, $PD = \frac{ab}{c}$.

d) Let $H$ be the homothety (or the parallel translation) that sends the circle with center $A$ and radius $r$ to circle $S$ (i.e., to the given circle with the marked center $O$). Since the radii of both circles are known, we can construct the image of any point $X$ under the mapping $H$. For this we have to draw through point $O$ the line parallel to line $AX$ and mark on it a segment equal to $\frac{r_s \cdot AX}{r}$, where $r_s$ is the radius of circle $S$.

We similarly construct the image of any point under the mapping $H^{-1}$. Hence, we can construct the line $l' = H(l)$ and find its intersection points with circle $S$ and then construct the images of these points under the map $H^{-1}$.

e) Let $A$ and $B$ be the centers of the given circles, $C$ one of the points to be constructed, $CH$ the height of triangle $ABC$. From Pythagoras theorem for triangles $ACH$ and $BCH$ we deduce that $AH = \frac{b^2 + c^2 - a^2}{2c}$. The quantities $a$, $b$ and $c$ are known, hence, we can construct point $H$ and the intersection points of line $CH$ with one of the given circles.

**8.80.** a) Let us draw lines parallel to lines $OA$ and $OB$, whose distance from the latter lines is equal to $a$ and which intersect the legs of the angles. The intersection point of these lines lies on the bisector to be constructed.

b) Let us draw the line parallel to $OB$, whose distance from $OB$ is equal to $a$ and which intersects ray $OA$ at a point $M$. Let us draw through points $O$ and $M$ another pair of parallel lines the distance between which is equal to $a$; the line that passes through point $O$ contains the leg of the angle to be found.

**8.81.** Let us draw through point $A$ an arbitrary line and then draw lines $l_1$ and $l_2$ parallel to it and whose distance from this line is equal to $a$; these lines intersect line $l$ at points $M_1$ and $M_2$. Let us draw through points $A$ and $M_1$ one more pair of parallel lines, $l_a$ and $l_m$, the distance between which is equal to $a$. The intersection point of lines $l_2$ and $l_m$ belongs to the perpendicular to be found.

**8.82.** Let us draw a line parallel to the given one at a distance of $a$. Now, we can make use of the results of Problems 8.77 and 8.74.

**8.83.** Let us draw through point $P$ lines $PA_1$ and $PB_1$ so that $PA_1 \parallel OA$ and $PB_1 \parallel OB$. Let line $PM$ divide the angle between lines $l$ and $PA_1$ in halves. The symmetry through line $PM$ sends line $PA_1$ to line $l$ and, therefore, line $PB_1$ turns under this symmetry into one of the lines to be constructed.

**8.84.** Let us complement triangle $ABM$ to parallelogram $ABMN$. Through point $N$ draw lines parallel to the bisectors of the angles between lines $l$ and $MN$. The intersection points of these lines with line $l$ are the ones to be found.

**8.85.** Let us draw line $l_1$ parallel to line $OA$ at a distance of $a$. On $l$, take an arbitrary point $B$. Let $B_1$ be the intersection point of lines $OB$ and $l_1$. Through point $B_1$ draw the line parallel to $AB$; this line intersects line $OA$ at point $A_1$. Now, let us draw through points $O$ and $A_1$ a pair of parallel lines the distance between which is equal to $a$.

There could be two pairs of such lines. Let $X$ and $X_1$ be the intersection points of the line that passes through point $O$ with lines $l$ and $l_1$. Since $OA_1 = OX_1$ and $\triangle OA_1X_1 \sim \triangle OAX$, point $X$ is the one to be found.

**8.86.** Let us erect perpendiculars to line $O_1O_2$ at points $O_1$ and $O_2$ and on the perpendiculars mark segments $O_1B_1 = O_2A_2$ and $O_2B_2 = O_1A_1$. Let us construct the midpoint $M$ of segment $B_1B_2$ and erect the perpendicular to $B_1B_2$ at point $M$. This perpendicular intersects line $O_1O_2$ at point $N$. Then $O_1N^2 + O_1B_1^2 = O_2N^2 + O_2B_2^2$ and, therefore, $O_1N^2 - O_1A_1^2 = O_2N^2 - O_2A_2^2$, i.e., point $N$ lies on the radical axis. It remains to erect the perpendicular to $O_1O_2$ at point $N$.

**8.87.** First, let us construct an arbitrary line $l_1$ perpendicular to line $l$ and then draw through point $A$ the line perpendicular to $l_1$.

**8.88.** a) Let us draw through points $A$ and $B$ lines $AB$ and $BQ$ perpendicular to line $AB$ and then draw an arbitrary perpendicular to line $AP$. As a result we get a rectangle. It remains to drop from the intersection point of its diagonals the perpendicular to line $AB$.

b) Let us raise from point $B$ perpendicular $l$ to line $AB$ and draw through point $A$ two perpendicular lines; they intersect line $l$ at points $M$ and $N$. Let us complement right triangle $MAN$ to rectangle $MANR$. The base of the perpendicular dropped from point $R$ to line $AB$ is point $C$ to be found.

**8.89.** a) Let us drop perpendicular $AP$ from point $A$ to line $OB$ and construct segment $AC$ whose midpoint is points $P$. Then angle $\angle AOC$ is the one to be found.

b) On line $OB$, take points $B$ and $B_1$ such that $OB = OB_1$. Let us place the right angle so that its sides would pass through points $B$ and $B_1$ and the vertex would lie on ray $OA$. If $A$ is the vertex of the right angle, then angle $\angle AB_1B$ is the one to be found.

**8.90.** Let us draw through point $O$ line $l'$ parallel to line $l$. Let us drop perpendiculars $BP$ and $BQ$ from point $B$ to lines $l'$ and $OA$, respectively, and then drop perpendicular $OX$ from point $O$ to line $PQ$. Then line $XO$ is the desired one (cf. Problem 2.3); if point $Y$ is symmetric to point $X$ through line $l'$, then line $YO$ is also the one to be found.

**8.91.** Let us complement triangle $OAB$ to parallelogram $OABC$ and then construct segment $CC_1$ whose midpoint is point $O$. Let us place the right angle so that its legs pass through points $C$ and $C_1$ and the vertex lies on line $l$. Then the vertex of the right angle coincides with point $X$ to be found.

**8.92.** Let us construct segment $AB$ whose midpoint is point $O$ and place the right angle so that its legs passes through points $A$ and $B$ and the vertex lies on line $l$. Then the vertex of the right angle coincides with the point to be found.

# Chapter 9. GEOMETRIC INEQUALITIES

## Background

1) For elements of a triangle the following notations are used:

$a$, $b$, $c$ are the lengths of sides $BC$, $CA$, $AB$, respectively;

$\alpha$, $\beta$, $\gamma$ the values of the angles at vertices $A$, $B$, $C$, respectively;

$m_a$, $m_b$, $m_c$ are the lengths of the medians drawn from vertices $A$, $B$, $C$, respectively;

$h_a$, $h_b$, $h_c$ are the lengths of the heights dropped from vertices $A$, $B$, $C$, respectively;

$l_a$, $l_b$, $l_c$ are the lengths of the bisectors drawn from vertices $A$, $B$, $C$, respectively;

$r$ and $R$ are the radii of the inscribed and circumscribed circles, respectively.

2) If $A$, $B$, $C$ are arbitrary points, then $AB \le AC + CB$ and the equality takes place only if point $C$ lies on segment $AB$ (*the triangle inequality*).

3) The median of a triangle is shorter than a half sum of the sides that confine it: $m_a < \frac{1}{2(b+c)}$ (Problem 9.1).

4) If one convex polygon lies inside another one, then the perimeter of the outer polygon is greater than the perimeter of the inner one (Problem 9.27 b).

5) The sum of the lengths of the diagonals of a convex quadrilateral is greater than the sum of the length of any pair of the opposite sides of the quadrilateral (Problem 9.14).

6) The longer side of a triangle subtends the greater angle (Problem 10.59).

7) The length of the segment that lies inside a convex polygon does not exceed either that of its longest side or that of its longest diagonal (Problem 10.64).

REMARK. While solving certain problems of this chapter we have to know various algebraic inequalities. The data on these inequalities and their proof are given in an appendix to this chapter; one should acquaint oneself with them but it should be taken into account that these inequalities are only needed in the solution of comparatively complicated problems; in order to solve simple problems we will only need the inequality $\sqrt{ab} \le \frac{1}{2}a + b$ and its corollaries.

## Introductory problems

**1.** Prove that $S_{ABC} \le \frac{1}{2}AB \cdot BC$.

**2.** Prove that $S_{ABCD} \le \frac{1}{2}(AB \cdot BC + AD \cdot DC)$.

**3.** Prove that $\angle ABC > 90°$ if and only if point $B$ lies inside the circle with diameter $AC$.

**4.** The radii of two circles are equal to $R$ and $r$ and the distance between the centers of the circles is equal to $d$. Prove that these circles intersect if and only if $|R - r| < d < R + r$.

**5.** Prove that any diagonal of a quadrilateral is shorter than the quadrilateral's semiperimeter.

## §1. A median of a triangle

**9.1.** Prove that $\frac{1}{2}(a + b - c) < m_c < \frac{1}{2}(a + b)$.

**9.2.** Prove that in any triangle the sum of the medians is greater than $\frac{3}{4}$ of the perimeter but less than the perimeter.

**9.3.** Given $n$ points $A_1, \ldots, A_n$ and a unit circle, prove that it is possible to find a point $M$ on the circle so that $MA_1 + \cdots + MA_n \geq n$.

**9.4.** Points $A_1, \ldots, A_n$ do not lie on one line. Let two distinct points $P$ and $Q$ have the following property

$$A_1 P + \cdots + A_n P = A_1 Q + \cdots + A_n Q = s.$$

Prove that $A_1 K + \cdots + A_n K < s$ for a point $K$.

**9.5.** On a table lies 50 working watches (old style, with hands); all work correctly. Prove that at a certain moment the sum of the distances from the center of the table to the endpoints of the minute's hands becomes greater than the sum of the distances from the center of the table to the centers of watches. (We assume that each watch is of the form of a disk.)

## §2. Algebraic problems on the triangle inequality

In problems of this section $a$, $b$ and $c$ are the lengths of the sides of an arbitrary triangle.

**9.6.** Prove that $a = y + z$, $b = x + z$ and $c = x + y$, where $x$, $y$ and $z$ are positive numbers.

**9.7.** Prove that $a^2 + b^2 + c^2 < 2(ab + bc + ca)$.

**9.8.** For any positive integer $n$, a triangle can be composed of segments whose lengths are $a^n$, $b^n$ and $c^n$. Prove that among numbers $a$, $b$ and $c$ two are equal.

**9.9.** Prove that

$$a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + 4abc > a^3 + b^3 + c^3.$$

**9.10.** Let $p = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $q = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$. Prove that $|p - q| < 1$.

**9.11.** Five segments are such that from any three of them a triangle can be constructed. Prove that at least one of these triangles is an acute one.

**9.12.** Prove that

$$(a + b - c)(a - b + c)(-a + b + c) \leq abc.$$

**9.13.** Prove that

$$a^2 b(a - b) + b^2 c(b - c) + c^2 a(c - a) \geq 0.$$

## §3. The sum of the lengths of quadrilateral's diagonals

**9.14.** Let $ABCD$ be a convex quadrilateral. Prove that $AB + CD < AC + BD$.

**9.15.** Let $ABCD$ be a convex quadrilateral and $AB + BD \leq AC + CD$. Prove that $AB < AC$.

**9.16.** Inside a convex quadrilateral the sum of lengths of whose diagonals is equal to $d$, a convex quadrilateral the sum of lengths of whose diagonals is equal to $d'$ is placed. Prove that $d' < 2d$.

**9.17.** Given closed broken line has the property that any other closed broken line with the same vertices (?) is longer. Prove that the given broken line is not a self-intersecting one.

**9.18.** How many sides can a convex polygon have if all its diagonals are of equal length?

**9.19.** In plane, there are $n$ red and $n$ blue dots no three of which lie on one line. Prove that it is possible to draw $n$ segments with the endpoints of distinct colours without common points.

**9.20.** Prove that the mean arithmetic of the lengths of sides of an arbitrary convex polygon is less than the mean arithmetic of the lengths of all its diagonals.

**9.21.** A convex $(2n+1)$-gon $A_1 A_3 A_5 \ldots A_{2n+1} A_2 \ldots A_{2n}$ is given. Prove that among all the closed broken lines with the vertices in the vertices of the given $(2n+1)$-gon the broken line $A_1 A_2 A_3 \ldots A_{2n+1} A_1$ is the longest.

## §4. Miscellaneous problems on the triangle inequality

**9.22.** In a triangle, the lengths of two sides are equal to 3.14 and 0.67. Find the length of the third side if it is known that it is an integer.

**9.23.** Prove that the sum of lengths of diagonals of convex pentagon $ABCDE$ is greater than its perimeter but less than the doubled perimeter.

**9.24.** Prove that if the lengths of a triangle's sides satisfy the inequality $a^2 + b^2 > 5c^2$, then $c$ is the length of the shortest side.

**9.25.** The lengths of two heights of a triangle are equal to 12 and 20. Prove that the third height is shorter than 30.

**9.26.** On sides $AB$, $BC$, $CA$ of triangle $ABC$, points $C_1$, $A_1$, $B_1$, respectively, are taken so that $BA_1 = \lambda \cdot BC$, $CB_1 = \lambda \cdot CA$ and $AC_1 = \lambda \cdot AB$, where $\frac{1}{2} < \lambda < 1$. Prove that the perimeter $P$ of triangle $ABC$ and the perimeter $P_1$ of triangle $A_1 B_1 C_1$ satisfy the following inequality: $(2\lambda - 1)P < P_1 < \lambda P$.

$$* * *$$

**9.27.** a) Prove that under the passage from a nonconvex polygon to its convex hull the perimeter diminishes. (The *convex hull* of a polygon is the smallest convex polygon that contains the given one.)

b) Inside a convex polygon there lies another convex polygon. Prove that the perimeter of the outer polygon is not less than the perimeter of the inner one.

**9.28.** Inside triangle $ABC$ of perimeter $P$, a point $O$ is taken. Prove that $\frac{1}{2}P < AO + BO + CO < P$.

**9.29.** On base $AD$ of trapezoid $ABCD$, a point $E$ is taken such that the perimeters of triangles $ABE$, $BCE$ and $CDE$ are equal. Prove that $BC = \frac{1}{2}AD$.

See also Problems 13.40, 20.11.

## §5. The area of a triangle does not exceed a half product of two sides

**9.30.** Given a triangle of area 1 the lengths of whose sides satisfy $a \le b \le c$. Prove that $b \ge \sqrt{2}$.

**9.31.** Let $E$, $F$, $G$ and $H$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$ of quadrilateral $ABCD$. Prove that

$$S_{ABCD} \le EG \cdot HF \le \frac{(AB + CD)(AD + BC)}{4}.$$

**9.32.** The perimeter of a convex quadrilateral is equal to 4. Prove that its area does not exceed 1.

**9.33.** Inside triangle $ABC$ a point $M$ is taken. Prove that

$$4S \le AM \cdot BC + BM \cdot AC + CM \cdot AB,$$

where $S$ is the area of triangle $ABC$.

**9.34.** In a circle of radius $R$ a polygon of area $S$ is inscribed; the polygon contains the center of the circle and on each of its sides a point is chosen. Prove that the perimeter of the convex polygon with vertices in the chosen points is not less than $\frac{2S}{R}$.

**9.35.** Inside a convex quadrilateral $ABCD$ of area $S$ point $O$ is taken such that $AO^2 + BO^2 + CO^2 + DO^2 = 2S$. Prove that $ABCD$ is a square and $O$ is its center.

### §6. Inequalities of areas

**9.36.** Points $M$ and $N$ lie on sides $AB$ and $AC$, respectively, of triangle $ABC$, where $AM = CN$ and $AN = BM$. Prove that the area of quadrilateral $BMNC$ is at least three times that of triangle $AMN$.

**9.37.** Areas of triangles $ABC, A_1B_1C_1, A_2B_2C_2$ are equal to $S, S_1, S_2$, respectively, and $AB = A_1B_1 + A_2B_2, AC = A_1C_1 + A_2B_2, BC = B_1C_1 + B_2C_2$. Prove that $S \leq 4\sqrt{S_1S_2}$.

**9.38.** Let $ABCD$ be a convex quadrilateral of area $S$. The angle between lines $AB$ and $CD$ is equal to $\alpha$ and the angle between $AD$ and $BC$ is equal to $\beta$. Prove that

$$AB \cdot CD \sin\alpha + AD \cdot BC \sin\beta \leq 2S \leq AB \cdot CD + AD \cdot BC.$$

**9.39.** Through a point inside a triangle three lines parallel to the triangle's sides are drawn.
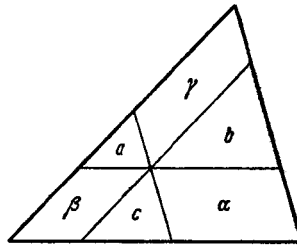


Figure 94 (9.39)

Denote the areas of the parts into which these lines divide the triangle as plotted on Fig. 94. Prove that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} \geq \frac{3}{2}$.

**9.40.** The areas of triangles $ABC$ and $A_1B_1C_1$ are equal to $S$ and $S_1$, respectively, and we know that triangle $ABC$ is not an obtuse one. The greatest of the ratios $\frac{a_1}{a}, \frac{b_1}{b}$ and $\frac{c_1}{c}$ is equal to $k$. Prove that $S_1 \leq k^2S$.

**9.41.** a) Points $B, C$ and $D$ divide the (smaller) arc $\smile AE$ of a circle into four equal parts. Prove that $S_{ACE} < 8S_{BCD}$.

b) From point $A$ tangents $AB$ and $AC$ to a circle are drawn. Through the midpoint $D$ of the (lesser) arc $\smile BC$ the tangent that intersects segments $AB$ and $AC$ at points $M$ and $N$, respectively is drawn. Prove that $S_{BCD} < 2S_{MAN}$.

**9.42.** All sides of a convex polygon are moved outwards at distance $h$ and extended to form a new polygon. Prove that the difference of areas of the polygons is more than $Ph + \pi h^2$, where $P$ is the perimeter.

**9.43.** A square is cut into rectangles. Prove that the sum of areas of the disks circumscribed about all these rectangles is not less than the area of the disk circumscribed about the initial square.

**9.44.** Prove that the sum of areas of five triangles formed by the pairs of neighbouring sides and the corresponding diagonals of a convex pentagon is greater than the area of the pentagon itself.

**9.45.** a) Prove that in any convex hexagon of area $S$ there exists a diagonal that cuts off the hexagon a triangle whose area does not exceed $\frac{1}{6}S$.

b) Prove that in any convex 8-gon of area $S$ there exists a diagonal that cuts off it a triangle of area not greater than $\frac{1}{8}S$.

See also Problem 17.19.

## §7. Area. One figure lies inside another

**9.46.** A convex polygon whose area is greater than 0.5 is placed in a unit square. Prove that inside the polygon one can place a segment of length 0.5 parallel to a side of the square.

**9.47.** Inside a unit square $n$ points are given. Prove that:

a) the area of one of the triangles some of whose vertices are in these points and some in vertices of the square does not exceed $\frac{1}{2(n+1)}$;

b) the area of one of the triangles with the vertices in these points does not exceed $\frac{1}{n-2}$.

**9.48.** a) In a disk of area $S$ a regular $n$-gon of area $S_1$ is inscribed and a regular $n$-gon of area $S_2$ is circumscribed about the disk. Prove that $S^2 > S_1 S_2$.

b) In a circle of length $L$ a regular $n$-gon of perimeter $P_1$ is inscribed and another regular $n$-gon of perimeter $P_2$ is circumscribed about the circle. Prove that $L^2 < P_1 P_2$.

**9.49.** A polygon of area $B$ is inscribed in a circle of area $A$ and circumscribed about a circle of area $C$. Prove that $2B \leq A + C$.

**9.50.** In a unit disk two triangles the area of each of which is greater than 1 are placed. Prove that these triangles intersect.

**9.51.** a) Prove that inside a convex polygon of area $S$ and perimeter $P$ one can place a disk of radius $\frac{S}{P}$.

b) Inside a convex polygon of area $S_1$ and perimeter $P_1$ a convex polygon of area $S_2$ and perimeter $P_2$ is placed. Prove that $\frac{2S_1}{P_1} > \frac{S_2}{P_2}$.

**9.52.** Prove that the area of a parallelogram that lies inside a triangle does not exceed a half area of the triangle.

**9.53.** Prove that the area of a triangle whose vertices lie on sides of a parallelogram does not exceed a half area of the parallelogram.

* * *

**9.54.** Prove that any acute triangle of area 1 can be placed in a right triangle of area $\sqrt{3}$.

**9.55.** a) Prove that a convex polygon of area $S$ can be placed in a rectangle of area not greater than $2S$.

b) Prove that in a convex polygon of area $S$ a parallelogram of area not less than $\frac{1}{2}S$ can be inscribed.

**9.56.** Prove that in any convex polygon of area 1 a triangle whose area is not less than a) $\frac{1}{4}$; b) $\frac{3}{8}$ can be placed.

**9.57.** A convex $n$-gon is placed in a unit square. Prove that there are three vertices $A, B$ and $C$ of this $n$-gon, such that the area of triangle $ABC$ does not exceed a) $\frac{8}{n^2}$; b) $\frac{16\pi}{n^3}$.

See also Problem 15.6.

## §8. Broken lines inside a square

**9.58.** Inside a unit square a non-self-intersecting broken line of length 1000 is placed. Prove that there exists a line parallel to one of the sides of the square that intersects this broken line in at least 500 points.

**9.59.** In a unit square a broken line of length $L$ is placed. It is known that each point of the square is distant from a point of this broken line less than by $\varepsilon$. Prove that $L \geq \frac{1}{2\varepsilon} - \frac{1}{2}\pi\varepsilon$.

**9.60.** Inside a unit square $n^2$ points are placed. Prove that there exists a broken line that passes through all these points and whose length does not exceed $2n$.

**9.61.** Inside a square of side 100 a broken line $L$ is placed. This broken line has the following property: the distance from any point of the square to $L$ does not exceed 0.5.

Prove that on $L$ there are two points the distance between which does not exceed 1 and the distance between which along $L$ is not less than 198.

## §9. The quadrilateral

**9.62.** In quadrilateral $ABCD$ angles $\angle A$ and $\angle B$ are equal and $\angle D > \angle C$. Prove that $AD < BC$.

**9.63.** In trapezoid $ABCD$, the angles at base $AD$ satisfy inequalities $\angle A < \angle D < 90°$. Prove that $AC > BD$.

**9.64.** Prove that if two opposite angles of a quadrilateral are obtuse ones, then the diagonal that connects the vertices of these angles is shorter than the other diagonal.

**9.65.** Prove that the sum of distances from an arbitrary point to three vertices of an isosceles trapezoid is greater than the distance from this point to the fourth vertex.

**9.66.** Angle $\angle A$ of quadrilateral $ABCD$ is an obtuse one; $F$ is the midpoint of side $BC$. Prove that $2FA < BD + CD$.

**9.67.** Quadrilateral $ABCD$ is given. Prove that $AC \cdot BD \leq AB \cdot CD + BC \cdot AD$. (*Ptolemy's inequality.*)

**9.68.** Let $M$ and $N$ be the midpoints of sides $BC$ and $CD$, respectively, of a convex quadrilateral $ABCD$. Prove that $S_{ABCD} < 4S_{AMN}$.

**9.69.** Point $P$ lies inside convex quadrilateral $ABCD$. Prove that the sum of distances from point $P$ to the vertices of the quadrilateral is less than the sum of pairwise distances between the vertices of the quadrilateral.

**9.70.** The diagonals divide a convex quadrilateral $ABCD$ into four triangles. Let $P$ be the perimeter of $ABCD$ and $Q$ the perimeter of the quadrilateral formed by the centers of the inscribed circles of the obtained triangles. Prove that $PQ > 4S_{ABCD}$.

**9.71.** Prove that the distance from one of the vertices of a convex quadrilateral to the opposite diagonal does not exceed a half length of this diagonal.

**9.72.** Segment $KL$ passes through the intersection point of diagonals of quadrilateral $ABCD$ and the endpoints of $KL$ lie on sides $AB$ and $CD$ of the quadrilateral. Prove that the length of segment $KL$ does not exceed the length of one of the diagonals of the quadrilateral.

**9.73.** Parallelogram $P_2$ is inscribed in parallelogram $P_1$ and parallelogram $P_3$ whose sides are parallel to the corresponding sides of $P_1$ is inscribed in parallelogram $P_2$. Prove that the length of at least one of the sides of $P_1$ does not exceed the doubled length of a parallel to it side of $P_3$.

See also Problems 13.19, 15.3 a).

## §10. Polygons

**9.74.** Prove that if the angles of a convex pentagon form an arithmetic progression, then each of them is greater than $36°$.

**9.75.** Let $ABCDE$ is a convex pentagon inscribed in a circle of radius 1 so that $AB = A$, $BC = b$, $CD = c$, $DE = d$, $AE = 2$. Prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

**9.76.** Inside a regular hexagon with side 1 point $P$ is taken. Prove that the sum of the distances from point $P$ to certain three vertices of the hexagon is not less than 1.

**9.77.** Prove that if the sides of convex hexagon $ABCDEF$ are equal to 1, then the radius of the circumscribed circle of one of triangles $ACE$ and $BDF$ does not exceed 1.

**9.78.** Each side of convex hexagon $ABCDEF$ is shorter than 1. Prove that one of the diagonals $AD$, $BE$, $CF$ is shorter than 2.

**9.79.** Heptagon $A_1 \ldots A_7$ is inscribed in a circle. Prove that if the center of this circle lies inside it, then the value of any angle at vertices $A_1$, $A_3$, $A_5$ is less than $450°$.

* * *

**9.80.** a) Prove that if the lengths of the projections of a segment to two perpendicular lines are equal to $a$ and $b$, then the segment's length is not less than $\frac{a+b}{\sqrt{2}}$.

b) The lengths of the projections of a polygon to coordinate axes are equal to $a$ and $b$. Prove that its perimeter is not less than $\sqrt{2}(a+b)$.

**9.81.** Prove that from the sides of a convex polygon of perimeter $P$ two segments whose lengths differ not more than by $\frac{1}{3}P$ can be constructed.

**9.82.** Inside a convex polygon $A_1 \ldots A_n$ a point $O$ is taken. Let $\alpha_k$ be the value of the angle at vertex $A_k, x_k = OA_k$ and $d_k$ the distance from point $O$ to line $A_k A_{k+1}$. Prove that $\sum x_k \sin \frac{\alpha_k}{2} \geq \sum d_k$ and $\sum x_k \cos \frac{\alpha_k}{2} \geq p$, where $p$ is the semiperimeter of the polygon.

**9.83.** Regular $2n$-gon $M_1$ with side $a$ lies inside regular $2n$-gon $M_2$ with side $2a$. Prove that $M_1$ contains the center of $M_2$.

**9.84.** Inside regular polygon $A_1 \ldots A_n$ point $O$ is taken.

Prove that at least one of the angles $\angle A_i O A_j$ satisfies the inequalities $\pi \left(1 - \frac{1}{n}\right) \leq \angle A_i O A_j \leq \pi$.

**9.85.** Prove that for $n \geq 7$ inside a convex $n$-gon there is a point the sum of distances from which to the vertices is greater than the semiperimeter of the $n$-gon.

**9.86.** a) Convex polygons $A_1 \ldots A_n$ and $B_1 \ldots B_n$ are such that all their corresponding sides except for $A_1 A_n$ and $B_1 B_n$ are equal and $\angle A_2 \geq \angle B_2, \ldots, \angle A_{n-1} \geq \angle B_{n-1}$, where at least one of the inequalities is a strict one. Prove that $A_1 A_n > B_1 B_n$.

b) The corresponding sides of nonequal polygons $A_1 \ldots A_n$ and $B_1 \ldots B_n$ are equal.

Let us write beside each vertex of polygon $A_1 \ldots A_n$ the sign of the difference $\angle A_i - \angle B_i$. Prove that for $n \geq 4$ there are at least four pairs of neighbouring vertices with distinct signs. (The vertices with the zero difference are disregarded: two vertices between which there only stand vertices with the zero difference are considered to be neighbouring ones.)

See also Problems 4.37, 4.53, 13.42.

## §11. Miscellaneous problems

**9.87.** On a segment of length 1 there are given $n$ points. Prove that the sum of distances from a point of the segment to these points is not less than $\frac{1}{2}n$.

**9.88.** In a forest, trees of cylindrical form grow. A communication service person has to connect a line from point $A$ to point $B$ through this forest the distance between the points being equal to $l$. Prove that to acheave the goal a piece of wire of length $1.6l$ will be sufficient.

**9.89.** In a forest, the distance between any two trees does not exceed the difference of their heights. Any tree is shorter than 100 m. Prove that this forest can be fenced by a fence of length 200 m.

**9.90.** A (not necessarily convex) paper polygon is folded along a line and both halves are glued together. Can the perimeter of the obtained lamina be greater than the perimeter of the initial polygon?

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**9.91.** Prove that a closed broken line of length 1 can be placed in a disk of radius 0.25.

**9.92.** An acute triangle is placed inside a circumscribed circle. Prove that the radius of the circle is not less than the radius of the circumscribed circle of the triangle.

Is a similar statement true for an obtuse triangle?

**9.93.** Prove that the perimeter of an acute triangle is not less than $4R$.

See also problems 14.23, 20.4.

## Problems for independent study

**9.94.** Two circles divide rectangle $ABCD$ into four rectangles. Prove that the area of one of the rectangles, the one adjacent to vertices $A$ and $C$, does not exceed a quarter of the area of $ABCD$.

**9.95.** Prove that if $AB + BD = AC + CD$, then the midperpendicular to side $BC$ of quadrilateral $ABCD$ intersects segment $AD$.

**9.96.** Prove that if diagonal $BD$ of convex quadrilateral $ABCD$ divides diagonal $AC$ in halves and $AB > BC$, then $AD < DC$.

**9.97.** The lengths of bases of a circumscribed trapezoid are equal to 2 and 11. Prove that the angle between the extensions of its lateral sides is an acute one.

**9.98.** The bases of a trapezoid are equal to $a$ and $b$ and its height is equal to $h$. Prove that the length of one of its diagonals is not less than $\sqrt{\frac{h^2+(b+a)^2}{4}}$.

**9.99.** The vertices of an $n$-gon $M_1$ are the midpoints of sides of a convex $n$-gon $M$. Prove that for $n \geq 3$ the perimeter of $M_1$ is not less than the semiperimeter of $M$ and for $n \geq 4$ the area of $M_1$ is not less than a half area of $M$.

**9.100.** In a unit circle a polygon the lengths of whose sides are confined between 1 and $\sqrt{2}$ is inscribed. Find how many sides does the polygon have.

## Supplement. Certain inequalities

**1.** *The inequality between the mean arithmetic and the mean geometric of two numbers* $\sqrt{ab} \leq \frac{1}{2}(a + b)$, where $a$ and $b$ are positive numbers, is often encountered. This inequality follows from the fact that $a - 2\sqrt{ab} + b = (\sqrt{a} - \sqrt{b})^2 \geq 0$, where the equality takes place only if $a = b$.

This inequality implies several useful inequalities, for example:

$$x(a - x) \quad \leq \left(\frac{x+a-x}{2}\right)^2 = \frac{a^2}{4};$$
$$a + \frac{1}{a} \quad \geq 2\sqrt{a \cdot \frac{1}{a}} = 2 \ \text{ for} a > 0.$$

**2.** *The inequality between the mean arithmetic and the mean geometric of $n$ positive numbers* $(a_1 a_2 \ldots a_n)^{\frac{1}{n}} \leq \frac{a_1+\cdots+a_n}{n}$ is sometimes used. In this inequality the equality takes place only if $a_1 = \cdots = a_n$.

First, let us prove this inequality for the numbers of the form $n = 2^m$ by induction on $m$. For $m = 1$ the equality was proved above.

Suppose that it is proved for $m$ and let us prove it for $m + 1$. Clearly, $a_k a_{k+2^m} \leq \left(\frac{a_k+a_{k+2^m}}{2}\right)^2$. Therefore,

$$(a_1 a_2 \ldots a_{2^{m+1}})^{\frac{1}{2^{m+1}}} \leq (b_1 b_2 \ldots b_{2^m})^{\frac{1}{2^m}},$$

where $b_k = \frac{1}{2}(a_k + a_{k+2^m})$ and by the inductive hypothesis

$$(b_1 \ldots b_{2^m})^{\frac{1}{2^m}} \leq \frac{1}{2^m}(b_1 + \cdots + b_{2^m}) = \frac{1}{2^{m+1}}(a_1 + \cdots + a_{2^{m+1}}).$$

Now, let $n$ be an arbitrary number. Then $n < 2^m$ for some $m$. Suppose $a_{n+1} = \cdots = a_{2^m} = \frac{a_1+\cdots+a_n}{n} = A$. Clearly,

$$(a_1 + \cdots + a_n) + (a_{n+1} + \cdots + a_{2^m}) = nA + (2^m - n)A = 2^m A$$

and $a_1 \ldots a_{2^m} = a_1 \ldots a_n \cdot A^{2^m - n}$. Hence,

$$a_1 \ldots a_n \cdot A^{2^m - n} \leq \left( \frac{2^m A}{2^m} \right)^{2^m} = A^{2^m}, \text{ i.e. } a_1 \ldots a_n \leq A^n.$$

The equality is attained only for $a_1 = \cdots = a_n$.

**3.** For arbitrary numbers $a_1, \ldots, a_n$ we have

$$(a + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2).$$

Indeed,

$$(a_1 + \cdots + a_n)^2 = \sum a_i^2 + 2 \sum_{i<j} a_i a_j \leq \sum a_i^2 + \sum_{i<j} (a_i^2 + a_j^2) = n \sum a_i^2.$$

**4.** Since $\int_0^\alpha \cos t \, dt = \sin \alpha$ and $\int_0^\alpha \sin t \, dt = 1 - \cos \alpha$, it follows that starting from the inequality $\cos t \leq 1$ we get: first, $\sin \alpha \leq \alpha$, then $1 - \cos \alpha \leq \frac{\alpha^2}{2}$ (i.e. $\cos \alpha \geq 1 - \frac{\alpha^2}{2}$), next, $\sin \alpha \geq \alpha - \frac{\alpha^3}{6}$, $\cos \alpha \leq 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}$, etc. (the inequalities are true for all $\alpha \geq 0$).

**5.** Let us prove that $\tan \alpha \geq \alpha$ for $0 \leq \alpha < \frac{\pi}{2}$. Let $AB$ be the tangent to the unit circle centered at $O$; let $B$ be the tangent point, $C$ the intersection point of ray $OA$ with the circle and $S$ the area of the disk sector $BOC$. Then $\alpha = 2S < 2S_{AOB} = \tan \alpha$.

**6.** On the segment $[0, \frac{\pi}{2}]$ the function $f(x) = \frac{x}{\sin x}$ monotonously grows because $f'(x) = \frac{\tan x - x}{\cos x \sin^2 x} > 0$. In particular, $f(\alpha) \leq f\left(\frac{\pi}{2}\right)$, i.e.,

$$\frac{\alpha}{\sin \alpha} \leq \frac{\pi}{2} \text{ for } 0 < \alpha < \frac{\pi}{2}.$$

**7.** If $f(x) = a \cos x + b \sin x$, then $f(x) \leq \sqrt{a^2 + b^2}$. Indeed, there exists an angle $\varphi$ such that $\cos \varphi = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin \varphi = \frac{b}{\sqrt{a^2+b^2}}$; hence,

$$f(x) = \sqrt{a^2 + b^2} \cos(\varphi - x) \leq \sqrt{a^2 + b^2}.$$

The equality takes place only if $\varphi = x + 2k\pi$, i.e., $\cos x = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin x = \frac{b}{\sqrt{a^2+b^2}}$.

## Solutions

**9.1.** Let $C_1$ be the midpoint of side $AB$. Then $CC_1 + C_1 A > CA$ and $BC_1 + C_1 C > BC$. Therefore, $2CC_1 + BA > CA + BC$, i.e., $m_c > \frac{1}{2}(a + b - c)$.

Let point $C'$ be symmetric to $C$ through point $C_1$. Then $CC_1 = C_1 C'$ and $BC' = CA$. Hence, $2m_c = CC' < CB + BC' = CB + CA$, i.e., $m_c < \frac{1}{2}(a + b)$.

**9.2.** The preceding problem implies that $m_a < \frac{1}{2}(b+c)$, $m_b < \frac{1}{2}(a+c)$ and $m_c < \frac{1}{2}(a+b)$ and, therefore, the sum of the lengths of medians does not exceed the perimeter.

Let $O$ be the intersection point of medians of triangle $ABC$. Then $BO + OA > BA$, $AO + OC > AC$ and $CO + OB > CB$. Adding these inequalities and taking into account that $AO = \frac{2}{3}m_a$, $BO = \frac{2}{3}m_b$, $CO = \frac{2}{3}m_c$ we get $m_a + m_b + m_c > \frac{3}{4}(a + b + c)$.

**9.3.** Let $M_1$ and $M_2$ be diametrically opposite points on a circle. Then $M_1 A_k + M_2 A_k \geq M_1 M_2 = 2$. Adding up these inequalities for $k = 1, \ldots, n$ we get

$$(M_1 A_1 + \cdots + M_1 A_n) + (M_2 A_1 + \cdots + M_2 A_n) \geq 2n.$$

Therefore, either $M_1 A_1 + \cdots + M_1 A_n \geq n$ and then we set $M = M_1$ or $M_2 A_1 + \cdots + M_2 A_n \geq n$ and then we set $M = M_2$.

**9.4.** For $K$ we can take the midpoint of segment $PQ$. Indeed, then $A_i K \leq \frac{1}{2}(A_i P + A_i Q)$ (cf. Problem 9.1), where at least one of the inequalities is a strict one because points $A_i$ cannot all lie on line $PQ$.

**9.5.** Let $A_i$ and $B_i$ be the positions of the minute hands of the $i$-th watch at times $t$ and $t + 30$ min, let $O_i$ be the center of the $i$-th watch and $O$ the center of the table. Then $OO_i \leq \frac{1}{2}(OA_i + OB_i)$ for any $i$, cf. Problem 9.1. Clearly, at a certain moment points $A_i$ and $B_i$ do not lie on line $O_iO$, i.e., at least one of $n$ inequalities becomes a strict one. Then either $OO_1 + \cdots + OO_n < OA_1 + \cdots + OA_n$ or $OO_1 + \cdots + OO_n < OB_1 + \cdots + OB_n$.

**9.6.** Solving the system of equations

$$x + y = c, \quad x + z = b, \quad y + z = a$$

we get

$$x = \frac{-a + b + c}{2}, \quad y = \frac{a - b + c}{2}, \quad z = \frac{a + b - c}{2}.$$

The positivity of numbers $x$, $y$ and $z$ follows from the triangle inequality.

**9.7.** Thanks to the triangle inequality we have

$$a^2 > (b - c)^2 = b^2 - 2bc + c^2, \quad b^2 > a^2 - 2ac + c^2, \quad c^2 > a^2 - 2ab + b^2.$$

Adding these inequalities we get the desired statement.

**9.8.** We may assume that $a \geq b \geq c$. Let us prove that $a = b$. Indeed, if $b < a$, then $b \leq \lambda a$ and $c \leq \lambda a$, where $\lambda < 1$. Hence, $b^n + c^n \leq 2\lambda^n a^n$. For sufficiently large $n$ we have $2\lambda^n < 1$ which contradicts the triangle inequality.

**9.9.** Since $c(a - b)^2 + 4abc = c(a + b)^2$, it follows that

$$a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + 4abc - a^3 - b^3 - c^3 =$$
$$a((b - c)^2 - a^2) + +b((c - a)^2 - b^2) + c((a + b)^2 - c^2) =$$
$$(a + b - c)(a - b + c)(-a + b + c).$$

The latter equality is subject to a direct verification. All three factors of the latter expression are positive thanks to the triangle inequality.

**9.10.** It is easy to verify that

$$abc|p - q| = |(b - c)(c - a)(a - b)|.$$

Since $|b - c| < a$, $|c - a| < b$ and $|a - b| < c$, we have $|(b - c)(c - a)(a - b)| < abc$.

**9.11.** Let us index the lengths of the segments so that $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. If all the triangles that can be composed of these segments are not acute ones, then $a_3^2 \geq a_1^2 + a_2^2$, $a_4^2 \geq a_2^2 + a_3^2$ and $a_5^2 \geq a_3^2 + a_4^2$. Hence,

$$a_5^2 \geq a_3^2 + a_4^2 \geq (a_1^2 + a_2^2) + (a_2^2 + a_3^2) \geq 2a_1^2 + 3a_2^2.$$

Since $a_1^2 + a_2^2 \geq 2a_1a_2$, it follows that

$$2a_1^2 + 3a_2^2 > a_1^2 + 2a_1a_2 + a_2^2 = (a_1 + a_2)^2.$$

We get the inequality $a_5^2 > (a_1 + a_2)^2$ which contradicts the triangle inequality.

**9.12. First solution.** Let us introduce new variables

$$x = -a + b + c, \quad y = a - b + c, \quad z = a + b - c.$$

Then $a = \frac{1}{2}(y + z)$, $b = \frac{1}{2}(x + z)$, $c = \frac{1}{2}(x + y)$, i.e., we have to prove that either

$$xyz \leq \frac{1}{8}(x + y)(y + z)(x + z)$$

or

$$6xyz \leq x(y^2 + z^2) + y(x^2 + z^2) + z(x^2 + y^2).$$

The latter inequality follows from the fact that $2xyz \leq x(y^2 + z^2)$, $2xyz \leq y(x^2 + z^2)$ and $2xyz \leq z(x^2 + y^2)$, because $x$, $y$, $z$ are positive numbers.

**Second solution.** Since $2S = ab\sin\gamma$ and $\sin\gamma = \frac{c}{2R}$, it follows that $abc = 2SR$. By Heron's formula

$$(a+b-c)(a-b+c)(-a+b+c) = \frac{8S^2}{p}.$$

Therefore, we have to prove that $\frac{8S^2}{p} \leq 4SR$, i.e., $2S \leq pR$. Since $S = pr$, we infer that $2r \leq R$, cf. Problem 10.26.

**9.13.** Let us introduce new variables

$$x = \frac{-a+b+c}{2}, \quad y = \frac{a-b+c}{2}, \quad z = \frac{a+b-c}{2}.$$

Then numbers $x$, $y$, $z$ are positive and

$$a = y+z, \quad b = x+z, \quad c = x+y.$$

Simple but somewhat cumbersome calculations show that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) = 2(x^3z + y^3x + z^3y - xyz(x+y+z)) =$$
$$2xyz\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} - x - y - z\right).$$

Since $2 \leq \frac{x}{y} + \frac{y}{x}$, it follows that

$$2x \leq x\left(\frac{x}{y} + \frac{y}{x}\right) = \frac{x^2}{y} + y.$$

Similarly,

$$2y \leq y\left(\frac{y}{z} + \frac{z}{y}\right) = \frac{y^2}{z} + z; \quad 2z \leq z\left(\frac{z}{x} + \frac{x}{z}\right) = \frac{z^2}{x} + x.$$

Adding these inequalities we get

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x + y + z.$$

**9.14.** Let $O$ be the intersection point of the diagonals of quadrilateral $ABCD$. Then

$$AC + BD = (AO + OC) + (BO + OD) = (AO + OB) + (OC + OD) > AB + CD.$$

**9.15.** By the above problem $AB + CD < AC + BD$. Adding this inequality to the inequality $AB + BD \leq AC + CD$ we get $2AB < 2AC$.

**9.16.** First, let us prove that if $P$ is the perimeter of convex quadrilateral $ABCD$ and $d_1$ and $d_2$ are the lengths of its diagonals, then $P > d_1 + d_2 > \frac{1}{2}P$. Clearly, $AC < AB + BC$ and $AC < AD + DC$; hence,

$$AC < \frac{AB + BC + CD + AD}{2} = \frac{P}{2}.$$

Similarly, $BD < \frac{1}{2}P$. Therefore, $AC + BD < P$. On the other hand, adding the inequalities

$$AB + CD < AC + BD \quad \text{and} \quad BC + AD < AC + BD$$

(cf. Problem 9.14) we get $P < 2(AC + BD)$.

Let $P$ be the perimeter of the outer quadrilateral, $P'$ the perimeter of the inner one. Then $d > \frac{1}{2}P$ and since $P' < P$ (by Problem 9.27 b)), we have $d' < P' < P < 2d$.

**9.17.** Let the broken line of the shortest length be a self-intersecting one. Let us consider two intersecting links. The vertices of these links can be connected in one of the following three ways: Fig. 95. Let us consider a new broken line all the links of which are the same

as of the initial one except that the two solid intersecting links are replaced by the dotted links (see Fig. 95).
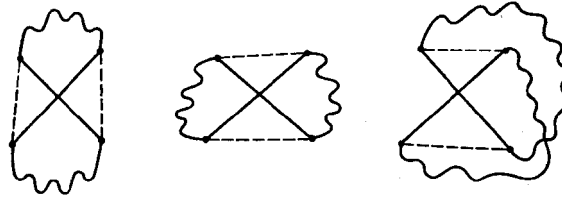


Figure 95 (Sol. 9.17)

Then we get again a broken line but its length is less than that of the initial one because the sum of the lengths of the opposite sides of a convex quadrilateral is less than the sum of the length of its diagonals. We have obtained a contradiction and, therefore, the closed broken line of the least length cannot have intersecting links.

**9.18.** Let us prove that the number of sides of such a polygon does not exceed 5. Suppose that all the diagonals of polygon $A_1 \ldots A_n$ are of the same length and $n \geq 6$. Then segments $A_1A_4$, $A_1A_5$, $A_2A_4$ and $A_2A_5$ are of equal length since they are the diagonals of this polygon. But in convex quadrilateral $A_1A_2A_4A_5$ segments $A_1A_5$ and $A_2A_4$ are opposite sides whereas $A_1A_4$ and $A_2A_5$ are diagonals. Therefore, $A_1A_5 + A_2A_4 < A_1A_4 + A_2A_5$. Contradiction.

It is also clear that a regular pentagon and a square satisfy the required condition.

**9.19.** Consider all the partitions of the given points into pairs of points of distinct colours. There are finitely many such partitions and, therefore, there exists a partition for which the sum of lengths of segments given by pairs of points of the partition is the least one. Let us show that in this case these segments will not intersect. Indeed, if two segments would have intersected, then we could have selected a partition with the lesser sum of lengths of segments by replacing the diagonals of the convex quadrilateral by its opposite sides as shown on Fig. 96.
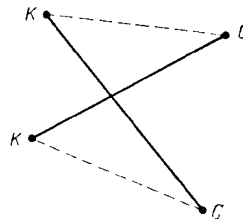


Figure 96 (Sol. 9.19)

**9.20.** Let $A_pA_{p+1}$ and $A_qA_{q+1}$ be nonadjacent sides of $n$-gon $A_1 \ldots A_n$ (i.e., $|p - q| \geq 2$). Then

$$A_pA_{p+1} + A_qA_{q+1} < A_pA_q + A_{p+1}A_{q+1}.$$

Let us write all such inequalities and add them. For each side there exist precisely $n - 3$ sides nonadjacent to it and, therefore, any side enters $n - 3$ inequality, i.e., in the left-hand side of the obtained sum there stands $(n - 3)p$, where $p$ is the sum of lengths of the $n$-gon's sides. Diagonal $A_mA_n$ enters two inequalities for $p = n$, $q = m$ and for $p = n - 1$, $q = m - 1$; hence, in the right-hand side stands $2d$, where $d$ is the sum of lengths of diagonals. Thus, $(n - 3)p < 2d$. Therefore, $\frac{p}{n} < \frac{d}{n(n-3)/2}$, as required.

**9.21.** Let us consider an arbitrary closed broken line with the vertices in vertices of the given polygon. If we have two nonintersecting links then by replacing these links by the

diagonals of the quadrilateral determined by them we enlarge the sum of the lengths of the links. In this process, however, one broken line can get split into two nonintersecting ones. Let us prove that if the number of links is odd then after several such operations we will still get in the end a closed broken line (since the sum of lengths of the links increases each time, there can be only a finite number of such operations). One of the obtained closed broken lines should have an odd number of links but then any of the remaining links does not intersect at least one of the links of this broken line (cf. Problem 23.1 a)); therefore, in the end we get just one broken line.
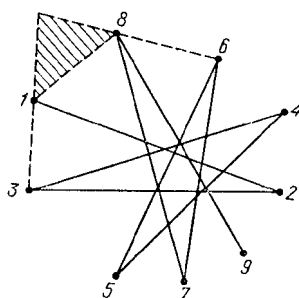


Figure 97 (Sol. 9.21)

Now, let us successively construct a broken line with pairwise intersecting links (Fig. 97). For instance, the 10-th vertex should lie inside the shaded triangle and therefore, the position of vertices is precisely as plotted on Fig. 97. Therefore, to convex polygon $A_1 A_3 A_5 \ldots A_{2n+1} A_2 \ldots A_{2n}$ the broken line $A_1 A_2 A_3 \ldots A_{2n+1} A_1$ corresponds.

**9.22.** Let the length of the third side be equal to $n$. From the triangle inequality we get $3.14 - 0.67 < n < 3.14 + 0.67$. Since $n$ is an integer, $n = 3$.

**9.23.** Clearly, $AB + BC > AC$, $BC + CD > BD$, $CD + DE > CE$, $DE + EA > DA$, $EA + AB > EB$. Adding these inequalities we see that the sum of the lengths of the pentagon's diagonals is shorter than the doubled perimeter.
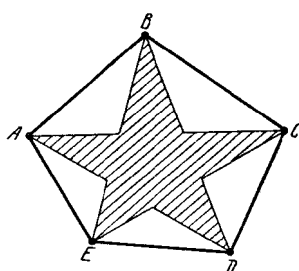


Figure 98 (Sol. 9.23)

The sum of the the diagonals' lengths is longer than the sum of lengths of the sides of the "rays of the star" and it, in turn, is greater than the perimeter of the pentagon (Fig. 98).

**9.24.** Suppose that $c$ is the length of not the shortest side, for instance, $a \le c$. Then $a^2 \le c^2$ and $b^2 < (a + c)^2 \le 4c^2$. Hence, $a^2 + b^2 < 5c^2$. Contradiction.

**9.25.** Since $c > |b - a|$ and $a = \frac{2S}{h_a}$, $c = \frac{2S}{h_c}$, it follows that $\frac{1}{h_c} > \left| \frac{1}{h_a} - \frac{1}{h_b} \right|$. Therefore, in our case $h_c < \frac{20 \cdot 12}{8} = 30$.

**9.26.** On sides $AB$, $BC$, $CA$ take points $C_2$, $A_2$, $B_2$, respectively, so that $A_1B_2 \parallel AB$, $B_1C_2 \parallel BC$, $CA_2 \parallel CA$ (Fig. 99). Then

$$A_1B_1 < A_1B_2 + B_2B_1 = (1 - \lambda)AB + (2\lambda - 1)CA.$$

Similarly,

$$BC_1 < (1 - \lambda)BC + (2\lambda - 1)AB \text{ and } C_1A_1 < (1 - \lambda)CA + (2\lambda - 1)BC.$$

Adding these inequalities we get $P_1 < \lambda P$.
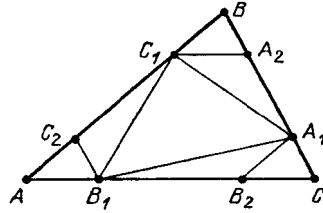


Figure 99 (Sol. 9.26)

Clearly, $A_1B_1 + AC > B_1C$, i.e.,

$$A_1B_1 + (1 - \lambda)BC > \lambda \cdot CA.$$

Similarly,

$$B_1C_1 + (1 - \lambda)CA > \lambda \cdot AB \quad \text{and} \quad C_1A_1 + (1 - \lambda)AB > \lambda \cdot BC.$$

Adding these inequalities we get $P_1 > (2\lambda - 1)P$.

**9.27.** a) Passing from a nonconvex polygon to its convex hull we replace certain broken lines formed by sides with segments of straight lines (Fig. 100). It remains to take into account that any brokenline is longer than the line segment with the same endpoints.
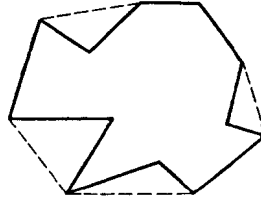


Figure 100 (Sol. 9.27 a))

b) On the sides of the inner polygon construct half bands directed outwards; let the parallel sides of half bands be perpendicular to the corresponding side of the polygon (Fig. 101).

Denote by $P$ the part of the perimeter of the outer polygon corresponding to the boundary of the polygon contained inside these half bands. Then the perimeter of the inner polygon does not exceed $P$ whereas the perimeter of the outer polygon is greater than $P$.

**9.28.** Since $AO + BO > AB$, $BO + OC > BC$ and $CO + OA > AC$, it follows that

$$AO + BO + CO > \frac{AB + BC + CA}{2}.$$

Since triangle $ABC$ contains triangle $ABO$, it follows that $AB + BO + OA < AB + BC + CA$ (cf. Problem 9.27 b)), i.e., $BO + OA < BC + CA$. Similarly,

$$AO + OC < AB + BC \text{ and } CO + OB < CA + AB.$$

Adding these inequalities we get $AO + BO + CO < AB + BC + CA$.
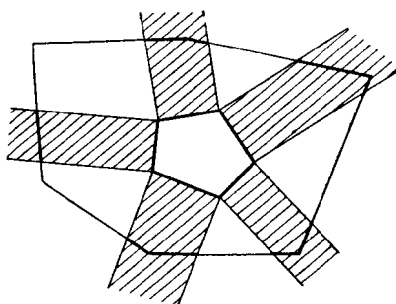
Figure 101 (Sol. 9.27 b))

**9.29.** It suffices to prove that $ABCE$ and $BCDE$ are parallelograms. Let us complement triangle $ABE$ to parallelogram $ABC_1E$. Then perimeters of triangles $BC_1E$ and $ABE$ are equal and, therefore, perimeters of triangles $BC_1E$ and $BCE$ are equal. Hence, $C_1 = C$ because otherwise one of the triangles $BC_1E$ and $BCE$ would have lied inside the other one and their perimeters could not be equal. Hence, $ABCE$ is a parallelogram. We similarly prove that $BCDE$ is a parallelogram.

**9.30.** Clearly, $2 = 2S = ab \sin \gamma \le ab \le b^2$, i.e., $b \ge \sqrt{2}$.

**9.31.** Since $EH$ is the midline of triangle $ABD$, it follows that $S_{AEH} = \frac{1}{4}S_{ABD}$. Similarly, $S_{CFG} = \frac{1}{4}S_{CBD}$. Therefore, $S_{AEH} + S_{CFG} = \frac{1}{4}S_{ABCD}$. Similarly, $S_{BFE} + S_{DGH} = \frac{1}{4}S_{ABCD}$. It follows that

$$S_{ABCD} = 2S_{EFGH} = EG \cdot HF \sin \alpha,$$

where $\alpha$ is the angle between lines $EG$ and $HF$. Since $\sin \alpha \le 1$, then $S_{ABCD} \le EG \cdot HF$.

Adding equalities

$$\overrightarrow{EG} = \overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CG} \ \text{ and } \ \overrightarrow{EG} = \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DG}$$

we obtain

$$2\overrightarrow{EG} = (\overrightarrow{EB} + \overrightarrow{EA}) + (\overrightarrow{BC} + \overrightarrow{AD}) + (\overrightarrow{DG} + \overrightarrow{CG}) = \overrightarrow{BC} + \overrightarrow{AD}.$$

Therefore, $EG \le \frac{1}{2}(BC + AD)$. Similarly, $HF \le \frac{1}{2}(AB + CD)$. It follows that

$$S_{ABCD} \le EG \cdot HF \le \frac{(AB + CD)(BC + AD)}{4}.$$

**9.32.** By Problem 9.31 $S_{ABCD} \le \frac{1}{4}(AB + CD)(BC + AD)$. Since $ab \le \frac{1}{4}(a + b)^2$, it follows that $S_{ABCD} \le \frac{1}{16}(AB + CD + AD + BC)^2 = 1$.

**9.33.** From points $B$ and $C$ drop perpendiculars $BB_1$ and $CC_1$ to line $AM$. Then

$$2S_{AMB} + 2S_{AMC} = AM \cdot BB_1 + AM \cdot CC_1 \le AM \cdot BC$$

because $BB_1 + CC_1 \le BC$. Similarly,

$$2S_{BMC} + 2S_{BMA} \le BM \cdot AC \ \text{ and } \ 2S_{CMA} + 2S_{CMB} \le CM \cdot AB.$$

Adding these inequalities we get the desired statement.

**9.34.** Let on sides $A_1A_2$, $A_2A_3$, ..., $A_nA_1$ points $B_1$, ..., $B_n$, respectively, be selected; let $O$ be the center of the circle. Further, let

$$S_k = S_{OB_kA_{k+1}B_{k+1}} = \frac{OA_{k+1} \cdot B_kB_{k+1} \sin \varphi}{2},$$

where $\varphi$ is the angle between $OA_{k+1}$ and $B_kB_{k+1}$. Since $OA_{k+1} = R$ and $\sin\varphi \leq 1$, it follows that $S_k \leq \frac{1}{2}R \cdot B_kB_{k+1}$. Hence,

$$S = S_1 + \cdots + S_n \leq \frac{R(B_1B_2 + \cdots + B_nB_1)}{2},$$

i.e., the perimeter of polygon $B_1B_2\ldots B_n$ is not less than $\frac{2S}{R}$.

**9.35.** We have $2S_{AOB} \leq AO \cdot OB \leq \frac{1}{2}(AO^2 + BO^2)$, where the equality is only possible if $\angle AOB = 90°$ and $AO = BO$. Similarly,

$$2S_{BOC} \leq \frac{BO^2 + CO^2}{2}, \quad 2S_{COD} \leq \frac{CO^2 + DO^2}{2} \quad \text{and} \quad 2S_{DOA} \leq \frac{DO^2 + AO^2}{2}.$$

Adding these inequalities we get

$$2S = 2(S_{AOB} + S_{BOC} + S_{COD} + S_{DOA}) \leq AO^2 + BO^2 + CO^2 + DO^2,$$

where the equality is only possible if $AO = BO = CO = DO$ and $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90°$, i.e., $ABCD$ is a square and $O$ is its center.

**9.36.** We have to prove that $\frac{S_{ABC}}{S_{AMN}} \geq 4$. Since $AB = AM + MB = AM + AN = AN + NC = AC$, it follows that

$$\frac{S_{ABC}}{S_{AMN}} = \frac{AB \cdot AC}{AM \cdot AN} = \frac{(AM + AN)^2}{AM \cdot AN} \geq 4.$$

**9.37.** Let us apply Heron's formula

$$S^2 = p(p-a)(p-b)(p-c).$$

Since $p - a = (p_1 - a_1) + (p_2 - a_2)$ and $(x+y)^2 \geq 4xy$, it follows that $(p-a)^2 \geq 4(p_1-a_1)(p_2-a_2)$. Similarly,

$$(p-b)^2 \geq 4(p_1 - b_1)(p_2 - b_2), \quad (p-c)^2 \geq 4(p_1 - c_1)(p_2 - c_2) \quad \text{and} \quad p^2 \geq 4p_1p_2.$$

Multiplying these inequalities we get the desired statement.

**9.38.** For definiteness, we may assume that rays $BA$ and $CD$, $BC$ and $AD$ intersect (Fig. 102). If we complement triangle $ADC$ to parallelogram $ADCK$, then point $K$ occurs inside quadrilateral $ABCD$. Therefore,

$$2S \geq 2S_{ABK} + 2S_{BCK} = AB \cdot AK \sin\alpha + BC \cdot CK \sin\beta =$$
$$AB \cdot CD \sin\alpha + BC \cdot AD \sin\beta.$$

The equality is obtained if point $D$ lies on segment $AC$.
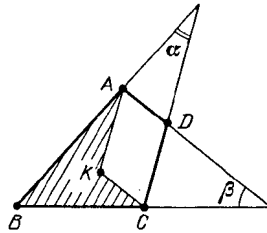


Figure 102 (Sol. 9.38)

Let point $D'$ be symmetric to point $D$ through the midperpendicular to segment $AC$. Then

$$2S = 2S_{ABCD'} = 2S_{ABD'} + 2S_{BCD'} \leq$$
$$AB \cdot AD' + BC \cdot CD' = AB \cdot CD + BC \cdot AD.$$

**9.39.** Thanks to the inequality between the mean geometric and the mean arithmetic, we have $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} \geq 3\sqrt[3]{\frac{abc}{\alpha\beta\gamma}} = \frac{3}{2}$ because $\alpha = 2\sqrt{bc}$, $\beta = 2\sqrt{ca}$ and $\gamma = 2\sqrt{ab}$, cf. Problem 1.33.

**9.40.** The inequalities $\alpha < \alpha_1$, $\beta < \beta_1$ and $\gamma < \gamma_1$ cannot hold simultaneously. Therefore, for instance, $\alpha_1 \leq \alpha \leq 90°$; hence, $\sin \alpha_1 \leq \sin \alpha$. It follows that $2S_1 = a_1 b_1 \sin \alpha_1 \leq k^2 ab \sin \alpha = 2k^2 S$.

**9.41.** a) Let chords $AE$ and $BD$ intersect diameter $CM$ at points $K$ and $L$, respectively. Then $AC^2 = CK \cdot CM$ and $BC^2 = CL \cdot CM$. It follows that $\frac{CK}{CL} = \frac{AC^2}{BC^2} < 4$. Moreover, $\frac{AE}{BD} = \frac{AE}{AC} < 2$. Therefore, $\frac{S_{ACE}}{S_{BCD}} = \frac{AE \cdot CK}{BD \cdot CL} < 8$.

b) Let $H$ be the midpoint of segment $BC$. Since $\angle CBD = \angle BCD = \angle ABD$, it follows that $D$ is the intersection point of the bisectors of triangle $ABC$. Hence, $\frac{AD}{DH} = \frac{AB}{BH} > 1$. Therefore, $S_{MAN} > \frac{1}{4}S_{ABC}$ and

$$S_{BCD} = \frac{BC \cdot DH}{2} < \frac{BC \cdot AH}{4} = \frac{S_{ABC}}{4}.$$

**9.42.** Let us cut off the obtained polygon rectangles with side $h$ constructed outwards on the sides of the initial polygon (Fig. 103). Then beside the initial polygon there will be left several quadrilaterals from which one can compose a polygon circumscribed about a circle of radius $h$. The sum of the areas of these quadrilaterals is greater than the area of the circle of radius $h$, i.e., greater than $\pi h^2$. It is also clear that the sum of areas of the cut off rectangles is equal to $Ph$.
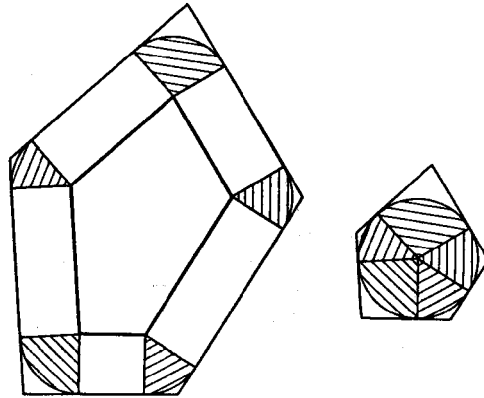


Figure 103 (Sol. 9.42)

**9.43.** Let $s, s_1, \ldots, s_n$ be the areas of the square and the rectangles that constitute it, respectively; $S, S_1, \ldots, S_n$ the areas of the disks circumscribed about the square and the rectangles, respectively. Let us prove that $s_k \leq \frac{2S_k}{\pi}$. Indeed, if the sides of the rectangle are equal to $a$ and $b$, then $s_k = ab$ and $S_k = \pi R^2$, where $R^2 = \frac{a^2}{4} + \frac{b^2}{4}$. Therefore, $s_k = ab \leq \frac{a^2 + b^2}{2} = \frac{2\pi R^2}{\pi} = \frac{2S_k}{\pi}$. It follows that

$$\frac{2S}{\pi} = s = s_1 + \cdots + s_n \leq \frac{2(S_1 + \cdots + S_n)}{\pi}.$$

**9.44.** Let, for definiteness, $ABC$ be the triangle of the least area. Denote the intersection point of diagonals $AD$ and $EC$ by $F$. Then $S_{ABCDE} < S_{AED} + S_{EDC} + S_{ABCF}$. Since point $F$ lies on segment $EC$ and $S_{EAB} \geq S_{CAB}$, it follows that $S_{EAB} \geq S_{FAB}$. Similarly, $S_{DCB} \geq S_{FCB}$. Therefore, $S_{ABCF} = S_{FAB} + S_{FCB} \leq S_{EAB} + S_{DCB}$. It follows that $S_{ABCDE} < S_{AED} + S_{EDC} + S_{EAB} + S_{DCB}$ and this is even a stronger inequality than the one required.

**9.45.** a) Denote the intersection points of diagonals $AD$ and $CF$, $CF$ and $BE$, $BE$ and $AD$ by $P$, $Q$, $R$, respectively (Fig. 104). Quadrilaterals $ABCP$ and $CDEQ$ have no common inner points since sides $CP$ and $QC$ lie on line $CF$ and segments $AB$ and $DE$ lie on distinct sides of it. Similarly, quadrilaterals $ABCP$, $CDEQ$ and $EFAR$ have no pairwise common inner points. Therefore, the sum of their areas does not exceed $S$.
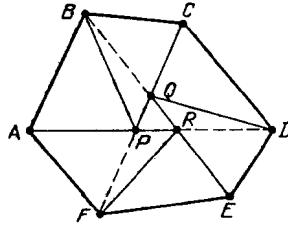


Figure 104 (Sol. 9.45 a))

It follows that the sum of the areas of triangles $ABP$, $BCP$, $CDQ$, $DEQ$, $EFR$, $FAR$ does not exceed $S$, i.e., the area of one of them, say $ABP$, does not exceed $\frac{1}{6}S$. Point $P$ lies on segment $CF$ and, therefore, one of the points, $C$ or $F$, is distant from line $AB$ not further than point $P$. Therefore, either $S_{ABC} \leq S_{ABP} \leq \frac{1}{6}S$ or $S_{ABF} \leq S_{ABP} \leq \frac{1}{6}S$.

b) Let $ABCDEFGH$ be a convex octagon. First, let us prove that quadrilaterals $ABEF$, $BCFG$, $CDGH$ and $DEHA$ have a common point. Clearly, a convex quadrilateral $KLMN$ (Fig. 105) is the intersection of $ABEF$ and $CDGH$. Segments $AF$ and $HC$ lie inside angles $\angle DAH$ and $\angle AHE$, respectively; hence, point $K$ lies inside quadrilateral $DEHA$. We similarly prove that point $M$ lies inside quadrilateral $DEHA$, i.e., the whole segment $KM$ lies inside it. Similarly, segment $LN$ lies inside quadrilateral $BCFG$. The intersection point of diagonals $KM$ and $LN$ belongs to all our quadrilaterals; denote it by $O$.
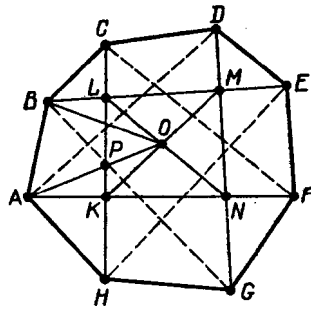


Figure 105 (Sol. 9.45 b))

Let us divide the 8-gon into triangles by connecting point $O$ with the vertices. The area of one of these triangles, say $ABO$, does not exceed $\frac{1}{8}S$. Segment $AO$ intersects side $KL$ at a point $P$, therefore, $S_{ABP} < S_{ABO} \leq \frac{1}{8}S$. Since point $P$ lies on diagonal $CH$, it follows that either $S_{ABC} \leq S_{ABP} \leq \frac{1}{8}S$ or $S_{ABH} \leq S_{ABP} \leq \frac{1}{8}S$.

**9.46.** Let us draw through all the vertices of the polygon lines parallel to one pair of sides of the square thus dividing the square into strips. Each such strip cuts off the polygon

either a trapezoid or a triangle. It suffices to prove that the length of one of the bases of these trapezoids is greater than 0.5. Suppose that the length of each base of all the trapezoids does not exceed 0.5. Then the area of each trapezoid does not exceed a half height of the strip that confines it. Therefore, the area of the polygon, equal to the sum of areas of trapezoids and triangles into which it is cut, does not exceed a half sum of heights of the strips, i.e., does not exceed 0.5. Contradiction.

**9.47.** a) Let $P_1$, ..., $P_n$ be the given points. Let us connect point $P_1$ with the vertices of the square. We will thus get four triangles. Next, for $k = 2, \ldots, n$ let us perform the following operation. If point $P_k$ lies strictly inside one of the triangles obtained earlier, then connect it with the vertices of this triangle.

If point $P_k$ lies on the common side of two triangles, then connect it with the vertices of these triangles opposite to the common side. Each such operation increases the total number of triangles by 2. As a result we get $2(n + 1)$ triangles. The sum of the areas of these triangles is equal to 1, therefore, the area of any of them does not exceed $\frac{1}{2(n+1)}$.

b) Let us consider the least convex polygon that contains the given points. Let is have $k$ vertices. If $k = n$ then this $k$-gon can be divided into $n - 2$ triangles by the diagonals that go out of one of its vertices. If $k < n$, then inside the $k$-gon there are $n - k$ points and it can be divided into triangles by the method indicated in heading a). We will thus get $k + 2(n - k - 1) = 2n - k - 2$ triangles. Since $k < n$, it follows that $2n - k - 2 > n - 2$.

The sum of the areas of the triangles of the partition is less than 1 and there are not less than $n - 2$ of them; therefore, the area of at least one of them does not exceed $\frac{1}{n-2}$.

**9.48.** a) We may assume that the circumscribed $n$-gon $A_1 \ldots A_n$ and the inscribed $n$-gon $B_1 \ldots B_n$ are placed so that lines $A_i B_i$ intersect at the center $O$ of the given circle. Let $C_i$ and $D_i$ be the midpoints of sides $A_i A_{i+1}$ and $B_i B_{i+1}$, respectively. Then

$$S_{OB_i C_i} = p \cdot OB_i \cdot OC_i, \quad S_{OB_i D_i} = p \cdot OB_i \cdot OD_i \text{ and } S_{OA_i C_i} = p \cdot OA_i \cdot OC_i,$$

where $p = \frac{1}{2} \sin \angle A_i OC_i$. Since $OA_i : OC_i = OB_i : OD_i$, it follows that $S^2_{OB_i C_i} = S_{OB_i D_i} S_{OA_i C_i}$. It remains to notice that the area of the part of the disk confined inside angle $\angle A_i OC_i$ is greater than $S_{OB_i C_i}$ and the areas of the parts of the inscribed and circumscribed $n$-gons confined inside this angle are equal to $S_{OB_i D_i}$ and $S_{OA_i C_i}$, respectively.

b) Let the radius of the circle be equal to $R$. Then $P_1 = 2nR \sin \frac{\pi}{n}$, $P_2 = 2nR \tan \frac{\pi}{n}$ and $L = 2\pi R$. We have to prove that $\sin x \tan x > x^2$ for $0 < x \leq \frac{1}{3}\pi$. Since

$$\left( \frac{\sin x}{x} \right)^2 \geq \left( 1 - \frac{x^2}{6} \right)^2 = 1 - \frac{x^2}{3} + \frac{x^4}{36}$$

and $0 < \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$ (see Supplement to this chapter), it remains to verify that $1 - \frac{x^2}{3} + \frac{x^4}{36} \geq 1 - \frac{x^2}{2} + \frac{x^4}{24}$, i.e., $12x^2 > x^4$. For $x \leq \frac{1}{3}\pi$ this inequality is satisfied.

**9.49.** Let $O$ be the center of homothety that sends the inscribed circle into the circumscribed one. Let us divide the plane by rays that exit from point $O$ and pass through the vertices of the polygon and the tangent points of its sides with the inscribed circle (Fig. 106).

It suffices to prove the required inequality for the parts of disks and the polygon confined inside each of the angles formed by these rays. Let the legs of the angle intersect the inscribed circle at points $P$, $Q$ and the circumscribed circle at points $R$, $S$ so that $P$ is the tangency point and $S$ is a vertex of the polygon. The areas of the parts of disks are greater than the areas of triangles $OPQ$ and $ORS$ and, therefore, it suffices to prove that $2S_{OPS} \leq S_{OPQ} + S_{ORS}$. Since $2S_{OPS} = 2S_{OPQ} + 2S_{PQS}$ and $S_{ORS} = S_{OPQ} + S_{PQS} + S_{PRS}$,
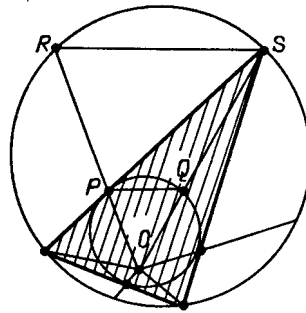
Figure 106 (Sol. 9.49)

it remains to prove that $S_{PQS} \leq S_{PRS}$. This inequality is obvious, because the heights of triangles $PQS$ and $PRS$ dropped to bases $PQ$ and $RS$, respectively, are equal and $PQ < RS$.

**9.50.** It suffices to prove that both triangles contain the center $O$ of the disk. Let us prove that if triangle $ABC$ placed in the disk of radius 1 does not contain the center of the disk, then its area is less than 1. Indeed, for any point outside the triangle there exists a line that passes through two vertices and separating this point from the third vertex. Let, for definiteness, line $AB$ separate points $C$ and $O$. Then $h_c < 1$ and $AB < 2$, hence, $S = \frac{1}{2}h_c \cdot AB < 1$.

**9.51.** a) On the sides of the polygon, construct inwards rectangles whose other side is equal to $R = \frac{S}{P}$. The rectangles will not cover the whole polygon (these rectangles overlap and can stick out beyond the limits of the polygon whereas the sum of their areas is equal to the area of the polygon). An uncovered point is distant from every side of the polygon further than by $R$, consequently, the disk of radius $R$ centered at this point entirely lies inside the polygon.

b) Heading a) implies that in the inner polygon a disk of radius $\frac{S_2}{P_2}$ can be placed. Clearly, this disk lies inside the outer polygon. It remains to prove that if inside a polygon a disk of radius $R$ lies, then $R \leq \frac{2S}{P}$. For this let us connect (with lines) the center $O$ of the disk with the vertices of the polygon. These lines split the polygon into triangles whose respective areas are equal to $\frac{1}{2}h_i a_i$, where $h_i$ is the distance from point $O$ to the $i$-th side and $a_i$ is the length of the $i$-th side. Since $h_i \geq R$, we deduce that $2S = \sum h_i a_i \geq \sum R a_i = RP$.

**9.52.** First, let us consider the case when two sides of a parallelogram lie on lines $AB$ and $AC$ and the fourth vertex $X$ lies on side $BC$. If $BX : CX = x : (1 - x)$, then the ratio of the area of the parallelogram to the area of the triangle is equal to $2x(1 - x) \leq \frac{1}{2}$.
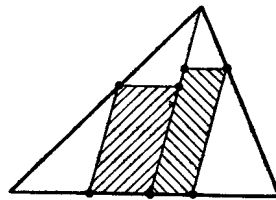


Figure 107 (Sol. 9.52)

In the general case let us draw parallel lines that contain a pair of sides of the given parallelogram (Fig. 107). The area of the given parallelogram does not exceed the sum of areas of the shaded parallelograms which fall in the case considered above. If lines that contain a pair of sides of the given parallelogram only intersect two sides of the triangle, then we can restrict ourselves to one shaded parallelogram only.

**9.53.** First, let us consider the following case: two vertices $A$ and $B$ of triangle $ABC$ lie on one side $PQ$ of the parallelogram. Then $AB \leq PQ$ and the height dropped to side $AB$ is not longer than the height of the parallelogram. Therefore, the area of triangle $ABC$ does not exceed a half area of the parallelogram.
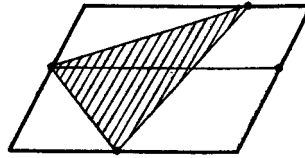


Figure 108 (Sol. 9.53)

If the vertices of the triangle lie on distinct sides of the parallelogram, then two of them lie on opposite sides. Let us draw through the third vertex of the triangle a line parallel to these sides (Fig. 108). This line cuts the parallelogram into two parallelograms and it cuts the triangle into two triangles so that two vertices of each of these triangles lie on sides of the parallelogram. We get the case already considered.

**9.54.** Let $M$ be the midpoint of the longest side $BC$ of the given acute triangle $ABC$. The circle of radius $MA$ centered at $M$ intersects rays $MB$ and $MC$ at points $B_1$ and $C_1$, respectively. Since $\angle BAC < 90°$, it follows that $MB < MB_1$. Let, for definiteness, $\angle AMB \leq \angle AMC$, i.e., $\angle AMB \leq 90°$. Then $AM^2 + MB^2 \leq AB^2 \leq BC^2 = 4MB^2$, i.e., $AM \leq \sqrt{3}BM$. If $AH$ is a height of triangle $ABC$, then $AH \cdot BC = 2$ and, therefore,

$$S_{AB_1C_1} = \frac{B_1C_1 \cdot AH}{2} = AM \cdot AH \leq \sqrt{3}BM \cdot AH = \sqrt{3}.$$

**9.55.** a) Let $AB$ be the longest of the diagonals and sides of the given polygon $M$. Polygon $M$ is confined inside the strip formed by the perpendiculars to segment $AB$ passing through points $A$ and $B$. Let us draw two baselines to $M$ parallel to $AB$. Let them intersect polygon $M$ at points $C$ and $D$. As a result we have confined $M$ into a rectangle whose area is equal to $2S_{ABC} + 2S_{ABD} \leq 2S$.
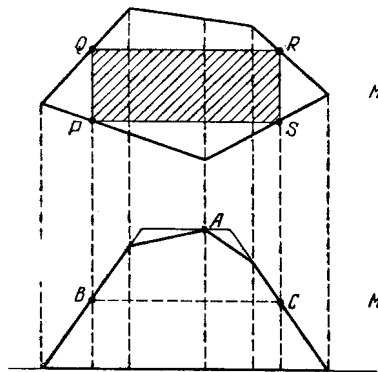


Figure 109 (Sol. 9.55)

b) Let $M$ be the initial polygon, $l$ an arbitrary line. Let us consider the polygon $M_1$ one of whose sides is the projection of $M$ to $l$ and the lengths of the sections of polygons $M$ and $M_1$ by any line perpendicular to $l$ are equal (Fig. 109). It is easy to verify that $M_1$ is also a convex polygon and its area is equal to $S$. Let $A$ be the most distant from $l$ point of $M_1$. The line equidistant from point $A$ and line $l$ intersects the sides of $M_1$ at points $B$ and $C$.

Let us draw base lines through points $B$ and $C$. As a result we will circumscribe a trapezoid about $M_1$ (through point $A$ a base line can also be drawn); the area of this trapezoid is no less than $S$. If the height of the trapezoid, i.e., the distance from $A$ to $l$ is equal to $h$ then its area is equal to $h \cdot BC$ and, therefore, $h \cdot BC \geq S$. Let us consider sections $PQ$ and $RS$ of polygon $M$ by lines perpendicular to $l$ and passing through $B$ and $C$. The lengths of these sections are equal to $\frac{1}{2}h$ and, therefore, $PQRS$ is a parallelogram whose area is equal to $\frac{1}{2}BC \cdot h \geq \frac{1}{2}S$.

**9.56.** a) Let us confine the polygon in the strip formed by parallel lines. Let us shift these lines parallelly until some vertices $A$ and $B$ of the polygon lie on them. Then let us perform the same for the strip formed by lines parallel to $AB$. Let the vertices that lie on these new lines be $C$ and $D$ (Fig. 110). The initial polygon is confined in a parallelogram and, therefore, the area of this parallelogram is not less than 1. On the other hand, the sum of areas of triangles $ABC$ and $ADB$ is equal to a half area of the parallelogram and, therefore, the area of one of these triangles is not less than $\frac{1}{4}$.
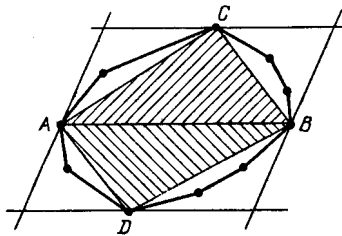


Figure 110 (Sol. 9.56 a))

b) As in heading a) let us confine the polygon in a strip formed by parallel lines so that some vertices, $A$ and $B$, lie on these lines. Let $d$ be the width of this strip. Let us draw three lines that divide this strip into equal strips of width $\frac{1}{4}d$. Let the first and the third lines intersect sides of the polygon at points $K$, $L$ and $M$, $N$, respectively (Fig. 111).
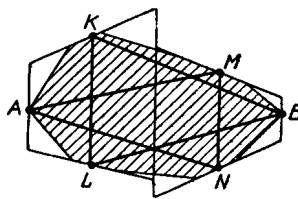


Figure 111 (Sol. 9.56 b))

Let us extend the sides on which points $K$, $L$, $M$ and $N$ lie to the intersection with the sides of the initial strip and with the line that divides it in halves. In this way we form two trapezoids with the midlines $KL$ and $MN$ and heights of length $\frac{1}{2}d$ each.

Since these trapezoids cover the whole polygon, the sum of their areas is not less than its area, i.e., $\frac{1}{2}(d \cdot KL + d \cdot MN) \geq 1$. The sum of areas of triangles $AMN$ and $BKL$ contained in the initial polygon is equal to $\frac{1}{8}(3d \cdot MN + 3d \cdot KL) \geq \frac{3}{4}$. Therefore, the area of one of these triangles is not less than $\frac{3}{8}$.

**9.57.** Let us prove that there exists even three last vertices satisfying the required condition. Let $\alpha_i$ be the angle between the $i$-th and $(i+1)$-th sides $\beta_i = \pi - \alpha_i$; let $a_i$ be the length of the $i$-th side.

a) The area of the triangle formed by the $i$-th and $(i+1)$-th sides is equal to $S_i = \frac{a_i a_{i+1} \sin \alpha_i}{2}$. Let $S$ be the least of these areas. Then $2S \leq a_i a_{i+1} \sin \alpha_i$; hence,

$$(2S)^n \leq (a_1^2 \ldots a_n^2)(\sin \alpha_1 \ldots \sin \alpha_n) \leq a_1^2 \ldots a_n^2.$$

By the inequality between the mean arithmetic and the mean geometric we have $(a_1 \ldots a_n)^{\frac{1}{n}} \leq \frac{a_1 + \cdots + a_n}{n}$ and, therefore,

$$2S \leq (a_1 \ldots a_n)^{\frac{2}{n}} \leq \frac{(a_1 + \cdots + a_n)^2}{n^2}.$$

Since $a_i \leq p_i + q_i$, where $p_i$ and $q_i$ are projections of the $i$-th side to a vertical and a horizontal sides of the square, it follows that

$$a_1 + \cdots + a_n \leq (p_1 + \cdots + p_n) + (q_1 + \cdots + q_n) \leq 4.$$

Hence, $2S \leq 16n^2$, i.e., $S \leq \frac{8}{n^2}$.

b) Let us make use of the inequality

$$2S \leq (a_1 \ldots a_n)^{\frac{2}{n}}(\sin \alpha_1 \ldots \sin \alpha_n)^{\frac{1}{n}} \leq \frac{16}{n^2}(\sin \alpha_1 \ldots \sin \alpha_n)$$

proved above. Since $\sin \alpha_i = \sin \beta_i$ and $\beta_1 + \cdots + \beta_n = 2\pi$, it follows that

$$(\sin \alpha_1 \ldots \sin \alpha_n)^{\frac{1}{n}} = (\sin \beta_1 \ldots \sin \beta_n)^{\frac{1}{n}} \leq \frac{\beta_1 + \cdots + \beta_n}{n} = \frac{2\pi}{n}.$$

Hence, $2S \leq \frac{32\pi}{n^3}$, i.e., $S \leq \frac{16\pi}{n^3}$.

**9.58.** Let $l_i$ be the length of the $i$-th link of the broken line; $a_i$ and $b_i$ the lengths of its projections to the sides of the square. Then $l_i \leq a_i + b_i$. It follows that

$$1000 = l_1 + \cdots + l_n \leq (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n),$$

i.e., either $a_1 + \cdots + a_n \geq 500$ or $b_1 + \cdots + b_n \geq 500$. If the sum of the lengths of the links' projections on a side of length 1 is not less than 500, then not fewer than 500 distinct lengths of the broken line are projected into one of the points of this side, i.e., the perpendicular to the side that passes through this point intersects the broken line at least at 500 points.

**9.59.** The locus of points distant from the given segment not further than by $\varepsilon$ is depicted on Fig. 112. The area of this figure is equal to $\pi\varepsilon^2 + 2\varepsilon l$, where $l$ is the length of the segment.
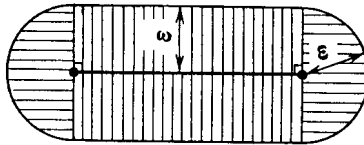


Figure 112 (Sol. 9.59)

Let us construct such figures for all $N$ links of the given broken lines. Since neighbouring figures have $N - 1$ common disks of radius $\varepsilon$ centered at vertices of the broken line which are not its endpoints, it follows that the area covered by these figures does not exceed

$$N\pi\varepsilon^2 + 2\varepsilon(l_1 + \cdots + l_n) - (N-1)\pi\varepsilon^2 = 2\varepsilon L + \pi\varepsilon^2.$$

This figure covers the whole square since any point of the square is distant from a point of the broken line by less than $\varepsilon$. Hence, $1 \leq 2\varepsilon L + \pi\varepsilon^2$, i.e., $L \geq \frac{1}{2\varepsilon} - \frac{\pi\varepsilon}{2}$.

**9.60.** Let us divide the square into $n$ vertical strips that contain $n$ points each. Inside each strip let us connect points downwards thus getting $n$ broken lines. These broken lines can be connected into one broken line in two ways: Fig. 113 a) and b).
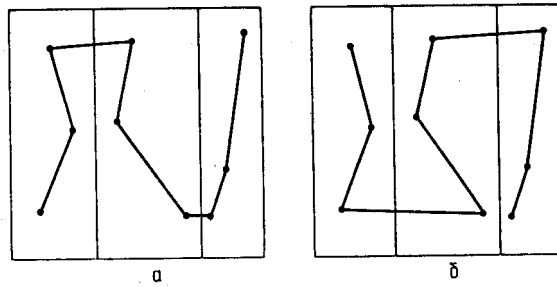
Figure 113 (Sol. 9.60)

Let us consider the segments that connect distinct bands. The union of all such segments obtained in both ways is a pair of broken lines such that the sum of the lengths of the horizontal projections of each of them does not exceed 1. Therefore, the sum of the lengths of horizontal projections of the connecting segments for one of these ways does not exceed 1.

Let us consider such a connection. The sum of the lengths of the horizontal projections for connecting links does not exceed 1 and for all the other links it does not exceed $(n - 1)(h_1 + \cdots + h_n)$, where $h_i$ is the width of the $i$-th strip. Clearly, $h_1 + \cdots + h_n = 1$. The sum of the vertical projections of all links of the broken line does not exceed $n$. As a result we deduce that the sum of the vertical and horizontal projections of all the links does not exceed $1 + (n - 1) + n = 2n$ and, therefore, the length of the broken line does not exceed $2n$.

**9.61.** Let $M$ and $N$ be the endpoints of the broken line. Let us traverse along the broken line from $M$ to $N$. Let $A_1$ be the first of points of the broken line that we meet whose distance from a vertex of the square is equal to 0.5. Let us consider the vertices of the square neighboring to this vertex. Let $B_1$ be the first after $A_1$ point of the broken line distant from one of these vertices by 0.5. Denote the vertices of the square nearest to points $A_1$ and $B_1$ by $A$ and $B$, respectively (Fig. 114).
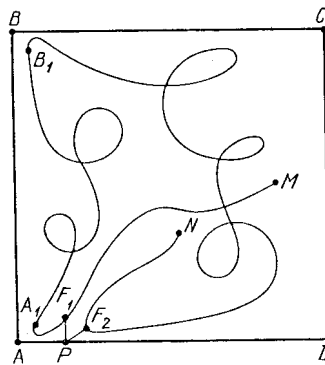


Figure 114 (Sol. 9.61)

Denote the part of the broken line from $M$ to $A_1$ by $L_1$ and the part from $A_1$ to $N$ by $L_2$. Let $X$ and $Y$ be the sets of points that lie on $AD$ and distant not further than by 0.5 from $L_1$ and $L_2$, respectively. By hypothesis, $X$ and $Y$ cover the whole side $AD$. Clearly, $A \in X$ and $D \notin X$; hence, $D \in Y$, i.e., both sets, $X$ and $Y$, are nonempty. But each of these sets consists of several segments and, therefore, they should have a common point $P$. Therefore, on $L_1$ and $L_2$, there are points $F_1$ and $F_2$ for which $PF_1 \le 0.5$ and $PF_2 \le 0.5$.

Let us prove that $F_1$ and $F_2$ are the points to be found. Indeed, $F_1F_2 \le F_1P + PF_2 \le 1$. On the other hand, while traversing from $F_1$ to $F_2$ we should pass through point $B$; and we

have $F_1B_1 \geq 99$ and $F_2B_1 \geq 99$ because point $B_1$ is distant from side $BC$ no further than by 0.5 while $F_1$ and $F_2$ are distant from side $AD$ not further than by 0.5.

**9.62.** Let $\angle A = \angle B$. It suffices to prove that if $AD < BC$; then $\angle D > \angle C$. On side $BC$, take point $D_1$ such that $BD_1 = AD$. Then $ABD_1D$ is an isosceles trapezoid. Hence, $\angle D > \angle D_1DA = \angle DD_1B \geq \angle C$.

**9.63.** Let $B_1$ and $C_1$ be the projections of points $B$ and $C$ on base $AD$. Since $\angle BAB_1 < \angle CDC_1$ and $BB_1 = CC_1$, it follows that $AB_1 > DC_1$ and, therefore, $B_1D < AC_1$. It follows that

$$BD^2 = B_1D^2 + B_1B^2 < AC_1^2 + CC_1^2 = AC^2.$$

**9.64.** Let angles $\angle B$ and $\angle D$ of quadrilateral $ABCD$ be obtuse ones. Then points $B$ and $D$ lie inside the circle with diameter $AC$. Since the distance between any two points that lie inside the circle is less than its diameter, $BD < AC$.

**9.65.** In an isosceles trapezoid $ABCD$ diagonals $AC$ and $BD$ are equal. Therefore,

$$BM + (AM + CM) \geq BM + AC = BM + BD \geq DM.$$

**9.66.** Let $O$ be the midpoint of segment $BD$. Point $A$ lies inside the circle with diameter $BD$, hence, $OA < \frac{1}{2}BD$. Moreover, $FO = \frac{1}{2}CD$. Therefore, $2FA \leq 2FO + 2OA < CD + BD$.

**9.67.** On rays $AB$, $AC$ and $AD$ mark segments $AB'$, $AC'$ and $AD'$ of length $\frac{1}{AB}$, $\frac{1}{AC}$ and $\frac{1}{AD}$. Then $AB : AC = AC' : AB'$, i.e., $\triangle ABC \sim \triangle AC'B'$. The similarity coefficient of these triangles is equal to $\frac{1}{AB \cdot AC}$ and therefore, $B'C' = \frac{BC}{AB \cdot AC}$. Analogously, $C'D' = \frac{CD}{AC \cdot AD}$ and $B'D' = \frac{BD}{AB \cdot AD}$. Substituting these expressions in the inequality $B'D' \leq B'C' + C'D'$ and multiplying both sides by $AB \cdot AC \cdot AD$, we get the desired statement.

**9.68.** Clearly,

$$S_{ABCD} = S_{ABC} + S_{ACD} = 2S_{AMC} + 2S_{ANC} = 2(S_{AMN} + S_{CMN}).$$

If segment $AM$ intersects diagonal $BD$ at point $A_1$, then $S_{CMN} = S_{A_1MN} < S_{AMN}$. Therefore, $S_{ABCD} < 4S_{AMN}$.

**9.69.** Diagonals $AC$ and $BD$ intersect at point $O$. Let, for definiteness, point $P$ lie in side of $AOB$. Then $AP + BP \leq AO + BO < AC + BD$ (cf. the solution of Problem 9.28) and $CP + DP < CB + BA + AD$.

**9.70.** Let $r_i$, $S_i$ and $p_i$ be the radii of the inscribed circles, the areas and semiperimeters of the obtained triangles, respectively. Then

$$Q \geq 2\sum r_i = 2\sum \left(\frac{S_i}{p_i}\right) > 4\sum \left(\frac{S_i}{P}\right) = \frac{4S}{P}.$$

**9.71.** Let $AC \leq BD$. Let us drop from vertices $A$ and $C$ perpendiculars $AA_1$ and $CC_1$ to diagonal $BD$. Then $AA_1 + CC_1 \leq AC \leq BD$ and, therefore, either $AA_1 \leq \frac{1}{2}BD$ or $CC_1 \leq \frac{1}{2}BD$.

**9.72.** Let us draw through the endpoints of segment $KL$ lines perpendicular to it and consider projections to these lines of the vertices of the quadrilateral. Consider also the intersection points of lines $AC$ and $BD$ with these lines, cf. Fig. 115.

Let, for definiteness, point $A$ lie inside the strip determined by these lines and point $B$ outside it. Then we may assume that $D$ lies inside the strip, because otherwise $BD > KL$ and the proof is completed. Since

$$\frac{AA'}{BB'} \leq \frac{A_1K}{B_1K} = \frac{C_1L}{D_1L} \leq \frac{CC'}{DD'},$$
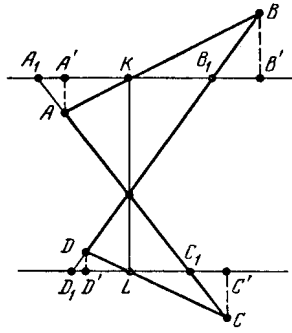
Figure 115 (Sol. 9.72)

then either $AA' \leq CC'$ (and, therefore, $AC > KL$) or $BB' \geq DD'$ (and, therefore, $BD > KL$).

**9.73.** Let us introduce the notations as plotted on Fig. 116. All the parallelograms considered have a common center (thanks to Problem 1.7). The lengths of the sides of parallelogram $P_3$ are equal to $a + a_1$ and $b + b_1$ and the lengths of the sides of parallelogram $P_1$ are equal to $a + a_1 + 2x$ and $b + b_1 + 2y$, consequently, we have to verify that either $a + a_1 + 2x \leq 2(a + a_1)$ or $b + b_1 + 2y \leq 2(b + b_1)$, i.e., either $2x \leq a + a_1$ or $2y \leq b + b_1$.
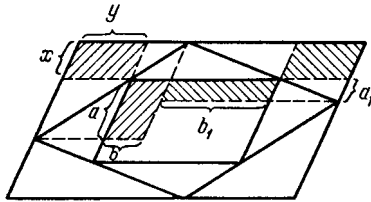


Figure 116 (Sol. 9.73)

Suppose that $a + a_1 < 2x$ and $b + b_1 < 2y$. Then $\sqrt{aa_1} \leq \frac{1}{2}(a + a_1) < x$ and $\sqrt{bb_1} < y$. On the other hand, the equality of the areas of shaded parallelograms (cf. Problem 4.19) shows that $ab = xy = a_1b$ and, therefore, $\sqrt{aa_1}\sqrt{bb_1} = xy$. Contradiction.

**9.74.** Let the angles of the pentagon be equal to $\alpha$, $\alpha + \gamma$, $\alpha + 2\gamma$, $\alpha + 3\gamma$, $\alpha + 4\gamma$, where $\alpha, \gamma \geq 0$. Since the sum of the angles of the pentagon is equal to $3\pi$, it follows that $5\alpha + 10\gamma = 3\pi$. Since the pentagon is a convex one, each of its angles is less than $\pi$, i.e., either $\alpha + 4\gamma < \pi$ or $-5\frac{1}{2}\alpha - 10\gamma > -\frac{1}{2}5\pi$. Taking the sum of the latter inequality with $5\alpha + 10\gamma = 3\pi$ we get $\frac{5\alpha}{2} > \frac{\pi}{2}$, i.e., $\alpha > \frac{\pi}{5} = 36°$.

**9.75.** Clearly,

$$4 = AE^2 = |\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}|^2 =$$
$$|\overrightarrow{AB} + \overrightarrow{BC}|^2 + 2(\overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{CD} + \overrightarrow{DE}) + |\overrightarrow{CD} + \overrightarrow{DE}|^2.$$

Since $\angle ACE = 90°$, we have

$$(\overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{CD} + \overrightarrow{DE}) = (\overrightarrow{AC}, \overrightarrow{CE}) = 0.$$

Hence,

$$4 = |\overrightarrow{AB} + \overrightarrow{BC}|^2 + |\overrightarrow{CD} + \overrightarrow{DE}|^2 =$$
$$AB^2 + BC^2 + CD^2 + DE^2 + 2(\overrightarrow{AB}, \overrightarrow{BC}) + 2(\overrightarrow{CD}, \overrightarrow{DE}),$$

i.e., it suffices to prove that

$$abc < 2(\overrightarrow{AB}, \overrightarrow{BC}) \text{ and } bcd < 2(\overrightarrow{CD}, \overrightarrow{DE}).$$

Since

$$2(\overrightarrow{AB}, \overrightarrow{BC}) = 2ab\cos(180° - \angle ABC) = 2ab\cos AEC = ab \cdot CE \text{ and } c < CE,$$

it follows that $abc < 2(\overrightarrow{AB}, \overrightarrow{BC})$.

The second inequality is similarly proved, because in notations $A_1 = E$, $B_1 = D$, $C_1 = C$, $a_1 = d$, $b_1 = c$, $c_1 = b$ the inequality $bcd < 2(\overrightarrow{CD}, \overrightarrow{DE})$ takes the form $a_1b_1c_1 < 2(\overrightarrow{A_1B_1}, \overrightarrow{B_1C_1})$.

**9.76.** Let $B$ be the midpoint of side $A_1A_2$ of the given hexagon $A_1 \dots A_6$ and $O$ its center. We may assume that point $P$ lies inside triangle $A_1OB$. Then $PA_3 \geq 1$ because the distance from point $A_3$ to line $BO$ is equal to 1; since the distances from points $A_4$ and $A_5$ to line $A_3A_6$ are equal to 1, we deduce that $PA_4 \geq 1$ and $PA_5 \geq 1$.

**9.77.** Suppose that the radii of the circumscribed circles of triangles $ACE$ and $BDF$ are greater than 1. Let $O$ be the center of the circumscribed circle of triangle $ACE$. Then $\angle ABC > \angle AOC$, $\angle CDE > \angle COE$ and $\angle EFA > \angle EOA$ and, therefore, $\angle B + \angle D + \angle F > 2\pi$. Similarly, $\angle A + \angle C + \angle E > 2\pi$, i.e., the sum of the angles of hexagon $ABCDEF$ is greater than $4\pi$. Contradiction.

REMARK. We can similarly prove that the radius of the circumscribed circle of one of triangles $ACE$ and $BDF$ is not less than 1.

**9.78.** We may assume that $AE \leq AC \leq CE$. By Problem 9.67

$$AD \cdot CE \leq AE \cdot CD + AC \cdot DE < AE + AC \leq 2CE,$$

i.e., $AD < 2$.

**9.79.** Since $\angle A_1 = 180° - \frac{1}{2} \smile A_2A_7$, $\angle A_3 = 180° - \frac{1}{2} \smile A_4A_2$ and $\angle A_5 = 180° - \frac{1}{2} \smile A_6A_4$, it follows that

$$\angle A_1 + \angle A_3 + \angle A_5 = 2 \cdot 180° + \frac{360° - \smile A_2A_7 - \smile A_4A_2 - \smile A_6A_4}{2} = $$
$$2 \cdot 180° + \frac{\smile A_7A_6}{2}.$$

Since the center of the circle lies inside the hexagon, it follows that $\smile A_7A_6 < 180°$ and, therefore, $\angle A_1 + \angle A_3 + \angle A_5 < 360° + 90° = 450°$.

**9.80.** a) We have to prove that if $c$ is the hypothenuse of the right triangle and $a$ and $b$ are its legs, then $c \geq \frac{a+b}{\sqrt{2}}$, i.e., $(a + b)^2 \leq 2(a^2 + b^2)$. Clearly,

$$(a + b)^2 = (a^2 + b^2) + 2ab \leq (a^2 + b^2) + (a^2 + b^2) = 2(a^2 + b^2).$$

b) Let $d_i$ be the length of the $i$-th side of the polygon; $x_i$ and $y_i$ the lengths of its projections to coordinate axes. Then $x_1 + \cdots + x_n \geq 2a$, $y_1 + \cdots + y_n \geq 2b$. By heading a) $d_i \geq \frac{x_i+y_i}{\sqrt{2}}$. Therefore,

$$d_1 + \cdots + d_n \geq \frac{x_1 + \cdots + x_n + y_1 + \cdots + y_n}{\sqrt{2}} \geq \sqrt{2}(a + b).$$

**9.81.** Let us take a segment of length $P$ and place the sides of the polygon on the segment as follows: on one end of the segment place the greatest side, on the other end place the second long side; place all the other sides between them. Since any side of the polygon is shorter than $\frac{1}{2}P$, the midpoint $O$ of the segment cannot lie on these two longest sides. The length of the side on which point $O$ lies, does not exceed $\frac{1}{3}P$ (otherwise the first two sides would also have been longer than $\frac{1}{3}P$ and the sum of the three sides would have been greater than $P$) and, therefore, one of its vertices is distant from $O$ not further than by $\frac{1}{6}P$. This vertex divides the segment into two segments to be found since the difference of their lengths does not exceed $\frac{2}{6}P = \frac{1}{3}P$.

**9.82.** Let $\beta_k = \angle OA_kA_{k+1}$. Then $x_k \sin \beta_k = d_k = x_{k+1} \sin(\alpha_{k+1} - \beta_{k+1})$. Hence,

$$2 \sum d_k = \sum x_k(\sin(\alpha_k - \beta_k) + \sin \beta_k) =$$
$$2 \sum x_k \sin \tfrac{\alpha_k}{2} \cos \left( \tfrac{\alpha_k}{2} - \beta_k \right) \le 2 \sum x_k \sin \tfrac{\alpha_k}{2}.$$

It is also clear that

$$A_kA_{k+1} = x_k \cos \beta_k + x_{k+1} \cos(\alpha_{k+1} - \beta_{k+1}).$$

Therefore,

$$2p = \sum A_kA_{k+1} = \sum x_k(\cos(\alpha_k - \beta_k) + \cos \beta_k) =$$
$$2 \sum x_k \cos \tfrac{\alpha_k}{2} \cos \left( \tfrac{\alpha_k}{2} - \beta_k \right) \le 2 \sum x_k \cos \tfrac{\alpha_k}{2}.$$

In both cases the equality is only attained if $\alpha_k = 2\beta_k$, i.e., $O$ is the center of the inscribed circle.

**9.83.** Suppose that the center $O$ of polygon $M_2$ lies outside polygon $M_1$. Then there exists a side $AB$ of polygon $M_1$ such that polygon $M_1$ and point $O$ lie on distinct sides of line $AB$. Let $CD$ be a side of $M_1$ parallel to $AB$. The distance between lines $AB$ and $CD$ is equal to the radius of the inscribed circle $S$ of polygon $M_2$ and, therefore, line $CD$ lies outside $S$. On the other hand, segment $CD$ lies inside $M_2$. Therefore, segment $CD$ is shorter than a half side of polygon $M_2$, cf. Problem 10.66. Contradiction.

**9.84.** Let $A_1$ be the nearest to $O$ vertex of the polygon. Let us divide the polygon into triangles by the diagonals that pass through vertex $A_1$. Point $O$ lies inside one of these triangles, say, in triangle $A_1A_kA_{k+1}$. If point $O$ lies on side $A_1A_k$, then $\angle A_1OA_k = \pi$ and the problem is solved.

Therefore, let us assume that point $O$ lies strictly inside triangle $A_1A_kA_{k+1}$. Since $A_1O \le A_kO$ and $A_1O \le A_{k+1}O$, it follows that $\angle A_1A_kO \le \angle A_kA_1O$ and $\angle A_1A_{k+1}O \le \angle A_{k+1}A_1O$. Hence,

$$\angle A_kOA_1 + \angle A_{k+1}OA_1 =$$
$$(\pi - \angle OA_1A_k - \angle OA_kA_1) + (\pi - \angle OA_1A_{k+1} - \angle OA_{k+1}A_1) \ge$$
$$2\pi - 2\angle OA_1A_k - 2\angle OA_1A_{k+1} = 2\pi - 2\angle A_kA_1A_{k+1} = 2\pi - \tfrac{2\pi}{n},$$

i.e., one of the angles $\angle A_kOA_1$ and $\angle A_{k+1}OA_1$ is not less than $\pi \left( 1 - \tfrac{1}{n} \right)$.

**9.85.** Let $d$ be the length of the longest diagonal (or side) $AB$ of the given $n$-gon. Then the perimeter of the $n$-gon does not exceed $\pi d$ (Problem 13.42). Let $A_i'$ be the projection of $A_i$ to segment $AB$. Then either $\sum AA_i' \ge \tfrac{1}{2}nd$ or $\sum BA_i' \ge \tfrac{1}{2}nd$ (Problem 9.87); let, for definiteness, the first inequality hold. Then $\sum AA_i > \sum AA_i' \ge \tfrac{1}{2}nd > \pi d \ge P$ because $\tfrac{1}{2}n \ge 3.5 > \pi$. Any point of the $n$-gon sufficiently close to vertex $A$ possesses the required property.

**9.86.** a) First, suppose that $\angle A_i > \angle B_i$ and for all the other considered pairs of angles an equality takes place. Let us arrange polygons so that vertices $A_1, \ldots, A_i$ coincide with $B_1, \ldots, B_i$. In triangles $A_1A_iA_n$ and $A_1A_iB_n$ sides $A_iA_n$ and $A_iB_n$ are equal and $\angle A_1A_iA_n > \angle A_1A_iB_n$; hence, $A_1A_n > A_1B_n$.

If several angles are distinct, then polygons $A_1 \ldots A_n$ and $B_1 \ldots B_n$ can be included in a chain of polygons whose successive terms are such as in the example considered above.

b) As we completely traverse the polygon we encounter the changes of minus sign by plus sign as often as the opposite change. Therefore, the number of pairs of neighbouring vertices with equal signs is an even one. It remains to verify that the number of sign changes cannot be equal to 2 (the number of sign changes is not equal to zero because the sums of the angles of each polygon are equal).

Suppose the number of sign changes is equal to 2. Let $P$ and $Q$, as well as $P'$ and $Q'$ be the midpoints of sides of polygons $A_1 \ldots A_n$ and $B_1 \ldots B_n$ on which a change of sign occurs. We can apply the statement of heading a) to pairs of polygons $M_1$ and $M_1'$, $M_2$ and $M_2'$
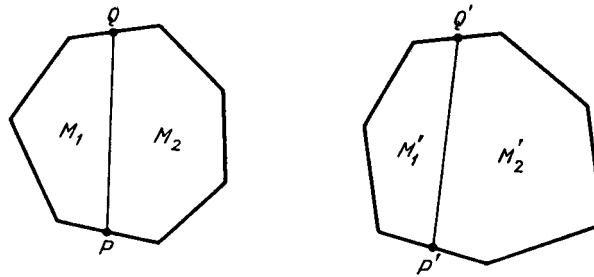
Figure 117 (Sol. 9.86)

(Fig. 117); we get $PQ > P'Q'$ in the one case, and $PQ < P'Q'$ in the other one, which is impossible.

**9.87.** Let $A$ and $B$ be the midpoints of the segment; $X_1, \ldots, X_n$ the given points. Since $AX_i + BX_i = 1$, it follows that $\sum AX_i + \sum BX_i = n$. Therefore, either $\sum AX_i \geq \frac{1}{2}n$ or $\sum BX_i \geq \frac{1}{2}n$.
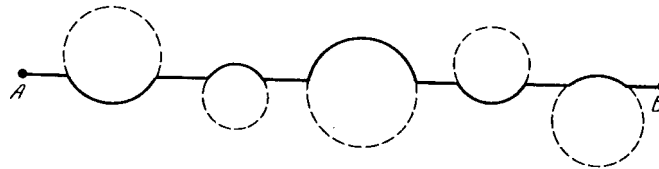


Figure 118 (Sol. 9.88)

**9.88.** Let us draw a wire along segment $AB$ circumventing the encountered trees along the shortest arc as on Fig. 118. It suffices to prove that the way along an arc of the circle is not more than 1.6 times longer than the way along the line. The ratio of the length of an arc with the angle value $2\varphi$ to the chord it subtends is equal to $\frac{\varphi}{\sin \varphi}$. Since $0 < \varphi \leq \frac{\pi}{2}$, it follows that $\frac{\varphi}{\sin \varphi} \leq \frac{\pi}{2} < 1.6$.

**9.89.** Let the trees of height $a_1 > a_2 > \cdots > a_n$ grow at points $A_1, \ldots, A_n$. Then by the hypothesis

$$A_1 A_2 \leq |a_1 - a_2| = a_1 - a_2, \ldots, A_{n-1} A_n \leq a_{n-1} - a_n.$$

It follows that the length of the broken line $A_1 A_2 \ldots A_n$ does not exceed

$$(a_1 - a_2) + (a_2 - a_3) + \cdots + (a_{n-1} - a_n) = a_1 - a_n < 100 \text{ m}.$$

This broken line can be fenced by a fence, whose length does not exceed 200 m (Fig. 119).
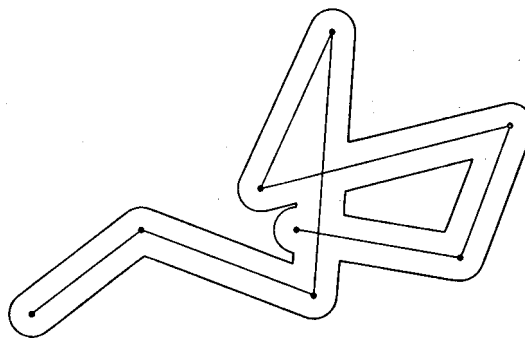


Figure 119 (Sol. 9.89)

**9.90.** In the obtained pentagon, distinguish the parts that were glued (on Fig. 120 these parts are shaded). All the sides that do not belong to the shaded polygons enter the

perimeters of the initial and the obtained polygons. The sides of the shaded polygons that lie on the line along which the folding was performed enter the perimeter of the obtained polygon whereas all the other sides enter the perimeter of the initial polygon.
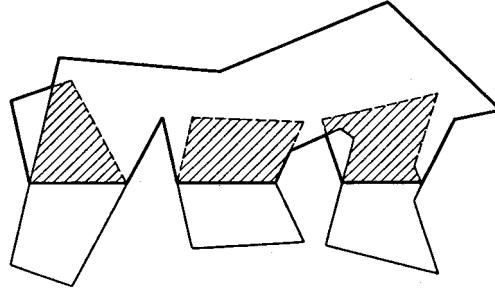


Figure 120 (Sol. 9.90)

Since for any polygon the sum of its sides that lie on a line is less than the sum of the other sides, the perimeter of the initial polygon is always longer than the perimeter of the obtained one.

**9.91.** On the broken line, take two points $A$ and $B$, that divide its perimeter in halves. Then $AB \leq \frac{1}{2}$. Let us prove that all the points of the broken line lie inside the circle of radius $\frac{1}{4}$ centered at the midpoint $O$ of segment $AB$. Let $M$ be an arbitrary point of the broken line and point $M_1$ be symmetric to $M$ through point $O$. Then

$$MO = \frac{M_1 M}{2} \leq \frac{M_1 A + AM}{2} = \frac{BM + AM}{2} \leq \frac{1}{4}$$

because $BM + AM$ does not exceed a half length of the broken line.

**9.92.** Let acute triangle $ABC$ be placed inside circle $S$. Let us construct the circumscribed circle $S_1$ of triangle $ABC$. Since triangle $ABC$ is an acute one, the angle value of the arc of circle $S_1$ that lies inside $S$ is greater than $180°$. Therefore, on this arc we can select diametrically opposite points, i.e., inside circle $S$ a diameter of circle $S_1$ is contained. It follows that the radius of $S$ is not shorter than the radius of $S_1$.

A similar statement for an acute triangle is false. An acute triangle lies inside a circle constructed on the longest side $a$ as on diameter. The radius of this circle is equal to $\frac{1}{2}a$ and the radius of the circle circumscribed about the triangle is equal to $\frac{a}{2\sin\alpha}$. Clearly, $\frac{1}{2}a < \frac{a}{2\sin\alpha}$.

**9.93. First solution.** Any triangle of perimeter $P$ can be placed in a disk of radius $\frac{1}{4}P$ and if an acute triangle is placed in a disk of radius $R_1$, then $R_1 \geq R$ (Problem 9.92). Hence, $\frac{1}{4}P = R_1 \geq R$.

**Second solution.** If $0 < x < \frac{\pi}{2}$, then $\sin x > \frac{2x}{\pi}$. Hence,

$$a + b + c = 2R(\sin\alpha + \sin\beta + \sin\gamma) > \frac{2R(2\alpha + 2\beta + 2\gamma)}{\pi} = 4R.$$

# Chapter 10. INEQUALITIES BETWEEN THE ELEMENTS OF A TRIANGLE

This chapter is in close connection with the preceding one. For background see the preceding chapter.

## §1. Medians

**10.1.** Prove that if $a > b$, then $m_a < m_b$.

**10.2.** Medians $AA_1$ and $BB_1$ of triangle $ABC$ intersect at point $M$. Prove that if quadrilateral $A_1MB_1C$ is a circumscribed one, then $AC = BC$.

**10.3.** Perimeters of triangles $ABM$, $BCM$ and $ACM$, where $M$ is the intersection point of medians of triangle $ABC$, are equal. Prove that triangle $ABC$ is an equilateral one.

**10.4.** a) Prove that if $a$, $b$, $c$ are the lengths of sides of an arbitrary triangle, then $a^2 + b^2 \geq \frac{1}{2}c^2$.

b) Prove that $m_a^2 + m_b^2 \geq \frac{2}{8}c^2$.

**10.5.** Prove that $m_a^2 + m_b^2 + m_c^2 \leq \frac{27}{4}R^2$.

b) Prove that $m_a + m_b + m_c \leq \frac{9}{2}R$.

**10.6.** Prove that $\frac{|a^2 - b^2|}{2c} < m_c \leq \frac{a^2 + b^2}{2c}$.

**10.7.** Let $x = ab + bc + ca$, $x_1 = m_a m_b + m_b m_c = m_c m_a$. Prove that $\frac{9}{20} < \frac{x_1}{x} < \frac{5}{4}$.

See also Problems 9.1, 10.74, 10.76, 17.17.

## §2. Heights

**10.8.** Prove that in any triangle the sum of the lengths of its heights is less than its semiperimeter.

**10.9.** Two heights of a triangle are longer than 1. Prove that its area is greater than $\frac{1}{2}$.

**10.10.** In triangle $ABC$, height $AM$ is not shorter than $BC$ and height $BH$ is not shorter than $AC$. Find the angles of triangle $ABC$.

**10.11.** Prove that $\frac{1}{2r} < \frac{1}{h_a} + \frac{1}{h_b} < \frac{1}{r}$.

**10.12.** Prove that $h_a + h_b + h_c \geq 9r$.

**10.13.** Let $a < b$. Prove that $a + h_a \leq b + h_b$.

**10.14.** Prove that $h_a \leq \sqrt{r_b r_c}$.

**10.15.** Prove that $h_a \leq \frac{a}{2} \cot \frac{\alpha}{2}$.

**10.16.** Let $a \leq b \leq c$. Prove that

$$h_a + h_b + h_c \leq \frac{3b(a^2 + ac + c^2)}{4pR}.$$

See also Problems 10.28, 10.55, 10.74, 10.79.

## §3. The bisectors

**10.17.** Prove that $l_a \leq \sqrt{p(p - a)}$.

**10.18.** Prove that $\frac{h_a}{l_a} \geq \sqrt{\frac{2r}{R}}$.

**10.19.** Prove that a) $l_a^2 + l_b^2 + l_c^2 \leq p^2$; b) $l_a + l_b + l_c \leq \sqrt{3}p$.

**10.20.** Prove that $l_a + l_b + m_c \leq \sqrt{3}p$.

See also Problems 6.38, 10.75, 10.94.

## §4. The lengths of sides

**10.21.** Prove that $\frac{9r}{2S} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{9R}{4S}$.

**10.22.** Prove that $\frac{2bc\cos\alpha}{b+c} < b + c - a < \frac{2bc}{a}$.

**10.23.** Prove that if $a$, $b$, $c$ are the lengths of sides of a triangle of perimeter 2, then $a^2 + b^2 + c^2 < 2(1 - abc)$.

**10.24.** Prove that $20Rr - 4r^2 \leq ab + bc + ca \leq 4(R + r)^2$.

## §5. The radii of the circumscribed, inscribed and escribed circles

**10.25.** Prove that $rr_c \leq \frac{c^2}{4}$.

**10.26.** Prove that $\frac{r}{R} \leq 2\sin\frac{\alpha}{2}\left(1 - \sin\frac{\alpha}{2}\right)$.

**10.27.** Prove that $6r \leq a + b$.

**10.28.** Prove that $\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 3$.

**10.29.** Prove that $27Rr \leq 2p^2 \leq \frac{1}{2}27R^2$.

**10.30.** Let $O$ be the centre of the inscribed circle of triangle $ABC$ and $OA \geq OB \geq OC$. Prove that $OA \geq 2r$ and $OB \geq r\sqrt{2}$.

**10.31.** Prove that the sum of distances from any point inside of a triangle to its vertices is not less than $6r$.

**10.32.** Prove that $3\left(\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c}\right) \geq 4\left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c}\right)$.

**10.33.** Prove that:

a) $5R - r \geq \sqrt{3}p$;

b) $4R - r_a \geq (p - a)\left[\sqrt{3} + \frac{a^2 + (b-c)^2}{2S}\right]$.

**10.34.** Prove that $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.

**10.35.** Prove that $r_a^2 + r_b^2 + r_c^2 \geq \frac{1}{4}27R^2$.

See also Problems 10.11, 10.12, 10.14, 10.18, 10.24, 10.55, 10.79, 10.82, 19.7.

## §6. Symmetric inequalities between the angles of a triangle

Let $\alpha$, $\beta$ and $\gamma$ be the angles of triangle $ABC$. In problems of this section you have to prove the inequalities indicated.

REMARK. If $\alpha$, $\beta$ and $\gamma$ are the angles of a triangle, then there exists a triangle with angles $\frac{\pi-\alpha}{2}$, $\frac{\pi-\beta}{2}$ and $\frac{\pi-\gamma}{2}$. Indeed, these numbers are positive and their sum is equal to $\pi$. It follows that if a symmetric inequality holds for sines, cosines, tangents and cotangents of the angles of any triangle then a similar inequality in which $\sin x$ is replaced with $\cos\frac{x}{2}$, $\cos x$ with $\sin\frac{x}{2}$, $\tan x$ with $\cot\frac{x}{2}$ and $\cot x$ with $\tan\frac{x}{2}$ is also true.

The converse passage from inequalities for halved angles to inequalities with whole angles is only possible for acute triangles. Indeed, if $\alpha' = \frac{1}{2}(\pi - \alpha)$, then $\alpha = \pi - 2\alpha'$. Therefore, for an acute triangle with angles $\alpha'$, $\beta'$, $\gamma'$ there exists a triangle with angles $\pi - 2\alpha'$, $\pi - 2\beta'$ and $\pi - 2\gamma'$. Under such a passage $\sin\frac{x}{2}$ turns into $\cos x$, etc., but the inequality obtained can only be true for acute triangles.

**10.36.** a) $1 < \cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$.

b) $1 < \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \leq \frac{3}{2}$.

**10.37.** a) $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3}{2}\sqrt{3}$.

b) $\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \leq \frac{3}{2}\sqrt{3}$.

**10.38.** a) $\cot\alpha + \cot\beta + \cot\gamma \geq \sqrt{3}$.

b) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq \sqrt{3}$.

**10.39.** $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} \geq 3\sqrt{3}$.

b) For an acute triangle $\tan\alpha + \tan\beta + \tan\gamma \geq 3\sqrt{3}$.

**10.40.** a) $\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \leq \frac{1}{8}$.

b) $\cos\alpha\cos\beta\cos\gamma \leq \frac{1}{8}$.

**10.41.** a) $\sin\alpha\sin\beta\sin\gamma \leq \frac{3\sqrt{3}}{8}$;

b) $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leq \frac{3}{8}\sqrt{3}$.

**10.42.** a) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \geq \frac{3}{4}$.

b) For an obtuse triangle

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma > 1.$$

**10.43.** $\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha \leq \frac{3}{4}$.

**10.44.** For an acute triangle

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha).$$

## §7. Inequalities between the angles of a triangle

**10.45.** Prove that $1 - \sin\frac{\alpha}{2} \leq 2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

**10.46.** Prove that $\sin\frac{\gamma}{2} \leq \frac{c}{a+b}$.

**10.47.** Prove that if $a + b < 3c$, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} < \frac{1}{2}$.

**10.48.** In an acute triangle, if $\alpha < \beta < \gamma$, then $\sin 2\alpha > \sin 2\beta > \sin 2\gamma$.

**10.49.** Prove that $\cos 2\alpha + \cos 2\beta - \cos 2\gamma \leq \frac{3}{2}$.

**10.50.** On median $BM$ of triangle $ABC$, point $X$ is taken. Prove that if $AB < BC$, then $\angle XAB < \angle XCB$.

**10.51.** The inscribed circle is tangent to sides of triangle $ABC$ at points $A_1$, $B_1$ and $C_1$. Prove that triangle $A_1B_1C_1$ is an acute one.

**10.52.** From the medians of a triangle whose angles are $\alpha$, $\beta$ and $\gamma$ a triangle whose angles are $\alpha_m$, $\beta_m$ and $\gamma_m$ is constructed. (Angle $\alpha_m$ subtends median $AA_1$, etc.) Prove that if $\alpha > \beta > \gamma$, then $\alpha > \alpha_m$, $\alpha > \beta_m$, $\gamma_m > \beta > \alpha_m$, $\beta_m > \gamma$ and $\gamma_m > \gamma$.

## §8. Inequalities for the area of a triangle

**10.53.** Prove that: a) $3\sqrt{3}r^2 \leq S \leq \frac{p^2}{3\sqrt{3}}$; b) $S \leq \frac{a^2+b^2+c^2}{4\sqrt{3}}$.

**10.54.** Prove that

$$a^2 + b^2 + c^2 - (a-b)^2 - (b-c)^2 - (c-a)^2 \geq 4\sqrt{3}S.$$

**10.55.** Prove that: a) $S^3 \leq \left(\frac{\sqrt{3}}{4}\right)^3 (abc)^2$; b) $\sqrt{h_a h_b h_c} \leq \sqrt[4]{3}\sqrt{S} \leq \sqrt[3]{r_a r_b r_c}$.

**\* \* \***

**10.56.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$, respectively, are taken so that $AA_1$, $BB_1$ and $CC_1$ meett at one point. Prove that $\frac{S_{A_1 B_1 C_1}}{S_{ABC}} \leq \frac{1}{4}$.

**10.57.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ arbitrary points $A_1$, $B_1$ and $C_1$ are taken. Let $a = S_{AB_1 C_1}$, $b = S_{A_1 BC_1}$, $c = S_{A_1 B_1 C}$ and $u = S_{A_1 B_1 C_1}$. Prove that

$$u^3 + (a + b + c)u^2 \geq 4abc.$$

**10.58.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ are taken. Prove that the area of one of the triangles $AB_1 C_1$, $A_1 BC_1$, $A_1 B_1 C$ does not exceed: a) $\frac{1}{4} S_{ABC}$; b) $S_{A_1 B_1 C_1}$.

See also Problems 9.33, 9.37, 9.40, 10.9, 20.1, 20.7.

## §9. The greater angle subtends the longer side

**10.59.** In a triangle $ABC$, prove that $\angle ABC < \angle BAC$ if and only if $AC < BC$, i.e., the longer side subtends the greater angle and the greater angle subtends the longer side.

**10.60.** Prove that in a triangle $ABC$ angle $\angle A$ is an acute one if and only if $m_b > \frac{1}{2}a$.

**10.61.** Let $ABCD$ and $A_1 B_1 C_1 D_1$ be two convex quadrilaterals with equal corresponding sides. Prove that if $\angle A > \angle A_1$, then $\angle B < \angle B_1$, $\angle C < \angle C_1$, $\angle D < \angle D_1$.

**10.62.** In an acute triangle $ABC$ the longest height $AH$ is equal to median $BM$. Prove that $\angle B \leq 60°$.

**10.63.** Prove that a convex pentagon $ABCDE$ with equal sides whose angles satisfy inequalities $\angle A \geq \angle B \geq \angle C \geq \angle D \geq \angle E$ is a regular one.

## §10. Any segment inside a triangle is shorter than the longest side

**10.64.** a) Segment $MN$ is placed inside triangle $ABC$. Prove that the length of $MN$ does not exceed the length of the longest side of the triangle.

b) Segment $MN$ is placed inside a convex polygon. Prove that the length of $MN$ does not exceed that of the longest side or of the greatest diagonal of this polygon.

**10.65.** Segment $MN$ lies inside sector $AOB$ of a disk of radius $R = AO = BO$. Prove that either $MN \leq R$ or $MN \leq AB$ (we assume that $\angle AOB < 180°$).

**10.66.** In an angle with vertex $A$, a circle tangent to the legs at points $B$ and $C$ is inscribed. In the domain bounded by segments $AB$, $AC$ and the shorter arc $\smile BC$ a segment is placed. Prove that the length of the segment does not exceed that of $AB$.

**10.67.** A convex pentagon lies inside a circle. Prove that at least one of the sides of the pentagon is not longer than a side of the regular pentagon inscribed in the circle.

**10.68.** Given triangle $ABC$ the lengths of whose sides satisfy inequalities $a > b > c$ and an arbitrary point $O$ inside the triangle. Let lines $AO$, $BO$, $CO$ intersect the sides of the triangle at points $P$, $Q$, $R$, respectively. Prove that $OP + OQ + OR < a$.

## §11. Inequalities for right triangles

In all problems of this section $ABC$ is a right triangle with right angle $\angle C$.

**10.69.** Prove that $c^n > a^n + b^n$ for $n > 2$.

**10.70.** Prove that $a + b < c + h_c$.

**10.71.** Prove that for a right triangle $0.4 < \frac{r}{h} < 0.5$, where $h$ is the height dropped from the vertex of the right angle.

**10.72.** Prove that $\frac{c}{r} \geq 2(1 + \sqrt{2})$.

**10.73.** Prove that $m_a^2 + m_b^2 > 29r^2$.

## §12. Inequalities for acute triangles

**10.74.** Prove that for an acute triangle
$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq 1 + \frac{R}{r}.$$

**10.75.** Prove that for an acute triangle
$$\frac{1}{l_a} + \frac{1}{l_b} + \frac{1}{l_c} \leq \sqrt{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

**10.76.** Prove that if a triangle is not an obtuse one, then $m_a + m_b + m_c \geq 4R$.

**10.77.** Prove that if in an acute triangle $h_a = l_b = m_c$, then this triangle is an equilateral one.

**10.78.** In an acute triangle $ABC$ heights $AA_1, BB_1$ and $CC_1$ are drawn. Prove that the perimeter of triangle $A_1B_1C_1$ does not exceed a semiperimeter of triangle $ABC$.

**10.79.** Let $h$ be the longest height of a non-obtuse triangle. Prove that $r + R \leq h$.

**10.80.** On sides $BC$, $CA$ and $AB$ of an acute triangle $ABC$, points $A_1$, $B_1$ and $C_1$, respectively, are taken. Prove that
$$2(B_1C_1 \cos\alpha + C_1A_1 \cos\beta + A_1B_1 \cos\gamma) \geq a\cos\alpha + b\cos\beta + c\cos\gamma.$$

**10.81.** Prove that a triangle is an acute one if and only if $a^2 + b^2 + c^2 > 8R^2$.

**10.82.** Prove that a triangle is an acute one if and only if $p > 2R + r$.

**10.83.** Prove that triangle $ABC$ is an acute one if and only if on its sides $BC$, $CA$ and $AB$ interior points $A_1$, $B_1$ and $C_1$, respectively, can be selected so that $AA_1 = BB_1 = CC_1$.

**10.84.** Prove that triangle $ABC$ is an acute one if and only if the lengths of its projections onto three distinct directions are equal.

See also Problems 9.93, 10.39, 10.44, 10.48, 10.62.

## §13. Inequalities in triangles

**10.85.** A line is drawn through the intersection point $O$ of the medians of triangle $ABC$. The line intersects the triangle at points $M$ and $N$. Prove that $NO \leq 2MO$.

**10.86.** Prove that if triangle $ABC$ lies inside triangle $A'B'C'$, then $r_{ABC} < r_{A'B'C'}$.

**10.87.** In triangle $ABC$ side $c$ is the longest and $a$ is the shortest. Prove that $l_c \leq h_a$.

**10.88.** Medians $AA_1$ and $BB_1$ of triangle $ABC$ are perpendicular. Prove that $\cot\angle A + \cot\angle B \geq \frac{2}{3}$.

**10.89.** Through vertex $A$ of an isosceles triangle $ABC$ with base $AC$ a circle tangent to side $BC$ at point $M$ and intersecting side $AB$ at point $N$ is drawn. Prove that $AN > CM$.

**10.90.** In an acute triangle $ABC$ bisector $AB$, median $BM$ and height $CH$ intersect at one point. What are the limits inside which the value of angle $A$ can vary?

**10.91.** In triangle $ABC$, prove that $\frac{1}{3}\pi \leq \pi a\alpha + b\beta + c\gamma a + b + c < \frac{1}{2}\pi$.

**10.92.** Inside triangle $ABC$ point $O$ is taken. Prove that
$$AO\sin\angle BOC + BO\sin\angle AOC + CO\sin\angle AOB \leq p.$$

**10.93.** On the extension of the longest side $AC$ of triangle $ABC$ beyond point $C$, point $D$ is taken so that $CD = CB$. Prove that angle $\angle ABD$ is not an acute one.

**10.94.** In triangle $ABC$ bisectors $AK$ and $CM$ are drawn. Prove that if $AB > BC$, then $AM > MK > KC$.

**10.95.** On sides $BC$, $CA$, $AB$ of triangle $ABC$ points $X$, $Y$, $Z$ are taken so that lines $AX$, $BY$, $CZ$ meet at one point $O$. Prove that of ratios $OA : OX$, $OB : OY$, $OC : OZ$ at least one is not greater than 2 and one is not less than 2.

**10.96.** Circle $S_1$ is tangent to sides $AC$ and $AB$ of triangle $ABC$, circle $S_2$ is tangent to sides $BC$ and $AB$ and, moreover, $S_1$ and $S_2$ are tangent to each other from the outside. Prove that the sum of radii of these circles is greater than the radius of the inscribed circle $S$.

See also Problems 14.24, 17.16, 17.18.

## Problems for independent study

**10.97.** In a triangle $ABC$, let $P = a+b+c$, $Q = ab+bc+ca$. Prove that $3Q < P^2 < 4Q$.

**10.98.** Prove that the product of any two sides of a triangle is greater than $4Rr$.

**10.99.** In triangle $ABC$ bisector $AA_1$ is drawn. Prove that $A_1C < AC$.

**10.100.** Prove that if $a > b$ and $a + h_a \leq b + h_b$, then $\angle C = 90°$.

**10.101.** Let $O$ be the centre of the inscribed circle of triangle $ABC$. Prove that $ab + bc + ca \geq (AO + BO + CO)^2$.

**10.102.** On sides of triangle $ABC$ equilateral triangles with centers at $D$, $E$ and $F$ are constructed outwards. Prove that $S_{DEF} \geq S_{ABC}$.

**10.103.** In plane, triangles $ABC$ and $MNK$ are given so that line $MN$ passes through the midpoints of sides $AB$ and $AC$ and the intersection of these triangles is a hexagon of area $S$ with pairwise parallel opposite sides. Prove that $3S < S_{ABC} + S_{MNK}$.

## Solutions

**10.1.** Let medians $AA_1$ and $BB_1$ meet at point $M$. Since $BC > AC$, points $A$ and $C$ lie on one side of the midperpendicular to segment $AB$ and therefore, both median $CC_1$ and its point $M$ lie on the same side. It follows that $AM < BM$, i.e., $m_a < m_b$.

**10.2.** Suppose that, for instance, $a > b$. Then $m < m_b$ (Problem 10.1). Since quadrilateral $A_1MB_1C$ is a circumscribed one, it follows that $\frac{1}{2}a + \frac{1}{3}m_b = \frac{1}{2}b + \frac{1}{3}m_a$, i.e., $\frac{1}{2}(a - b) = \frac{1}{3}(m_a - m_b)$. Contradiction.

**10.3.** Let, for instance, $BC > AC$. Then $MA < MB$ (cf. Problem 10.1); hence, $BC + MB + MC > AC + MA + MC$.

**10.4.** a) Since $c \leq a + b$, it follows that $c^2 \leq (a + b)^2 = a^2 + b^2 + 2ab \leq 2(a^2 + b^2)$.

b) Let $M$ be the intersection point of medians of triangle $ABC$. By heading a) $MA^2 + MB^2 \geq \frac{1}{2}AB^2$, i.e., $\frac{4}{9}m_a^2 + \frac{4}{9}m_b^2 \geq \frac{1}{2}c^2$.

**10.5.** a) Let $M$ be the intersection point of medians, $O$ the center of the circumscribed circle of triangle $ABC$. Then

$$AO^2 + BO^2 + CO^2 =$$
$$(\overrightarrow{AM} + \overrightarrow{MO})^2 + (\overrightarrow{BM} + \overrightarrow{MO})^2 + (\overrightarrow{CM} + \overrightarrow{MO})^2 =$$
$$AM^2 + BM^2 + CM^2 + 2(\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM}, \overrightarrow{MO}) + 3MO^2.$$

Since $\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM} = \overrightarrow{0}$, it follows that

$$AO^2 + BO^2 + CO^2 = AM^2 + BM^2 + CM^2 + 3MO^2 \geq AM^2 + BM^2 + CM^2,$$

i.e., $3R^2 \geq \frac{4}{9}(m_a^2 + m_b^2 + m_c^2)$.

b) It suffices to notice that $(m_a + m_b + m_c)^2 \leq 3(m_a^2 + m_b^2 + m_c^2)$, cf. Supplement to Ch. 9.

**10.6.** Heron's formula can be rewritten as

$$16S^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$$

Since $m_c^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2)$ (Problem 12.11 a)), it follows that the inequalities

$$m_c^2 \leq \left(\frac{a^2 + b^2}{2c}\right)^2 ; \quad m_c^2 \geq \left(\frac{a^2 - b^2}{2c}\right)^2$$

are equivalent to the inequalities $16S^2 \leq 4a^2b^2$ and $16S^2 > 0$, respectively.

**10.7.** Let $y = a^2 + b^2 + c^2$ and $y_1 = m_a^2 + m_b^2 + m_c^2$. Then $3y = 4y_1$ (Problem 12.11, b), $y < 2x$ (Problem 9.7) and $2x_1 + y_1 < 2x + y$ because $(m_a + m_b + m_c)^2 < (a + b + c)^2$ (cf. Problem 9.2). By adding $8x_1 + 4y_1 < 8x + 4y$ to $3y = 4y_1$ we get $8x_1 < y + 8x < 10x$, i.e., $\frac{x_1}{x} < \frac{5}{4}$.

Let $M$ be the intersection point of the medians of triangle $ABC$. Let us complement triangle $AMB$ to parallelogram $AMBN$. Applying the above-proved statement to triangle $AMN$ we get $\frac{(x/4)}{(4x_1/9)} < \frac{5}{4}$, i.e., $\frac{x}{x_1} < \frac{20}{9}$.

**10.8.** Clearly, $h_a \leq b$, $h_b \leq c$, $h_c \leq a$, where at least one of these inequalities is a strict one. Hence, $h_a + h_b + h_c < a + b + c$.

**10.9.** Let $h_a > 1$ and $h_b > 1$. Then $a \geq h_b > 1$. Hence, $S = \frac{1}{2}ah_a > \frac{1}{2}$.

**10.10.** By the hypothesis $BH \geq AC$ and since the perpendicular is shorter than a slanted line, $BH \geq AC \geq AM$. Similarly, $AM \geq BC \geq BH$. Hence, $BH = AM = AC = BC$. Since $AC = AM$, segments $AC$ and $AM$ coincide, i.e., $\angle C = 90°$; since $AC = BC$, the angles of triangle $ABC$ are equal to $45°$, $45°$, $90°$.

**10.11.** Clearly, $\frac{1}{h_a} + \frac{1}{h_b} = \frac{a+b}{2S} = \frac{a+b}{(a+b+c)r}$ and $a + b + c < 2(a + b) < 2(a + b + c)$.

**10.12.** Since $ah_a = r(a + b + c)$, it follows that $h_a = r\left(1 + \frac{b}{a} + \frac{c}{a}\right)$. Adding these equalities for $h_a$, $h_b$ and $h_c$ and taking into account that $\frac{x}{y} + \frac{y}{x} \geq 2$ we get the desired statement.

**10.13.** Since $h_a - h_b = 2S\left(\frac{1}{a} - \frac{1}{b}\right) = 2S\frac{b-a}{ab}$ and $2S \leq ab$, it follows that $h_a - h_b \leq b - a$.

**10.14.** By Problem 12.21 $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c}$. Moreover, $\frac{1}{r_b} + \frac{1}{r_c} \geq \frac{2}{\sqrt{r_b r_c}}$.

**10.15.** Since

$$2\sin\beta\sin\gamma = \cos(\beta - \gamma) - \cos(\beta + \gamma) \leq 1 + \cos\alpha,$$

we have

$$\frac{h_a}{a} = \frac{\sin\beta\sin\gamma}{\sin\alpha} \leq \frac{1 + \cos\alpha}{2\sin\alpha} = \frac{1}{2}\cot\frac{\alpha}{2}.$$

**10.16.** Since $\frac{b}{2R} = \sin\beta$, then multiplying by $2p$ we get

$$(a + b + c)(h_a + h_b + h_c) \leq 3\sin\beta(a^2 + ac + c^2).$$

Subtracting $6S$ from both sides we get

$$a(h_b + h_c) + b(h_a + h_c) + c(h_a + h_b) \leq 3\sin\beta(a^2 + c^2).$$

Since, for instance, $ah_b = a^2\sin\gamma = \frac{a^2c}{2R}$, we obtain $a(b^2 + c^2) - 2b(a^2 + c^2) + c(a^2 + b^2) \leq 0$. To prove the latter inequality let us consider the quadratic exoression

$$f(x) = x^2(a + c) - 2x(a^2 + c^2) + ac(a + c).$$

It is easy to verify that $f(a) = -a(a - c)^2 \leq 0$ and $f(c) = -c(a - c)^2 \leq 0$. Since the coefficient of $x$ is positive and $a \leq b \leq c$, it follows that $f(b) \leq 0$.

**10.17.** By Problem 12.35 a) $l_a^2 = \frac{4bcp(p-a)}{(b+c)^2}$. Moreover, $4bc \leq (b + c)^2$.

**10.18.** Clearly, $\frac{h_a}{l_a} = \cos \frac{1}{2}(\beta - \gamma)$. By Problem 12.36 a)

$$\frac{2r}{R} = 8 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 4 \sin \frac{\alpha}{2} \left[ \cos \frac{\beta - \gamma}{2} - \cos \frac{\beta + \gamma}{2} \right] = 4x(q - x),$$

$$\text{where } x = \sin \frac{\alpha}{2} \text{ and } q = \cos \frac{\beta - \gamma}{2}.$$

It remains to notice that $4x(q - x) \le q^2$.

**10.19.** a) By Problem 10.17 $l_a^2 \le p(p - a)$. Adding three similar inequalities we get the desired statement.

b) For any numbers $l_a$, $l_b$ and $l_c$ we have $(l_a + l_b + l_c)^2 \le 3(l_a^2 + l_b^2 + l_c^2)$.

**10.20.** It suffices to prove that $\sqrt{p(p - a)} + \sqrt{p(p - b)} + m_c \le \sqrt{3p}$. We may assume that $p = 1$; let $x = 1 - a$ and $y = 1 - b$. Then

$$m_c^2 = \frac{2a^2 + 2b^2 - c^2}{4} = 1 - (x + y) + \frac{(x - y)^2}{4} = m(x, y).$$

Let us consider the function

$$f(x, y) = \sqrt{x} + \sqrt{y} + \sqrt{m(x, y)}.$$

We have to prove that $f(x, y) \le \sqrt{3}$ for $x, y \ge 0$ and $x + y \le 1$. Let

$$g(x) = f(x, x) = 2\sqrt{x} + \sqrt{1 - 2x}.$$

Since $g'(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1 - 2x}}$, it follows that as $x$ grows from 0 to $\frac{1}{3}$ and $g(x)$ grows from 1 to $\sqrt{3}$ and as $x$ grows from $\frac{1}{3}$ to $\frac{1}{2}$; we also see that $g(x)$ diminishes from $\sqrt{3}$ to $\sqrt{2}$. Introduce new variables: $d = x - y$ and $q = \sqrt{x} + \sqrt{y}$. It is easy to verify that $(x - y)^2 - 2q^2(x + y) + q^4 = 0$, i.e., $x + y = \frac{d^2 + q^4}{2q^2}$. Hence,

$$f(x, y) = q + \sqrt{1 - \frac{q^2}{2} - \frac{d^2(2 - q^2)}{4q^2}}.$$

Now, observe that $q^2 = (\sqrt{x} + \sqrt{y})^2 \le 2(x + y) \le 2$, i.e., $\frac{d^2(2 - q^2)}{4q^2} \ge 0$. It follows that for a fixed $q$ the value of function $f(x, y)$ is the maximal one for $d = 0$, i.e., $x = y$; the case $x = y(?)$ is the one considered above.

**10.21.** Clearly, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{h_a + h_b + h_c}{2S}$. Moreover, $9r \le h_a + h_b + h_c$ (Problem 10.12) and $h_a + h_b + h_c \le m_a + m_b + m_c \le \frac{9}{2}R$ (Problem 10.5 b)).

**10.22.** First, let us prove that $b + c - a < \frac{2bc}{a}$. Let $2x = b + c - a$, $2y = a + c - b$ and $2z = a + b - c$. We have to prove that

$$2x < \frac{2(x + y)(x + z)}{y + z}, \quad \text{i.e., } xy + xz < xy + xz + x^2 + yz.$$

The latter inequality is obvious.

Since

$$2bc \cos \alpha = b^2 + c^2 - a^2 = (b + c - a)(b + c + a) - 2bc,$$

it follows that

$$\frac{2bc \cos \alpha}{b + c} = b + c - a + \left[ \frac{(b + c - a)a}{b + c} - \frac{2bc}{b + c} \right].$$

The expression in square brackets is negative because $b + c - a < \frac{2bc}{a}$.

**10.23.** By Problem 12.30 we have

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ac) = 4p^2 - 2r^2 - 2p^2 - 8rR = 2p^2 - 2r^2 - 8rR$$

and $abc = 4prR$. Thus, we have to prove that

$$2p^2 - 2r^2 - 8rR < 2(1 - 4prR), \quad \text{where } p = 1.$$

This inequality is obvious.

**10.24.** By Problem 12.30, $ab + bc + ca = r^2 + p^2 + 4Rr$. Moreover, $16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$ (Problem 10.34).

**10.25.** Since

$$r(\cot \alpha + \cot \beta) = c = r_c(\tan \alpha + \tan \beta),$$

it follows that

$$c^2 = rr_c \left( 2 + \frac{\tan \alpha}{\tan \beta} + \frac{\tan \beta}{\tan \beta} \right) \ge 4rr_c.$$

**10.26.** It suffices to apply the results of Problems 12.36 a) and 10.45. Notice also that $x(1 - x) \le \frac{1}{4}$, i.e., $\frac{r}{R} \le \frac{1}{2}$.

**10.27.** Since $h_c \le a$ and $h_c \le b$, it follows that $4S = 2ch_c \le c(a + b)$. Hence,

$$6r(a + b + c) = 12S \le 4ab + 4S \le (a + b)^2 + c(a + b) = (a + b)(a + b + c).$$

**10.28.** Since $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c}$ (Problem 12.21), it follows that $\frac{r_a}{h_a} = \frac{1}{2} \left( \frac{r_a}{r_b} + \frac{r_a}{r_c} \right)$. Let us write similar equalities for $\frac{r_b}{h_b}$ and $\frac{r_c}{h_c}$ and add them. Taking into account that $\frac{x}{y} + \frac{y}{x} \ge 2$ we get the desired statement.

**10.29.** Since $Rr = \frac{PS}{p} = \frac{abc}{4p}$ (cf. Problem 12.1), we obtain $27abc \le 8p^3 = (a + b + c)^3$.

Since $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$ for any numbers $a$, $b$ and $c$, we have

$$p^2 \le \frac{3}{4}(a^2 + b^2 + c^2) = m_a^2 + m_b^2 + m_c^2$$

(cf. Problem 12.11 b)). It remains to notice that $m_a^2 + m_b^2 + m_c^2 \le \frac{27}{4}R^2$ (Problem 10.5 a)).

**10.30.** Since $OA = \frac{r}{\sin \frac{A}{2}}$, $OB = \frac{r}{\sin \frac{\angle B}{2}}$ and $OC = \frac{r}{\sin \frac{\angle X}{2}}$ and since angles $\frac{1}{2}\angle A$, $\frac{1}{2}\angle B$ and $\frac{1}{2}\angle C$ are acute ones, it follows that $\angle A \le \angle B \le \angle C$. Hence, $\angle A \le 60°$ and $\angle B \le 90°$ and, therefore, $\sin \frac{\angle A}{2} \le \frac{1}{2}$ and $\sin \frac{\angle B}{2} \le \frac{1}{\sqrt{2}}$.

**10.31.** If $\angle C \ge 120°$, then the sum of distances from any point inside the triangle to its vertices is not less than $a + b$ (Problem 11.21); moreover, $a + b \ge 6r$ (Problem 10.27).

If each angle of the triangle is less than 120°, then at a point the sum of whose distances from the vertices of the triangle is the least one the square of this sum is equal to $\frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}S$ (Problem 18.21 b)). Further, $\frac{1}{2}(a^2 + b^2 + c^2) \ge 2\sqrt{3}S$ (Problem 10.53 b)) and $4\sqrt{3}S \ge 36r^2$ (Problem 10.53 a)).

**10.32.** Let $\alpha = \cos \frac{\angle A}{2}$, $\beta = \cos \frac{\angle B}{2}$ and $\gamma = \cos \frac{\angle C}{2}$. By Problem 12.17 b) $\frac{a}{r_a} = \frac{\alpha}{\beta\gamma}$, $\frac{b}{r_b} = \frac{\beta}{\gamma\alpha}$ and $\frac{c}{r_c} = \frac{\gamma}{\alpha\beta}$. Therefore, multiplying by $\alpha\beta\gamma$ we express the inequality to be proved in the form

$$3(\alpha^2 + \beta^2 + \gamma^2) \ge 4(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2).$$

Since $\alpha^2 = \frac{1 + \cos \angle A}{2}$, $\beta^2 = \frac{1 + \cos \angle B}{2}$ and $\gamma^2 = \frac{1 + \cos \angle C}{2}$, we obtain the inequality

$$\cos \angle A + \cos \angle B + \cos \angle C + 2(\cos \angle A \cos \angle B + \cos \angle B \cos \angle C + \cos \angle C \cos \angle A) \le 3.$$

It remains to make use of results of Problems 10.36 and 10.43.

**10.33.** a) Adding equality $4R + r = r_a + r_b + r_c$ (Problem 12.24) with inequality $R - 2r \ge 0$ (Problem 10.26) we get

$$5R - r \ge r_a + r_b + r_c = pr((p - a)^{-1} + (p - b)^{-1} + (p - c)^{-1}) =$$
$$\frac{p(ab + bc + ca - p^2)}{S} = = \frac{p(2(ab + bc + ca) - a^2 - b^2 - c^2)}{4S}.$$

It remains to observe that

$$2(ab + bc + ca) - a^2 - b^2 - c^2 \geq 4\sqrt{3}S$$

(Problem 10.54).

b) It is easy to verify that

$$4R - r_a = r_b + r_c - r = \frac{pr}{p-b} + \frac{pr}{p-c} - \frac{pr}{p} = \frac{(p-a)(p^2-bc)}{S}.$$

It remains to observe that

$$4(p^2 - bc) = a^2 + b^2 + c^2 + 2(ab - bc + ca) =$$
$$= 2(ab + bc + ca) =$$
$$-a^2 - b^2 - c^2 + 2(a^2 + b^2 + c^2 - 2bc) \geq 4\sqrt{3}S + 2(a^2 + (b-c)^2).$$

**10.34.** Let $a$, $b$ and $c$ be the lengths of the sides of the triangle, $F = (a-b)(b-c)(c-a) = A - B$, where $A = ab^2 + bc^2 + ca^2$ and $B = a^2b + b^2c + c^2a$. Let us prove that the required inequalities can be obtained by a transformation of an obvious inequality $F^2 \geq 0$. Let $\sigma_1 = a + b + c = 2p$, $\sigma_2 = ab + bc + ca = r^2 + p^2 + 4rR$ and $\sigma_3 = abc = 4prR$, cf. Problem 12.30. It is easy to verify that

$$F^2 = \sigma_1^2 \sigma_2^2 - 4\sigma_2^3 - 4\sigma_1^3 \sigma_3 + 18\sigma_1\sigma_2\sigma_3 - 27\sigma_3^2.$$

Indeed,

$$(\sigma_1\sigma_2)^2 - F^2 = (A + B + 3abc)^2 - (A - B)^2 = 4AB + 6(A + B)\sigma_3 + 9\sigma_3^2 =$$
$$4(a^3b^3 + \dots) + 4(a^4bc + \dots) + 6(A + B)\sigma_3 + 21\sigma_3^2.$$

It is also clear that

$$4\sigma_2^3 = 4(a^3b^3 + \dots) + 12(A + B)\sigma_3 + 24\sigma_3^2,$$
$$4\sigma_1^3\sigma_3 = 4(a^4bc + \dots) + 12(A + B)\sigma_3 + 24\sigma_3^2,$$
$$18\sigma_1\sigma_2\sigma_3 = 18(A + B)\sigma_3 + 54\sigma_3^2.$$

Expressing $\sigma_1$, $\sigma_2$ and $\sigma_3$ via $p$, $r$ and $R$, we obtain

$$F^2 = -4r^2[(p^2 - 2R^2 - 10Rr + r^2)^2 - 4R(R - 2r)^3] \geq 0.$$

Thus, we obtain

$$p^2 \geq 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} =$$
$$[(R - 2r) - \sqrt{R(R - 2r)}]^2 + 16Rr - 5r^2 \geq 16Rr - 5r^2$$
$$p^2 \leq 2R^2 + 10Rr + r^2 + 2(R - 2r)\sqrt{R(R - 2r)} =$$
$$4R^2 + 4Rr + 3r^2 - [(R - 2r) - \sqrt{R(R - 2r)}]^2 \leq$$
$$4R^2 + 4Rr + 3r^2.$$

**10.35.** Since $r_a + r_b + r_c = 4R + r$ and $r_a r_b + r_b r_c + r_c r_a = p^2$ (Problems 12.24 and 12.25), it follows that $r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - 2p^2$. By Problem 10.34 $p^2 \leq 4R^2 + 4Rr + 3r^2$; hence, $r_a^2 + r_b^2 + r_c^2 = 8R^2 - 5r^2$. It remains to notice that $r \leq \frac{1}{2}R$ (Problem 10.26).

**10.36.** a) By Problem 12.38 $\cos\alpha + \cos\beta + \cos\gamma = \frac{R+r}{R}$. Moreover, $r \leq \frac{1}{2}R$ (Problem 10.26).

b) Follows from heading a), cf. Remark.

**10.37.** a) Clearly, $\sin\alpha + \sin\beta + \sin\gamma = \frac{p}{R}$. Moreover, $p \leq \frac{3}{2}\sqrt{3}R$ (Problem 10.29).

b) Follows from heading a), cf. Remark.

**10.38.** a) By Problem 12.44 a)

$$\cot\alpha + \cot\beta + \cot\gamma = \frac{a^2 + b^2 + c^2}{4S}.$$

Moreover, $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$ (Problem 10.53 b)).

b) Follows from heading a), cf. Remark.

**10.39.** a) By Problem 12.45 a)

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \frac{p}{r}.$$

Moreover, $p \geq 3\sqrt{3}r$ (Problem 10.53 a)).

b) Follows from heading a), cf. Remark. For an acute triangle $\tan\alpha + \tan\beta + \tan\gamma < 0$; cf., for instance, Problem 12.46.

**10.40.** a) By Problem 12.36 a)

$$\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} = \frac{r}{4R}.$$

Moreover, $r \leq \frac{1}{2}R$ (Problem 10.26).

b) For an acute triangle it follows from heading a), cf. Remark. For an obtuse triangle $\cos\alpha\cos\beta\cos\gamma < 0$.

**10.41.** a) Since $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, we see that making use of results of Problems 12.36 a) and 12.36 c) we obtain $\sin\alpha\sin\beta\sin\gamma = \frac{pr}{2R^2}$. Moreover, $p \leq \frac{3}{2}\sqrt{3}R$ (Problem 10.29) and $r \leq \frac{1}{2}R$ (Problem 10.26).

b) Follows from heading a), cf. Remark.

**10.42.** By Problem 12.39 b)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 - 2\cos\alpha\cos\beta\cos\gamma.$$

It remains to notice that $\cos\alpha\cos\beta\cos\gamma \leq \frac{1}{8}$ (Problem 10.40 b)) and for an obtuse triangle $\cos\alpha\cos\beta\cos\gamma < 0$.

**10.43.** Clearly,

$$2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha) =$$
$$(\cos\alpha + \cos\beta + \cos\gamma)^2 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma.$$

It remains to notice that $\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$ (Problem 10.36 a)) and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \geq \frac{3}{4}$ (Problem 10.42).

**10.44.** Let the extensions of bisectors of acute triangle $ABC$ with angles $\alpha$, $\beta$ and $\gamma$ intersect the circumscribed circle at points $A_1$, $B_1$ and $C_1$, respectively. Then

$$S_{ABC} = \frac{R^2(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)}{2};$$

$$S_{A_1B_1C_1} = \frac{R^2(\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha))}{2}.$$

It remains to make use of results of Problems 12.72 and 10.26.

**10.45.** Clearly,

$$2\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \cos\frac{\beta - \gamma}{2} - \cos\frac{\beta + \gamma}{2} \leq 1 - \sin\frac{\alpha}{2}.$$

**10.46.** Let us drop perpendiculars $AA_1$ and $BB_1$ from vertices $A$ and $B$ to the bisector of angle $\angle ACB$. Then $AB \geq AA_1 + BB_1 = b\sin\frac{\gamma}{2} + a\sin\frac{\gamma}{2}$.

**10.47.** By Problem 12.32 $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{a+b-c}{a+b+c}$. Since $a+b < 3c$, it follows that $a+b-c < \frac{1}{2}(a+b+c)$.

**10.48.** Since $\pi - 2\alpha > 0$, $\pi - 2\beta > 0$, $\pi - 2\gamma > 0$ and $(\pi - 2\alpha) + (\pi - 2\beta) + (\pi - 2\gamma) = \pi$, it follows that there exists a triangle whose angles are $\pi - 2\alpha$, $\pi - 2\beta$, $\pi - 2\gamma$. The lengths of sides opposite to angles $\pi - 2\alpha$, $\pi - 2\beta$, $\pi - 2\gamma$ are proportional to $\sin(\pi - 2\alpha) = \sin 2\alpha$, $\sin 2\beta$, $\sin 2\gamma$, respectively. Since $\pi - 2\alpha > \pi - 2\beta > \pi - 2\gamma$ and the greater angle subtends the longer side, $\sin 2\alpha > \sin 2\beta > \sin 2\gamma$.

**10.49.** First, notice that

$$\cos 2\gamma = \cos 2(\pi - \alpha - \beta) = \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta.$$

Hence,

$$\cos 2\alpha + \cos 2\beta - \cos 2\gamma = \cos 2\alpha + \cos 2\beta - \cos 2\alpha \cos 2\beta + \sin 2\alpha \sin 2\beta.$$

Since $a \cos \varphi + b \sin \varphi \leq \sqrt{a^2 + b^2}$ (cf. Supplement to Ch. 9), it follows that

$$(1 - \cos 2\beta) \cos 2\alpha + \sin 2\beta \sin 2\alpha + \cos 2\beta \leq$$
$$\sqrt{(1 - \cos 2\beta)^2 + \sin^2 2\beta} + \cos 2\beta = 2|\sin \beta| + 1 - 2\sin^2 \beta.$$

It remains to notice that the greatest value of the quadratic $2t + 1 - 2t^2$ is attained at point $t = \frac{1}{2}$ and this value is equal to $\frac{3}{2}$. The maximal value corresponds to angles $\alpha = \beta = 30°$ and $\gamma = 120°$.

**10.50.** Since $AB < CB$, $AX < CX = S_{ABX} = S_{BCX}$, it follows that $\sin \angle XAB > \sin \angle XCB$. Taking into account that angle $\angle XCB$ is an acute one, we get the desired statement.

**10.51.** If the angles of triangle $ABC$ are equal to $\alpha$, $\beta$ and $\gamma$, then the angles of triangle $A_1B_1C_1$ are equal to $\frac{1}{2}(\beta + \gamma)$, $\frac{1}{2}(\gamma + \alpha)$ and $\frac{1}{2}(\alpha + \beta)$.

**10.52.** Let $M$ be the intersection point of medians $AA_1$, $BB_1$ and $CC_1$. Complementing triangle $AMB$ to parallelogram $AMBN$ we get $\angle BMC_1 = \alpha_m$ and $\angle AMC_1 = \beta_m$. It is easy to verify that $\angle C_1CB < \frac{1}{2}\gamma$ and $\angle B_1BC < \frac{1}{2}\beta$. It follows that $\alpha_m = \angle C_1CB + \angle B_1BC < \frac{1}{2}(\beta + \gamma) < \beta$. Similarly, $\gamma_m = \angle A_1AB + \angle B_1BA > \frac{1}{2}(\alpha + \beta) > \beta$.

First, suppose that triangle $ABC$ is an acute one. Then the heights' intersection point $H$ lies inside triangle $AMC_1$. Hence, $\angle AMB < \angle AHB$, i.e., $\pi - \gamma_m < \pi - \gamma$ and $\angle CMB < \angle CHB$, i.e., $\pi - \alpha_m > \pi - \alpha$. Now, suppose that angle $\alpha$ is an obtuse one. Then angle $CC_1B$ is also an obtuse one and therefore, angle $\alpha_m$ is an acute one, i.e., $\alpha_m < \alpha$. Let us drop perpendicular $MX$ from point $M$ to $BC$. Then $\gamma_m > \angle XMB > 180° - \angle HAB > \gamma$.

Since $\alpha > \alpha_m$, it follows that $\alpha + (\pi - \alpha_m) > \pi$, i.e., point $M$ lies inside the circumscribed circle of triangle $AB_1C_1$. Therefore, $\gamma = \angle AB_1C_1 < \angle AMC_1 = \beta_m$. Similarly, $\alpha = \angle CB_1A_1 > \angle CMA_1 = \beta_m$ because $\gamma + (\pi - \gamma_m) < \pi$.

**10.53** a) Clearly,

$$\frac{S^2}{p} = (p - a)(p - b)(p - c) \leq \left(\frac{p - a + p - b + p - c}{3}\right)^2 = \frac{p^3}{27}.$$

Hence, $pr = S \leq \frac{p^2}{3\sqrt{3}}$, i.e., $r \leq \frac{p}{3\sqrt{3}}$. By multiplying the latter inequality by $r$ we get the desired statement.

b) Since $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, it follows that

$$S \leq \frac{p^2}{3\sqrt{3}} = \frac{(a + b + c)^2}{12\sqrt{3}} \leq \frac{a^2 + b^2 + c^2}{4\sqrt{3}}.$$

**10.54.** Let $x = p - a$, $y = p - b$, $z = p - c$. Then

$$(a^2 - (b - c)^2) + (b^2 - (a - c)^2) + (c^2 - (a - b)^2) =$$
$$4(p - b)(p - c) + 4(p - a)(p - c) + 4(p - a)(p - b) =$$
$$4(yz + zx + xy)$$

and

$$4\sqrt{3}S = 4\sqrt{3p(p - a)(p - b)(p - c)} = 4\sqrt{3(x + y + z)xyz}.$$

Thus, we have to prove that $xy + yz + zx \geq \sqrt{3(x + y + z)xyz}$. After squaring and simplification we obtain

$$x^2y^2 + y^2z^2 + z^2x^2 \geq x^2yz + y^2xz + z^2xy.$$

Adding inequalities

$$x^2yz \leq \frac{x^2(y^2 + z^2)}{2}, \quad y^2xz \leq \frac{y^2(x^2 + z^2)}{2} \quad \text{and} \quad z^2xy \leq \frac{z^2(x^2 + y^2)}{2}$$

we get the desired statement.

**10.55.** a) By multiplying three equalities of the form $S = \frac{1}{2}ab\sin\gamma$ we get $S^3 = \frac{1}{8}(abc)^2 \sin\gamma \sin\beta \sin\alpha$. It remains to make use of a result of Problem 10.41.

b) Since $(h_a h_b h_c)^2 = \frac{(2S)^6}{(abc)^2}$ and $(abc)^2 \geq \left(\frac{4}{\sqrt{3}}\right)^3 S^3$, it follows that $(h_a h_b h_c)^2 \leq \frac{(2S)^6(\sqrt{3}/4)^3}{S^3} = (\sqrt{3}S)^3$.

Since $(r_a r_b r_c)^2 = \frac{S^4}{r^2}$ (Problem 12.18, c) and $r^2(\sqrt{3})^3 \leq S$ (Problem 10.53 a), it follows that $(r_a r_b r_c)^2 \geq (\sqrt{3}S)^3$.

**10.56.** Let $p = \frac{BA}{BC}$, $q = \frac{CB_1}{CA}$ and $r = \frac{AC_1}{AC}$. Then

$$\frac{S_{A_1 B_1 C_1}}{S_{ABC}} = 1 - p(1 - r) - q(1 - p) - r(1 - q) = 1 - (p + q + r) + (pq + qr + rp).$$

By Cheva's theorem (Problem 5.70) $pqr = (1 - p)(1 - q)(1 - r)$, i.e., $2pqr = 1 - (p + q + r) + (pq + qr + rp)$. Moreover,

$$(pqr)^2 = p(1 - p)(1 - q)r(1 - r) \leq \left(\frac{1}{4}\right)^3.$$

Therefore, $\frac{S_{A_1 B_1 C_1}}{S_{ABC}} = 2pqr \leq \frac{1}{4}$.

**10.57.** We can assume that the area of triangle $ABC$ is equal to 1. Then $a + b + c = 1$ and, therefore, the given inequality takes the form $u^2 \geq 4abc$. Let $x = \frac{BA_1}{BC}$, $y = \frac{CB_1}{CA}$ and $z = \frac{AC_1}{AB}$. Then

$$u = 1 - (x + y + z) + xy + yz + zx \quad \text{and} \quad abc = xyz(1 - x)(1 - y)(1 - z) = v(u - v),$$

where $v = xyz$. Therefore, we pass to inequality $u^2 \geq 4v(u - v)$, i.e., $(u - 2v)^2 \geq 0$ which is obvious.

**10.58.** a) Let $x = \frac{BA_1}{BC}$, $y = \frac{BC_1}{BA}$ and $z = \frac{AC_1}{AB}$. We may assume that the area of triangle $ABC$ is equal to 1. Then $S_{AB_1 C_1} = z(1 - y)$, $S_{A_1 BC_1} = x(1 - z)$ and $S_{A_1 B_1 C} = y(1 - x)$. Since $x(1 - x) \leq \frac{1}{4}$, $y(1 - y) \leq \frac{1}{4}$ and $z(1 - z) \leq \frac{1}{4}$, it follows that the product of numbers $S_{AB_1 C_1}$, $S_{A_1 BC_1}$ and $S_{A_1 B_1 C}$ does not exceed $\left(\frac{1}{4}\right)^3$; hence, one of them does not exceed $\frac{1}{4}$.

b) Let, for definiteness, $x \geq \frac{1}{2}$. If $y \leq \frac{1}{2}$, then the homothety with center $C$ and coefficient 2 sends points $A_1$ and $B_1$ to inner points on sides $BC$ and $AC$, consequently, $S_{A_1 B_1 C} \leq S_{A_1 B_1 C_1}$. Hence, we can assume that $y \geq \frac{1}{2}$ and, similarly, $z \geq \frac{1}{2}$. Let $x = \frac{1}{2}(1 + \alpha)$, $y = \frac{1}{2}(1 + \beta)$ and $z = \frac{1}{2}(1 + \gamma)$. Then $S_{AB_1 C_1} = \frac{1}{4}(1 + \gamma - \beta - \beta\gamma)$, $S_{A_1 B_1 C_1} = \frac{1}{4}(1 + \alpha - \gamma - \alpha\gamma)$ and $S_{A_1 B_1 C} = \frac{1}{4}(1 + \beta - \alpha - \alpha\beta)$; hence, $S_{A_1 B_1 C_1} = \frac{1}{4}(1 + \alpha\beta + \beta\gamma + \alpha\gamma) \geq \frac{1}{4}$ and $S_{AB_1 C_1} + S_{A_1 BC_1} + S_{A_1 B_1 C} \leq \frac{3}{4}$.

**10.59.** It suffices to prove that if $AC < BC$, then $\angle ABC < \angle BAC$. Since $AC < BC$, on side $BC$ point $A_1$ can be selected so that $A_1C = AC$. Then $\angle BAC < \angle AAC = \angle AA_1C > \angle ABC$.

**10.60.** Let $A_1$ be the midpoint of side $BC$. If $AA_1 < \frac{1}{2}BC = BA_1 = A_1C$, then $\angle BAA_1 > \angle ABA_1$ and $\angle CAA_1 > \angle ACA_1$; hence, $\angle A = \angle BAA_1 + \angle CAA_1 < \angle B + \angle C$, i.e., $\angle A > 90°$. Similarly, if $AA_1 > \frac{1}{2}BC$ then $\angle A > 90°$.

**10.61.** If we fix two sides of the triangle, then the greater the angle between these sides the longer the third side. Therefore, inequality $\angle A > \angle A_1$ implies that $BD > B_1D_1$, i.e., $\angle C < \angle C_1$. Now, suppose that $\angle B \geq \angle B_1$. Then $AC \geq A_1C_1$, i.e., $\angle D \geq \angle D_1$. Hence,

$$360° = \angle A + \angle B + \angle C + \angle D > \angle A_1 + \angle B_1 + \angle C_1 + \angle D_1 = 360°.$$

Contradiction; therefore, $\angle B < \angle B_1$ and $\angle D < \angle D_1$.

**10.62.** Let point $B_1$ be symmetric to $B$ through point $M$. Since the height dropped from point $M$ to side $BC$ is equal to a half of $AH$, i.e., to a half of $BM$, it follows that $\angle MBC = 30°$. Since $AH$ is the longest of heights, $BC$ is the shortest of sides. Hence, $AB_1 = BC \leq AB$, i.e., $\angle ABB_1 \leq \angle AB_1B = \angle MBC = 30°$. Therefore, $\angle ABC = \angle ABB_1 + \angle MBC \leq 30° + 30° = 60°$.

**10.63.** First, let us suppose that $\angle A > \angle D$. Then $BE > EC$ and $\angle EBA < \angle ECD$. Since in triangle $EBC$ side $BE$ is longer than side $EC$, it follows that $\angle EBC < \angle ECB$. Therefore,

$$\angle B = \angle ABE + \angle EBC < \angle ECD + \angle ECB = \angle C$$

which contradicts the hypothesis. Thus, $\angle A = \angle B = \angle C = \angle D$. Similarly, the assumption $\angle B > \angle E$ leads to inequality $\angle C < \angle D$. Hence, $\angle B = \angle C = \angle D = \angle E$.

**10.64.** Let us carry out the proof for the general case. Let line $MN$ intersect the sides of the polygon at points $M_1$ and $N_1$. Clearly, $MN \leq M_1N_1$. Let point $M_1$ lie on side $AB$ and point $N_1$ lie on $PQ$. Since $\angle AM_1N_1 + \angle BM_1N_1 = 180°$, one of these angles is not less than $90°$. Let, for definiteness, $\angle AM_1N_1 \geq 90°$. Then $AN_1 \geq M_1N_1$ because the longer side subtends the greater angle.

We similarly prove that either $AN_1 \leq AP$ or $AN_1 \leq AQ$. Therefore, the length of segment $MN$ does not exceed the length of a segment with the endpoints at vertices of the polygon.

**10.65.** The segment can be extended to its intersection with the boundary of the sector because this will only increase its length. Therefore, we may assume that points $M$ and $N$ lie on the boundary of the disk sector. The following three cases are possible:

1) Points $M$ and $N$ lie on an arc of the circle. Then

$$MN = 2R\sin\frac{\angle MON}{2} \leq 2R\sin\frac{\angle AOB}{2} = AB$$

because $\frac{1}{2}\angle MON \leq \frac{1}{2}\angle AOB \leq 90°$.

2) Points $M$ and $N$ lie on segments $AO$ and $BO$, respectively. Then $MN$ is not longer than the longest side of triangle $AOB$.

3) One of points $M$ and $N$ lies on an arc of the circle, the other one on one of segments $AO$ or $BO$. Let, for definiteness, $M$ lie on $AO$ and $N$ on an arc of the circle. Then the length of $MN$ does not exceed that of the longest side of triangle $ANO$. It remains to notice that $AO = NO = R$ and $AN \leq AB$.

**10.66.** If the given segment has no common points with the circle, then a homothety with center $A$ (and coefficient greater than 1) sends it into a segment that has a common point $X$ with arc $AB$ and lies in our domain. Let us draw through point $X$ tangent $DE$ to the circle (points $D$ and $E$ lie on segments $AB$ and $AC$). Then segments $AD$ and $AE$ are

shorter than $AB$ and $DE < \frac{1}{2}(DE + AD + AE) = AB$, i.e., each side of triangle $ADE$ is shorter than $AB$. Since our segment lies inside triangle $ADE$ (or on its side $DE$), its length does not exceed that of $AB$.

**10.67.** First, suppose that the center $O$ of the circle lies inside the given pentagon $A_1A_2A_3A_4A_5$. Consider angles $\angle A_1OA_2$, $\angle A_2OA_3$, ..., $\angle A_5OA_1$. The sum of these five angles is equal to $2\pi$; hence, one of them, say, $\angle A_1OA_2$, does not exceed $\frac{2}{5}\pi$. Then segment $A_1A_2$ can be placed in disk sector $OBC$, where $\angle BOC = \frac{2}{5}\pi$ and points $B$ and $C$ lie on the circle. In triangle $OBC$, side $BC$ is the longest one; hence, $A_1A_2 \le BC$.

If point $O$ does not belong to the given pentagon, then the union of angles $\angle A_1OA_2$, ..., $\angle A_5OA_1$ is less than $\pi$ and each point of the angle — the union — is covered twice by these angles. Therefore, the sum of these five angles is less than $2\pi$, i.e., one of them is less than $\frac{2}{5}\pi$. The continuation of the proof is similar to the preceding case.

If point $O$ lies on a side of the pentagon, then one of the considered angles is not greater than $\frac{1}{4}\pi$ and if it is its vertex, then one of them is not greater than $\frac{1}{3}\pi$. Clearly, $\frac{1}{4}\pi < \frac{1}{3}\pi < \frac{2}{5}\pi$.
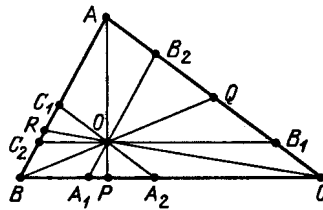


Figure 121 (Sol. 10.68)

**10.68.** On sides $BC$, $CA$, $AB$ take points $A_1$ and $A_2$, $B_1$ and $B_2$, $C_1$ and $C_2$, respectively, so that $B_1C_2 \parallel BC$, $C_1A_2 \parallel CA$, $A_1B_2 \parallel AB$ (Fig. 121). In triangles $A_1A_2O$, $B_1B_2O$, $C_1C_2O$ sides $A_1A_2$, $B_1O$, $C_2O$, respectively, are the longest ones. Hence, $OP < A_1A_2$, $OQ < BO$, $OR \le C_2O$, i.e.,

$$OP + OQ + OR < A_1A_2 + B_1O + C_2O = A_1A_2 + CA_2 + BA_1 = BC.$$

**10.69.** Since $c^2 = a^2 + b^2$, it follows that

$$c^n = (a^2 + b^2)c^{n-2} = a^2c^{n-2} + b^2c^{n-2} > a^n + b^n.$$

**10.70.** The height of any of the triangles considered is longer than $2r$. Moreover, in a right triangle $2r = a + b - c$ (Problem 5.15).

**10.71.** Since $ch = 2S = r(a + b + c)$ and $c = \sqrt{a^2 + b^2}$, it follows that $\frac{r}{h} = \frac{\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}} = \frac{1}{x+1}$, where $x = \frac{a+b}{\sqrt{a^2+b^2}} = \sqrt{1 + \frac{2ab}{a^2+b^2}}$. Since $0 < \frac{2ab}{a^2+b^2} \le 1$, it follows that $1 < x \le \sqrt{2}$. Hence, $\frac{2}{5} < \frac{1}{1+\sqrt{2}} \le \frac{r}{h} < \frac{1}{2}$.

**10.72.** Clearly, $a + b \ge 2\sqrt{ab}$ and $c^2 + a^2 + b^2 \ge 2ab$. Hence,

$$\frac{c^2}{r^2} = \frac{(a + b + c)^2 c^2}{a^2b^2} \ge \frac{(2\sqrt{ab} + \sqrt{2ab})^2 \cdot 2ab}{a^2b^2} = 4(1 + \sqrt{2})^2.$$

**10.73.** By Problem 12.11 a) $m_a^2 + m_b^2 = \frac{1}{4}(4c^2 + a^2 + b^2) = \frac{5}{4}c^2$. Moreover,

$$\frac{5c^2}{4} \ge 5(1 + \sqrt{2})^2 r^2 = (15 + 10\sqrt{2})r^2 > 29r^2,$$

cf. Problem 10.72.

**10.74.** Let $O$ be the center of the circumscribed circle, $A_1$, $B_1$, $C_1$ the midpoints of sides $BC$, $CA$, $AB$, respectively. Then $m_a = AA_1 \le AO + OA_1 = R + OA_1$. Similarly,

$m_b \leq R + OB_1$ and $m_c \leq R + OC_1$. Hence,

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq R\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) + \frac{OA_1}{h_a} + \frac{OB_1}{h_b} + \frac{OC_1}{h_c}.$$

It remains to make use of the result of Problem 12.22 and the solution of Problem 4.46.

**10.75.** By Problem 4.47 $\frac{1}{b} + \frac{1}{c} = \frac{2\cos(\alpha/2)}{l_a} \geq \frac{\sqrt{2}}{l_a}$. Adding three analogous inequalities we get the required statement.

**10.76.** Denote the intersection point of medians by $M$ and the center of the circumscribed circle by $O$. If triangle $ABC$ is not an obtuse one, then point $O$ lies inside it (or on its side); let us assume, for definiteness, that it lies inside triangle $AMB$. Then $AO + BO \leq AM + BM$, i.e., $2R \leq \frac{2}{3}m_a + \frac{2}{3}m_b$ or, which is the same, $m_a + m_b \geq 3R$. It remains to notice that since angle $\angle COC_1$ (where $C_1$ is the midpoint of $AB$) is obtuse, it follows that $CC_1 \geq CO$, i.e., $m_c \geq R$.

The equality is attained only for a degenerate triangle.

**10.77.** In any triangle $h_b \leq l_b \leq m_b$ (cf. Problem 2.67); hence, $h_a = l_b \geq h_b$ and $m_c = l_b \leq m_b$. Therefore, $a \leq b$ and $b \leq c$ (cf. Problem 10.1), i.e., $c$ is the length of the longest side and $\gamma$ is the greatest angle.

The equality $h_a = m_c$ yields $\gamma \leq 60°$ (cf. Problem 10.62). Since the greatest angle $\gamma$ of triangle $ABC$ does not exceed $60°$, all the angles of the triangle are equal to $60°$.

**10.78.** By Problem 1.59 the ratio of the perimeters of triangles $A_1B_1C_1$ and $ABC$ is equal to $\frac{r}{R}$. Moreover, $r \leq \frac{R}{2}$ (Problem 10.26).

REMARK. Making use of the result of Problem 12.72 it is easy to verify that $\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{r_1}{2R_1} \leq \frac{1}{4}$.

**10.79.** Let $90° \geq \alpha \geq \beta \geq \gamma$, then $CH$ is the longest height. Denote the centers of the inscribed and circumscribed circles by $I$ and $O$, the tangent points of the inscribed circle with sides $BC$, $CA$, $AB$ by $K$, $L$, $M$, respectively (Fig. 122).
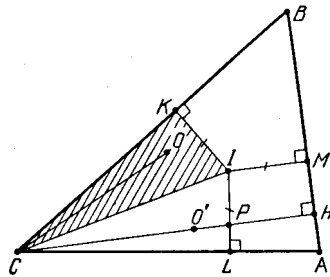


Figure 122 (Sol. 10.79)

First, let us prove that point $O$ lies inside triangle $KCI$. For this it suffices to prove that $CK \geq KB$ and $\angle BCO \leq \angle BCI$. Clearly, $CK = r\cot\frac{\gamma}{2} \geq r\cot\frac{\beta}{2} = KB$ and

$$2\angle BCO = 180° - \angle BOC = 180° - 2\alpha \leq 180° - \alpha - \beta = \gamma = 2\angle BCI.$$

Since $\angle BCO = 90° - \alpha = \angle ACH$, the symmetry through $CI$ sends line $CO$ to line $CH$. Let $O'$ be the image of $O$ under this symmetry and $P$ the intersection point of $CH$ and $IL$. Then $CP \geq CO' = CO = R$. It remains to prove that $PH \geq IM = r$. It follows from the fact that $\angle MIL = 180° - \alpha \geq 90°$.

**10.80.** Let $B_2C_2$ be the projection of segment $B_1C_1$ on side $BC$. Then

$$BC_1 \geq B_2C_2 = BC - BC_1\cos\beta - CB_1\cos\gamma.$$

Similarly,

$$A_1C_1 \geq AC - AC_1 \cos\alpha - CA_1 \cos\gamma;$$

$$A_1B_1 \geq AB - AB_1 \cos\alpha - BA_1 \cos\beta.$$

Let us multiply these inequalities by $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, respectively, and add them; we get

$$B_1C_1 \cos\alpha + C_1A_1 \cos\beta + AB_1 \cos\gamma \geq a\cos\alpha + b\cos\beta + c\cos\gamma - $$
$$-(a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma + c\cos\alpha\cos\beta).$$

Since $c = a\cos\beta + b\cos\alpha$, it follows that $c\cos\gamma = a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma$. Write three analogous inequalities and add them; we get

$$a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma + c\cos\alpha\cos\beta = \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{2}.$$

**10.81.** Since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1$$

(Problem 12.39 b)), it follows that triangle $ABC$ is an acute one if and only if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma < 1$, i.e., $\sin^2\alpha + \sin^2\beta + \sin^2\gamma > 2$. Multiplying both sides of the latter inequality by $4R^2$ we get the desired statement.

**10.82.** It suffices to notice that

$$p^2 - (2R + r)^2 = 4R^2 \cos\alpha\cos\beta\cos\gamma$$

(cf. Problem 12.41 b).

**10.83.** Let $\angle A \leq \angle B \leq \angle C$. If triangle $ABC$ is not an acute one, then $CC_1 < AC < AA_1$ for any points $A_1$ and $C_1$ on sides $BC$ and $AB$, respectively. Now, let us prove that for an acute triangle we can select points $A_1$, $B_1$ and $C_1$ with the required property. For this it suffices to verify that there exists a number $x$ satisfying the following inequalities:

$$h_a \leq x < \max(b, c) = c, \quad h_b \leq x < \max(a, c) = c \quad \text{and} \quad h_c \leq x < \max(a, b) = b.$$

It remains to notice that $\max(h_a, h_b, h_c) = h_a$, $\min(b, c) = b$ and $h_a < h$.

**10.84.** Let $\angle A \leq \angle B \leq \angle C$. First, suppose that triangle $ABC$ is an acute one. As line $l$ that in its initial position is parallel to $AB$ rotates, the length of the triangle's projection on $l$ first varies monotonously from $c$ to $h_b$, then from $h_b$ to $a$, then from $a$ to $h_c$, next from $h_c$ to $b$, then from $b$ to $h_a$ and, finally, from $h_a$ to $c$. Since $h_b < a$, there exists a number $x$ such that $h_b < x < a$. It is easy to verify that a segment of length $x$ is encountered on any of the first four intervals of monotonity.

Now, suppose that triangle $ABC$ is not an acute one. As line $l$ that in its initial position is parallel to $AB$ rotates, the length of the triangle's projection on $l$ monotonously decreases first from $c$ to $h_b$, then from $h_b$ to $h_c$; after that it monotonously increases, first, from $h_c$ to $h_a$, then from $h_a$ to $c$. Altogether we have two intervals of monotonity.

**10.85.** Let points $M$ and $N$ lie on sides $AB$ and $AC$, respectively. Let us draw through vertex $C$ the line parallel to side $AB$. Let $N_1$ be the intersection point of this line with $MN$. Then $N_1O : MO = 2$ but $NO \leq N_1O$; hence, $NO : MO \leq 2$.

**10.86.** Sircle $S$ inscribed in triangle $ABC$ lies inside triangle $A'B'C'$. Draw the tangents to this circle parallel to sides of triangle $A'B'C'$; we get triangle $A''B''C''$ similar to triangle $A'B'C'$ and $S$ is the inscribed circle of triangle $A''B''C''$. Hence, $r_{ABC} = r_{A''B''C''} < r_{A'B'C'}$.

**10.87.** The bisector $l_c$ divides triangle $ABC$ into two triangles whose doubled areas are equal to $al_c \sin\frac{\gamma}{2}$ and $bl_c \sin\frac{\gamma}{2}$. Hence, $ah_a = 2S = l_c(a + b)\sin\frac{\gamma}{2}$. The conditions of the problem imply that $\frac{a}{a+b} \leq \frac{1}{2} \leq \sin\frac{\gamma}{2}$.

**10.88.** Clearly, $\cot \angle A + \cot \angle B = \frac{c}{h_c} \geq \frac{c}{m_c}$. Let $M$ be the intersection point of the medians, $N$ the midpoint of segment $AB$. Since triangle $AMB$ is a right one, $MN = \frac{1}{2}AB$. Therefore, $c = 2MN = \frac{2}{3}m_c$.

**10.89.** Since $BN \cdot BA = BM^2$ and $BM < BA$, it follows that $BN < BM$ and, therefore, $AN > CN$.

**10.90.** Let us draw through point $B$ the perpendicular to side $AB$. Let $F$ be the intersection point of this perpendicular with the extension of side $AC$ (Fig. 123). Let us prove that bisector $AD$, median $BM$ and height $CH$ intersect at one point if and only if $AB = CF$. Indeed, let $L$ be the intersection point of $BM$ and $CH$. Bisector $AD$ passes through point $L$ if and only if $BA : AM = BL : LM$ but $BL : LM = FC : CM = FC : AM$.
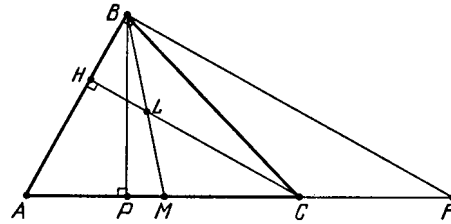


Figure 123 (Sol. 10.90)

If on side $AF$ of right triangle $ABF$ ($\angle ABF = 90°$) segment $CF$ equal to $AB$ is marked, then angles $\angle BAC$ and $\angle ABC$ are acute ones. It remains to find out when angle $\angle ACB$ is acute.

Let us drop perpendicular $BP$ from point $B$ to side $AF$. Angle $ACB$ is an acute one if $FP > FC = AB$, i.e., $BF \sin \angle A > BF \cot \angle A$. Therefore, $1 - \cos^2 \angle A = \sin^2 \angle A > \cos \angle A$, i.e., $\cos A < \frac{1}{2}(\sqrt{5} - 1)$. Finally, we see that

$$90° > \angle A > \arccos \frac{\sqrt{5} - 1}{2} \approx 51°50'.$$

**10.91.** Since the greater angle subtends the longer side,

$$(a - b)(\alpha - \beta) \geq 0, \ (b - c)(\beta - \gamma) \geq 0 \text{ and } (a - c)(\alpha - \gamma) \geq 0.$$

Adding these inequalities we get

$$2(a\alpha + b\beta + c\gamma) \geq a(\beta + \gamma) + b(\alpha + \gamma) + c(\alpha + \beta) =$$
$$(a + b + c)\pi - a\alpha - b\beta - c\gamma,$$

i.e., $\frac{1}{3}\pi \leq \frac{a\alpha + b\beta + c\gamma}{a + b + c}$. The triangle inequality implies that

$$\alpha(b + c - a) + \beta(a + c - b) + \gamma(a + b - c) > 0,$$

i.e.,

$$a(\beta + \gamma - \alpha) + b(\alpha + \gamma - \beta) + c(\alpha + \beta - \gamma) > 0.$$

Since $\alpha + \beta + \gamma = \pi$, it follows that $a(\pi - 2\alpha) + b(\pi - 2\beta) = c(\pi - 2\gamma) > 0$, i.e., $\frac{a\alpha + b\beta + c\gamma}{a + b + c} < \frac{1}{2}\pi$.

**10.92.** On rays $OB$ and $OC$, take points $C_1$ and $B_1$, respectively, such that $OC_1 = OC$ and $OB_1 = OB$. Let $B_2$ and $C_2$ be the projections of points $B_1$ and $C_1$, respectively, on a line perpendicular to $AO$. Then

$$BO \sin \angle AOC + CO \sin \angle AOB = B_2C_2 \leq BC.$$

Adding three analogous inequalities we get the desired statement. It is also easy to verify that the conditions $B_1C_1 \perp AO$, $C_1A_1 \perp BO$ and $A_1B_1 \perp CO$ are equivalent to the fact that $O$ is the intersection point of the bisectors.

**10.93.** Since $\angle CBD = \frac{1}{2}\angle C$ and $\angle B \leq \angle A$, it follows that $\angle ABD = \angle B + \angle CBD \geq \frac{1}{2}(\angle A + \angle B + \angle C) = 90°$.

**10.94.** By the bisector's property, $BM : MA = BC : CA$ and $BK : KC = BA : AC$. Hence, $BM : MA < BK : KC$, i.e.,

$$\frac{AB}{AM} = 1 + \frac{BM}{MA} < 1 + \frac{BK}{KC} = \frac{CB}{CK}.$$

Therefore, point $M$ is more distant from line $AC$ than point $K$, i.e., $\angle AKM < \angle KAC = \angle KAM$ and $\angle KMC < \angle MCA = \angle MCK$. Hence, $AM > MK$ and $MK > KC$, cf. Problem 10.59.

**10.95.** Suppose that all the given ratios are less than 2. Then

$$S_{ABO} + S_{AOC} < 2S_{XBO} + 2S_{XOC} = 2S_{OBC},$$

$$S_{ABO} + S_{OBC} < 2S_{AOC}, \ S_{AOC} + S_{OBC} < 2S_{ABO}.$$

Adding these inequalities we come to a contradiction. We similarly prove that one of the given ratios is not greater than 2.

**10.96.** Denote the radii of the circles $S$, $S_1$ and $S_2$ by $r$, $r_1$ and $r_2$, respectively. Let triangles $AB_1C_1$ and $A_2BC_2$ be similar to triangle $ABC$ with similarity coefficients $\frac{r_1}{r}$ and $\frac{r_2}{r}$, respectively. Circles $S_1$ and $S_2$ are the inscribed circles of triangles $AB_1C_1$ and $A_2BC_2$, respectively. Therefore, these triangles intersect because otherwise circles $S_1$ and $S_2$ would not have had common points. Hence, $AB_1 + A_2B > AB$, i.e., $r_1 + r_2 > r$.

# Chapter 11. PROBLEMS ON MAXIMUM AND MINIMUM

## Background

1) Geometric problems on maximum and minimum are in close connection with geometric inequalities because in order to solve these problems we always have to prove a corresponding geometric inequality and, moreover, to prove that sometimes it turns into an equality. Therefore, before solving problems on maximum and minimum we have to skim through Supplement to Ch. 9 once again with the special emphasis on the conditions under which strict inequalities become equalities.

2) For elements of a triangle we use the standard notations.

3) Problems on maximum and minimum are sometimes called *extremal problems* (from Latin *extremum*).

## Introductory problems

**1.** Among all triangles $ABC$ with given sides $AB$ and $AC$ find the one with the greatest area.

**2.** Inside triangle $ABC$ find the vertex of the smallest angle that subtends side $AB$.

**3.** Prove that among all triangles with given side $a$ and height $h_a$. an isosceles triangle is the one with the greatest value of angle $\alpha$.

**4.** Among all triangles with given sides $AB$ and $AC$ ($AB < AC$), find the one for which the radius of the circumscribed circle is maximal.

**5.** The iagonals of a convex quadrilateral are equal to $d_1$ and $d_2$. What the greatest value the quadrilateral's area can attain?

## §1. The triangle

**11.1.** Prove that among all the triangles with fixed angle $\alpha$ and area $S$, an isosceles triangle with base $BC$ has the shortest length of side $BC$.

**11.2.** Prove that among all triangles with fixed angle $\alpha$ and semiperimeter $p$, an isosceles triangle with base $BC$ is of the greatest area.

**11.3.** Prove that among all the triangles with fixed semiperimeter $p$, an equilateral triangle has the greatest area.

**11.4.** Consider all the acute triangles with given side $a$ and angle $\alpha$. What is the maximum of $b^2 + c^2$?

**11.5.** Among all the triangles inscribed in a given circle find the one with the maximal sum of squared lengths of the sides.

**11.6.** The perimeter of triangle $ABC$ is equal to $2p$. On sides $AB$ and $AC$ points $M$ and $N$, respectively, are taken so that $MN \parallel BC$ and $MN$ is tangent to the inscribed circle of triangle $ABC$. Find the greatest value of the length of segment $MN$.

**11.7.** Into a given triangle place a centrally symmetric polygon of greatest area.

**11.8.** The area of triangle $ABC$ is equal to 1. Let $A_1$, $B_1$, $C_1$ be the midpoints of sides $BC$, $CA$, $AB$, respectively. On segments $AB_1$, $CA_1$, $BC_1$, points $K$, $L$, $M$, respectively, are taken. What is the least area of the common part of triangles $KLM$ and $A_1B_1C_1$?

**11.9.** What least width From an infinite strip of paper any triangle of area 1 can be cut. What is the least width of such a strip?

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**11.10.** Prove that triangles with the lengths of sides $a$, $b$, $c$ and $a_1$, $b_1$, $c_1$, respectively, are similar if and only if

$$\sqrt{aa_1} + \sqrt{bb_1} + \sqrt{cc_1} = \sqrt{(a+b+c)(a_1+b_1+c_1)}.$$

**11.11.** Prove that if $\alpha$, $\beta$, $\gamma$ and $\alpha_1$, $\beta_1$, $\gamma_1$ are the respective angles of two triangles, then

$$\frac{\cos \alpha_1}{\sin \alpha} + \frac{\cos \beta_1}{\sin \beta} + \frac{\cos \gamma_1}{\sin \gamma} \leq \cot \alpha + \cot \beta + \cot \gamma.$$

**11.12.** Let $a$, $b$ and $c$ be the lengths of the sides of a triangle of area $S$; let $\alpha_1$, $\beta_1$ and $\gamma_1$ be the angles of another triangle. Prove that

$$a^2 \cot \alpha_1 + b^2 \cot \beta_1 + c^2 \cot \gamma_1 \geq 4S,$$

where the equality is attained only if the considered triangles are similar.

**11.13.** In a triangle $a \geq b \geq c$; let $x$, $y$ and $z$ be the angles of another triangle. Prove that

$$bc + ca - ab < bc \cos x + ca \cos y + ab \cos z \leq \frac{a^2 + b^2 + c^2}{2}.$$

See also Problem 17.21.

## §2. Extremal points of a triangle

**11.14.** On hypothenuse $AB$ of right triangle $ABC$ point $X$ is taken; $M$ and $N$ are the projections of $X$ on legs $AC$ and $BC$, respectively.

a) What is the position of $X$ for which the length of segment $MN$ is the smallest one?

b) What is the position of point $X$ for which the area of quadrilateral $CMXN$ is the greatest one?

**11.15.** From point $M$ on side $AB$ of an acute triangle $ABC$ perpendiculars $MP$ and $MQ$ are dropped to sides $BC$ and $AC$, respectively. What is the position of point $M$ for which the length of segment $PQ$ is the minimal one?

**11.16.** Triangle $ABC$ is given. On line $AB$ find point $M$ for which the sum of the radii of the circumscribed circles of triangles $ACM$ and $ACN$ is the least possible one.

**11.17.** From point $M$ of the circumscribed circle of triangle $ABC$ perpendiculars $MP$ and $MQ$ are dropped on lines $AB$ and $AC$, respectively. What is the position of point $M$ for which the length of segment $PQ$ is the maximal one?

**11.18.** Inside triangle $ABC$, point $O$ is taken. Let $d_a$, $d_b$, $d_c$ be distances from it to lines $BC$, $CA$, $AB$, respectively. What is the position of point $O$ for which the product $d_a d_b d_c$ is the greatest one?

**11.19.** Points $A_1$, $B_1$ and $C_1$ are taken on sides $BC$, $CA$ and $AB$, respectively, of triangle $ABC$ so that segments $AA_1$, $BB_1$ and $CC_1$ meet at one point $M$. For what position of point $M$ the value of $\frac{MA_1}{AA_1} \cdot \frac{MB_1}{BB_1} \cdot \frac{MC_1}{CC_1}$ is the maximal one?

**11.20.** From point $M$ inside given triangle $ABC$ perpendiculars $MA_1$, $MB_1$, $MC_1$ are dropped to lines $BC$, $CA$, $AB$, respectively. What are points $M$ inside the given triangle $ABC$ for which the quantity $\frac{a}{MA_1} + \frac{b}{MB_1} + \frac{c}{MC_1}$ takes the least possible value?

**11.21.** Triangle $ABC$ is given. Find a point $O$ inside of it for which the sum of lengths of segments $OA$, $OB$, $OC$ is the minimal one. (Take a special heed to the case when one of the angles of the triangle is greater than $120°$.)

**11.22.** Inside triangle $ABC$ find a point $O$ for which the sum of squares of distances from it to the sides of the triangle is the minimal one.

See also Problem 18.21 a).

## §3. The angle

**11.23.** On a leg of an acute angle points $A$ and $B$ are given. On the other leg construct point $C$ the vertex of the greatest angle that subtends segment $AB$.

**11.24.** Angle $\angle XAY$ and point $O$ inside it are given. Through point $O$ draw a line that cuts off the given angle a triangle of the least area.

**11.25.** Through given point $P$ inside angle $\angle AOB$ draw line $MN$ so that the value $OM + ON$ is minimal (points $M$ and $N$ lie on legs $OA$ and $OB$, respectively).

**11.26.** Angle $\angle XAY$ and a circle inside it are given. On the circle construct a point the sum of the distances from which to lines $AX$ and $AY$ is the least.

**11.27.** A point $M$ inside acute angle $\angle BAC$ is given. On legs $BA$ and $AC$ construct points $X$ and $Y$, respectively, such that the perimeter of triangle $XYM$ is the least.

**11.28.** Angle $\angle XAY$ is given. The endpoints $B$ and $C$ of unit segments $BO$ and $CO$ move along rays $AX$ and $AY$, respectively. Construct quadrilateral $ABOC$ of the greatest area.

## §4. The quadrilateral

**11.29.** Inside a convex quadrilateral find a point the sum of distances from which to the vertices were the least one.

**11.30.** The diagonals of convex quadrilateral $ABCD$ intersect at point $O$. What least area can this quadrilateral have if the area of triangle $AOB$ is equal to 4 and the area of triangle $COD$ is equal to 9?

**11.31.** Trapezoid $ABCD$ with base $AD$ is cut by diagonal $AC$ into two triangles. Line $l$ parallel to the base cuts these triangles into two triangles and two quadrilaterals. What is the position of line $l$ for which the sum of areas of the obtained triangles is the minimal one?

**11.32.** The area of a trapezoid is equal to 1. What is the least value the length of the longest diagonal of this trapezoid can attain?

**11.33.** On base $AD$ of trapezoid $ABCD$ point $K$ is given. On base $BC$ find point $M$ for which the area of the common part of triangles $AMD$ and $BKC$ is maximal.

**11.34.** Prove that among all quadrilaterals with fixed lengths of sides an inscribed quadrilateral has the greatest area.

See also Problems 9.35, 15.3 b).

## §5. Polygons

**11.35.** A polygon has a center of symmetry, $O$. Prove that the sum of the distances from a point to the vertices attains its minimum at point $O$.

**11.36.** Among all the polygons inscribed in a given circle find the one for which the sum of squared lengths of its sides is minimal.

**11.37.** A convex polygon $A_1 \ldots A_n$ is given. Prove that a point of the polygon for which the sum of distances from it to all the vertices is maximal is a vertex.

See also Problem 6.69.

## §6. Miscellaneous problems

**11.38.** Inside a circle centered at $O$ a point $A$ is given. Find point $M$ on the circle for which angle $\angle OMA$ is maximal.

**11.39.** In plane, line $l$ and points $A$ and $B$ on distinct sides of $l$ are given. Construct a circle that passes through points $A$ and $B$ so that line $l$ intercepts on the circle a shortest chord.

**11.40.** Line $l$ and points $P$ and $Q$ lying on one side of $l$ are given. On line $l$, take point $M$ and in triangle $PQM$ draw heights $PP'$ and $QQ'$. What is the position of point $M$ for which segment $P'Q'$ is the shortest?

**11.41.** Points $A$, $B$ and $O$ do not lie on one line. Through point $O$ draw line $l$ so that the sum of distances from it to points $A$ and $B$ were: a) maximal; b) minimal.

**11.42.** If five points in plane are given, then considering all possible triples of these points we can form 30 angles. Denote the least of these angles by $\alpha$. Find the greatest value of $\alpha$.

**11.43.** In a town there are 10 streets parallel to each other and 10 streets that intersect them at right angles. A closed bus route passes all the road intersections. What is the least number of turns such a bus route can have?

**11.44.** What is the greatest number of cells on a $8 \times 8$ chessboard that one straight line can intersect? (An intersection should have a common *inner* point.)

**11.45.** What is the greatest number of points that can be placed on a segment of length 1 so that on any segment of length $d$ contained in this segment not more than $1 + 1000d^2$ points lie?

See also Problems 15.1, 17.20.

## §7. The extremal properties of regular polygons

**11.46.** a) Prove that among all $n$-gons circumscribed about a given circle a regular $n$-gon has the least area.

b) Prove that among all the $n$-gons circumscribed about a given circle a regular $n$-gon has the least perimeter.

**11.47.** Triangles $ABC_1$ and $ABC_2$ have common base $AB$ and $\angle AC_1B = \angle AC_2B$. Prove that if $|AC_1 - C_1B| < |AC_2 - C_2B|$, then:

a) the area of triangle $ABC_1$ is greater than the area of triangle $ABC_2$;

b) the perimeter of triangle $ABC_1$ is greater than the perimeter of triangle $ABC_2$.

**11.48.** a) Prove that among all the $n$-gons inscribed in a given circle a regular $n$-gon has the greatest area.

b) Prove that among all $n$-gons inscribed in a given circle a regular $n$-gon has the greatest perimeter.

## Problems for independent study

**11.49.** On a leg of an acute angle with vertex $A$ point $B$ is given. On the other leg construct point $X$ such that the radius of the circumscribed circle of triangle $ABX$ is the least possible.

**11.50.** Through a given point inside a (given?) circle draw a chord of the least length.

**11.51.** Among all triangles with a given sum of lengths of their bisectors find a triangle with the greatest sum of lengths of its heights.

**11.52.** Inside a convex quadrilateral find a point the sum of squared distances from which to the vertices is the least possible.

**11.53.** Among all triangles inscribed in a given circle find the one for which the value $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is the least possible.

**11.54.** On a chessboard with the usual coloring draw a circle of the greatest radius so that it does not intersect any white field.

**11.55.** Inside a square, point $O$ is given. Any line that passes through $O$ cuts the square into two parts. Through point $O$ draw a line so that the difference of areas of these parts were the greatest possible.

**11.56.** What is the greatest length that the shortest side of a triangle inscribed in a given square can have?

**11.57.** What greatest area can an equilateral triangle inscribed in a given square can have?

## Solutions

**11.1.** By the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha = (b - c)^2 + 2bc(1 - \cos \alpha) =$$

$$(b - c)^2 + \frac{4S(1 - \cos \alpha)}{\sin \alpha}.$$

Since the last summand is constant, $a$ is minimal if $b = c$.

**11.2.** Let an escribed circle be tangent to sides $AB$ and $AC$ at points $K$ and $L$, respectively. Since $AK = AL = p$, the escribed circle $S_a$ is fixed. The radius $r$ of the inscribed circle is maximal if it is tangent to circle $S_a$, i.e., triangle $ABC$ is an isosceles one. It is also clear that $S = pr$.

**11.3.** By Problem 10.53 a) we have $S \leq \frac{p^2}{3\sqrt{3}}$, where the equality is only attained for an equilateral triangle.

**11.4.** By the law of cosines $b^2 + c^2 = a^2 + 2bc \cos \alpha$. Since $2bc \leq b^2 + c^2$ and $\cos \alpha > 0$, it follows that $b^2 + c^2 \leq a^2 + (b^2 + c^2) \cos \alpha$, i.e., $b^2 + c^2 \leq \frac{a^2}{1 - \cos \alpha}$. The equality is attained if $b = c$.

**11.5.** Let $R$ be the radius of the given circle, $O$ its center; let $A$, $B$ and $C$ be the vertices of the triangle; $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$. Then
$$AB^2 + BC^2 + CA^2 = |\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2 =$$
$$2(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) - -2(\mathbf{a}, \mathbf{b}) - 2(\mathbf{b}, \mathbf{c}) - 2(\mathbf{c}, \mathbf{a}).$$

Since
$$|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}, \mathbf{b}) + 2(\mathbf{b}, \mathbf{c}) + 2(\mathbf{c}, \mathbf{a}),$$

it follows that

$$AB^2 + BC^2 + CA^2 = 3(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) - |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 \leq$$

$$3(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) = 9R^2,$$

where the equality is only attained if $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. This equality means that triangle $ABC$ is an equilateral one.

**11.6.** Denote the length of the height dropped on side $BC$ by $h$. Since $\triangle AMN \sim \triangle ABC$, it follows that $\frac{MN}{BC} = \frac{h - 2r}{h}$, i.e. $MN = a \left(1 - \frac{2r}{h}\right)$. Since $r = \frac{S}{p} = \frac{ah}{2p}$, we deduce that $MN = a \left(1 - \frac{a}{p}\right)$. The maximum of the quadratic expression $a \left(1 - \frac{a}{p}\right) = \frac{a(p - a)}{p}$ in $a$

is attained for $a = \frac{1}{2}p$. This maximum is equal to $\frac{p}{4}$. It remains to notice that there exists a triangle of perimeter $2p$ with side $a = \frac{1}{2}p$ (set $b = c = \frac{3}{4}p$).

**11.7.** Let $O$ be the center of symmetry of polygon $M$ lying inside triangle $T$, let $S(T)$ be the image of triangle $T$ under the symmetry through point $O$. Then $M$ lies both in $T$ and in $S(T)$. Therefore, among all centrally symmetric polygons with the given center of symmetry lying in $T$ the one with the greatest area is the intersection of $T$ and $S(T)$. Point $O$ lies inside triangle $T$ because the intersection of $T$ and $S(T)$ is a convex polygon and a convex polygon always contains its center of symmetry.
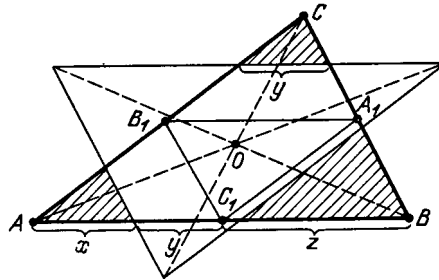


Figure 124 (Sol. 11.7)

Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$, respectively, of triangle $T = ABC$. First, let us suppose that point $O$ lies inside triangle $A_1B_1C_1$. Then the intersection of $T$ and $S(T)$ is a hexagon (Fig. 124). Let side $AB$ be divided by the sides of triangle $S(T)$ in the ratio of $x : y : z$, where $x + y + z = 1$. Then the ratio of the sum of areas of the shaded triangles to the area of triangle $ABC$ is equal to $x^2 + y^2 + z^2$ and we have to minimize this expression. Since

$$1 = (x + y + z)^2 = 3(x^2 + y^2 + z^2) - (x - y)^2 - (y - z)^2 - (z - x)^2,$$

it follows that $x^2 + y^2 + z^2 \geq \frac{1}{3}$, where the equality is only attained for $x = y = z$; the latter equality means that $O$ is the intersection point of the medians of triangle $ABC$.

Now, consider another case: point $O$ lies inside one of the triangles $AB_1C_1$, $A_1BC_1$, $A_1B_1C$; for instance, inside $AB_1C_1$. In this case the intersection of $T$ and $S(T)$ is a parallelogram and if we replace point $O$ with the intersection point of lines $AO$ and $B_1C_1$, then the area of this parallelogram can only increase. If point $O$ lies on side $B_1C_1$, then this is actually the case that we have already considered (set $x = 0$).

The polygon to be found is a hexagon with vertices at the points that divide the sides of the triangles into three equal parts. Its area is equal to $\frac{2}{3}$ of the area of the triangle.

**11.8.** Denote the intersection point of lines $KM$ and $BC$ by $T$ and the intersection points of the sides of triangles $A_1B_1C_1$ and $KLM$ as shown on Fig. 125.



Figure 125 (Sol. 11.8)

Then $TL : RZ = KL : KZ = LC : ZB_1$. Since $TL \geq BA_1 = A_1C \geq LC$, it follows that $RZ \geq ZB_1$, i.e., $S_{RZQ} \geq S_{ZB_1Q}$. Similarly, $S_{QYP} \geq S_{YA_1P}$ and $S_{PXR} \geq S_{XC_1R}$. Adding all these inequalities and the inequality $S_{PQR} > 0$ we see that the area of hexagon $PXRZQY$ is not less than the area of the remaining part of triangle $A_1B_1C_1$, i.e., its area is not less than $\frac{S_{A_1B_1C_1}}{2} = \frac{1}{8}$. The equality is attained, for instance, if point $K$ coincides with $B_1$ and point $M$ with $B$.
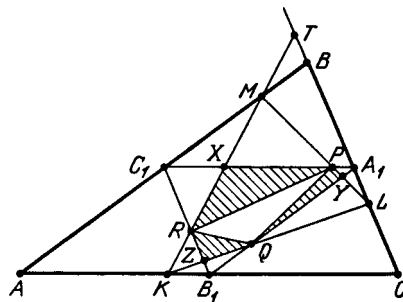
**11.9.** Since the area of an equilateral triangle with side $a$ is equal to $\frac{a^2\sqrt{3}}{4}$, the side of an equilateral triangle of area 1 is equal to $\frac{2}{\sqrt[4]{3}}$ and its height is equal to $\sqrt[4]{3}$. Let us prove that it is impossible to cut an equilateral triangle of area 1 off a strip of width less than $\sqrt[4]{3}$.

Let equilateral triangle $ABC$ lie inside a strip of width less than $\sqrt[4]{3}$. Let, for definiteness, the projection of vertex $B$ on the boundary of the strip lie between the projections of vertices $A$ and $C$. Then the line drawn through point $B$ perpendicularly to the boundary of the strip intersects segment $AC$ at a point $M$. The length of a height of triangle $ABC$ does not exceed $BM$ and $BM$ is not greater than the width of the strip and, therefore, a height of triangle $ABC$ is shorter than $\sqrt[4]{3}$, i.e., its area is less than 1.

It remains to prove that any triangle of area 1 can be cut off a strip of width $\sqrt[4]{3}$. Let us prove that any triangle of area 1 has a height that does not exceed $\sqrt[4]{3}$. For this it suffices to prove that it has a side not shorter than $\frac{2}{\sqrt[4]{3}}$. Suppose that all sides of triangle $ABC$ are shorter than $2\sqrt[4]{3}$. Let $\alpha$ be the smallest angle of this triangle. Then $\alpha \leq 60°$ and

$$S_{ABC} = \frac{AB \cdot AC \sin \alpha}{2} < \left(\frac{2}{\sqrt[4]{3}}\right)^2 \left(\frac{\sqrt{3}}{4}\right) = 1.$$

We have obtained a contradiction. A triangle that has a height not exceeding $\sqrt[4]{3}$ can be placed in a strip of width $\sqrt[4]{3}$: place the side to which this height is dropped on a boundary of the strip.

**11.10.** Squaring both sides of the given equality we easily reduce the equality to the form

$$(\sqrt{ab_1} - \sqrt{a_1b})^2 + (\sqrt{ca_1} - \sqrt{c_1a})^2 + (\sqrt{bc_1} - \sqrt{cb_1})^2 = 0,$$

i.e., $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.

**11.11.** Fix angles $\alpha$, $\beta$ and $\gamma$. Let $A_1B_1C_1$ be a triangle with angles $\alpha_1$, $\beta_1$ and $\gamma_1$. Consider vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ codirected with vectors $\overrightarrow{B_1C_1}$, $\overrightarrow{C_1A_1}$ and $\overrightarrow{A_1B_1}$ and of length $\sin\alpha$, $\sin\beta$ and $\sin\gamma$, respectively. Then

$$\frac{\cos\alpha_1}{\sin\alpha} + \frac{\cos\beta_1}{\sin\beta} + \frac{\cos\gamma_1}{\sin\gamma} = -\frac{[(\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a})]}{\sin\alpha\sin\beta\sin\gamma}.$$

Since

$$2[(\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a})] = |\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2 - |\mathbf{c}|^2,$$

the quantity $(\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a})$ attains its minimum when $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, i.e., $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $\gamma_1 = \gamma$.

**11.12.** Let $x = \cot\alpha_1$ and $y = \cot\beta_1$. Then $x + y > 0$ (since $\alpha_1 + \beta_1 < \pi$) and

$$\cot\gamma_1 = \frac{1 - xy}{x + y} = \frac{x^2 + 1}{x + y} - x.$$

Therefore,

$$a^2 \cot\alpha_1 + b^2 \cot\beta_1 + c^2 \cot\gamma_1 = (a^2 - b^2 - c^2)x + b^2(x + y) + c^2\frac{x^2 + 1}{x + y}.$$

For a fixed $x$ this expression is minimal for a $y$ such that $b^2(x+y) = c^2\frac{x^2+1}{x+y}$, i.e.,

$$\frac{c}{b} = \frac{x+y}{\sqrt{1+x^2}} = \sin\alpha_1(\cot\alpha_1 + \cot\beta_1) = \frac{\sin\gamma_1}{\sin\beta_1}.$$

Similar arguments show that if $a : b : c = \sin\alpha_1 : \sin\beta_1 : \sin\gamma_1$, then the considered expression is minimal. In this case triangles are similar and $a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma = 4S$, cf. Problem 12.44 b).

**11.13.** Let $f = bc\cos x + ca\cos y + ab\cos z$. Since $\cos x = -\cos y\cos z + \sin y\sin z$, it follows that

$$f = c(a - b\cos z)\cos y + bc\sin y\sin z + ab\cos z.$$

Consider a triangle the lengths of whose two sides are equal to $a$ and $b$ and the angle between them is equal to $z$; let $\xi$ and $\eta$ be the angles subtending sides $a$ and $b$; let $t$ be the length of the side that subtends angle $z$. Then

$$\cos z = \frac{a^2 + b^2 - t^2}{2ab} \quad \text{and} \quad \cos\eta = \frac{t^2 + a^2 - b^2}{2at};$$

hence, $\frac{a - b\cos z}{t} = \cos\eta$. Moreover, $\frac{b}{t} = \frac{\sin\eta}{\sin z}$. Therefore, $f = ct\cos(\eta - y) + \frac{1}{2}(a^2 + b^2 - t^2)$.

Since $\cos(\eta - y) \le 1$, it follows that $f \le \frac{1}{2}(a^2 + b^2 + c^2) - \frac{1}{2}((c-t)^2) \le \frac{1}{2}(a^2 + b^2 + c^2)$. Since $a \ge b$, it follows that $\xi \ge \eta$, consequently, $-\xi \le -\eta < y - \eta < \pi - z - \eta = \xi$, i.e., $\cos(y - \psi) > \cos\xi$. Hence,

$$f > ct\cos\xi + \frac{a^2 + b^2 - t^2}{2} = \frac{c-b}{2b}t^2 + \frac{c(b^2 - a^2)}{2b} + \frac{a^2 + b^2}{2} = g(t).$$

The coefficient of $t^2$ is either negative or equal to zero; moreover, $t < a + b$. Hence, $g(t) \ge g(a+b) = bc + ca - ab$.

**11.14.** a) Since $CMXN$ is a rectangle, $MN = CX$. Therefore, the length of segment $MN$ is the least possible if $CX$ is a height.

b) Let $S_{ABC} = S$. Then $S_{AMX} = \frac{AX^2 \cdot S}{AB^2}$ and $S_{BNX} = \frac{BX^2 \cdot S}{AB^2}$. Since $AX^2 + BX^2 \ge \frac{1}{2}AB^2$ (where the equality is only attained if $X$ is the midpoint of segment $AB$), it follows that $S_{CMXN} = S - S_{AMX} - S_{BNX} \le \frac{1}{2}S$. The area of quadrilateral $CMXN$ is the greatest if $X$ is the midpoint of side $AB$.

**11.15.** Points $P$ and $Q$ lie on the circle constructed on segment $CM$ as on the diameter. In this circle the constant angle $C$ intercepts chord $PQ$, therefore, the length of chord $PQ$ is minimal if the diameter $CM$ of the circle is minimal, i.e., if $CM$ is a height of triangle $ABC$.

**11.16.** By the law of sines the radii of the circumscribed circles of triangles $ACM$ and $BCM$ are equal to $\frac{AC}{2\sin AMC}$ and $\frac{BC}{2\sin BMC}$, respectively. It is easy to verify that $\sin AMC = \sin BMC$. Therefore,

$$\frac{AC}{2\sin AMC} + \frac{BC}{2\sin BMC} = \frac{AC + BC}{2\sin BMC}.$$

The latter expression is minimal if $\sin BMC = 1$, i.e., $CM \perp AB$.

**11.17.** Points $P$ and $Q$ lie on the circle with diameter $AM$, hence, $PQ = AM\sin PAQ = AM\sin A$. It follows that the length of segment $PQ$ is maximal if $AM$ is a diameter of the circumscribed circle.

**11.18.** Clearly, $2S_{ABC} = ad_a + bd_b + cd_c$. Therefore, the product $(ad_a)(bd_b)(cd_c)$ takes its greatest value if $ad_a = bd_b = cd_c$ (cf. Supplement to Ch. 9, the inequality between the mean arithmetic and the mean geometric). Since the value $abc$ is a constant, the product $(ad_a)(bd_b)(cd_c)$ attains its greatest value if and only if the product $d_a d_b d_c$ takes its greatest value.

Let us show that equality $ad_a = bd_b = cd_c$ means that $O$ is the intersection point of the medians of triangle $ABC$. Denote the intersection point of lines $AO$ and $BC$ by $A_1$. Then

$$BA_1 : A_1C = S_{ABA_1} : S_{ACA_1} = S_{ABO} : S_{ACO} = (cd_c) : (bd_b) = 1,$$

i.e., $AA_1$ is a median. We similarly prove that point $O$ lies on medians $BB_1$ and $CC_1$.

**11.19.** Let $\alpha = \frac{MA_1}{AA_1}$, $\beta = \frac{MB_1}{BB_1}$ and $\gamma = \frac{MC_1}{CC_1}$. Since $\alpha + \beta + \gamma = 1$ (cf. Problem 4.48 a)), we have $\sqrt{\alpha\beta\gamma} \leq \frac{1}{3}(\alpha + \beta + \gamma) = \frac{1}{3}$, where the equality is attained when $\alpha = \beta = \gamma = \frac{1}{3}$, i.e., $M$ is the intersection point of the medians.

**11.20.** Let $x = MA_1$, $y = MB_1$ and $z = MC_1$. Then

$$ax + by + cz = 2S_{BMC} + 2S_{AMC} + 2S_{AMB} = 2S_{ABC}.$$

Hence,

$$\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right) \cdot 2S_{ABC} =$$
$$\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)(ax + by + cz) = a^2 + b^2 + c^2 + ab\left(\frac{x}{y} + \frac{y}{x}\right) + bc\left(\frac{y}{z} + \frac{z}{y}\right) + ac\left(\frac{z}{x} + \frac{x}{z}\right) \geq$$
$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ac,$$

where the equality is only attained if $x = y = z$, i.e., $M$ is the center of the inscribed circle of triangle $ABC$.

**11.21.** First, suppose that all the angles of triangle $ABC$ are less than $120°$. Then inside triangle $ABC$ there exists a point $O$ — the vertex of angles of $120°$ that subtend each side. Let us draw through vertices $A$, $B$ and $C$ lines perpendicular to segments $OA$, $OB$ and $OC$, respectively. These lines form an equilateral triangle $A_1B_1C_1$ (Fig. 126).
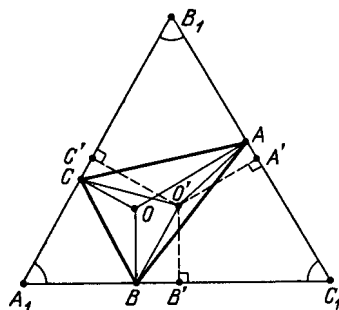


Figure 126 (Sol. 11.21)

Let $O'$ be any point that lies inside triangle $ABC$ and is distinct from $O$. Let us prove then that $O'A + O'B + O'C > OA + OB + OC$, i.e., $O$ is the desired point. Let $A'$, $B'$ and $C'$ be the bases of the perpendiculars dropped from point $O'$ on sides $B_1C_1$, $C_1A_1$ and $A_1B_1$, respectively, $a$ the length of the side of equilateral triangle $A_1B_1C_1$. Then

$$O'A' + O'B' + O'C' = \frac{2(S_{O'B_1C_1} + S_{O'A_1C_1})}{a} =$$
$$\frac{2S_{A_1B_1C_1}}{a} = OA + OB + OC.$$

Since a slanted line is longer than the perpendicular,

$$O'A + O'B + O'C > O'A' + O'B' + O'C' = OA + OB + OC.$$
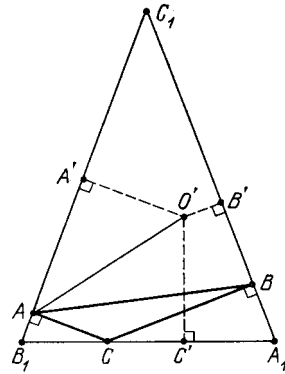
Figure 127 (Sol. 11.21)

Now, let one of the angles of triangle $ABC$, say $\angle C$, be greater than 120°. Let us draw through points $A$ and $B$ perpendiculars $B_1C_1$ and $C_1A_1$ to segments $CA$ and $CB$ and through point $C$ line $A_1B_1$ perpendicular to the bisector of angle $\angle ACB$ (Fig. 127).

Since $\angle AC_1B = 180° - \angle ACB < 60°$, it follows that $B_1C_1 > A_1B_1$. Let $O'$ be any point that lies inside triangle $A_1B_1C_1$. Since

$$B_1C_1 \cdot O'A' + C_1A_1 \cdot O'B' + A_1C_1 \cdot O'C' = 2S_{A_1B_1C_1},$$

it follows that

$$(O'A' + O'B' + O'C') \cdot B_1C_1 = 2S_{A_1B_1C_1} + (B_1C_1 - A_1B_1) \cdot O'C'.$$

Since $B_1C_1 > A_1B_1$, the sum $O'A' + O'B' + O'C'$ is minimal for points that lie on side $B_1A_1$. It is also clear that

$$O'A + O'B + O'C \geq O'A' + O'B' + O'C'.$$

Therefore, vertex $C$ is the point to be found.

**11.22.** Let the distances from point $O$ to sides $BC$, $CA$ and $AB$ be equal to $x$, $y$ and $z$, respectively. Then

$$ax + by + cz = 2(S_{BOC} + S_{COA} + S_{AOB}) = 2S_{ABC}.$$

It is also clear that

$$x : y : z = \left(\frac{S_{BOC}}{a}\right) : \left(\frac{S_{COA}}{b}\right) : \left(\frac{S_{AOB}}{c}\right).$$

Equation $ax + by + cz = 2S$ determines a plane in 3-dimensional space with coordinates $x$, $y$, $z$; vector $(a, b, c)$ is perpendicular to this plane because if $ax_1 + by_1 + cz_1 = 2S$ and $ax_2 + by_2 + cz_2 = 2S$, then $a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) = 0$.

We have to find a point $(x_0, y_0, z_0)$ on this plane at which the minimum of expression $x^2 + y^2 + z^2$ is attained and verify that an inner point of the triangle corresponds to this point. Since $x^2 + y^2 + z^2$ is the squared distance from the origin to point $(x, y, z)$, it follows that the base of the perpendicular dropped from the origin to the plane is the desired point, i.e., $x : y : z = a : b : c$. It remains to verify that inside the triangle there exists point $O$ for which $x : y : z = a : b : c$. This equality is equivalent to the condition

$$\left(\frac{S_{BOC}}{a}\right) : \left(\frac{S_{COA}}{b}\right) : \left(\frac{S_{AOB}}{c}\right) = a : b : c,$$

i.e., $S_{BOC} : S_{COA} : S_{AOB} = a^2 : b^2 : c^2$. Since the equality $S_{BOC} : S_{AOB} = a^2 : c^2$ follows from equalities $S_{BOC} : S_{COA} = a^2 : b^2$ and $S_{COA} : S_{AOB} = b^2 : c^2$, the desired point is the intersection point of lines $CC_1$ and $AA_1$ that divide sides $AB$ and $BC$, respectively, in the ratios of $BC_1 : C_1A = a^2 : b^2$ and $CA_1 : A_1B = b^2 : c^2$, respectively.

**11.23.** Let $O$ be the vertex of the given angle. Point $C$ is the tangent point of a leg with the circle that passes through points $A$ and $B$, i.e., $OC^2 = OA \cdot OB$. To find the length of segment $OC$, it suffices to draw the tangent to any circle that passes through points $A$ and $B$.

**11.24.** Let us consider angle $\angle X'A'Y'$ symmetric to angle $\angle XAY$ through point $O$. Let $B$ and $C$ be the intersection points of the legs of these angles. Denote the intersection points of the line that passes through point $O$ with the legs of angles $\angle XAY$ and $\angle X'A'Y'$ by $B_1$, $C_1$ and $B_1'$, $C_1'$, respectively (Fig. 128).
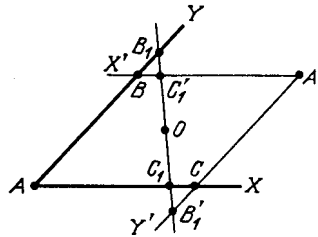


Figure 128 (Sol. 11.24)

Since $S_{AB_1C_1} = S_{A'B_1'C_1'}$, it follows that $S_{AB_1C_1} = \frac{1}{2}(S_{ABA'C} + S_{BB_1C_1'} + S_{CC_1B_1'})$. The area of triangle $AB_1C_1$ is the least if $B_1 = B$ and $C_1 = C$, i.e., line $BC$ is the one to be found.

**11.25.** On legs $OA$ and $OB$, take points $K$ and $L$ so that $KP \parallel OB$ and $LP \parallel OA$. Then $KM : KP = PL : LN$ and, therefore,

$$KM + LN \geq 2\sqrt{KM \cdot LN} = 2\sqrt{KP \cdot PL} = 2\sqrt{OK \cdot OL}$$

where the equality is attained when $KM = LN = \sqrt{OK \cdot OL}$. It is also clear that $OM + ON = (OK + OL) + (KM + LN)$.

**11.26.** On rays $AX$ and $AY$, mark equal segments $AB$ an $AC$. If point $M$ lies on segment $BC$, then the sum of distances from it to lines $AB$ and $AC$ is equal to $\frac{2(S_{ABM}+S_{ACM})}{AB} = \frac{2S_{ABC}}{AB}$. Therefore, the sum of distances from a point to lines $AX$ and $AY$ is the lesser, the lesser is the distance between point $A$ and the point's projection on the bisector of angle $\angle XAY$.

**11.27.** Let points $M_1$ and $M_2$ be symmetric to $M$ through lines $AB$ and $AC$, respectively. Since $\angle BAM_1 = \angle BAM$ and $\angle CAM_2 = \angle CAM$, it follows that $\angle M_1AM_2 = 2\angle BAC < 180°$. Hence, segment $M_1M_2$ intersects rays $AB$ and $AC$ at certain points $X$ and $Y$ (Fig. 129). Let us prove that $X$ and $Y$ are the points to be found.
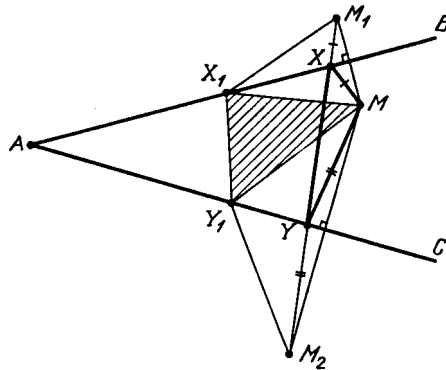


Figure 129 (Sol. 11.27)

Indeed, if points $X_1$ and $Y_1$ lie on rays $AB$ and $AC$, respectively, then $MX_1 = M_1X_1$ and $MY_1 = M_2Y_1$, i.e., the perimeter of triangle $MX_1Y_1$ is equal to the length of the broken

line $MX_1Y_1M_2$. Of all the broken lines with the endpoints at $M_1$ and $M_2$ segment $M_1M_2$ is the shortest one.

**11.28.** Quadrilateral $ABOC$ of the greatest area is a convex one. Among all the triangles $ABC$ with the fixed angle $\angle A$ and side $BC$ an isosceles triangle with base $BC$ has the greatest area. Therefore, among all the considered quadrilaterals $ABOC$ with fixed diagonal $BC$ the quadrilateral with $AB = AC$, i.e., for which point $O$ lies on the bisector of angle $\angle A$, is of greatest area.

Further, let us consider triangle $ABO$ in which angle $\angle BAO$ equal to $\frac{1}{2}\angle A$ and side $BO$ are fixed. The area of this triangle is maximal when $AB = AO$.

**11.29.** Let $O$ be the intersection point of the diagonals of convex quadrilateral $ABCD$ and $O_1$ any other point. Then $AO_1 + CO_1 \geq AC = AO + CO$ and $BO_1 + DO_1 \geq BD = BO + DO$, where at least one of the inequalities is a strict one. Therefore, $O$ is the point to be found.

**11.30.** Since $S_{AOB} : S_{BOC} = AO : OC = S_{AOD} : S_{DOC}$, it follows that $S_{BOC} \cdot S_{AOD} = S_{AOB} \cdot S_{DOC} = 36$. Therefore, $S_{BOC} + S_{AOD} \geq 2\sqrt{S_{BOC} \cdot S_{AOD}} = 12$, where the equality takes place if $S_{BOC} = S_{AOD}$, i.e., $S_{ABC} = S_{ABD}$. This implies that $AB \parallel CD$. In this case the area of the triangle is equal to 4+9+12=25.

**11.31.** Let $S_0$ and $S$ be the considered sums of areas of triangles for line $l_0$ that passes through the intersection point of the diagonals of the trapezoid and for another line $l$. It is easy to verify that $S = S_0 + s$, where $s$ is the area of the triangle formed by diagonals $AC$ and $BD$ and line $l$. Hence, $l_0$ is the line to be found.

**11.32.** Denote the lengths of the diagonals of the trapezoid by $d_1$ and $d_2$ and the lengths of their projections on the bottom base by $p_1$ and $p_2$, respectively; denote the lengths of the bases by $a$ and $b$ and that of the height by $h$. Let, for definiteness, $d_1 \geq d_2$. Then $p_1 \geq p_2$. Clearly, $p_1 + p_2 \geq a + b$. Hence, $p_1 \geq \frac{a+b}{2} = \frac{S}{h} = \frac{1}{h}$. Therefore, $d_1^2 = p_1^2 + h^2 \geq \frac{1}{h^2} + h^2 \geq 2$, where the equality is attained only if $p_1 = p_2 = h = 1$. In this case $d_1 = \sqrt{2}$.

**11.33.** Let us prove that point $M$ that divides side $BC$ in the ratio of $BM : NC = AK : KD$ is the desired one. Denote the intersection points of segments $AM$ and $BK$, $DM$ and $CK$ by $P$ and $Q$, respectively. Then $KQ : QC = KD : MC = KA : MB = KP : PB$, i.e., line $PQ$ is parallel to the basis of the trapezoid.

Let $M_1$ be any other point on side $BC$. For definiteness, we may assume that $M_1$ lies on segment $BM$. Denote the intersection points of $AM_1$ and $BK$, $DM_1$ and $CK$, $AM_1$ and $PQ$, $DM_1$ and $PQ$, $AM$ and $DM_1$ by $P_1$, $Q_1$, $P_2$, $Q_2$, $O$, respectively (Fig. 130).
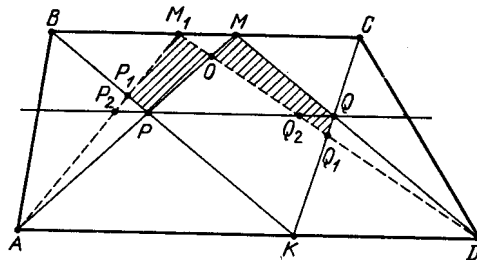

Figure 130 (Sol. 11.33)

We have to prove that $S_{MPKQ} > S_{M_1P_1KQ_1}$, i.e., $S_{MOQ_1Q} > S_{M_1OPP_1}$. Clearly, $S_{MOQ_1Q} > S_{MOQ_2Q} = S_{M_1OPP_2} > S_{M_1OPP_1}$.

**11.34.** By Problem 4.45 a) we have

$$S^2 = (p - a)(p - b)(p - c)(p - d) - abcd \cos^2 \frac{\angle B + \angle D}{2}.$$

This quantity takes its maximal value when $\cos\frac{\angle B+\angle D}{2}=0$, i.e., $\angle B + \angle D = 180°$.

**11.35.** If $A$ and $A'$ are vertices of the polygon symmetric through point $O$, then the sum of distances from any point of segment $AA'$ to points $A$ and $A'$ is the same whereas for any other point it is greater. Point $O$ belongs to all such segments.

**11.36.** If in triangle $ABC$, angle $\angle B$ is either obtuse or right, then by the law of sines $AC^2 \geq AB^2 + BC^2$. Therefore, if in a polygon the angle at vertex $B$ is not acute, then deleting vertex $B$ we obtain a polygon with the sum of squared lengths of the sides not less than that of the initial polygon. Since for $n \geq 3$ any $n$-gon has a nonacute angle, it follows that by repeating such an operation we eventually get a triangle. Among all the triangles inscribed in the given circle an equilateral triangle has the greatest sum of squared lengths of the sides, cf. Problem 11.5.

**11.37.** If point $X$ divides segment $PQ$ in the ratio of $\lambda : (1 - \lambda)$, then $\overrightarrow{A_iX} = (1 - \lambda)\overrightarrow{A_iP} + \lambda\overrightarrow{A_iQ}$; hence, $A_iX \leq (1 - \lambda)A_iP + \lambda A_iQ$. Therefore,

$$f(X) = \sum A_iX \leq (1 - \lambda)\sum A_iP + \lambda\sum A_iQ = (1 - \lambda)f(P) + \lambda f(Q).$$

Let, for instance, $f(P) \leq f(Q)$, then $f(X) \leq f(Q)$; hence, on segment $PQ$ the function $f$ attains its maximal value at one of the endpoints; more precisely, inside the segment there can be no point of strict maximum of $f$. Hence, if $X$ is any point of the polygon, then $f(X) \leq f(Y)$, where $Y$ is a point on a side of the polygon and $f(Y) \leq f(Z)$, where $Z$ is a vertex.

**11.38.** The locus of points $X$ for which angle $\angle OXA$ is a constant consists of two arcs of circles $S_1$ and $S_2$ symmetric through line $OA$.

Consider the case when the diameter of circles $S_1$ and $S_2$ is equal to the radius of the initial circle, i.e., when these circles are tangent to the initial circle at points $M_1$ and $M_2$ for which $\angle OAM_1 = \angle OAM_2 = 90°$. Points $M_1$ and $M_2$ are the desired ones because if $\angle OXA > \angle OM_1A = \angle OM_2A$, then point $X$ lies strictly inside the figure formed by circles $S_1$ and $S_2$, i.e., cannot lie on the initial circle.

**11.39.** Let us denote the intersection point of line $l$ and segment $AB$ by $O$. Let us consider an arbitrary circle $S$ that passes through points $A$ and $B$. It intersects $l$ at certain points $M$ and $N$. Since $MO \cdot NO = AO \cdot BO$ is a constant,

$$MN = MO + NO \geq 2\sqrt{MO \cdot NO} = 2\sqrt{AO \cdot BO},$$

where the equality is only attained if $MO = NO$. In the latter case the center of $S$ is the intersection point of the midperpendicular to $AB$ and the perpendicular to $l$ that passes through point $O$.

**11.40.** Let us construct the circle with diameter $PQ$. If this circle intersects with $l$, then any of the intersection points is the desired one because in this case $P' = Q'$. If the circle does not intersect with $l$, then for any point $M$ on $l$ angle $\angle PMQ$ is an acute one and $\angle P'PQ' = 90° \pm \angle PMQ$. Now it is easy to establish that the length of chord $P'Q'$ is minimal if angle $\angle PMQ$ is maximal.

To find point $M$ it remains to draw through points $P$ and $Q$ circles tangent to $l$ (cf. Problem 8.56 a)) and select the needed point among the tangent points.

**11.41.** Let the sum of distances from points $A$ and $B$ to line $l$ be equal to $2h$. If $l$ intersects segment $AB$ at point $X$, then $S_{AOB} = h \cdot OX$ and, therefore, the value of $h$ is extremal when the value of $OX$ is extremal, i.e., when line $OX$ corresponds to a side or a height of triangle $AOB$.

If line $l$ does not intersect segment $AB$, then the value of $h$ is equal to the length of the midline of the trapezoid confined between the perpendiculars dropped from points $A$ and $B$

on line $l$. This quantity is an extremal one when $l$ is either perpendicular to median $OM$ of triangle $AOB$ or corresponds to a side of triangle $AOB$. Now it only remains to select two of the obtained four straight lines.

**11.42.** First, suppose that the points are the vertices of a convex pentagon. The sum of angles of the pentagon is equal to $540°$; hence, one of its angles does not exceed $\frac{540°}{5} = 108°$. The diagonals divide this angle into three angles, hence, one of them does not exceed $\frac{108°}{3} = 36°$. In this case $\alpha \leq 36°$.

If the points are not the vertices of a convex pentagon, then one of them lies inside the triangle formed by some other three points. One of the angles of this triangle does not exceed $60°$. The segment that connects the corresponding vertex with an inner point divides this angle into two angles, hence, one of them does not exceed $30°$. In this case $\alpha \leq 30°$. In all the cases $\alpha \leq 36°$. Clearly, for a regular pentagon $\alpha = 36°$.

**11.43.** A closed route that passes through all the road crossings can have 20 turns (Fig. 131). It remains to prove that such a route cannot have less than 20 turns. After each turn a passage from a horizontal street to a vertical one or from a vertical street to a horizontal one occurs.
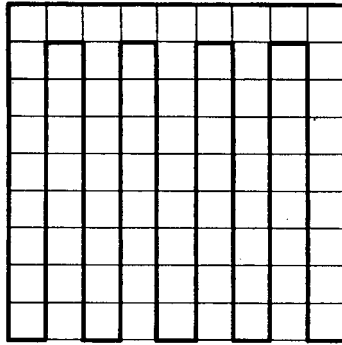


Figure 131 (Sol. 11.43)

Hence, the number of horizontal links of a closed route is equal to the number of vertical links and is equal to half the number of turns. Suppose that a closed route has less than 20 turns. Then there are streets directed horizontally, as well as streets directed vertically, along which the route does not pass. Therefore, the route does not pass through the intersection point of these streets.

**11.44.** A line can intersect 15 cells (Fig. 132). Let us prove now that a line cannot intersect more than 15 cells. The number of cells that the line intersects is by 1 less than the number of intersection points of the line with the segments that determine the sides of the cells. Inside a square there are 14 such segments.

Hence, inside a square there are not more than 14 intersection points of the line with sides of cells. No line can intersect the boundary of the chessboard at more than 2 points; hence, the number of intersection points of the line with the segments does not exceed 16. Hence, the maximal number of cells on the chessboard of size $8 \times 8$ that can be intersected by one line is equal to 15.

**11.45.** First, let us prove that 33 points are impossible to place in the required way. Indeed, if on a segment of length 1 there are 33 points, then the distance between some two of them does not exceed $\frac{1}{32}$. The segment with the endpoints at these points contains two points and it should contain not more than $1 + \frac{1000}{32^2}$ points, i.e., not less than two points.

Now, let us prove that it is possible to place 32 points. Let us take 32 points that divide the segment into equal parts (the endpoints of the given segment should be among these 32
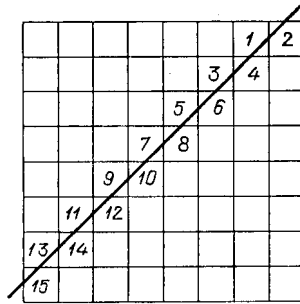
Figure 132 (Sol. 11.44)

points). Then a segment of length $d$ contains either $[31d]$ or $[31d] + 1$ points. (Recall that $[x]$ denotes the integer part of the number $x$, i.e., the greatest integer that does not exceed $x$.) We have to prove that $[31d] \leq 1000d^2$. If $31d < 1$, then $[31d] = 0 < 1000d^2$. If $31d \geq 1$, then $[31d] \leq 31d \leq (31d)^2 = 961d^2 < 1000d^2$.

**11.46.** a) Let a non-regular $n$-gon be circumscribed about circle $S$. Let us circumscribe a regular $n$-gon about this circle and let us circumscribe circle $S_1$ about this regular $n$-gon (Fig. 133). Let us prove that the area of the part of the non-regular $n$-gon confined inside $S_1$ is greater than the area of the regular $n$-gon.
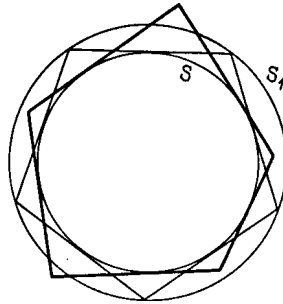


Figure 133 (Sol. 11.46)

All the tangents to $S$ cut off $S_1$ equal segments. Hence, the sum of areas of the segments cut off $S_1$ by the sides of the regular $n$-gon is equal to the sum of segments cut off $S_1$ by the sides of the non-regular $n$-gon or by their extensions.

But for the regular $n$-gon these segments do not intersect (more exactly, they do not have common interior points) and for the non-regular $n$-gon some of them must overlap, hence, the area of the union of these segments for a regular-gon is greater than for a non-regular one. Therefore, the area of the part of the non-regular $n$-gon confined inside $S_1$ is greater than the area of the regular $n$-gon and the area of the whole non-regular $n$-gon is still greater than the area of the regular one.

b) This heading follows from heading a) because the perimeter of the polygon circumscribed about a circle of radius $R$ is equal to $\frac{2S}{R}$, where $S$ is the area of the polygon.

**11.47.** The sides of triangle $ABC$ are proportional to $\sin \alpha$, $\sin \beta$ and $\sin \gamma$. If angle $\gamma$ is fixed, then the value of

$$|\sin \alpha - \sin \beta| = 2 \left| \sin \frac{\alpha - \beta}{2} \sin \frac{\gamma}{2} \right|$$

is the greater the greater is $\varphi = |\alpha - \beta|$. It remains to observe that quantities

$$S = 2R^2 \sin\alpha \sin\beta \sin\gamma = R^2 \sin\gamma(\cos\alpha - \beta + \cos\gamma) =$$
$$R^2 \sin\gamma(\cos\varphi + \cos\gamma)$$
$$\text{and} \quad \sin\alpha + \sin\beta = 2\cos\tfrac{\gamma}{2}\cos\tfrac{\varphi}{2}$$

monotonously decrease as $\varphi$ increases.

**11.48.** a) Denote the length of the side of a regular $n$-gon inscribed in the given circle by $a_n$. Consider an arbitrary non-regular $n$-gon inscribed in the same circle. It will necessarily have a side shorter than $a_n$.

On the other hand, it can have no side longer than $a_n$ and in such a case such a polygon can be confined in a segment cut off a side of the regular $n$-gon. Since the symmetry through a side of a regular $n$-gon sends the segment cut off this side inside the $n$-gon, the area of the $n$-gon is greater than the area of the segment(??). Therefore, we may assume that the considered $n$-gon has a side shorter than $a_n$ and a side longer than $a_n$.

We can replace neighbouring sides of the $n$-gon, i.e., replace $A_1A_2A_3 \ldots A_n$ with polygon $A_1A_2'A_3 \ldots A_n$, where point $A_2'$ is symmetric to $A_2$ through the midperpendicular to segment $A_1A_3$ (Fig. 134). Clearly, both polygons are inscribed in the same circle and their areas are equal. It is also clear that with the help of this operation we can make any two sides of the polygon neighbouring ones. Therefore, let us assume that for the $n$-gon considered, $A_1A_2 > a_n$ and $A_2A_3 < a_n$.
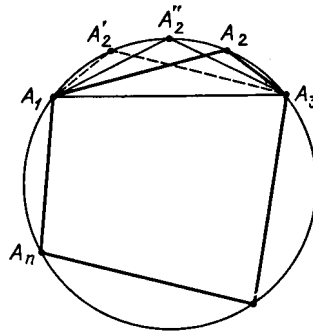


Figure 134 (Sol. 11.48)

Let $A_2'$ be the point symmetric to $A_2$ through the midperpendicular to segment $A_1A_3$. If point $A_2''$ lies on arc $\smile A_2A_2'$, then the difference of the angles at the base $A_1A_3$ of triangle $A_1A_2''A_3$ is less than that of triangle $A_1A_2A_3$ because the values of angles $\angle A_1A_2A_2''$ and $\angle A_3A_1A_2''$ are confined between the values of angles $\angle A_1A_3A_2$ and $\angle A_3A_1A_2$.

Since $A_1A_2' < a_n$ and $A_1A_2 > a_n$, on arc $\smile A_2A_2'$ there exists a point $A_2''$ for which $A_1A_2'' = a_n$. The area of triangle $A_1A_2''A_3$ is greater than the area of triangle $A_1A_2A_3$, cf. Problem 11.47 a). The area of polygon $A_1A_2''A_3 \ldots A_n$ is greater than the area of the initial polygon and it has at least by 1 more sides equal to $a_n$.

After finitely many steps we get a regular $n$-gon and at each step the area increases. Therefore, the area of any non-regular $n$-gon inscribed in a circle is less than the area of a regular $n$-gon inscribed in the same circle.

b) Proof is similar to the proof of heading a); one only has to make use of the result of Problem 11.47 b) instead of that of Problem 11.47 a).

# Chapter 12. CALCULATIONS AND METRIC RELATIONS

## Introductory problems

**1.** Prove the law of cosines:
$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos \angle A.$$

**2.** Prove the law of sines:
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R.$$

**3.** Prove that the area of a triangle is equal to $\sqrt{p(p-a)(p-b)(p-c)}$, where $p$ is semiperimeter (*Heron's formula.*)

**4.** The sides of a parallelogram are equal to $a$ and $b$ and its diagonals are equal to $d$ and $e$. Prove that $2(a^2 + b^2) = d^2 + e^2$.

**5.** Prove that for convex quadrilateral $ABCD$ with the angle $\varphi$ between the diagonals we have $S_{ABCD} = \frac{1}{2}AC \cdot BD \sin \varphi$.

## §1. The law of sines

**12.1.** Prove that the area $S$ of triangle $ABC$ is equal to $\frac{abc}{4R}$.

**12.2.** Point $D$ lies on base $AC$ of equilateral triangle $ABC$. Prove that the radii of the circumscribed circles of triangles $ABD$ and $CBD$ are equal.

**12.3.** Express the area of triangle $ABC$ in terms of the length of side $BC$ and the value of angles $\angle B$ and $\angle C$.

**12.4.** Prove that $\frac{a+b}{c} = \frac{\cos \frac{\alpha - \beta}{2}}{\sin \frac{\gamma}{2}}$ and $\frac{a-b}{c} = \frac{\sin \frac{\alpha - \beta}{2}}{\cos \frac{\gamma}{2}}$.

**12.5.** In an acute triangle $ABC$ heights $AA_1$ and $CC_1$ are drawn. Points $A_2$ and $C_2$ are symmetric to $A_1$ and $C_1$ through the midpoints of sides $BC$ and $AB$, respectively. Prove that the line that connects vertex $B$ with the center $O$ of the circumscribed circle divides segment $A_2C_2$ in halves.

**12.6.** Through point $S$ lines $a$, $b$, $c$ and $d$ are drawn; line $l$ intersects them at points $A$, $B$, $C$ and $D$. Prove that the quantity $\frac{AC \cdot BD}{BC \cdot AD}$ does not depend on the choice of line $l$.

**12.7.** Given lines $a$ and $b$ that intersect at point $O$ and an arbitrary point $P$. Line $l$ that passes through point $P$ intersects lines $a$ and $b$ at points $A$ and $B$. Prove that the value of $\frac{\frac{OA}{OB}}{\frac{PA}{PB}}$ does not depend on the choice of line $l$.

**12.8.** Denote the vertices and the intersection points of links of a (non-regular) five-angled star as shown on Fig. 135. Prove that
$$A_1C \cdot B_1D \cdot C_1E \cdot D_1A \cdot E_1B = A_1D \cdot B_1E \cdot C_1A \cdot D_1B \cdot E_1C.$$

**12.9.** Two similar isosceles triangles have a common vertex. Prove that the projections of their bases on the line that connects the midpoints of the bases are equal.

**12.10.** On the circle with diameter $AB$, points $C$ and $D$ are taken. Line $CD$ and the tangent to the circle at point $B$ intersect at point $X$. Express $BX$ in terms of the radius $R$ of the circle and angles $\varphi = \angle BAC$ and $\psi = \angle BAD$.
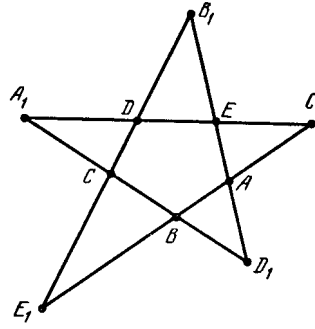
Figure 135 (12.8)

## §2. The law of cosines

**12.11.** Prove that:
a) $m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}$;
b) $m_a^2 + m_b^2 + m_c^2 = \frac{3(a^2 + b^2 + c^2)}{4}$.
**12.12.** Prove that $4S = \left(a^2 - (b-c)^2\right) \cot \frac{\alpha}{2}$.
**12.13.** Prove that

$$\cos^2 \frac{\alpha}{2} = \frac{p(p-a)}{bc} \quad \text{and} \quad \sin^2 \frac{\alpha}{2} = \frac{(p-b)(p-c)}{bc}.$$

**12.14.** The lengths of sides of a parallelogram are equal to $a$ and $b$; the lengths of the diagonals are equal to $m$ and $n$. Prove that $a^4 + b^4 = m^2 n^2$ if and only if the acute angle of the parallelogram is equal to $45°$.
**12.15.** Prove that medians $AA_1$ and $BB_1$ of triangle $ABC$ are perpendicular if and only if $a^2 + b^2 = 5c^2$.
**12.16.** Let $O$ be the center of the circumscribed circle of scalane triangle $ABC$, let $M$ be the intersection point of the medians. Prove that line $OM$ is perpendicular to median $CC_1$ if and only if $a^2 + b^2 = 2c^2$.

## §3. The inscribed, the circumscribed and escribed circles; their radii

**12.17.** Prove that:
a) $a = r \left( \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} \right) = \frac{r \cos \frac{\alpha}{2}}{\sin \frac{\beta}{2} \sin \frac{\gamma}{2}}$;
b) $a = r_a \left( \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} \right) = \frac{r_a \cos \frac{\alpha}{2}}{\cos \frac{\beta}{2} \cos \frac{\gamma}{2}}$;
c) $p - b = r \cot \frac{\beta}{2} = r_a \tan \frac{\gamma}{2}$;
d) $p = r_a \cot \frac{\alpha}{2}$.
**12.18.** Prove that:
a) $rp = r_a(p-a)$, $rr_a = (p-b)(p-c)$ and $r_b r_c = p(p-a)$;
b) $S^2 = p(p-a)(p-b)(p-c)$; (*Heron's formula.*)
c) $S^2 = r r_a r_b r_c$.
**12.19.** Prove that $S = r_c^2 \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \cot \frac{\gamma}{2}$.
**12.20.** Prove that $S = \frac{c r_a r_b}{r_a + r_b}$.
**12.21.** Prove that $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c}$.
**12.22.** Prove that $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = 1r$.
**12.23.** Prove that

$$\frac{1}{(p-a)(p-b)} + \frac{1}{(p-b)(p-c)} + \frac{1}{(p-c)(p-a)} = \frac{1}{r^2}.$$

**12.24.** Prove that $r_a + r_b + r_c = 4R + r$.

**12.25.** Prove that $r_a r_b + r_b r_c + r_c r_a = p^2$.

**12.26.** Prove that $\frac{1}{r^3} - \frac{1}{r_a^3} - \frac{1}{r_b^3} - \frac{1}{r_c^2} = \frac{12R}{S^2}$.

**12.27.** Prove that

$$a(b + c) = (r + r_a)(4R + r - r_a) \text{ and } a(b - c) = (r_b - r_c)(4R - r_b - r_c).$$

**12.28.** Let $O$ be the center of the inscribed circle of triangle $ABC$. Prove that $\frac{OA^2}{bc} + \frac{OB^2}{ac} + \frac{OC^2}{ab} = 1$.

**12.29.** a) Prove that if for a triangle we have $p = 2R + r$, then this triangle is a right one.

b) Prove that if $p = 2R \sin \varphi + r \cot \frac{\varphi}{2}$, then $\varphi$ is one of the angles of the triangle (we assume here that $0 < \varphi < \pi$).

## §4. The lengths of the sides, heights, bisectors

**12.30.** Prove that $abc = 4prR$ and $ab + bc + ca = r^2 + p^2 + 4rR$.

**12.31.** Prove that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$.

**12.32.** Prove that $\frac{a+b-c}{a+b+c} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$.

**12.33.** Prove that $h_a = \frac{bc}{2R}$.

**12.34.** Prove that

$$h_a = \frac{2(p - a) \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} = \frac{2(p - b) \sin \frac{\beta}{2} \cos \frac{\gamma}{2}}{\sin \frac{\alpha}{2}}.$$

**12.35.** Prove that the length of bisector $l_a$ can be computed from the following formulas:

a) $l_a = \sqrt{\frac{4p(p-a)bc}{(b+c)^2}}$;

b) $l_a = \frac{2bc \cos \frac{\alpha}{2}}{b+c}$;

c) $l_a = \frac{2R \sin \beta \sin \gamma}{\cos \frac{\beta-\gamma}{2}}$;

d) $l_a = \frac{4p \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\sin \beta + \sin \gamma}$.

## §5. The sines and cosines of a triangle's angles

Let $\alpha$, $\beta$ and $\gamma$ be the angles of triangle $ABC$. In the problems of this section one should prove the relations indicated.

**12.36.** a) $\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{r}{4R}$;

b) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} = \frac{r}{p}$;

c) $\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} = \frac{p}{4R}$.

**12.37.** a) $\cos \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{p-a}{4R}$;

b) $\sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} = \frac{r_a}{4R}$.

**12.38.** $\cos \alpha + \cos \beta + \cos \gamma = \frac{R+r}{R}$.

**12.39.** a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 4 \cos \alpha \cos \beta \cos \gamma + 1 = 0$;

b) $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma = 1$.

**12.40.** $\sin 2\alpha + \cos 2\beta + \cos 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma$.

**12.41.** a) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{p^2 - r^2 - 4rR}{2R^2}$.

b) $4R^2 \cos \alpha \cos \beta \cos \gamma = p^2 - (2R + r)^2$.

**12.42.** $ab \cos \gamma + bc \cos \alpha + ca \cos \beta = \frac{a^2 + b^2 + c^2}{2}$.

**12.43.** $\frac{\cos^2 \frac{\alpha}{2}}{a} + \frac{\cos^2 \frac{\beta}{2}}{b} + \frac{\cos^2 \frac{\gamma}{2}}{c} = \frac{p}{4Rr}$.

## §6. The tangents and cotangents of a triangle's angles

In problems of this section one has to prove the relations indicated between the values $\alpha$, $\beta$ and $\gamma$ of the angles of triangle $ABC$.

**12.44.** a) $\cot \alpha + \cot \beta + \cot \gamma = \frac{a^2+b^2+c^2}{4S}$;

b) $a^2 \cot \alpha + b^2 \cot \beta + c^2 \cot \gamma = 4S$.

**12.45.** a) $\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = \frac{p}{r}$;

b) $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = \frac{1}{2} \left( \frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \right)$.

**12.46.** $\tan \alpha + \tan \beta + \tan \gamma = \tan \sigma \tan \beta \tan \gamma$.

**12.47.** $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1$.

**12.48.** a) $\cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \alpha \cot \gamma = 1$;

b) $\cot \alpha + \cot \beta + \cot \gamma - \cot \alpha \cot \beta \cot \gamma = \frac{1}{\sin \alpha \sin \beta \sin \gamma}$.

**12.49.** For a non-right triangle we have

$$\tan \sigma + \tan \beta + \tan \gamma = \frac{4S}{a^2 + b^2 + c^2 - 8R^2}.$$

## §7. Calculation of angles

**12.50.** Two intersecting circles, each of radius $R$ with the distance between their centers greater than $R$ are given. Prove that $\beta = 3\alpha$ (Fig. 136).
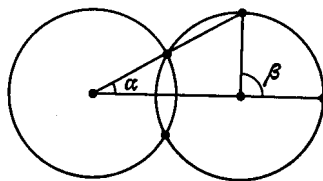


Figure 136 (12.50)

**12.51.** Prove that if $\frac{1}{b} + \frac{1}{c} = \frac{1}{l_a}$, then $\angle A = 120°$.

**12.52.** In triangle $ABC$ height $AH$ is equal to median $BM$. Find angle $\angle MBC$.

**12.53.** In triangle $ABC$ bisectors $AD$ and $BE$ are drawn. Find the value of angle $\angle C$ if it is given that $AD \cdot BC = BE \cdot AC$ and $AC \neq BC$.

**12.54.** Find angle $\angle B$ of triangle $ABC$ if the length of height $CH$ is equal to a half length of side $AB$ and $\angle BAC = 75°$.

**12.55.** In right triangle $ABC$ with right angle $\angle A$ the circle is constructed with height $AD$ of the triangle as a diameter; the circle intersects leg $AB$ at point $K$ and leg $AC$ at point $M$. Segments $AD$ and $KM$ intersect at point $L$. Find the acute angles of triangle $ABC$ if $AK : AL = AL : AM$.

**12.56.** In triangle $ABC$, angle $\angle C = 2\angle A$ and $b = 2a$. Find the angles of triangle $ABC$.

**12.57.** In triangle $ABC$ bisector $BE$ is drawn and on side $BC$ point $K$ is taken so that $\angle AKB = 2\angle AEB$. Find the value of angle $\angle AKE$ if $\angle AEB = \alpha$.

* * *

**12.58.** In an isosceles triangle $ABC$ with base $BC$ angle at vertex $A$ is equal to $80°$. Inside triangle $ABC$ point $M$ is taken so that $\angle MBC = 30°$ and $\angle MCB = 10°$. Find the value of angle $\angle AMC$.

**12.59.** In an isosceles triangle $ABC$ with base $AC$ the angle at vertex $B$ is equal to $20°$. On sides $BC$ and $AB$ points $D$ and $E$, respectively, are taken so that $\angle DAC = 60°$ and $\angle ECA = 50°$. Find angle $\angle ADE$.

**12.60.** In an acute triangle $ABC$ segments $BO$ and $CO$, where $O$ is the center of the circumscribed circle, are extended to their intersection at points $D$ and $E$ with sides $AC$ and $AB$, respectively. It turned out that $\angle BDE = 50°$ and $\angle CED = 30°$. Find the value of the angles of triangle $ABC$.

## §8. The circles

**12.61.** Circle $S$ with center $O$ on base $BC$ of isosceles triangle $ABC$ is tangent to equal sides $AB$ and $AC$. On sides $AB$ and $AC$, points $P$ and $Q$, respectively, are taken so that segment $PQ$ is tangent to $S$. Prove that $4PB \cdot CQ = BC^2$.

**12.62.** Let $E$ be the midpoint of side $AB$ of square $ABCD$ and points $F$ and $G$ are taken on sides $BC$ and $CD$, respectively, so that $AG \parallel EF$. Prove that segment $FG$ is tangent to the circle inscribed in square $ABCD$.

**12.63.** A chord of a circle is distant from the center by $h$. A square is inscribed in each of the disk segments subtended by the chord so that two neighbouring vertices of the square lie on an arc and two other vertices lie either on the chord or on its extension (Fig. 137). What is the difference of lengths of sides of these squares?
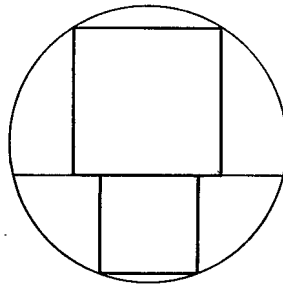


Figure 137 (12.63)

**12.64.** Find the ratio of sides of a triangle one of whose medians is divided by the inscribed circle into three equal parts.

\* \* \*

**12.65.** In a circle, a square is inscribed; in the disk segment cut off the disk by one of the sides of this square another square is inscribed. Find the ratio of the lengths of the sides of these squares.

**12.66.** On segment $AB$, point $C$ is taken and on segments $AC$, $BC$ and $AB$ as on diameters semicircles are constructed lying on one side of line $AB$. Through point $C$ the line perpendicular to $AB$ is drawn and in the obtained curvilinear triangles $ACD$ and $BCD$ circles $S_1$ and $S_2$ are inscribed (Fig. 138). Prove that the radii of these circles are equal.

**12.67.** The centers of circles with radii 1, 3 and 4 are positioned on sides $AD$ and $BC$ of rectangle $ABCD$. These circles are tangent to each other and lines $AB$ and $CD$ as shown on Fig. 139. Prove that there exists a circle tangent to all these circles and line $AB$.

## §9. Miscellaneous problems

**12.68.** Find all the triangles whose angles form an arithmetic projection and sides form a) an arithmetic progression; b) a geometric progression.
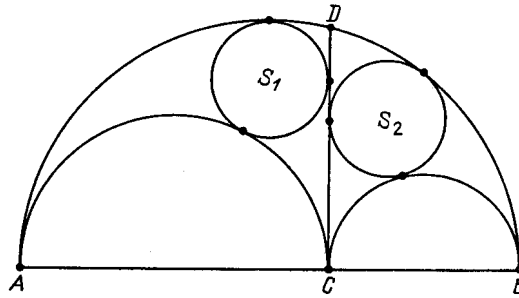
Figure 138 (12.66)
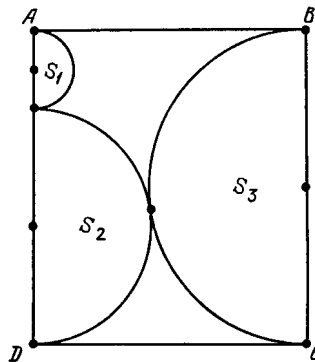


Figure 139 (12.67)

**12.69.** Find the height of a trapezoid the lengths of whose bases $AB$ and $CD$ are equal to $a$ and $b$ ($a < b$), the angle between the diagonals is equal to $90°$, and the angle between the extensions of the lateral sides is equal to $45°$.

**12.70.** An inscribed circle is tangent to side $BC$ of triangle $ABC$ at point $K$. Prove that the area of the triangle is equal to $BK \cdot KC \cot \frac{\alpha}{2}$.

**12.71.** Prove that if $\cot \frac{\alpha}{2} = \frac{a+b}{a}$, then the triangle is a right one.

**12.72.** The extensions of the bisectors of triangle $ABC$ intersect the circumscribed circle at points $A_1$, $B_1$ and $C_1$. Prove that $\frac{S_{ABC}}{S_{A_1B_1C_1}} = \frac{2r}{R}$, where $r$ and $R$ are the radii of the inscribed and circumscribed circles, respectively, of triangle $ABC$.

**12.73.** Prove that the sum of cotangents of the angles of triangle $ABC$ is equal to the sum of cotangents of the angles of the triangle formed by the medians of triangle $ABC$.

**12.74.** Let $A_4$ be the orthocenter of triangle $A_1A_2A_3$. Prove that there exist numbers $\lambda_1, \ldots, \lambda_4$ such that $A_iA_j^2 = \lambda_i + \lambda_j$ and if the triangle is not a right one, then $\sum \frac{1}{\lambda_i} = 0$.

## §10. The method of coordinates

**12.75.** Coordinates of the vertices of a triangle are rational numbers. Prove that then the coordinates of the center of the circumscribed circle are also rational.

**12.76.** Diameters $AB$ and $CD$ of circle $S$ are perpendicular. Chord $EA$ intersects diameter $CD$ at point $K$, chord $EC$ intersects diameter $AB$ at point $L$. Prove that if $CK : KD = 2 : 1$, then $AL : LB = 3 : 1$.

**12.77.** In triangle $ABC$ angle $\angle C$ is a right one. Prove that under the homothety with center $C$ and coefficient 2 the inscribed circle turns into a circle tangent to the circumscribed circle.

**12.78.** A line $l$ is fixed. Square $ABCD$ is rotated about its center. Find the locus of the midpoints of segments $PQ$, where $P$ is the base of the perpendicular dropped from point $D$ on $l$ and $Q$ is the midpoint of side $AB$.

See also Problems 7.6, 7.14, 7.47, 22.15.

## Problems for independent study

**12.79.** Each of two circles is tangent to both sides of the given right angle. Find the ratio of the circles' radii if it is known that one of the circles passes through the center of the other one.

**12.80.** Let the extensions of sides $AB$ and $CD$, $BC$ and $AD$ of convex quadrilateral $ABCD$ intersect at points $K$ and $M$, respectively. Prove that the radii of the circles circumscribed about triangles $ACM$, $BDK$, $ACK$, $BDM$ are related by the formula $R_{ACM} \cdot R_{BDK} = R_{ACK} \cdot R_{BDM}$.

**12.81.** Three circles of radii 1, 2, 3 are tangent to each other from the outside. Find the radius of the circle that passes through the tangent points of these circles.

**12.82.** Let point $K$ lie on side $BC$ of triangle $ABC$. Prove that

$$AC^2 \cdot BK + AB^2 \cdot CK = BC(AK^2 + BK \cdot KC).$$

**12.83.** Prove that the length of the bisector of an outer angle $\angle A$ of triangle $ABC$ is equal to $\frac{2bc \sin \frac{\alpha}{2}}{|b-c|}$.

**12.84.** Two circles of radii $R$ and $r$ are placed so that their common inner tangents are perpendicular. Find the area of the triangle formed by these tangents and their common outer tangent.

**12.85.** Prove that the sum of angles at rays of any (nonregular) five-angled star is equal to $180°$.

**12.86.** Prove that in any triangle $S = (p-a)^2 \tan \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$.

**12.87.** Let $a < b < c$ be the lengths of sides of a triangle; $l_a$, $l_b$, $l_c$ and $l'_a$, $l'_b$, $l'_c$ the lengths of its bisectors and the bisectors of its outer angles, respectively. Prove that $\frac{1}{al_a l'_a} + \frac{1}{cl_c l'_c} = \frac{1}{bl_b l'_b}$.

**12.88.** In every angle of a triangle a circle tangent to the inscribed circle of the triangle is inscribed. Find the radius of the inscribed circle if the radii of these smaller circles are known.

**12.89.** The inscribed circle is tangent to sides $AB$, $BC$, $CA$ at points $K$, $L$, $M$, respectively. Prove that:

a) $S = \frac{1}{2}\left(\frac{MK^2}{\sin \alpha} + \frac{KL^2}{\sin \beta} + \frac{LM^2}{\sin \gamma}\right)$;

b) $S^2 = \frac{1}{4}(bcMK^2 + caKL^2 + abLM^2)$;

c) $\frac{MK^2}{h_b h_c} + \frac{KL^2}{h_c h_a} + \frac{LM^2}{h_a h_b} = 1$.

## Solutions

**12.1.** By the law of sines $\sin \gamma = c2R$; hence, $S = \frac{1}{2}ab \sin \gamma = \frac{abc}{4R}$.

**12.2.** The radii of the circumscribed circles of triangles $ABD$ and $CBD$ are equal to $\frac{AB}{2\sin \angle ADB}$ and $\frac{BC}{2\sin \angle BDC}$. It remains to notice that $AB = BC$ and $\sin \angle ADB = \sin \angle BDC$.

**12.3.** By the law of sines $b = \frac{a \sin \beta}{\sin \alpha} = \frac{a \sin \beta}{\sin(\beta+y)}$ and, therefore, $S = \frac{1}{2}ab \sin \gamma = \frac{a^2 \sin \beta \sin \gamma}{2\sin(\beta+\gamma)}$.

**12.4.** By the law of sines $\frac{1}{2}(a+b) = \frac{\sin \alpha + \sin \beta}{\sin \gamma}$. Moreover,

$$\sin \alpha + \sin \beta = 2\sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} = 2\cos \frac{\gamma}{2} \cos \frac{\alpha-\beta}{2}$$

and $\sin \gamma = 2\sin \frac{\gamma}{2} \cos \frac{\gamma}{2}$. The second equality is similarly proved.

**12.5.** In triangle $A_2BC_2$, the lengths of sides $A_2B$ and $BC_2$ are equal to $b\cos\gamma$ and $b\cos\alpha$; line $BO$ divides angle $\angle A_2BC_2$ into angles of $90° - \gamma$ and $90° - \alpha$. Let line $BO$ intersect segment $A_2C_2$ at point $M$. By the law of sines

$$A_2M = \frac{A_2B\sin\angle A_2BM}{\sin\angle A_2MB} = \frac{b\cos\gamma\cos\alpha}{\sin\angle C_2MB} = C_2M.$$

**12.6.** Let $\alpha = \angle(a,c)$, $\beta = \angle(c,d)$ and $\gamma = \angle(d,b)$. Then

$$\frac{\frac{AC}{AS}}{\frac{BC}{BS}} = \frac{\sin\alpha}{\sin(\beta+\gamma)} \quad\text{and}\quad \frac{\frac{BD}{BS}}{\frac{AD}{AS}} = \frac{\sin\gamma}{\sin(\alpha+\beta)}.$$

Hence

$$\frac{(AC\cdot BD)}{(BC\cdot AD)} = \frac{\sin\alpha\sin\gamma}{\sin(\alpha+\beta)\sin(\beta+\gamma)}.$$

**12.7.** Since $\frac{OA}{PA} = \frac{\sin\angle OPA}{\sin\angle POA}$ and $\frac{OB}{PB} = \frac{\sin\angle OPB}{\sin\angle POB}$, it follows that

$$\frac{(OA:OB)}{(PA:PB)} = \frac{\sin\angle POB}{\sin\angle POA}.$$

**12.8.** It suffices to multiply five equalities of the form $\frac{D_1A}{D_1B} = \frac{\sin\angle B}{\sin\angle A}$.

**12.9.** Let $O$ be the common vertex of the given triangles, $M$ and $N$ the midpoints of the bases, $k$ the ratio of the lengths of the bases to that of heights. The projections of the bases of given triangles on line $MN$ are equal to $k\cdot OM\sin\angle OMN$ and $k\cdot ON\sin\angle ONM$. It remains to notice that $\frac{OM}{\sin\angle ONM} = \frac{ON}{\sin\angle OMN}$.

**12.10.** By the law of sines

$$\frac{BX}{\sin\angle BDX} = \frac{BD}{\sin\angle BXD} = \frac{2R\sin\psi}{\sin\angle BXD}.$$

Moreover, $\sin\angle BDX = \sin\angle BDC = \sin\varphi$ and the value of angle $\angle BXD$ is easy to calculate: if points $C$ and $D$ lie on one side of $AB$, then $\angle BXD = \pi - \varphi - \psi$ and if they lie on distinct sides, then $\angle BXD = |\varphi - \psi|$. Hence, $BX = \frac{2R\sin\varphi\sin\psi}{\sin|\varphi\pm\psi|}$.

**12.11.** a) Let $A_1$ be the midpoint of segment $BC$. Adding equalities

$$AB^2 = AA_1^2 + A_1B^2 - 2AA_1\cdot BA_1\cos\angle BA_1A$$

and

$$AC^2 = AA_1^2 + A_1C^2 - 2AA_1\cdot A_1C\cos\angle CA_1A$$

and taking into account that $\cos\angle BA_1A = -\cos\angle CA_1A$ we get the statement desired.

b) Follows in an obvious way from heading a).

**12.12.** By the law of cosines

$$a^2 - (b-c)^2 = 2bc(1-\cos\alpha) = \frac{4S(1-\cos\alpha)}{\sin\alpha} = 4S\tan\frac{\alpha}{2}.$$

**12.13.** By the law of cosines $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$. It remains to make use of the formulas $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$ and $\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$.

**12.14.** Let $\alpha$ be the angle at a vertex of the parallelogram. By the law of cosines

$$m^2 = a^2 + b^2 + 2ab\cos\alpha \quad\text{and}\quad n^2 = a^2 + b^2 - 2ab\cos\alpha.$$

Hence,

$$m^2n^2 = (a^2+b^2)^2 - (2ab\cos\alpha)^2 = a^4 + b^4 + 2a^2b^2(1-2\cos^2\alpha).$$

Therefore, $m^2n^2 = a^4 + b^4$ if and only if $\cos^2\alpha = \frac{1}{2}$.

**12.15.** Let $M$ be the intersection point of medians $AA_1$ and $BB_1$. Angle $\angle AMB$ is a right one if and only if $AM^2 + BM^2 = AB^2$, i.e. $\frac{4}{9}(m_a^2 + m_b^2) = c^2$. By Problem 12.11 $m_a^2 + m_b^2 = \frac{4c^2 + a^2 + b^2}{4}$.

**12.16.** Let $m = C_1M$ and $\varphi = \angle C_1 MO$. Then

$$OC_1^2 = C_1 M^2 + OM^2 - 2OM \cdot C_1 M \cos \varphi$$

and

$$BO^2 = CO^2 = OM^2 + MC^2 + 2OM \cdot CM \cos \varphi = OM^2 + 4C_1 M^2 + 4OM \cdot C_1 M \cos \varphi.$$

Hence,

$$BC_1^2 = BO^2 - OC_1^2 = 3C_1 M^2 + 6OM \cdot C_1 M \cos \varphi,$$

i.e.,

$$c^2 = 4BC_1^2 = 12m^2 + 24OM \cdot C_1 M \cos \varphi.$$

It is also clear that $18m^2 = 2m_c^2 = a^2 + b^2 - \frac{c^2}{2}$, cf. Problem 12.11. Therefore, equality $a^2 + b^2 = 2c^2$ is equivalent to the fact that $18m^2 = \frac{3c^2}{2}$, i.e., $c^2 = 12m^2$. Since $c^2 = 12m^2 + 24OM \cdot C_1 M \cos \varphi$, equality $a^2 + b^2 = 2c^2$ is equivalent to the fact that $\angle C_1 MO = \varphi = 90°$, i.e., $CC_1 \perp OM$.

**12.17.** Let the inscribed circle be tangent to side $BC$ at point $K$ and the escribed one at point $L$. Then

$$BC = BK + KC = t \cot \frac{\beta}{2} + r \cot \frac{\gamma}{2}$$

and

$$BC = BL + LC = r_a \cot LBO_a + r_a \cot LCO_a = r_a \tan \frac{\beta}{2} + r_a \tan \frac{\gamma}{2}.$$

Moreover, $\cos \frac{\alpha}{2} = \sin \left( \frac{\beta}{2} + \frac{\gamma}{2} \right)$.

By Problem 3.2, $p - b = BK = r \cot \frac{\beta}{2}$ and $p - b = CL = r_a \tan \frac{\gamma}{2}$.

If the inscribed circle is tangent to the extensions of sides $AB$ and $AC$ at points $P$ and $Q$, respectively, then $p = AP = AQ = r_a \cot \frac{\alpha}{2}$.

**12.18.** a) By Problem 12.17,

$$p = r_a \cot \frac{\alpha}{2} \quad \text{and} \quad r \cot \frac{\alpha}{2} = p - a;$$
$$r \cot \frac{\beta}{2} = p - b \quad \text{and} \quad r_a \tan \frac{\beta}{2} = p - c;$$
$$r_c \tan \frac{\beta}{2} = p - a \quad \text{and} \quad r_b \cot \frac{\beta}{2} = p.$$

By multiplying these pairs of equalities we get the desired statement.

b) By multiplying equalities $rp = r_a(p - a)$ and $rr_a = (p - b)(p - c)$ we get $r^2 p = (p - a)(p - b)(p - c)$. It is also clear that $S^2 = p(r^2 p)$.

c) It suffices to multiply $rr_a = (p - b)(p - c)$ and $r_b r_c = p(p - a)$ and make use of Heron's formula.

**12.19.** By Problem 12.17, $r = r_c \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$ and $p = r_c \cot \frac{\gamma}{2}$.

**12.20.** By Problem 12.18 a), $r_a = \frac{rp}{p-a}$ and $r_b = \frac{rp}{p-b}$. Hence,

$$cr_a r_b = \frac{cr^2 p^2}{(p-a)(p-b)} \quad \text{and} \quad r_a + r_b = \frac{rpc}{(p-a)(p-b)}$$

and, therefore, $\frac{cr_a r_b}{r_a + r_b} = rp = S$.

**12.21.** By Problem 12.18 a), $\frac{1}{r_b} = \frac{p-b}{pr}$ and $\frac{1}{r_c} = \frac{p-c}{pr}$, hence, $\frac{1}{r_b} + \frac{1}{r_c} = \frac{a}{pr} = \frac{a}{S} = \frac{2}{h_a}$.

**12.22.** It is easy to verify that $\frac{1}{h_a} = \frac{a}{2pr}$ and $\frac{1}{r_a} = \frac{p-a}{pr}$. Adding similar equalities we get the desired statement.

**12.23.** By Problem 12.18 a) $\frac{1}{(p-b)(p-c)} = \frac{1}{rr_a}$. It remains to add similar equalities and make use of the result of Problem 12.22.

**12.24.** By Problem 12.1, $4SR = abc$. It is also clear that

$$abc = p(p-b)(p-c) + p(p-c)(p-a) + p(p-a)(p-b) - (p-a)(p-b)(p-c) =$$
$$\frac{S^2}{p-a} + \frac{S^2}{p-b} + \frac{S^2}{p-c} - S^2 p = S(r_a + r_b + r_c - r).$$

**12.25.** By Problem 12.18 a)

$$r_a r_b = p(p-c), \ r_b r_c = p(p-a) \text{ and } r_c r_a = p(p-b).$$

Adding these equalities we get the desired statement.

**12.26.** Since

$$S = rp = r_a(p-a) = r_b(p-b) = r_c(p-c),$$

the right-most expression is equal to

$$\frac{p^3 - (p-a)^3 - (p-b)^3 - (p-c)^3}{S^3} = \frac{3abc}{S^3}.$$

It remains to observe that $\frac{abc}{S} = 4R$ (Problem 12.1).

**12.27.** Let the angles of triangle $ABC$ be equal to $2\alpha$, $2\beta$ and $2\gamma$. Thanks to Problems 12.36 a) and 12.37 b) we have $r = 4R\sin\alpha\sin\beta\sin\gamma$ and $r_a = 4R\sin\alpha\cos\beta\cos\gamma$. Therefore,

$$(r + r_a)(4R + r - r_a) =$$
$$16R^2\sin\alpha \cdot (\sin\beta\sin\gamma + \cos\beta\cos\gamma)(1 + \sin\alpha(\sin\beta\sin\gamma - \cos\beta\cos\gamma)) =$$
$$16R^2\sin\alpha\cos(\beta - \gamma)(1 - \sin\alpha\cos(\beta + \gamma)) =$$
$$16R^2\sin\alpha\cos(\beta - \gamma)\cos^2\alpha.$$

It remains to notice that $4R\sin\alpha\cos\alpha = a$ and

$$4R\sin(\beta + \gamma)\cos(\beta - \gamma) = 2R(\sin 2\beta + \sin 2\gamma) = b + c.$$

The second equality is similarly proved.

**12.28.** Since $OA = \frac{r}{\sin\frac{\alpha}{2}}$ and $bc = \frac{2S}{\sin\alpha}$, it follows that

$$\frac{OA^2}{bc} = \frac{r^2\cot\frac{\alpha}{2}}{S} = \frac{r(p-a)}{S},$$

cf. Problem 12.17 c). It remains to notice that $r(p - a + p - b + p - c) = rp = S$.

**12.29.** Let us solve heading b); heading a) is its particular case. Since $\cot\frac{\varphi}{2} = \frac{\sin\varphi}{1-\cos\varphi}$, it follows that

$$p^2(1-x)^2 = (1-x^2)(2R(1-x) + r)^2, \text{ where } x = \cos\varphi.$$

The root $x_0 = 1$ of this equation is of no interest to us because in this case $\cot\frac{\varphi}{2}$ is undefined; therefore, by dividing both parts of the equation by $1 - x$ we get a cubic equation. Making use of results of Problems 12.38, 12.41 b) and 12.39 b) we can verify that this equation coincides with the equation

$$(x - \cos\alpha)(x - \cos\beta)(x - \cos\gamma) = 0,$$

where $\alpha$, $\beta$ and $\gamma$ are the angles of the triangle. Therefore the cosine of $\varphi$ is equal to the cosine of one of the angles of the triangle; moreover, the cosine is monotonous on the interval $[0, \pi]$.

**12.30.** It is clear that $2pr = 2S = ab\sin\gamma = \frac{abc}{2R}$, i.e., $4prR = abc$. To prove the second equality make use of Heron's formula: $S^2 = p(p-a)(p-b)(p-c)$, i.e.,

$$pr^2 = (p-a)(p-b)(p-c) = p^3 - p^2(a+b+c) + p(ab+bc+ca) - abc =$$
$$= -p^3 + p(ab+bc+ca) - 4prR.$$

By dividing by $p$ we get the desired equality.

**12.31.** Since $abc = 4RS$ (Problem 12.1), the expression in the left-hand side is equal to $\frac{c+a+b}{4RS} = \frac{2p}{4Rpr} = \frac{1}{2Rr}$.

**12.32.** It suffices to observe that $\frac{p-c}{p} = \frac{r}{r_c}$ (Problem 12.18 a)),

$$r = \frac{c \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos \frac{\gamma}{2}} \quad \text{and} \quad r_c = \frac{c \cos \frac{\alpha}{2} \cos \frac{\beta}{2}}{\cos \frac{\gamma}{2}}$$

(Problem 12.17).

**12.33.** By Problem 12.1, $S = \frac{abc}{4R}$. On the other hand, $S = \frac{ah_a}{2}$. Hence, $h_a = \frac{bc}{2R}$.

**12.34.** Since $ah_a = 2S = 2(p-a)r_a$ and $\frac{r_a}{a} = \frac{\cos \frac{\beta}{2} \cos \frac{\gamma}{2}}{\cos \frac{\alpha}{2}}$ (Problem 12.17 b)), we have

$$h_a = \frac{2(p-a) \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}{\cos \frac{\alpha}{2}}.$$

Taking into account that $(p-a) \cot \frac{\beta}{2} = r_c = (p-b) \cot \frac{\alpha}{2}$ (Problem 12.17 c)), we get $h_a = \frac{2(p-b) \sin \frac{\beta}{2} \cos \frac{\gamma}{2}}{\sin \frac{\alpha}{2}}$.

**12.35.** a) Let the extension of bisector $AD$ intersect the circumscribed circle of triangle $ABC$ at point $M$. Then $AD \cdot DM = BD \cdot DC$ and since $\triangle ABC \sim \triangle AMC$, it follows that

$$AB \cdot AC = AD \cdot AM = AD(AD + DM) = AD^2 + BD \cdot DC.$$

Moreover, $BD = \frac{ac}{b+c}$ and $DC = \frac{ab}{b+c}$. Hence,

$$AD^2 = bc - \frac{bca^2}{(b+c)^2} = \frac{4p(p-a)bc}{(b+c)^2}.$$

b) See the solution of Problem 4.47.

c) Let $AD$ be a bisector, $AH$ a height of triangle $ABC$. Then $AH = c \sin \beta = 2R \sin \beta \sin \gamma$. On the other hand,

$$AH = AD \sin \angle ADH = l_a \sin \left(\beta + \frac{\alpha}{2}\right) = l_a \sin \frac{\pi + \beta - \gamma}{2} = l_a \cos \frac{\beta - \gamma}{2}.$$

d) Taking into account that $p = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$ (Problem 12.36 c)) and

$$\sin \beta + \sin \gamma = 2 \sin \frac{\beta + \gamma}{2} \cos \frac{\beta - \gamma}{2} = 2 \cos \frac{\alpha}{2} \cos \frac{\beta - \gamma}{2}$$

we arrive at the formula of heading c).

**12.36.** a) Let $O$ be the center of the inscribed circle, $K$ the tangent point of the inscribed circle with side $AB$. Then

$$2R \sin \gamma = AB = AK + KB = r \left(\cot \frac{\alpha}{2} + \cot \frac{\beta}{2}\right) = r \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha}{2} \sin \frac{\beta}{2}.$$

Taking into account that $\sin \gamma = 2 \sin \frac{\gamma}{2} \cos \frac{\gamma}{2}$ and $\sin \frac{\alpha + \beta}{2} = \cos \frac{\gamma}{2}$ we get the desired statement.

b) By Problem 3.2, $p - a = AK = r \cot \frac{\alpha}{2}$. Similarly, $p - b = r \cot \frac{\beta}{2}$ and $p - c = r \cot \frac{\gamma}{2}$. By multiplying these equalities and taking into account that $p(p-a)(p-b)(p-c) = S^2 = (pr)^2$ we get the desired statement.

c) Obviously follows from headings a) and b).

**12.37.** a) By multiplying equalities $r \cos \frac{\alpha}{2} \sin \frac{\alpha}{2} = p - a$ and

$$\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{r}{4R}$$

(cf. Problems 12.17 c) and 12.36 a)) we get the desired statement.

b) By Problem 12.17 c), $r_a \tan \frac{\gamma}{2} = p - b = r \cot \frac{\beta}{2}$. By multiplying this equality by $\frac{r}{4R} = \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$ we get the desired statement.

**12.38.** By adding equalities

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \gamma = - \cos(\alpha + \beta) = -2 \cos^2 \frac{\alpha + \beta}{2} + 1$$

and taking into account that

$$\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} = 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2}$$

we get

$$\cos \alpha + \cos \beta + \cos \gamma = 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + 1 = \frac{r}{R} + 1,$$

cf. Problem 12.36 a).

**12.39.** a) Adding equalities

$$\cos 2\alpha + \cos 2\beta = 2 \cos(\alpha + \beta) \cos(\alpha - \beta) = -2 \cos \gamma \cos(\alpha - \beta);$$
$$\cos 2\gamma = 2 \cos^2 \gamma - 1 = -2 \cos \gamma \cos(\alpha + \beta) - 1$$

and taking into account that

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$$

we get the desired statement.

b) It suffices to substitute expressions of the form $\cos 2\alpha = 2 \cos^2 \alpha - 1$ in the equality obtained in heading a).

**12.40.** Adding equalities

$$\sin 2\alpha + \sin 2\beta = 2 \sin(\alpha + \beta) \cos(\alpha - \beta) = 2 \sin \gamma \cos(\alpha - \beta);$$
$$\sin 2\gamma = 2 \sin \gamma \cos \gamma = -2 \sin \gamma \cos(\alpha + \beta)$$

and taking into account that

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta$$

we get the desired statement.

**12.41.** a) Clearly,

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{a^2 + b^2 + c^2}{4R}$$

and

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 4p^2 - 2(r^2 + p^2 + 4rR),$$

cf. Problem 12.30.

b) By Problem 12.39 b)

$$2 \cos \alpha \cos \beta \cos \gamma = \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma - 2.$$

It remains to make use of a result of heading a).

**12.42.** The law of cosines can be expressed as $ab \cos \gamma = \frac{a^2 + b^2 - c^2}{2}$. By adding three similar equalities we get the desired statement.

**12.43.** By Problem 12.13 $\frac{\cos^2 \frac{\alpha}{2}}{a} = \frac{p(p-a)}{abc}$. It remains to notice that $p(p-a) + p(p-b) + p(p-c) = p^2$ and $abc = 4SR = 4prR$.

**12.44.** a) Since $bc \cos \alpha = 2S \cot \alpha$, it follows that $a^2 = b^2 + c^2 - 4S \cot \alpha$. By adding three similar equalities we get the desired statement.

b) For an acute triangle $a^2 \cot \alpha = 2R^2 \sin 2\alpha = 4S_{BOC}$, where $O$ is the center of the circumscribed circle. It remains to add three analogous equalities. For a triangle with an obtuse angle $\alpha$ the quality $S_{BOC}$ should be taken with the minus sign.

**12.45.** By Problem 12.17 $\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} = \frac{c}{r}$ and $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = \frac{c}{r_c}$. It remains to add such equalities for all pairs of angles of the triangle.

**12.46.** Clearly,
$$\tan \gamma = -\tan(\alpha + \beta) = -\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$
By multiplying both sides of equality by $1 - \tan \alpha \tan \beta$ we get the desired statement.

**12.47.**
$$\tan \frac{\gamma}{2} = \cot \left( \frac{\alpha}{2} + \frac{\beta}{2} \right) = \left[ 1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right] \left[ \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} \right].$$
It remains to multiply both sides of the equality by $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}$.

**12.48.** a) Let us multiply both sides of the equality by $\sin \alpha \sin \beta \sin \gamma$. Further on:
$$\cos \gamma (\sin \alpha \cos \beta + \sin \beta \cos \alpha) + \sin \gamma (\cos \alpha \cos \beta - \sin \alpha \sin \beta) =$$
$$\cos \gamma \sin(\alpha + \beta) + \sin \gamma \cos(\alpha + \beta) =$$
$$\cos \gamma \sin \gamma - \sin \gamma \cos \gamma = 0.$$

b) Let us multiply both sides of the equality by $\sin \alpha \sin \beta \sin \gamma$. Further on:
$$\cos \alpha (\sin \beta \sin \gamma - \cos \beta \cos \gamma) + \sin \alpha (\cos \beta \sin \gamma + \cos \gamma \sin \beta) = \cos^2 \alpha + \sin^2 \alpha = 1.$$

**12.49.** Since
$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma - 2 = 2 \cos \alpha \cos \beta \cos \gamma$$
(see Problem 12.39 b) and $S = 2R^2 \sin \alpha \sin \beta \sin \gamma$, it remains to verify that
$$(\tan \alpha + \tan \beta + \tan \gamma) \cos \alpha \cos \beta \cos \gamma = \sin \gamma \sin \beta \sin \alpha.$$

The latter equality is proved in the solution of Problem 12.48 a).

**12.50.** Let $A$ and $B$ be the vertices of angles $\alpha$ and $\beta$, let $P$ be the intersection point of non-coinciding legs of these angles, $Q$ the common point of the given circles that lies on segment $PA$. Triangle $AQB$ is an isosceles one, hence, $\angle PQB = 2\alpha$. Since $\angle PQB + \angle QPB = \beta + \angle QBA$, it follows that $\beta = 3\alpha$.

**12.51.** By Problem 4.47, $\frac{1}{b} + \frac{1}{c} = \frac{2 \cos \frac{\alpha}{2}}{l_a}$, hence, $\cos \frac{\alpha}{2} = \frac{1}{2}$, i.e., $\alpha = 120°$.

**12.52.** Let us drop perpendicular $MD$ from point $M$ to line $BC$. Then $MD = \frac{1}{2} AH = \frac{1}{2} BM$. In right triangle $BDM$, leg $MD$ is equal to a half hypothenuse $BM$. Hence, $\angle MBC = \angle MBD = 30°$.

**12.53.** The quantities $AD \cdot BC \sin ADB$ and $BE \cdot AC \sin AEB$ are equal because each of them is equal to the doubled area of triangle $ABC$. Hence, $\sin ADB = \sin AEB$. Two cases are possible:

1) $\angle ADB = \angle AEB$. In this case points $A$, $E$, $D$, $B$ lie on one circle; hence, $\angle EAD = \angle EBD$, i.e., $\angle A = \angle B$ which contradicts the hypothesis.

2) $\angle ADB + \angle AEB = 180°$. In this case $\angle ECD + \angle EOD = 180°$, where $O$ is the intersection point of bisectors. Since $\angle EOD = 90° + \frac{\angle C}{2}$ (Problem 5.3), it follows that $\angle C = 60°$.

**12.54.** Let $B'$ be the intersection point of the midperpendicular to segment $AC$ with line $AB$. Then $AB' = CB'$ and $\angle AB'C = 180° - 2 \cdot 75° = 30°$. Hence, $AB' = CB' = 2CH = AB$, i.e., $B' = B$ and $\angle B = 30°$.

**12.55.** Clearly, $AKDM$ is a rectangle and $L$ the intersection point of its diagonals. Since $AD \perp BC$ and $AM \perp BA$, it follows that $\angle DAM = \angle ABC$. Similarly, $\angle KAD = \angle ACB$. Let us drop perpendicular $AP$ from point $A$ to line $KM$. Let, for definiteness, $\angle B < \angle C$. Then point $P$ lies on segment $KL$. Since $\triangle AKP \sim \triangle MKA$, it follows that $AK : AP = MK : MA$. Hence, $AK \cdot AM = AP \cdot MK = AP \cdot AD = 2AP \cdot AL$. By the hypothesis $AL^2 = AK \cdot AM$; hence, $AL = 2AP$, i.e., $\angle ALP = 30°$. Clearly, $\angle KMA = \frac{\angle ALP}{2} = 15°$. Therefore, the acute angles of triangle $ABC$ are equal to $15°$ and $75°$.

**12.56.** Let $CD$ be a bisector. Then $BD = \frac{ac}{a+b}$. On the other hand, $\triangle BDC \sim \triangle BCA$, consequently, $BD : BC = BC : BA$, i.e., $BD = \frac{a^2}{c}$. Hence $c^2 = a(a+b) = 3a^2$. The lengths of the sides of triangle $ABC$ are equal to $a$, $2a$ and $\sqrt{3}a$; hence, its angles are equal to $30°$, $90°$ and $60°$, respectively.

**12.57.** Let $\angle ABC = 2x$. Then the outer angle $\angle A$ of triangle $ABE$ is equal to $\angle ABE + \angle AEB = x + \alpha$. Further,

$$\angle KAE = \angle BAE - \angle BAK = (180° - x - \alpha) - (180° - 2x - 2\alpha) = x + \alpha.$$

Therefore, $AE$ is the bisector of the outer angle $\angle A$ of triangle $ABK$. Since $BE$ is the bisector of the inner angle $\angle B$ of triangle $ABK$, it follows that $E$ is the center of its escribed circle tangent to side $AK$. Hence, $\angle AKE = \frac{1}{2}\angle AKC = 90° - \alpha$.

**12.58.** Let $A_1 \ldots A_{18}$ be a regular 18-gon. For triangle $ABC$ we can take triangle $A_{14}A_1A_9$. By Problem 6.59 b) the diagonals $A_1A_{12}$, $A_2A_{14}$ and $A_9A_{18}$ meet at one point, hence, $\angle AMC = \frac{1}{2}(\smile A_{18}A_2 + \smile A_9A_{14}) = 70°$.

**12.59.** Let $A_1 \ldots A_{18}$ be a regular 18-gon, $O$ its center. For triangle $ABC$ we can take triangle $A_1OA_{18}$. The diagonals $A_2A_{14}$ and $A_{18}A_6$ are symmetric through diameter $A_1A_{10}$; diagonal $A_2A_{14}$ passes through the intersection point of diagonals $A_1A_{12}$ and $A_9A_{18}$ (cf. the solution of Problem 12.58), therefore, $\angle ADE = \frac{1}{2}(\smile A_1A_2 + \smile A_{12}A_{14}) = 30°$.

**12.60.** Since $\angle BDE = 50°$ and $\angle CDE = 30°$, it follows that $\angle BOC = \angle EOD = 180° - 50° - 30° = 100°$. Let us assume that diameters $BB'$ and $CC'$ of the circle are fixed, $\angle BOC = 100°$ and point $A$ moves along arc $\smile B'C'$. Let $D$ be the intersection point of $BB'$ and $AC$, $E$ the intersection point of $CC'$ and $AB$ (Fig. 140). As point $A$ moves from $B'$ to $C'$, segment $OE$ increases while $OD$ decreases, consequently, angle $\angle OED$ decreases and angle $\angle ODE$ increases. Therefore, there exists a unique position of point $A$ for which $\angle CED = \angle OED = 30°$ and $\angle BDE = \angle ODE = 50°$.
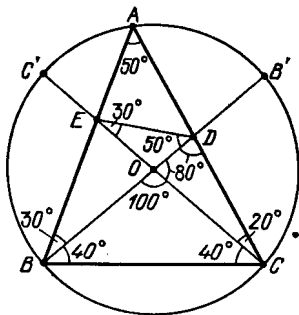


Figure 140 (Sol. 12.60)

Now, let us prove that triangle $ABC$ with angles $\angle A = 50°$, $\angle B = 70°$, $\angle C = 60°$ possesses the required property. Let $A_1 \ldots A_{18}$ be a regular 18-gon. For triangle $ABC$ we can take triangle $A_2A_{14}A_9$. Diagonal $A_1A_{12}$ passes through point $E$ (cf. solution of Problem 12.58). Let $F$ be the intersection point of lines $A_1A_{12}$ and $A_5A_{14}$; line $A_9A_{16}$ is symmetric to line $A_1A_{12}$ through line $A_5A_{14}$ and, therefore, it passes through point $F$. In triangle $CDF$,

ray $CE$ is the bisector of angle $\angle C$ and line $FE$ is the bisector of the outer angle at vertex $F$. Hence, $DE$ is the bisector of angle $\angle ADB$, i.e., $\angle ODE = \frac{1}{4}(\smile A_2A_{14} + \smile A_5A_9) = 50°$.

**12.61.** Let $D$, $E$ and $F$ be the tangent points of the circle with $BP, PQ$ and $QC$, respectively; $\angle BOD = 90° - \angle B = 90° - \angle C = \angle COF = \alpha$, $\angle DOP = \angle POE = \beta$ and $\angle EOQ = \angle QOF = \gamma$. Then $180° = \angle BOC = 2\alpha + 2\beta + 2\gamma$, i.e., $\alpha + \beta + \gamma = 90°$. Since $\angle BPO = \frac{1}{2}\angle DPE = \frac{1}{2}(180° - \angle DOE) = 90° - \beta$ and $\angle QOC = \gamma + \alpha = 90° - \beta$, it follows that $\angle BPO = \angle COQ$. It is also clear that $\angle PBO = \angle OCQ$. Hence, $\triangle BPO \sim \triangle COQ$, i.e., $PB \cdot CQ = BO \cdot CO = \frac{1}{4}BC^2$.

**12.62.** Let $P$ and $Q$ be the midpoints of sides $BC$ and $CD$, respectively. Points $P$ and $Q$ are the tangent points of the inscribed circle with sides $BC$ and $CD$. Therefore, it suffices to verify that $PF + GQ = FG$. Indeed, if $F'G'$ is the segment parallel to $FG$ and tangent to the inscribed circle, then $PF' + G'Q = F'G'$; hence, $F' = F$ and $G' = G$.

We may assume that the side of the square is equal to 2. Let $GD = x$. Since $BF : EB = AD : GD$, then $BF = \frac{2}{x}$. Therefore, $CG = 2 - x$, $GQ = x - 1$, $CF = 2 - \frac{2}{x}$, $FP = \frac{2}{x} - 1$, i.e., $PF + GQ = x + \frac{2}{x} - 2$ and

$$FG^2 = CG^2 + CF^2 = (2 - x)^2 + \left(2 - \frac{2}{x}\right)^2 =$$

$$4 - 4x + x^2 + 4 - \frac{8}{x} + \frac{4}{x^2} = \left(x + \frac{2}{x} - 2\right)^2 = (PF + GQ)^2.$$

**12.63.** Denote the vertices of the squares as shown on Fig. 141. Let $O$ be the center of the circle, $H$ the midpoint of the given chord, $K$ the midpoint of segment $AA_1$.
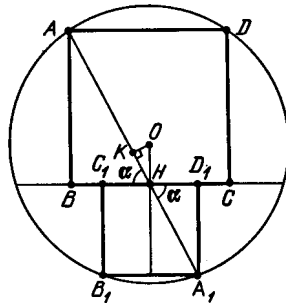


Figure 141 (Sol. 12.63)

Since $\tan AHB = 2 = \tan A_1HD_1$, point $H$ lies on line $AA_1$. Let $\alpha = \angle AHB = \angle A_1HD_1$, then

$$AB - A_1D_1 = (AH - A_1H) \cdot \sin \alpha = 2KH \sin \alpha = 2OH \sin^2 \alpha.$$

Since $\tan \alpha = 2$ and $1 + \cot^2 \alpha = \frac{1}{\sin^2 \alpha}$, it follows that $\sin^2 \alpha = \frac{4}{5}$. Therefore, the difference of the lengths of the squares' sides is equal to $\frac{8}{5}h$.

**12.64.** Let median $BM$ of triangle $ABC$ intersect the inscribed circle at points $K$ and $L$, where $BK = KL = LM = x$. Let, for definiteness, the tangent point of the inscribed circle with side $AC$ lie on segment $MC$. Then since the symmetry through the midperpendicular to segment $BM$ interchanges points $B$ and $M$ and fixes the inscribed circle, tangent $MC$ turns into tangent $BC$. Therefore, $BC = MC = \frac{1}{2}AC$, i.e., $b = 2a$.

Since $BM^2 = \frac{2a^2 + 2c^2 - b^2}{4}$ by Problem 12.11 a), we have $9x^2 = \frac{2a^2 + 2c^2 - 4a^2}{4} = \frac{c^2 - a^2}{2}$. Let $P$ be the tangent point of the inscribed circle with side $BC$. Then $BP = \frac{a+c-b}{2} = \frac{c-a}{2}$. On the other hand, by a property of the tangent, $BP^2 = BK \cdot BL$, i.e., $BP^2 = 2x^2$. Hence,

$2x^2 = \left(\frac{c-a}{2}\right)^2$. Multiplying inequalities $9x^2 = \frac{c^2-a^2}{2}$ and $\left(\frac{c-a}{2}\right)^2 = 2x^2$ we get $\frac{c+a}{c-a} = \frac{9}{4}$, i.e., $c : a = 13 : 5$. As a result we get $a : b : c = 5 : 10 : 13$.

**12.65.** Let $2a$ and $2b$ be the length of the side of the first and second squares, respectively. Then the distance from the center of the circle to any of the vertices of the second square that lie on the circle is equal to $\sqrt{(a + 2b)^2 + b^2}$. On the other hand, this distance is equal to $\sqrt{2}a$. Therefore, $(a + 2b)^2 + b^2 = 2a^2$, i.e., $a = 2b \pm \sqrt{4b^2 + 5b^2} = (2 \pm 3)b$. Only the solution $a = 5b$ is positive.

**12.66.** Let $P$ and $Q$ be the midpoints of segments $AC$ and $AB$, respectively, $R$ the center of circle $S_1$; $a = \frac{1}{2}AC$, $b = \frac{1}{2}BC$, $x$ the radius of circle $S_1$. It is easy to verify that $PR = a + x$, $QR = a + b - x$ and $PQ = b$. In triangle $PQR$, draw height $RH$. The distance from point $R$ to line $CD$ is equal to $x$, hence, $PH = a - x$, consequently, $QH = |b - a + x|$. It follows that

$$(a + x)^2 - (a - x)^2 = RH^2 = (a + b - x)^2 - (b - a + x)^2,$$

i.e., $ax = b(a - x)$. As a result we get $x = \frac{ab}{a+b}$.

For the radius of circle $S_2$ we get precisely the same expression.

**12.67.** Let $x$ be the radius of circle $S$ tangent to circles $S_1$ and $S_2$ and ray $AB$, let $y$ be the radius of circle $S'$ tangent to circles $S_2$ and $S_3$ and ray $BA$. The position of the circle tangent to circle $S_1$ and ray $AB$ (resp. $S_3$ and $BA$) is uniquely determined by its radius, consequently, it suffices to verify that $x = y$.

By equating two expressions for the squared distance from the center of circle $S$ to line $AD$ we get

$$(x + 1)^2 - (x - 1)^2 = (3 + x)^2 - (5 - x)^2, \text{ i.e., } x = \frac{4}{3}.$$

Considering circles $S_2$ and $S_3$ it is easy to verify that $AB^2 = (3 + 4)^2 - 1^2 = 48$. On the other hand, the squared distances from the center of circle $S'$ to lines $AD$ and $BC$ are equal to $(y + 3)^2 - (5 - y)^2 = 16(y - 1)$ and $(4 + y)^2 - (4 - y)^2 = 16y$, respectively. Therefore, $4\sqrt{y - 1} + 4\sqrt{y} = \sqrt{48}$, i.e., $y = \frac{4}{3}$.

**12.68.** If the angles of a triangle form an arithmetic progression, then they are equal to $\alpha - \gamma$, $\alpha$, $\alpha + \gamma$, where $\gamma \geq 0$. Since the sum of the angles of a triangle is equal to $180°$, we deduce that $\alpha = 60°$. The sides of this triangle are equal to $2R\sin(\alpha - \gamma)$, $2R\sin\alpha$, $2R\sin(\alpha + \gamma)$. Since the greater side subtends the greater angle, $\sin(\alpha - \gamma) \leq \sin\alpha \leq \sin(\alpha + \gamma)$.

a) If the numbers $\sin(\alpha - \gamma) \leq \sin\alpha \leq \sin(\alpha + \gamma)$ form an arithmetic progression, then $\sin\alpha = \frac{1}{2}(\sin(\alpha + \gamma) + \sin(\alpha - \gamma)) = \sin\alpha\cos\gamma$, i.e., either $\cos\gamma = 1$ or $\gamma = 0$. Therefore, each of the triangle's angles is equal to $60°$.

b) If the numbers $\sin(\alpha - \gamma) \leq \sin\alpha \leq \sin(\alpha + \gamma)$ form a geometric progression, then

$$\sin^2\alpha = \sin(\alpha - \gamma)\sin(\alpha + \gamma) = \sin^2\alpha\cos^2\gamma - \sin^2\gamma\cos^2\alpha \leq \sin^2\alpha\cos^2\gamma.$$

Hence, $\cos\gamma = 1$, i.e., each of the triangle's angles is equal to $60°$.

**12.69.** Let us complement triangle $ABC$ to parallelogram $ABCE$ (Fig. 142). Let $BC = x$ and $AD = y$. Then $(b - a)h = 2S_{AED} = xy\sin 45°$ and

$$(b - a)^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - 2xy\sin 45°.$$

By Pythagoras theorem

$$a^2 + b^2 = (AO^2 + BO^2) + (CO^2 + DO^2) = (BO^2 + CO^2) + (DO^2 + AO^2) = x^2 + y^2.$$

Therefore,

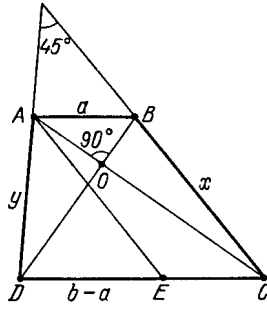$$(b - a)^2 = x^2 + y^2 - 2xy\sin 45° = a^2 + b^2 - 2(b - a)h,$$

Figure 142 (Sol. 12.69)

i.e., $h = \frac{ab}{b-a}$.

**12.70.** Since $BK = \frac{1}{2}(a + c - b)$ and $KC = \frac{1}{2}(a + b - c)$ (cf. Problem 3.2), it follows that $BK \cdot KC = \frac{a^2 - (b-c)^2}{4} = S \tan \frac{\alpha}{2}$, cf. Problem 12.12.

**12.71.** Since $\frac{b+c}{a} = \frac{\cos \frac{\beta - \gamma}{2}}{\sin \frac{\alpha}{2}}$ (Problem 12.4), it follows that $\cos \frac{\beta - \gamma}{2} = \cos \frac{\alpha}{2}$, i.e., $\beta - \gamma = \pm \alpha$. If $\beta = \gamma + \alpha$, then $\beta = 90°$ and if $\beta + \alpha = \gamma$, then $\gamma = 90°$.

**12.72.** It is easy to verify that $S_{ABC} = 2R^2 \sin \alpha \sin \beta \sin \gamma$. Analogously,

$$S_{A_1 B_1 C_1} = 2R^2 \sin \frac{\beta + \gamma}{2} \sin \frac{\alpha + \gamma}{2} \sin \frac{\alpha + \beta}{2} = 2R^2 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.$$

Hence,

$$\frac{S_{ABC}}{S_{A_1 B_1 C_1}} = 8 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \frac{2r}{R},$$

cf. Problem 12.36 a).

**12.73.** The sum of cotangents of the angles of a triangle is equal to $\frac{a^2 + b^2 + c^2}{4S}$, cf. Problem 12.44 a). Moreover, $m_a^2 + m_b^2 + m_c^2 = \frac{3(a^2 + b^2 + c^2)}{4}$ (by Problem 12.11 b)) and the area of the triangle formed by the medians of triangle $ABC$ is equal to $\frac{3}{4} S_{ABC}$ (Problem 1.36).

**12.74.** One of the points $A_i$ lies inside the triangle formed by the other three points; hence, we can assume that triangle $A_1 A_2 A_3$ is an acute one (or a right one). Numbers $\lambda_1$, $\lambda_2$ and $\lambda_3$ are easy to obtain from the corresponding system of equations; as a result we get

$$\lambda_1 = \frac{b^2 + c^2 - a^2}{2}, \ \lambda_2 = \frac{a^2 + c^2 - b^2}{2} \text{ and } \lambda_3 = \frac{a^2 + b^2 - c^2}{2},$$

where $a = A_2 A_3, b = A_1 A_3$ and $c = A_1 A_2$. By Problem 5.45 b) $A_1 A_4^2 = 4R^2 - a^2$, where $R$ is the radius of the circumscribed circle of triangle $A_1 A_2 A_3$. Hence,

$$\lambda_4 = A_1 A_4^2 - \lambda_1 = 4R^2 - \frac{a^2 + b^2 + c^2}{2} = A_2 A_4^2 - \lambda_2 = A_3 A_4^2 - \lambda_3.$$

Now, let us verify that $\sum \frac{1}{\lambda_i} = 0$. Since $\frac{b^2 + c^2 - a^2}{2} = bc \cos \alpha = 2S \cot \alpha$, it follows that $\frac{1}{\lambda_1} = \tan \frac{\alpha}{2S}$. It remains to observe that

$$\frac{2}{a^2 + b^2 + c^2 - 8R^2} = \frac{\tan \alpha + \tan \beta + \tan \gamma}{2S}$$

?Problem 12.49.

**12.75.** Let $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$ be the coordinates of the triangle's vertices. The coordinates of the center of the circumscribed circle of the triangle are given by the system of equations

$$(x - a_1)^2 + (y - b_1)^2 = (x - a_2)^2 + (y - b_2)^2,$$
$$(x - a_1)^2 + (y - b_1)^2 = (x - a_3)^2 + (y - b_3)^2.$$

It is easy to verify that these equations are actually linear ones and, therefore, the solution of the considered system is a rational one.

**12.76.** On segments $AB$ and $CD$, take points $K$ and $L$ that divide the segments in the ratios indicated. It suffices to prove that the intersection point of lines $AK$ and $CL$ lies on circle $S$. Let us take the coordinate system with the origin at the center $O$ of circle $S$ and axes $Ox$ and $Oy$ directed along rays $OB$ and $OD$. The radius of circle $S$ can be assumed to be equal to 1. Lines $AK$ and $CL$ are given by equations $y = \frac{x+1}{3}$ and $y = 2x - 1$, respectively. Therefore, the coordinates of their intersection point are $x_0 = \frac{4}{5}$ and $y_0 = \frac{3}{5}$. Clearly, $x_0^2 + y_0^2 = 1$.

**12.77.** Let $d$ be the distance between the center of the circumscribed circle and the image of the center of the inscribed circle under the considered homothety. It suffices to verify that $R = d + 2r$. Let $(0,0)$, $(2a, 0)$ and $(0, 2b)$ be the coordinates of the vertices of the given triangle. Then $(a, b)$ are the coordinates of the center of the circumscribed circle, $(r, r)$ the coordinates of the center of the inscribed circle, where $r = a + b - R$. Therefore,

$$d^2 = (2r - a)^2 + (2r - b)^2 = a^2 + b^2 - 4r(a + b - r) + 4r^2 = (R - 2r)^2$$

because $a^2 + b^2 = R^2$.

**12.78.** Let us consider the coordinate system with the origin at the center of the square and the $Ox$-axis parallel to line $l$. Let the coordinates of the vertices of the square be $(A(x, y)$, $B(y, -x)$, $C(-x, -y)$ and $D(-y, x)$; let line $l$ be given by the equation $y = a$. Then the coordinates of point $Q$ are $\left(\frac{x+y}{2}, \frac{y-x}{2}\right)$ and those of $P$ are $(-y, a)$. Therefore, the locus to be found consists of points $\left(t, -t + \frac{1}{2}a\right)$, where $t = \frac{x-y}{4}$. It remains to observe that the quantity $x - y$ varies from $-\sqrt{2(x^2 + y^2)} = -AB$ to $AB$.

# Chapter 13. VECTORS

## Background

1. We will make use of the following notations:

a) $\overrightarrow{AB}$ and $\mathbf{a}$ denote vectors;

b) $AB$ and $|\mathbf{a}|$ denote the lengths of these vectors; sometimes the length of vector $\mathbf{a}$ will be denoted by $a$; a *unit* vector is a vector of unit length;

c) $(\overrightarrow{AB}, \overrightarrow{CD})$, $(\mathbf{a}, \mathbf{b})$ and $(\overrightarrow{AB}, \mathbf{a})$ denote the inner products of the vectors;

d) $(x, y)$ is the vector with coordinates $x$, $y$;

e) $\overrightarrow{0}$ or $\mathbf{0}$ denotes the zero vector.

2. The *oriented angle between the nonzero vectors* $\mathbf{a}$ and $\mathbf{b}$ (notation $\angle(\mathbf{a}, \mathbf{b})$) is the angle through which one should rotate the vector $\mathbf{a}$ counterclockwise to make it directed as $\mathbf{b}$ is. The angles that differ by 360 degrees are assumed to be equal. It is easy to verify the following properties of oriented angles between vectors:

a) $\angle(\mathbf{a}, \mathbf{b}) = -\angle(\mathbf{b}, \mathbf{a})$;

b) $\angle(\mathbf{a}, \mathbf{b}) + \angle(\mathbf{b}, \mathbf{c}) = \angle(\mathbf{a}, \mathbf{c})$;

c) $\angle(-\mathbf{a}, \mathbf{b}) = \angle(\mathbf{a}, \mathbf{b}) + 180°$.

3. The *inner product of vectors* $\mathbf{a}$ and $\mathbf{b}$ is the number

$$(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| \cdot |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b})$$

(if one of these vectors is the zero one, then by definition $(\mathbf{a}, \mathbf{b}) = 0$). The following properties of the inner product are easily verified:

a) $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$;

b) $|(\mathbf{a}, \mathbf{b})| \leq |\mathbf{a}| \cdot |\mathbf{b}|$;

c) $(\lambda \mathbf{a} + \mu \mathbf{b}, \mathbf{c}) = \lambda(\mathbf{a}, \mathbf{c}) + \mu(\mathbf{b}, \mathbf{c})$;

d) if $\mathbf{a}, \mathbf{b} \neq 0$ then $(\mathbf{a}, \mathbf{b}) = 0$ if and only if $\mathbf{a} \perp \mathbf{b}$.

4. Many of vector inequalities can be proved with the help of the following fact.

*Given two sets of vectors such that the sum of lengths of projections of the vectors of the first set to any straight line does not exceed the sum of the lengths of projections of the vectors from the second set to the same line, the sum of the lengths of the vectors from the first set does not exceed the sum of the lengths of the vectors of the second set*, cf. Problem 13.39.

In this way a problem on a plane reduces to a problem on a straight line which is usually easier.

## Introductory problems

**1.** Let $AA_1$ be the median of triangle $ABC$. Prove that $\overrightarrow{AA_1} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$.

**2.** Prove that $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$.

**3.** Prove that if vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular, then $|\mathbf{a}| = |\mathbf{b}|$.

**4.** Let $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{0}$ and $OA = OB = OC$. Prove that $ABC$ is an equilateral triangle.

**5.** Let $M$ and $N$ be the midpoints of segments $AB$ and $CD$, respcetively. Prove that $\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{BD})$.

## §1. Vectors formed by polygons' (?) sides

**13.1.** a) Prove that from the medians of a triangle one can construct a triangle.

b) From the medians of triangle $ABC$ one constructed triangle $A_1B_1C_1$ and from the medians of triangle $A_1B_1C_1$ one constructed triangle $A_2B_2C_2$. Prove that triangles $ABC$ and $A_2B_2C_2$ are similar with simlarity coefficient $\frac{3}{4}$.

**13.2.** The sides of triangle $T$ are parallel to the respective medians of triangle $T_1$. Prove that the medians of $T$ are parallel to the corresponding sides of $T_1$.

**13.3.** Let $M_1$, $M_2, \ldots, M_6$ be the midpoints of a convex hexagon $A_1A_2 \ldots A_6$. Prove that there exists a triangle whose sides are equal and parallel to the segments $M_1M_2$, $M_3M_4$, $M_5M_6$.

**13.4.** From a point inside a convex $n$-gon, the rays are drawn perpendicular to the sides and intersecting the sides (or their continuations). On these rays the vectors $\mathbf{a}_1$, $\ldots$, $\mathbf{a}_n$ whose lengths are equal to the lengths of the corresponding sides are drawn. Prove that $\mathbf{a}_1 + \cdots + \mathbf{a}_n = \mathbf{0}$.

**13.5.** The sum of four unit vectors is equal to zero. Prove that the vectors can be divided into two pairs of opposite vectors.

**13.6.** Let $E$ and $F$ be the midpoints of sides $AB$ and $CD$ of quadrilateral $ABCD$ and $K$, $L$, $M$ and $N$ are the midpoints of segments $AF$, $CE$, $BF$ and $DE$, respectively. Prove that $KLMN$ is a parallelogram.

**13.7.** Consider $n$ pairwise noncodirected vectors ($n \geq 3$) whose sum is equal to zero. Prove that there exists a convex $n$-gon such that the set of vectors formed by its sides coincides with the given set of vectors.

**13.8.** Given four pairwise nonparallel vectors whose sum is equal to zero. Prove that we can construct from them:

a) a nonconvex quadrilateral;

b) a self-intersecting broken line of four links.

**13.9.** Given four pairwise nonparallel vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ whose sum is equal to zero, prove that

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| > |\mathbf{a} + \mathbf{b}| + |\mathbf{a} + \mathbf{c}| + |\mathbf{a} + \mathbf{d}|.$$

**13.10.** In a convex pentagon $ABCDE$ side $BC$ is parallel to diagonal $AD$, in addition we have $CD \parallel BE$, $DE \parallel AC$ and $AE \parallel BD$. Prove that $AB \parallel CE$.

## §2. Inner product. Relations

**13.11.** Prove that if the diagonals of quadrilateral $ABCD$ are perpendicular to each other, then the diagonals of any other quadrilateral with the same lengths of its sides are perpendicular to each other.

**13.12.** a) Let $A$, $B$, $C$ and $D$ be arbitrary points on a plane. Prove that

$$(\overrightarrow{AB}, \overrightarrow{CD}) + (\overrightarrow{BC}, \overrightarrow{AD}) + (\overrightarrow{CA}, \overrightarrow{BD}) = 0.$$

b) Prove that the hights of a triangle intersect at one point.

**13.13.** Let $O$ be the center of the circle inscribed in triangle $ABC$ and let point $H$ satisfy $OH = OA + OB + OC$. Prove that $H$ is the intersection point of heights of triangle $ABC$.

**13.14.** Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be vectors formed by the sides of an $n$-gon, $\varphi_{ij} = \angle(\mathbf{a}_i, \mathbf{a}_j)$. Prove that

$$a_1^2 = a_2^2 + \cdots + a_n^2 + 2 \sum_{i>j>1} a_i a_j \cos \varphi_{ij}, \text{ where } a_i = |\mathbf{a}_i|.$$

**13.15.** Given quadrilateral $ABCD$ and the numbers

$$u = AD^2, \;\; v = BD^2, \;\; w = CD^2, \;\; U = BD^2 + CD^2 - BC^2,$$
$$V = AD^2 + CD^2 - AC^2, \;\; W = AD^2 + BD^2 - AB^2.$$

Prove that

((Gauss).) $$uU^2 + vV^2 + wW^2 = UVW + 4uvw.$$

**13.16.** Points $A$, $B$, $C$ and $D$ are such that for any point $M$ the numbers $(\overrightarrow{MA}, \overrightarrow{MB})$ and $(\overrightarrow{MC}, \overrightarrow{MD})$ are distinct. Prove that $\overrightarrow{AC} = \overrightarrow{DB}$.

**13.17.** Prove that in a convex $k$-gon the sum of distances from any inner point to the sides of the $k$-gon is constant if and only if the sum of vectors of unit exterior normals to the sides is equal to zero.

**13.18.** In a convex quadrilateral the sum of distances from a vertex to the sides is the same for all vertices. Prove that this quadrilateral is a parallelogram.

## §3. Inequalities

**13.19.** Given points $A$, $B$, $C$ and $D$. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2,$$

where the equality is attained only if $ADCD$ is a parallelogram.

**13.20.** Prove that from any five vectors one can always select two so that the length of their sum does not exceed the length of the sum of the remaining three vectors.

**13.21.** Ten vectors are such that the length of the sum of any nine of them is smaller than the length of the sum of all the ten vectors. Prove that there exists an axis such that the projection of every of the ten vectors to the axis is positive.

**13.22.** Points $A_1, \ldots, A_n$ lie on a circle with center $O$ and $\overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$. Prove that for any point $X$ we have

$$XA_1 + \cdots + XA_n \geq nR,$$

where $R$ is the radius of the circle.

**13.23.** Given eight real numbers $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$. Prove that at least one of the six numbers

$$ac + bd, \;\; ae + bf, \;\; ag + bh, \;\; ce + df, \;\; cg + dh, \;\; eg + fh$$

is nonnegative.

**13.24.** On the circle of radius 1 with center $O$ there are given $2n+1$ points $P_1, \ldots, P_{2n+1}$ which lie on one side of a diameter. Prove that

$$|\overrightarrow{OP_1} + \cdots + \overrightarrow{OP_{2n+1}}| \geq 1.$$

**13.25.** Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be vectors whose length does not exceed 1. Prove that in the sum

$$\mathbf{c} = \pm\mathbf{a}_1 \pm \mathbf{a}_2 \pm \cdots \pm \mathbf{a}_n$$

we can select signs so that $|\mathbf{c}| \leq \sqrt{2}$.

**13.26.** Point $O$ is the beginning point of $n$ unit vectors such that in any half plane bounded by a straight line through $O$ there are contained not less than $k$ vectors (we assume that the boundary line belongs to the half-plane). Prove that the length of the sum of these vectors does not exceed $n - 2k$.

## §4. Sums of vectors

**13.27.** Prove that point $X$ belongs to line $AB$ if and only if

$$\overrightarrow{OX} = t\overrightarrow{OA} + (1 - t)\overrightarrow{OB}$$

for some $t$ and any point $O$.

**13.28.** We are given several points and for several pairs $(A, B)$ of these points the vectors $AB$ are taken in such a way that as many vectors exit from every point as terminate in it. Prove that the sum of all the selected vectors is equal to $\mathbf{0}$.

**13.29.** Inside triangle $ABC$, point $O$ is taken. Prove that

$$S_{BOC} \cdot \overrightarrow{OA} + S_{AOC} \cdot \overrightarrow{OB} + S_{AOB} \cdot \overrightarrow{OC} = \overrightarrow{0}.$$

**13.30.** Points $A$ and $B$ move along two fixed rays with common origin $O$ so that $\frac{p}{OA} + \frac{q}{OB}$ is a constant. Prove that line $AB$ passes through a fixed point.

**13.31.** Through the intersection point $M$ of medians of triangle $ABC$ a straight line is drawn intersecting $BC$, $CA$ and $AB$ at points $A_1$, $B_1$ and $C_1$, respectively. Prove that

$$(\frac{1}{\overline{MA_1}}) + (\frac{1}{\overline{MB_1}}) + (\frac{1}{\overline{MC_1}}) = 0.$$

(Segments $MA_1$, $MB_1$ and $MC_1$ are assumed to be oriented.)

**13.32.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$, respectively, are taken. Segments $BB_1$ and $CC_1$, $CC_1$ and $AA_1$, $AA_1$ and $BB_1$ intersect at points $A_1$, $B_2$ and $C_2$, respectively. Prove that if $\overrightarrow{AA_2} + \overrightarrow{BB_2} + \overrightarrow{CC_2} = \overrightarrow{0}$, then

$$AB_1 : B_1C = CA_1 : A_1B = BC_1 : C_1A.$$

**13.33.** Quadrilateral $ABCD$ is an inscribed one. Let $H_a$ be the orthocenter of $BCD$, let $M_a$ be the midpoint of $AH_a$; let points $M_b$, $M_c$ and $M_d$ be similarly defined. Prove that points $M_a$, $M_b$, $M_c$ and $M_d$ coincide.

**13.34.** Quadrilateral $ABCD$ is inscribed in a circle of radius $R$.

a) Let $S_a$ be the circle of radius $R$ with center at the orthocenter of triangle $BCD$; let circles $S_b$, $S_c$ and $S_d$ be similarly defined. Prove that these four circles intersect at one point.

b) Prove that the circles of nine points of triangles $ABC$, $BCD$, $CDA$ and $DAB$ intersect at one point.

## §5. Auxiliary projections

**13.35.** Point $X$ belongs to the interior of triangle $ABC$; let $\alpha = S_{BXC}$, $\beta = S_{CXA}$ and $\gamma = S_{AXB}$. Let $A_1$, $B_1$ and $C_1$ be the projections of points $A$, $B$ and $C$, respectively, on an arbitrary line $l$. Prove that the length of vector $\alpha\overrightarrow{AA_1} + \beta\overrightarrow{BB_1} + \gamma\overrightarrow{CC_1}$ is equal to $(\alpha + \beta + \gamma)d$, where $d$ is the distance from $X$ to $l$.

**13.36.** A convex $2n$-gon $A_1A_2 \ldots A_{2n}$ is inscribed into a unit circle. Prove that

$$|\overrightarrow{A_1A_2} + \overrightarrow{A_3A_4} + \cdots + \overrightarrow{A_{2n-1}A_{2n}}| \leq 2.$$

**13.37.** Let $a$, $b$ and $c$ be the lengths of the sides of triangle $ABC$; let $\mathbf{n}_a$, $\mathbf{n}_b$ and $\mathbf{n}_c$ be unit vectors perpendicular to the corresponding sides and directed outwards. Prove that

$$a^3\mathbf{n}_a + b^2\mathbf{n}_b + c^2\mathbf{n}_c = 12S \cdot \overrightarrow{MO},$$

where $S$ is the area, $M$ the intersection point of the medians, $O$ the center of the circle inscribed into triangle $ABC$.

**13.38.** Let $O$ and $R$ be the center and the radius, respectively, of an escribed circle of triangle $ABC$; let $Z$ and $r$ be the center and the radius of the inscribed circle, $K$ the intersection point of the medians of the triangle with vertices at the tangent points of the inscribed circle of triangle $ABC$ with the sides of triangle $ABC$. Prove that $Z$ belongs to segment $OK$ and

$$OZ : ZK = 3R : r.$$

## §6. The method of averaging

**13.39.** Given two sets of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ such that the sum of the lengths of the projections of the vectors from the first set to any straight line does not exceed the sum of the lengths of the projections of the vectors from the second set to the same straight line. Prove that the sum of the lengths of the vectors from the first set does not exceed the sum of the lengths of the vectors from the second set.

**13.40.** Prove that if one convex polygon lies inside another one, then the perimeter of the inner polygon does not exceed the perimeter of the outer one.

**13.41.** The sum of the length of several vectors on a plane is equal to $L$. Prove that from these vectors one can select several vectors (perhaps, just one) so that the length of their sum is not less than $\frac{L}{\pi}$.

**13.42.** Prove that if the lengths of any side and diagonal of a convex polygon are shorter than $d$, then its perimeter is shorter than $\pi d$.

**13.43.** On the plane, there are given four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ whose sum is equal to zero. Prove that

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| \geq |\mathbf{a} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}|.$$

**13.44.** Inside a convex $n$-gon $A_1 A_2 \ldots A_n$ a point $O$ is selected so that $\overrightarrow{OA_1} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$. Let $d = OA_1 + \cdots + OA_n$. Prove that the perimeter of the polygon is not shorter than $\frac{4d}{n}$ for $n$ even and not shorter than $\frac{4dn}{n^2-1}$ for $n$ odd.

**13.45.** The length of the projection of a closed convex curve to any line is equal to 1. Prove that its length is equal to $\pi$.

**13.46.** Given several convex polygons so that it is impossible to draw a line which does not intersect any of the polygons and at least one polygon would lie on both sides of it. Prove that all the polygons are inside a polygon whose perimeter does not exceed the sum of the perimeters of the given polygons.

## §7. Pseudoinner product

The *pseudoinner product* of nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$c = |\mathbf{a}| \cdot |\mathbf{b}| \sin \angle(\mathbf{a}, \mathbf{b});$$

the pseudoinner product is equal to 0 if at least one of the vectors $\mathbf{a}$ or $\mathbf{b}$ is zero. The pseudoinner product is denoted by $c = \mathbf{a} \vee \mathbf{b}$. Clearly, $\mathbf{a} \vee \mathbf{b} = -\mathbf{b} \vee \mathbf{a}$.

The absolute value of the pseudoinner product of $\mathbf{a}$ and $\mathbf{b}$ is equal to the area of the parallelogram spanned by these vectors. In this connection the *oriented area* of the triple of points $A$, $B$ and $C$ is the number

$$S(A, B, C) = \frac{1}{2}(\overrightarrow{AB} \vee \overrightarrow{AC}).$$

The absolute value of $S(A, B, C)$ is equal to the area of triangle $ABC$.

**13.47.** Prove that:

a) $(\lambda \mathbf{a}) \vee \mathbf{b} = \lambda(\mathbf{a}, \mathbf{b})$;

b) $\mathbf{a} \vee (\mathbf{b} + \mathbf{c}) = \mathbf{a} \vee \mathbf{b} + \mathbf{a} \vee \mathbf{c}$.

**13.48.** Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Prove that

$$\mathbf{a} \vee \mathbf{b} = a_1 b_2 - a_2 b_1.$$

**13.49.** a) Prove that

$$S(A, B, C) = -S(B, A, C) = S(B, C, A).$$

b) Prove that for any points $A$, $B$, $C$ and $D$ we have

$$S(A, B, C) = S(D, A, B) + S(D, B, C) + S(D, C, A).$$

**13.50.** Three runners $A$, $B$ and $C$ run along the parallel lanes with constant speeds. At the initial moment the area of triangle $ABC$ is equal to 2 in 5 seconds it is equal to 3. What might be its value after 5 more seconds?

**13.51.** Three pedestrians walk at constant speeds along three straight roads. At the initial moment the pedestrians were not on one straight line. Prove that the pedestrians can occure on one straight line not more than twice.

**13.52.** Prove Problem 4.29 b) with the help of a pseudoinner product.

**13.53.** Points $P_1$, $P_2$ and $P_3$ not on one line are inside a convex $2n$-gon $A_1 \ldots A_{2n}$. Prove that if the sum of the areas of triangles $A_1 A_2 P_i$, $A_3 A_4 P_i$, $\ldots$, $A_{2n-1} A_{2n} P_i$ is equal to the same number $c$ for $i = 1$, 2, 3, then for any inner point $P$ the sum of the areas of these triangles is equal to $c$.

**13.54.** Given triangle $ABC$ and point $P$. Let point $Q$ be such that $CQ \parallel AP$ and point $R$ be such that $AR \parallel BQ$ and $CR \parallel BP$. Prove that $S_{ABC} = S_{PQR}$.

**13.55.** Let $H_1$, $H_2$ and $H_3$ be the orthocenters of triangles $A_2 A_3 A_4$, $A_1 A_3 A_4$ and $A_1 A_2 A_4$. Prove that the areas of triangles $A_1 A_2 A_3$ and $H_1 H_2 H_3$ are equal.

**13.56.** In a convex 5-gon $ABCDE$ whose area is equal to $S$ the areas of triangles $ABC$, $BCD$, $CDE$, $DEA$ and $EAB$ are equal to $a$, $b$, $c$, $d$ and $e$, respectively. Prove that

$$S^2 - S(a + b + c + d + e) + ab + bc + cd + de + ea = 0.$$

## Problems for independent study

**13.57.** Let $M$ and $N$ be the midpoints of segments $AB$ and $AC$, respectively, $P$ the midpoint of $MN$ and $O$ an arbitrary point. Prove that $2\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 4\overrightarrow{OP}$.

**13.58.** Points $A$, $B$ and $C$ move uniformly with the same angle velocities along the three circles in the same direction. Prove that the intersection point of the medians of triangle $ABC$ moves along a circle.

**13.59.** Let $A$, $B$, $C$, $D$ and $E$ be arbitrary points. Is there a point $O$ such that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE}$? Find all such points, if any.

**13.60.** Let $P$ and $Q$ be the midpoints of the diagonals of a convex quadrilateral $ABCD$. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

**13.61.** The midpoints of segments $AB$ and $CD$ are connected by a segment; so are the midpoints of segments $BC$ and $DE$. The midpoints of the segments obtained are also connected by a segment. Prove that the last segment is parallel to segment $AE$ and its length is equal to $\frac{1}{4} AE$.

**13.62.** The inscribed circle is tangent to sides $BC$, $CA$ and $AB$ of triangle $ABC$ at points $A_1$, $B_1$ and $C_1$, respectively. Prove that if $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}$, then triangle $ABC$ is an equilateral one.

**13.63.** Quadrilaterals $ABCD$, $AEFG$, $ADFH$, $FIJE$ and $BIJC$ are parallelograms. Prove that quadrilateral $AFHG$ is also a parallelogram.

## Solutions

**13.1.** a) Let $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$; let $AA'$, $BB'$ and $CC'$ be medians of triangle $ABC$. Then $\overrightarrow{AA'} = \frac{1}{2}(\mathbf{c} - \mathbf{b})$, $\overrightarrow{BB'} = \frac{1}{2}(\mathbf{a} - \mathbf{c})$ and $\overrightarrow{CC'} = \frac{1}{2}(\mathbf{b} - \mathbf{c})$. Therefore, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0}$.

b) Let $\mathbf{a}_1 = \overrightarrow{AA'}$, $\mathbf{b}_1 = \overrightarrow{BB'}$ and $\mathbf{c} = \overrightarrow{CC'}$. Then $\frac{1}{2}(\mathbf{c}_1 - \mathbf{b}_1) = \frac{1}{4}(\mathbf{b} - \mathbf{a} - \mathbf{a} + \mathbf{c}) = -\frac{3}{4}\mathbf{a}$ is the vector of one of the sides of triangle $A_2 B_2 C_2$.

**13.2.** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors of the sides of $T$. Then $\frac{1}{2}(\mathbf{b} - \mathbf{a})$, $\frac{1}{2}(\mathbf{a} - \mathbf{c})$ and $\frac{1}{2}(\mathbf{c} - \mathbf{b})$ are the vectors of its medians. We may assume that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the vectors directed from the intersection point of the medians of triangle $T_1$ to its vertices. Then $\mathbf{b} - \mathbf{a}$, $\mathbf{a} - \mathbf{c}$ and $\mathbf{c} - \mathbf{a}$ are the vectors of its sides.

**13.3.** It is clear that $2\overrightarrow{M_1 M_2} = \overrightarrow{A_1 A_2} + \overrightarrow{A_2 A_3} = \overrightarrow{A_1 A_3}$, $2\overrightarrow{M_3 M_4} = \overrightarrow{A_3 A_5}$ and $2\overrightarrow{M_5 M_6} = \overrightarrow{A_5 A_1}$. Therefore, $\overrightarrow{M_1 M_2} + \overrightarrow{M_3 M_4} + \overrightarrow{M_5 M_6} = \overrightarrow{0}$.

**13.4.** After rotation through $90°$ the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ turn into the vectors of sides of the $n$-gon.

**13.5.** From given vectors one can construct a convex quadrilateral. The lengths of all the sides of this quadrilateral are equal to 1, therefore, this quadrilateral is a rhombus; the pairs of its opposite sides provide us with the division desired.

**13.6.** Let $\mathbf{a} = \overrightarrow{AE}$, $\mathbf{b} = \overrightarrow{DF}$ and $\mathbf{v} = \overrightarrow{AD}$. Then $2\overrightarrow{AK} = \mathbf{b} + \mathbf{v}$ and $2\overrightarrow{AL} = \mathbf{a} + \mathbf{v} + 2\mathbf{b}$ and, therefore, $\overrightarrow{KL} = \overrightarrow{AL} - \overrightarrow{AK} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. Similarly, $\overrightarrow{NM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

**13.7.** Let us draw the given vectors from one point and index them clockwise: $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Consider a closed broken line $A_1 \ldots A_n$, where $\overrightarrow{A_i A_{i+1}} = \mathbf{a}_i$. Let us prove that $A_1 \ldots A_n$ is a convex polygon. Introduce a coordinate system and direct the $Ox$-axis along $\mathbf{a}_1$. Let the vectors $\mathbf{a}_2, \ldots, \mathbf{a}_k$ lie on one side of $Ox$-axis and the vectors $\mathbf{a}_{k+1}, \ldots, \mathbf{a}_n$ lie on the other side (if there is a vector directed opposite to $\mathbf{a}_1$ it can be referred to either of these two groups).

The projections of the vectors from the first group on the $Oy$-axis are of one sign and the projections of the vectors of the other group are of the opposite sign. Therefore, the second coordinate of the points $A_2$, $A_3, \ldots, A_{k+1}$ and the points $A_{k+1}, \ldots, A_n$, $A_1$ vary monotonously: for the first group from 0 to a quantity $d$, for the second group they decrease from $d$ to 0. Since there are two intervals of monotonity, all the vertices of the polygon lie on one side of the line $A_1 A_2$.

For the other lines passing through the sides of the polygon the proof is similar.

**13.8.** Thanks to Problem 13.7 the given vectors form a convex quadrilateral. The rest is clear from Fig. 143.
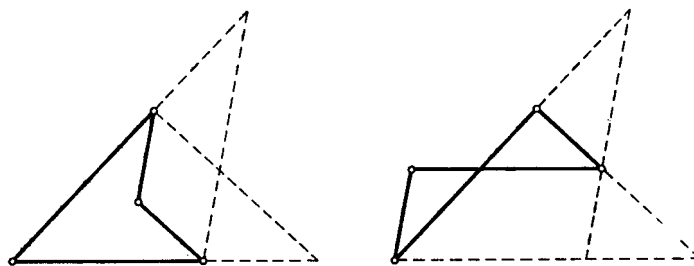


Figure 143 (Sol. 13.8)

**13.9.** By Problem 13.8 b) from the given vectors we can construct a self-intersecting broken line of four links; this broken line can be viewed as the two diagonals and two opposite sides of a convex quadrilateral. Two cases are possible: the vector **a** can be either a side or a diagonal of this quadrilateral.

But in both cases the sum in the left-hand side of the inequality is the sum of lengths of two opposite sides and two diagonals of the quadrilateral and the sum in the right-hand side is constituted by the length of the sum of vectors of the same opposite sides and the lengths of the two other opposite sides. It only remains to observe that the sum of lengths of two vectors is not shorter than the length of their sum and the sum of the length of diagonals of a convex quadrilateral is longer than the sum of lengths of the two opposite sides: cf. Problem 19.14.

**13.10.** Let diagonal $BE$ intersect diagonals $AD$ and $AC$ at points $F$ and $G$, respectively. The respective sides of triangles $AFE$ and $BCD$ are parallel; hence, the triangles are similar and $AF : FE = BC : CD$. Therefore,

$$AD : BE = (AF + BC) : (EF + CD) = BC : CD.$$

Similarly, $AE : BD = DE : AC$. From the similarity of $BED$ and $EGA$ we deduce that $AE : DB = EG : BE = CD : BE$. Thus,

$$\frac{BC}{AD} = \frac{CD}{BE} = \frac{AE}{BD} = \frac{DE}{AC} = \lambda.$$

Clearly,

$$\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} + \overrightarrow{AB} = \overrightarrow{0},$$

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CA} + \overrightarrow{DB} + \overrightarrow{EC} = \overrightarrow{0}$$

and

$$\overrightarrow{BC} = \lambda\overrightarrow{AD}, \;\; \overrightarrow{CD} = \lambda\overrightarrow{BE}, \;\; \overrightarrow{DE} = \lambda\overrightarrow{CA}, \;\; \overrightarrow{EA} = \lambda\overrightarrow{DB}.$$

It follows that

$$\overrightarrow{0} = \lambda(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CA} + \overrightarrow{DB}) + \overrightarrow{AB} = -\lambda\overrightarrow{EC} + \overrightarrow{AB},$$

i.e., $\overrightarrow{AB} = \lambda\overrightarrow{EC}$. Hence, $AB \parallel EC$.

**13.11.** Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$, $\mathbf{c} = \overrightarrow{CD}$ and $\mathbf{d} = \overrightarrow{DA}$. It suffices to verify that $AC \perp BD$ if and only if $a^2 + c^2 = b^2 + d^2$. Clearly,

$$d^2 = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = a^2 + b^2 + c^2 + 2[(\mathbf{a}, \mathbf{b}) + (\mathbf{b}, \mathbf{c}) + (\mathbf{c}, \mathbf{a})].$$

Therefore, the condition $AC \perp BD$, i.e.,

$$0 = (\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}) = b^2 + (\mathbf{b}, \mathbf{c}) + (\mathbf{a}, \mathbf{c}) + (\mathbf{a}, \mathbf{b})$$

is equivalent to the fact that

$$d^2 = a^2 + b^2 + c^2 - 2b^2.$$

**13.12.** a) Let us express all the vectors that enter the formula through $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$, i.e., let us write $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, $\overrightarrow{CA} = -\overrightarrow{AB} - \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$. After simplification we get the statement desired.

b) Let $D$ be the intersection point of heights drawn from vertices $A$ and $C$ of triangle $ABC$. Then in the formula proved in heading a) the first two summands are zero and, therefore, the last summand is also zero, i.e., $BD \perp AC$.

**13.13.** Let us prove that $AH \perp BC$. Indeed, $\overrightarrow{AH} = \overrightarrow{AO} + \overrightarrow{OH} = \overrightarrow{AO} + \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OC}$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{OB} + \overrightarrow{OC}$ and, therefore,

$$(\overrightarrow{AH}, \overrightarrow{BC}) = OC^2 - OB^2 = R^2 - R^2 = 0$$

because $O$ is the center of the circumscribed circle. We similarly prove that $BH \perp AC$ and $CH \perp AB$.

**13.14.** Let $\alpha_i = \angle(\mathbf{a}_i, \mathbf{a}_1)$. Considering the projections to the straight line parallel to $\mathbf{a}_1$ and the straight line perpendicular to $\mathbf{a}_1$ we get $a_1 = \sum_{i>1} a_i \cos \alpha_i$ and $0 = \sum_{i>1} a_i \sin \alpha_i$, respectively. Squaring these equalities and summing we get

$$\begin{aligned} a_1^2 &= \sum_{i>1} a_i^2(\cos^2 \alpha_i + \sin^2 \alpha_i) + 2\sum_{i>j>1} a_i a_j(\cos \alpha_i \cos \alpha_j + \sin \alpha_i \sin \alpha_j) = \\ & a_2^2 + \cdots + a_n^2 + 2\sum_{i>j>1} a_i a_j \cos(\alpha_i - \alpha_j). \end{aligned}$$

It remains to notice that $\alpha_i - \alpha_j = \angle(\mathbf{a}_i, \mathbf{a}_1) - \angle(\mathbf{a}_j, \mathbf{a}_1) = \angle(\mathbf{a}_i, \mathbf{a}_j) = \varphi_{ij}$.

**13.15.** Let $\mathbf{a} = \overrightarrow{AD}$, $\mathbf{b} = \overrightarrow{BD}$ and $\mathbf{c} = \overrightarrow{CD}$. Since $BC^2 = |\mathbf{b} - \mathbf{c}|^2 = BD^2 + CD^2 - 2(\mathbf{b}, \mathbf{c})$, it follows that $U = 2(\mathbf{b}, \mathbf{c})$. Similarly, $V = 2(\mathbf{a}, \mathbf{c})$ and $W = 2(\mathbf{a}, \mathbf{b})$. Let $\alpha = \angle(\mathbf{a}, \mathbf{b})$ and $\beta = \angle(\mathbf{b}, \mathbf{c})$. Multiplying the equality

$$\cos^2 \alpha + \cos^2 \beta + \cos^2(\alpha + \beta) = 2\cos \alpha \cos \beta \cos(\alpha + \beta) + 1$$

(cf. Problem 12.39 b)) by $4uvw = 4|a|^2|b|^2|c|^2$ we get the statement desired.

**13.16.** Fix an arbitrary point $O$. Let $\mathbf{m} = \overrightarrow{OM}$, $\mathbf{a} = \overrightarrow{OA}, \ldots, \mathbf{d} = \overrightarrow{OD}$. Then

$$\begin{aligned} (\overrightarrow{MA}, \overrightarrow{MB}) - (\overrightarrow{MC}, \overrightarrow{MD}) &= (\mathbf{a} - \mathbf{m}, \mathbf{b} - \mathbf{m}) - (\mathbf{c} - \mathbf{m}, \mathbf{d} - \mathbf{m}) = \\ & (\mathbf{c} + \mathbf{d} - \mathbf{a} - \mathbf{b}, \mathbf{m}) + (\mathbf{a}, \mathbf{b}) - (\mathbf{c}, \mathbf{d}). \end{aligned}$$

If $\mathbf{v} = \mathbf{c} + \mathbf{d} - \mathbf{a} - \mathbf{b} \neq \mathbf{0}$, then as the point $M$ runs over the plane the value $(\mathbf{v}, \mathbf{m})$ attains all the real values, in particular, it takes the value $(\mathbf{c}, \mathbf{d}) - (\mathbf{a}, \mathbf{b})$. Hence, $\mathbf{v} = \mathbf{0}$, i.e., $\overrightarrow{OC} + \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB}$ and, therefore, $\overrightarrow{AC} = \overrightarrow{DB}$.

**13.17.** Let $\mathbf{n}_1, \ldots, \mathbf{n}_k$ be the unit exterior normals to the sides and let $M_1, \ldots, M_k$ be arbitrary points on these sides. For any point $X$ inside the polygon the distance from $X$ to the $i$-th side is equal to $(\overrightarrow{XM_i}, \mathbf{n}_i)$. Therefore, the sums of distances from the inner points $A$ and $B$ to the sides of the polygon are equal if and only if

$$\sum_{i=1}^{k} (\overrightarrow{AM_i}, \mathbf{n}_i) = \sum_{i=1}^{k} (\overrightarrow{BM_i}, \mathbf{n}_i) =$$

$$\sum_{i=1}^{k} (\overrightarrow{BA}, \mathbf{n}_i) + \sum_{i=1}^{k} (\overrightarrow{AM_i}, \mathbf{n}_i),$$

i.e., $(\overrightarrow{BA}, \sum_{i=1}^{k} \mathbf{n}_i) = 0$. Hence, the sum of distances from any inner point of the polygon to the sides is constant if and only if $\sum \mathbf{n}_i = \mathbf{0}$.

**13.18.** Let $l$ be an arbitrary line, $\mathbf{n}$ the unit vector perpendicular to $l$. If points $A$ and $B$ belong to the same half-plane given by the line $l$ the vector $\mathbf{n}$ belongs to, then $\rho(B, l) - \rho(A, l) = (\overrightarrow{AB}, \mathbf{n})$, where $\rho(X, l)$ is the distance from $X$ to $l$.

Let $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ and $\mathbf{n}_4$ be unit vectors perpendicular to the consecutive sides of quadrilateral $ABCD$ and directed inwards. Denote the sum of distances from point $X$ to the sides

of quadrilateral $ABCD$ by $\sum(X)$. Then

$$0 = \sum(B) - \sum(A) = (\overrightarrow{AB}, \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4).$$

Similarly,

$$(\overrightarrow{BC}, \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4) = 0.$$

Since points $A$, $B$ and $C$ do not belong to the same line, $\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4 = \mathbf{0}$. It remains to make use of the result of Problem 13.5.

**13.19.** Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$ and $\mathbf{c} = \overrightarrow{CD}$. Then $\overrightarrow{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$, $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$ and $\overrightarrow{BD} = \mathbf{b} + \mathbf{c}$. It is also clear that

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 - |\mathbf{a} + \mathbf{b}|^2 - |\mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2 + 2(\mathbf{a}, \quad \mathbf{c}) + |\mathbf{c}|^2 = |\mathbf{a} + \mathbf{c}|^2 \geq 0.$$

The equality is only attained if $\mathbf{a} = -\mathbf{c}$, i.e., $ABCD$ is a parallelogram.

**13.20.** Consider five vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$ and suppose that the length of the sum of any two of them is longer than the length of the sum of the three remaining ones. Since $|\mathbf{a}_1 + \mathbf{a}_2| > |\mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5|$, it follows that

$$|\mathbf{a}_1|^2 + 2(\mathbf{a}_1, \mathbf{a}_2) + |\mathbf{a}_2|^2 > |\mathbf{a}_3|^2 + |\mathbf{a}_4|^2 + |\mathbf{a}_5|^2 + 2(\mathbf{a}_3, \mathbf{a}_4) + 2(\mathbf{a}_4, \mathbf{a}_5) + 2(\mathbf{a}_3, \mathbf{a}_5).$$

Adding such inequalities for all ten pairs of vectors we get

$$4(|\mathbf{a}_1|^2 + \dots) + 2((\mathbf{a}_1, \mathbf{a}_2) + \dots) > 6(|\mathbf{a}_1|^2 + \dots) + 6((\mathbf{a}_1, \mathbf{a}_2) + \dots)$$

i.e., $|\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5|^2 < 0$. Contradiction.

**13.21.** Denote the given vectors by $\mathbf{e}_1, \dots, \mathbf{e}_{10}$. Let $\overrightarrow{AB} = \mathbf{e}_1 + \cdots + \mathbf{e}_{10}$. Let us prove that the ray $AB$ determines the required axis. Clearly, $|\overrightarrow{AB} - \mathbf{e}_i|^2 = AB^2 - 2(\overrightarrow{AB}, \mathbf{e}_i) + |\mathbf{e}_i|^2$, i.e., $(\overrightarrow{AB}, \mathbf{e}_i) = \frac{1}{2}(AB^2 + |\mathbf{e}_i|^2 - |\overrightarrow{AB} - \mathbf{e}_i|^2)$. By the hypothesis $AB > |\overrightarrow{AB} - \mathbf{e}_i|$ and, therefore, $(\overrightarrow{AB}, \mathbf{e}_i) > 0$, i.e., the projection of $\mathbf{e}_i$ to $AB$ is positive.

**13.22.** Let $\mathbf{a}_i = \overrightarrow{OA_i}$ and $\mathbf{x} = \overrightarrow{OX}$. Then $|\mathbf{a}_i| = R$ and $\overrightarrow{XA_i} = \mathbf{a}_i - \mathbf{x}$. Therefore,

$$\sum XA_i = \sum |\mathbf{a}_i - \mathbf{x}| = \sum \frac{|\mathbf{a}_i - \mathbf{x}| \cdot |\mathbf{a}_i|}{R} \geq$$

$$\sum \frac{\mathbf{a}_i - \mathbf{x}, \mathbf{a}_i}{R} = \sum \frac{\mathbf{a}_i, \mathbf{a}_i}{R} - \frac{(\mathbf{x}, \sum \mathbf{a}_i)}{R}.$$

It remains to observe that $(\mathbf{a}_i, \mathbf{a}_i) = R^2$ and $\sum \mathbf{a}_i = \mathbf{0}$.

**13.23.** On the plane, consider four vectors $(a, b)$, $(c, d)$, $(e, f)$ and $(g, h)$. One of the angles between these vectors does not exceed $\frac{360°}{4} = 90°$. If the angle between the vectors does not exceed $90°$, then the inner product is nonnegative.

The given six numbers are inner products of all the pairs of our four vectors and, therefore, at least one of them is nonnegative.

**13.24.** Let us prove this statement by induction. For $n = 0$ the statement is obviously true. Let us assume that the statement is proved for $2n + 1$ vectors. In a system of $2n + 3$ vectors consider two extreme vectors (i.e., the vectors the angle between which is maximal).

For definiteness sake, suppose that these are vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_{2n+3}}$. By the inductive hypothesis the length of $\overrightarrow{OR} = \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_{2n+2}}$ is not shorter than 1. The vector $\overrightarrow{OR}$ belongs to the interior of angle $\angle P_1 O P_{2n+3}$ and, therefore, it forms an acute angle with the vector $\overrightarrow{OS} = \overrightarrow{OP_1} + \overrightarrow{OP_{2n+3}}$. Hence, $|\overrightarrow{OS} + \overrightarrow{OR}| \geq OR \geq 1$.

**13.25.** First, let us prove that if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors whose length does not exceed 1, then at least one of the vectors $\mathbf{a} \pm \mathbf{b}$, $\mathbf{a} \pm \mathbf{c}$, $\mathbf{b} \pm \mathbf{c}$ is not longer than 1.

Indeed, two of the vectors $\pm\mathbf{a}$, $\pm\mathbf{b}$, $\pm\mathbf{c}$ form an angle not greater than 60° and, therefore, the difference of these two vectors is not longer than 1 (if (??) in triangle $ABC$ we have $AB \leq 1$, $BC \leq 1$ and $\angle ABC \leq 60°$, then $AC$ is not the greatest side and $AC \leq 1$).

Thus, we can reduce the discussion to two vectors $\mathbf{a}$ and $\mathbf{b}$. Then either the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ or between vectors $\mathbf{a}$ and $-\mathbf{b}$ does not exceed 90°; hence, either $|\mathbf{a}-\mathbf{b}| \leq \sqrt{2}$ or $|\mathbf{a}+\mathbf{b}| \leq \sqrt{2}$.

**13.26.** We can assume that the sum $\mathbf{a}$ of the given vectors is nonzero because otherwise the statement of the problem is obvious.
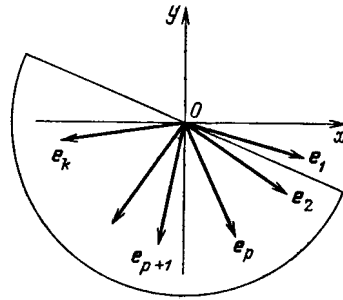


Figure 144 (Sol. 13.26)

Let us introduce a coordinate system directing $Oy$-axis along $\mathbf{a}$. Let us enumerate the vectors of the lower half-plane clockwise: $\mathbf{e}_1$, $\mathbf{e}_2, \ldots$ as on Fig. 144. By the hypothesis there are not less than $k$ of these vectors. Let us prove that among the given vectors there are also vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ such that the second coordinate of the vector $\mathbf{v}_i + \mathbf{e}_i$ is nonpositive for any $i = 1, \ldots, k$. This will prove the required statement.

Indeed, the length of the sum of the given vectors is equal to the sum of the second coordinates (the coordinate system was introduced just like this). The second coordinate of the sum of the vectors $\mathbf{e}_1$, $\mathbf{v}_1, \ldots, \mathbf{e}_k$, $\mathbf{v}_k$ is nonpositive and the second coordinate of any of the remaining vectors does not exceed 1. Therefore, the second coordinate of the sum of all the given vectors does not exceed $n - 2k$.

Let vectors $\mathbf{e}_1, \ldots, \mathbf{e}_p$ belong to the fourth quadrant. Let us start assigning to them the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$. Let us rotate the lower half plane that consists of points with nonpositive second coordinate by rotating the $Ox$-axis clockwise through an angle between 0° and 90°. If one of the two vectors that belongs to the half plane rotated this way lies in the fourth quadrant, then their sum has a nonpositive second coordinate. As the $Ox$-axis rotates beyond vector $\mathbf{e}_1$, at least one vector that belongs to the half plane should be added to the vectors $\mathbf{e}_2, \ldots, \mathbf{e}_k$; hence, the vector which follows $\mathbf{e}_k$ should be taken for $\mathbf{v}_1$.

Similarly, while the $Ox$-axis is rotated beyond $\mathbf{e}_2$ we get vector $\mathbf{v}_2$, and so on. These arguments remain valid until the $Ox$-axis remains in the fourth quadrant. For the vectors $\mathbf{e}_{p+1}, \ldots, \mathbf{e}_k$ which belong to the third quadrant the proof is given similarly (if the first coordinate of the vector $\mathbf{e}_{p+1}$ is zero, then we should first disregard it; then take any of the remaining vectors for its(whose?) partner).

**13.27.** Point $X$ belongs to line $AB$ if and only if $\overrightarrow{AX} = \lambda\overrightarrow{AB}$, i.e.,

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = (1 - \lambda)\overrightarrow{OA} + \lambda\overrightarrow{OB}.$$

**13.28.** Let us take an arbitrary point $O$ and express all the selected vectors in the form $\overrightarrow{A_iA_j} = \overrightarrow{OA_j} - \overrightarrow{OA_i}$. By the hypothesis every vector $\overrightarrow{OA_i}$ enters the sum of all the chosen vectors with the "plus" sign as many times as with the "minus" sign.

**13.29.** Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be unit vectors codirected with vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, respectively; let $\alpha = \angle BOC$, $\beta = \angle COA$ and $\gamma = \angle AOB$. We have to prove that

$$\mathbf{e}_1 \sin \alpha + \mathbf{e}_2 \sin \beta + \mathbf{e}_3 \sin \gamma = \overrightarrow{0}.$$

Consider triangle $A_1B_1C_1$ whose sides are parallel to lines $OC$, $OA$ and $OB$. Then

$$\overrightarrow{0} = \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} + \overrightarrow{C_1A_1} = \pm 2R(\mathbf{e}_1 \sin \alpha + \mathbf{e}_2 \sin \beta + \mathbf{e}_3 \sin \gamma),$$

where $R$ is the radius of the circumscribed circle of triangle $ABC$.

**13.30.** Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors codirected with rays $OA$ and $OB$, let $\lambda = OA$ and $\mu = OB$. Line $AB$ consists of all points $X$ such that

$$\overrightarrow{OX} = t\overrightarrow{OA} + (1-t)\overrightarrow{OB} = t\lambda\mathbf{a} + (1-t)\mu\mathbf{b}.$$

We have to find numbers $x_0$ and $y_0$ such that $\frac{x_0}{\lambda} = t = 1 - \frac{y_0}{\mu}$ for all the considered values of $\lambda$ and $\mu$. It remains to set $x_0 = \frac{p}{c}$ and $y_0 = \frac{q}{c}$. As a result we see that if $\frac{p}{OA} + \frac{q}{OB} = c$, then line $AB$ passes through a point $X$ such that $\overrightarrow{OX} = \frac{p\mathbf{a} + q\mathbf{b}}{c}$.

**13.31.** Let $\mathbf{a} = \overrightarrow{MA}$, $\mathbf{b} = \overrightarrow{MB}$ and $\mathbf{c} = \overrightarrow{MC}$. Then $\mathbf{e} = \overrightarrow{MC_1} = p\mathbf{a} + (1-p)\mathbf{b}$ and

$$\overrightarrow{MA_1} = q\mathbf{c} + (1-q)\mathbf{b} = -q\mathbf{a} + (1-2q)\mathbf{b}.$$

On the other hand, $\overrightarrow{MA_1} = \alpha\mathbf{e}$. Similarly,

$$\beta\mathbf{e} = \overrightarrow{MB_1} = -r\mathbf{b} + (1-2r)\mathbf{a}.$$

We have to show that $1 + \frac{1}{\alpha} + \frac{1}{\beta} = 0$. Since $\alpha p\mathbf{a} + \alpha(1-p)\mathbf{b} = \alpha\mathbf{e} = -q\mathbf{a} + (1-2q)\mathbf{b}$, it follows that $\alpha p = 1 - 2r$ and $\alpha(1-p) = 1 - 2q$ and, therefore, $\frac{1}{\alpha} = 1 - 3p$. Similarly, $\beta p = 1 - 2r$ and $\beta(1-p) = -r$ and, therefore, $\frac{1}{\beta} = 3p - 2$.

**13.32.** Summing up the equalities $\overrightarrow{AA_2} + \overrightarrow{BB_2} + \overrightarrow{CC_2} = \overrightarrow{0}$ and $\overrightarrow{A_2B_2} + \overrightarrow{B_2C_2} + \overrightarrow{C_2A_2} = \overrightarrow{0}$ we get $\overrightarrow{AB_2} + \overrightarrow{BC_2} + \overrightarrow{CA_2} = \overrightarrow{0}$. It follows that $\overrightarrow{AB_2} = \lambda\overrightarrow{C_2B_2}, \overrightarrow{BC_2} = \lambda\overrightarrow{A_2C_2}$ and $\overrightarrow{CA_2} = \lambda\overrightarrow{B_2A_2}$. Let $E$ be a point on line $BC$ such that $A_2E \parallel AA_1$. Then $\overrightarrow{BA_1} = \lambda\overrightarrow{EA_1}$ and $\overrightarrow{EC} = \lambda\overrightarrow{EA_1}$; hence, $\overrightarrow{A_1C} = \overrightarrow{EC} - \overrightarrow{EA_1} = (\lambda - 1)\overrightarrow{EA_1}$. Therefore, $\frac{A_1C}{BA_1} = \frac{\lambda - 1}{\lambda}$. Similarly, $\frac{AB_1}{B_1C} = \frac{BC_1}{C_1A} = \frac{\lambda - 1}{\lambda}$.

**13.33.** Let $O$ be the center of the inscribed circle of the given quadrilateral, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$ and $\mathbf{d} = \overrightarrow{OD}$. If $H_a$ is the orthocenter of triangle $BCD$, then $\overrightarrow{OH_a} = \mathbf{b} + \mathbf{c} + \mathbf{d}$ (cf. Problem 13.13). Therefore,

$$\overrightarrow{OM_a} = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \overrightarrow{OM_b} = \overrightarrow{OM_c} = \overrightarrow{OM_d}.$$

**13.34.** Let $O$ be the center of the circumscribed circle of the given quadrilateral; $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$ and $\mathbf{d} = \overrightarrow{OD}$. If $H_d$ is the orthocenter of triangle $ABC$, then $\overrightarrow{OH_d} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ (Problem 13.13).

a) Take a point $K$ such that $\overrightarrow{OK} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$. Then

$$KH_d = |\overrightarrow{OK} - \overrightarrow{OH_d}| = |\mathbf{d}| = R,$$

i.e., $K$ belongs to circle $S_d$. We similarly prove that $K$ belongs to circles $S_a$, $S_b$ and $S_c$.

b) Let $O_d$ be the center of the circle of nine points of triangle $ABC$, i.e., the midpoint of $OH_d$. Then $\overrightarrow{OO_d} = \overrightarrow{OH_d}/2 = (\mathbf{a} + \mathbf{b} + \mathbf{c})/2$. Take point $X$ such that $\overrightarrow{OX} = (\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})/2$. Then $XO_d = \frac{1}{2}|\mathbf{d}| = \frac{1}{2}R$, i.e., $X$ belongs to the circle of nine points of triangle $ABC$. We

similarly prove that $X$ belongs to the circles of nine points of triangles $BCD$, $CDA$ and $DAB$.

**13.35.** Let $X_1$ be the projection of $X$ on $l$. Vector $\alpha\overrightarrow{AA_1}+\beta\overrightarrow{BB_1}+\gamma\overrightarrow{CC_1}$ is the projection of vector $\alpha\overrightarrow{AX_1} + \beta\overrightarrow{BX_1} + \gamma\overrightarrow{CX_1}$ to a line perpendicular to $l$. Since

$$\alpha\overrightarrow{AX_1} + \beta\overrightarrow{BX_1} + \gamma\overrightarrow{CX_1} = \alpha\overrightarrow{AX} + \beta\overrightarrow{BX} + \gamma\overrightarrow{CX} + (\alpha + \beta + \gamma)\overrightarrow{XX_1}$$

and $\alpha\overrightarrow{AX} + \beta\overrightarrow{BX} + \gamma\overrightarrow{CX} = \overrightarrow{0}$ (by Problem 13.29), we get the statement required.

(?)**13.36.** Let $\mathbf{a} = \overrightarrow{A_1A_2} + \overrightarrow{A_3A_4} + \cdots + \overrightarrow{A_{2n-1}A_{2n}}$ and $\mathbf{a} \neq \mathbf{0}$. Introduce the coordinate system directing the $Ox$-axis along vector $\mathbf{a}$. Since the sum of projections of vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_3A_4}$, ..., $\overrightarrow{A_{2n-1}A_{2n}}$ on $Oy$ is zero, it follows that the length of $\mathbf{a}$ is equal to the absolute value of the difference between the sum of the lengths of positive projections of these vectors to the $Ox$-axis and the sum of lengths of their negative projections.

Therefore, the length of $\mathbf{a}$ does not exceed either the sum of the lengths of the positive projections or the sum of the lengths of the negative projections.

It is easy to verify that the sum of the lengths of positive projections as well as the sum of the lengths of negative projections of the given vectors on any axis does not exceed the diameter of the circle, i.e., does not exceed 2.

**13.37.** In the proof of the equality of vectors it suffices to verify the equality of their projections (minding the sign) on lines $BC$, $CA$ and $AB$. Let us carry out the proof, for example, for the projections on line $BC$, where the direction of ray $BC$ will be assumed to be the positive one. Let $P$ be the projection of point $A$ on line $BC$ and $N$ the midpoint of $BC$. Then

$$\overrightarrow{PN} = \overrightarrow{PC} + \overrightarrow{CN} = \frac{b^2 + a^2 - c^2}{2a} - \frac{a}{2} = \frac{b^2 - c^2}{2a}$$

($PC$ is found from the equation $AB^2 - BP^2 = AC^2 - CP^2$). Since $NM : NA = 1 : 3$, the projection of $\overrightarrow{MO}$ on line $BC$ is equal to $\frac{1}{3}\overrightarrow{PN} = \frac{b^2-c^2}{6a}$. It remains to notice that the projection of vector $a^3\mathbf{n}_a + b^3\mathbf{n}_b + c^3\mathbf{n}_c$ on $BC$ is equal to

$$b^3 \sin\gamma - c^3 \sin\beta = \frac{b^3c - c^3b}{2R} = \frac{abc}{2R} \cdot \frac{b^2 - c^2}{a} = 2S\frac{b^2 - c^2}{a}.$$

**13.38.** Let the inscribed circle be tangent to sides $AB$, $BC$ and $CA$ at points $U$, $V$ and $W$, respectively. We have to prove that $\overrightarrow{OZ} = \frac{3R}{r}\overrightarrow{ZK}$, i.e., $\overrightarrow{OZ} = \frac{R}{r}(\overrightarrow{ZU} + \overrightarrow{ZV} + \overrightarrow{ZW})$. Let us prove, for example, that the (oriented) projections of these vectors on line $BC$ are equal; the direction of ray $BC$ will be assumed to be the positive one.

Let $N$ be the projection of point $O$ on line $BC$. Then the projection of vector $OZ$ on line $BC$ is equal to

$$\overrightarrow{NV} = \overrightarrow{NC} + \overrightarrow{CV} = (\frac{a}{2}) - \frac{(a + b - c)}{2} = \frac{(c - b)}{2}.$$

The projection of vector $\overrightarrow{ZU} + \overrightarrow{ZV} + \overrightarrow{ZW}$ on this line is equal to the projection of vector $\overrightarrow{ZU} + \overrightarrow{ZW}$, i.e., it is equal to

$$-r \sin VZU + r \sin VZW = -r \sin B + r \sin C = \frac{r(c - b)}{2R}.$$

**13.39.** Introduce the coordinate system $Oxy$. Let $l_\varphi$ be the straight line through $O$ and constituting an angle of $\varphi$ $(0 < \varphi < \pi)$ with the $Ox$-axis, i.e., if point $A$ belongs to $l_\varphi$ and the second coordinate of $A$ is positive, then $\angle AOX = \varphi$; in particular, $l_0 = l_\pi = Ox$.

If vector $\mathbf{a}$ forms an angle of $\alpha$ with the $Ox$-axis (the angle is counted counterclockwise from the $Ox$-axis to the vector $\mathbf{a}$), then the length of the projection of $\mathbf{a}$ on $l_\varphi$ is equal to $|\mathbf{a}| \cdot |\cos(\varphi - \alpha)|$. The integral $\int_o^\pi |\mathbf{a}| \cdot |\cos(\varphi - \alpha)| d\varphi = 2|\mathbf{a}|$ does not depend on $\alpha$.

Let vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$; $\mathbf{b}_1, \ldots, \mathbf{b}_m$ constitute angles of $\alpha_1, \ldots, \alpha_n$; $\beta_1, \ldots, \beta_n$, respectively, with the $Ox$-axis. Then by the hypothesis

$$|\mathbf{a}_1| \cdot |\cos(\varphi - \alpha_1)| + \cdots + |\mathbf{a}_n| \cdot |\cos(\varphi - \alpha_n)| \leq$$
$$|\mathbf{b}_1| \cdot |\cos(\varphi - \beta_1)| + \cdots + |\mathbf{b}_m| \cdot |\cos(\varphi - \beta_m)|$$

for any $\varphi$. Integrating these inequalities over $\varphi$ from 0 to $\pi$ we get

$$|\mathbf{a}_1| + \cdots + |\mathbf{a}_n| \leq |\mathbf{b}_1| + \cdots + |\mathbf{b}_m|.$$

REMARK. The value $\frac{1}{b-a} \int_a^b f(x) dx$ is called the *mean value* of the function $f$ on the segment $[a, b]$. The equality

$$\int_0^\pi |\mathbf{a}| \cdot |\cos(\varphi - \alpha)| d\varphi = 2|\mathbf{a}|$$

means that the mean value of the length of the projection of vector $\mathbf{a}$ is equal to $\frac{2}{\pi}|\mathbf{a}|$; more precisely, the mean value of the function $f(\varphi)$ equal to the length of the projection of $\mathbf{a}$ to $l_\varphi$ on the segment $[0, \pi]$ is equal to $\frac{2}{\pi}|\mathbf{a}|$.

**13.40.** The sum of the lengths of the projections of a convex polygon on any line is equal to twice the length of the projection of the polygon on this line. Therefore, the sum of the lengths of the projections of vectors formed by edges on any line is not longer for the inner polygon than for the outer one. Hence, by Problem 13.39 the sum of the lengths of vectors formed by the sides, i.e., the perimeter of the inner polygon, is not longer than that of the outer one.

**13.41.** If the sum of the lengths of vectors is equal to $L$, then by Remark to Problem 13.39 the mean value of the sum of the lengths of projections of these vectors is equal to $2L/\pi$.

The value of function $f$ on segment $[a, b]$ cannot be always less than its mean value $c$ because otherwise

$$c = \frac{1}{a - b} \int_a^b f(x) dx < \frac{(b - a)c}{b - a} = c.$$

Therefore, there exists a line $l$ such that the sum of the lengths of the projections of the initial vectors on $l$ is not shorter than $2L/\pi$.

On $l$, select a direction. Then either the sum of the lengths of the positive projections to this directed line or the sum of the lengths of the negative projections is not shorter than $L/\pi$. Therefore, either the length of the sum of vectors with positive projections or the length of the sum of vectors with negative porjections is not shorter than $L/\pi$.

**13.42.** Let $AB$ denote the projection of the polygon on line $l$. Clearly, points $A$ and $B$ are projections of certain vertices $A_1$ and $B_1$ of the polygon. Therefore, $A_1B_1 \geq AB$, i.e., the length of the projection of the polygon is not longer than $A_1B_1$ and $A_1B_1 < d$ by the hypothesis. Since the sum of the lengths of the projections of the sides of the polygon on $l$ is equal to $2AB$, it does not exceed $2d$.

The mean value of the sum of the lengths of the projections of sides is equal to $\frac{2}{\pi}P$, where $P$ is a perimeter (see Problem 13.39). The mean value does not exceed the maximal one; hence, $\frac{2}{\pi}P < 2d$, i.e., $P < \pi d$.

**13.43.** By Problem 13.39 it suffices to prove the inequality

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| \geq |\mathbf{a} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}| + |\mathbf{c} + \mathbf{d}|$$

for the projections of the vectors on a line, i.e., we may assume that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors parallel to one line, i.e., they are just numbers such that $a + b + c + d = 0$. Let us assume that $d \geq 0$ because otherwise we can change the sign of all the numbers.

We can assume that $a \leq b \leq c$. We have to consider three cases:

1) $a, b, c \leq 0$;

2) $a \leq 0$ and $b, c \geq 0$;

3) $a, b \leq 0, c \geq 0$.

All arising inequalities are quite easy to verify. In the third case we have to consider separately the subcases $|d| \leq |b|$, $|b| \leq |d| \leq |a|$ and $|a| \leq |d|$ (in the last subcase we have to take into account that $|d| = |a| + |b| - |c| \leq |a| + |b|$).

**13.44.** By Problem 13.39 it suffices to prove the inequality for the projections of vectors on any line. Let the projections of $\overrightarrow{OA}_1, \dots, \overrightarrow{OA}_n$ on a line $l$ be equal (up to a sign) to $a_1, \dots, a_n$. Let us divide the numbers $a_1, \dots, a_n$ into two groups: $x_1 \geq x_2 \geq \cdots \geq x_k > 0$ and $y'_1 \leq y'_2 \leq \cdots \leq y'_{n-k} \leq 0$. Let $y_i = -y'_i$. Then $x_1 + \cdots + x_k = y_1 + \cdots + y_{n-k} = a$ and, therefore, $x_1 \geq \frac{a}{k}$ and $y_1 \geq \frac{a}{n-k}$. To the perimeter the number $2(x_1 + y_1)$ in the projection corresponds. To the sum of the vectors $\overrightarrow{OA}_i$ the number $x_1 + \cdots + x_k + y_1 + \cdots + y_{n-k} = 2a$ in the projection corresponds. And since

$$\frac{2(x_1 + y_1)}{x_1 + \cdots + y_{n-k}} \geq \frac{2((a/k) + (a/(n-k)))}{2a} = \frac{n}{k(n-k)},$$

it remains to notice that the quantity $k(n-k)$ is maximal for $k = n/2$ if $n$ is even and for $k = (n \pm 1)/2$ if $n$ is odd.

**13.45.** By definition the *length of a curve* is the limit of perimeters of the polygons inscribed in it. [Vo vvedenie]

Consider an inscribed polygon with perimeter $P$ and let the length of the projection on line $l$ be equal to $d_i$. Let $1 - \varepsilon < d_i < 1$ for all lines $l$. The polygon can be selected so that $\varepsilon$ is however small. Since the polygon is a convex one, the sum of the lengths of the projections of its sides on $l$ is equal to $2d_i$.

By Problem 13.39 the mean value of the quantity $2d_i$ is equal to $\frac{2}{\pi}P$ (cf. Problem 13.39) and, therefore, $2 - 2\varepsilon < \frac{2}{\pi}P < 2$, i.e., $\pi - \pi\varepsilon < P < \pi$. Tending $\varepsilon$ to zero we see that the length of the curve is equal to $\pi$.

**13.46.** Let us prove that the perimeter of the convex hull of all the vertices of given polygons does not exceed the sum of their perimeters. To this end it suffices to notice that by the hypothesis the projections of given polygons to any line cover the projection of the convex hull.

**13.47.** a) If $\lambda < 0$, then

$$(\lambda\mathbf{a}) \vee \mathbf{b} = -\lambda|\mathbf{a}| \cdot |\mathbf{b}| \sin \angle(-\mathbf{a}, \mathbf{b}) = \lambda|\mathbf{a}| \cdot |\mathbf{a}| \sin \angle(\mathbf{a}, \mathbf{b}) = \lambda(\mathbf{a} \vee \mathbf{b}).$$

For $\lambda > 0$ the proof is obvious.

b) Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Introduce the coordinate system directing the $Oy$-axis along ray $OA$. Let $A = (0, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$. Then

$$\mathbf{a} \vee \mathbf{b} = x_2 y_1, \mathbf{a} \vee \mathbf{c} = x_3 y_1; \quad \mathbf{a} \vee (\mathbf{b} + \mathbf{c}) = (x_2 + x_3) y_1 = \mathbf{a} \vee \mathbf{b} + \mathbf{a} \vee \mathbf{c}.$$

**13.48.** Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit vectors directed along the axes $Ox$ and $Oy$. Then $\mathbf{e}_1 \vee \mathbf{e}_2 = -\mathbf{e}_2 \vee \mathbf{e}_1 = 1$ and $\mathbf{e}_1 \vee \mathbf{e}_1 = \mathbf{e}_2 \vee \mathbf{e}_2 = 0$; hence,

$$\mathbf{a} \vee \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \vee (b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = a_1 b_2 - a_2 b_1.$$

**13.49.** a) Clearly,

$$\overrightarrow{AB} \vee \overrightarrow{AC} = \overrightarrow{AB} \vee (\overrightarrow{AB} + \overrightarrow{BC}) = -\overrightarrow{BA} \vee \overrightarrow{BC} = \overrightarrow{BC} \vee \overrightarrow{BA}.$$

b) In the proof it suffices to make use of the chain of inequalities

$$\overrightarrow{AB} \vee \overrightarrow{AC} = (\overrightarrow{AD} + \overrightarrow{DB}) \vee (\overrightarrow{AD} + \overrightarrow{DC}) =$$
$$\overrightarrow{AD} \vee \overrightarrow{DC} + \overrightarrow{DB} \vee \overrightarrow{AD} + \overrightarrow{DB} \vee \overrightarrow{DC} =$$
$$= \overrightarrow{DC} \vee \overrightarrow{DA} + \overrightarrow{DA} \vee \overrightarrow{DB} + \overrightarrow{DB} \vee \overrightarrow{DC}.$$

**13.50.** Let at the initial moment, i.e., at $t = 0$ we have $\overrightarrow{AB} = \mathbf{v}$ and $\overrightarrow{AC} = \mathbf{w}$. Then at the moment $t$ we get $\overrightarrow{AB} = \mathbf{v} + t(\mathbf{a} - \mathbf{b})$ and $\overrightarrow{AC} = \mathbf{w} + t(\mathbf{c} - \mathbf{a})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the velocity vectors of the runners $A$, $B$ and $C$, respectively. Since vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are parallel, it follows that $(\mathbf{b} - \mathbf{a}) \vee (\mathbf{c} - \mathbf{a}) = 0$ and, therefore, $|S(A, B, C)| = \frac{1}{2}|\overrightarrow{AB} \vee \overrightarrow{AC}| = |x + ty|$, where $x$ and $y$ are some constants.

Solving the system $|x| = 2$, $|x + 5y| = 3$ we get two solutions with the help of which we express the dependence of the area of triangle $ABC$ of time $t$ as $|2 + \frac{t}{5}|$ or $|2 - t|$. Therefore, at $t = 10$ the value of the area can be either 4 or 8.

**13.51.** Let $\mathbf{v}(t)$ and $\mathbf{w}(t)$ be the vectors directed from the first pedestrian to the second and the third ones, respectively, at time $t$. Clearly, $\mathbf{v}(t) = t\mathbf{a} + \mathbf{b}$ and $\mathbf{w}(t) = t\mathbf{c} + \mathbf{d}$. The pedestrians are on the same line if and only if $\mathbf{v}(t) \parallel \mathbf{w}(t)$, i.e., $\mathbf{v}(t) \vee \mathbf{w}(t) = 0$. The function

$$f(t) = \mathbf{v}(t) \vee \mathbf{w}(t) = t^2 \mathbf{a} \vee \mathbf{c} + t(\mathbf{a} \vee \mathbf{d} + \mathbf{b} \vee \mathbf{c}) + \mathbf{b} \vee \mathbf{d}$$

is a quadratic and $f(0) \neq 0$. We know that a quadratic not identically equal to zero has not more than 2 roots.

**13.52.** Let $\overrightarrow{OC} = \mathbf{a}$, $\overrightarrow{OB} = \lambda\mathbf{a}$, $\overrightarrow{OD} = \mathbf{b}$ and $\overrightarrow{OA} = \mu\mathbf{b}$. Then

$$\pm 2S_{OPQ} = \overrightarrow{OP} \vee \overrightarrow{OQ} = \frac{\mathbf{a} + \mu\mathbf{b}}{2} \vee \frac{\lambda\mathbf{a} + \mathbf{b}}{2} = \frac{1 - \lambda\mu}{4}(\mathbf{a} \vee \mathbf{b})$$

and

$$\pm S_{ABCD} = \pm 2(S_{COD} - S_{AOB}) = \pm(\mathbf{a} \vee \mathbf{b} - \lambda\mathbf{a} \vee \mu\mathbf{b}) = \pm(1 - \lambda\mu)\mathbf{a} \vee \mathbf{b}.$$

**13.53.** Let $\mathbf{a}_j = \overrightarrow{P_1 A_j}$. Then the doubled sum of the areas of the given triangles is equal for any inner point $P$ to

$$(\mathbf{x} + \mathbf{a}_1) \vee (\mathbf{x} + \mathbf{a}_2) + (\mathbf{x} + \mathbf{a}_3) \vee (\mathbf{x} + \mathbf{a}_4) + \cdots + (\mathbf{x} + \mathbf{a}_{2n-1}) \vee (\mathbf{x} + \mathbf{a}_{2n}),$$

where $\mathbf{x} = \overrightarrow{PP_1}$ and it differs from the doubled sum of the areas of these triangles for point $P_1$ by

$$\mathbf{x} \vee (\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \cdots + \mathbf{a}_{2n-1} - \mathbf{a}_{2n}) = \mathbf{x} \vee \mathbf{a}.$$

By the hypothesis $\mathbf{x} \vee \mathbf{a} = 0$ for $\mathbf{x} = \overrightarrow{P_1 P_1}$ and $\mathbf{x} = \overrightarrow{P_3 P_1}$ and these vectors are not parallel. Hence, $\mathbf{a} = \mathbf{0}$, i.e., $\mathbf{x} \vee \mathbf{a} = 0$ for any $\mathbf{x}$.

**13.54.** Let $\mathbf{a} = \overrightarrow{AP}$, $\mathbf{b} = \overrightarrow{BQ}$ and $\mathbf{c} = \overrightarrow{CR}$. Then $\overrightarrow{QC} = \alpha\mathbf{a}$, $\overrightarrow{RA} = \beta\mathbf{b}$ and $\overrightarrow{PB} = \gamma\mathbf{c}$; we additionally have

$$(1 + \alpha)\mathbf{a} + (1 + \beta)\mathbf{b} + (1 + \gamma)\mathbf{c} = \mathbf{0}.$$

It suffices to verify that $\overrightarrow{AB} \vee \overrightarrow{CA} = \overrightarrow{PQ} \vee \overrightarrow{RP}$. The difference between these quantities is equal to

$$(\mathbf{a} + \gamma\mathbf{c}) \vee (\mathbf{c} + \beta\mathbf{b}) - (\gamma\mathbf{c} + \mathbf{b}) \vee (\mathbf{a} + \beta\mathbf{b}) = \mathbf{a} \vee \mathbf{c} + \beta\mathbf{a} \vee \mathbf{b} + \mathbf{a} \vee \mathbf{b} + \gamma\mathbf{a} \vee \mathbf{c} =$$
$$= \mathbf{a} \vee [(1 + \gamma)\mathbf{c} + (1 + \beta)\mathbf{b}] = -\mathbf{a} \vee (1 + \alpha)\mathbf{a} = 0.$$

**13.55.** Let $\mathbf{a}_i = \overrightarrow{A_4A_i}$ and $\mathbf{w}_i = \overrightarrow{A_4H_i}$. By Problem 13.49 b) it suffices to verify that

$$\mathbf{a}_1 \vee \mathbf{a}_2 + \mathbf{a}_2 \vee \mathbf{a}_3 + \mathbf{a}_3 \vee \mathbf{a}_1 = \mathbf{w}_1 \vee \mathbf{w}_2 + \mathbf{w}_2 \vee \mathbf{w}_3 + \mathbf{w}_3 \vee \mathbf{w}_1.$$

Vectors $\mathbf{a}_1 - \mathbf{w}_2$ and $\mathbf{a}_2 - \mathbf{w}_1$ are perpendicular to vector $\mathbf{a}_3$ and, therefore, they are parallel to each other, i.e., $(\mathbf{a}_1 - \mathbf{w}_2) \vee (\mathbf{a}_2 - \mathbf{w}_1) = 0$. Adding this equality to the equalities $(\mathbf{a}_2 - \mathbf{w}_3) \vee (\mathbf{a}_3 - \mathbf{w}_2) = 0$ and $(\mathbf{a}_3 - \mathbf{w}_1) \vee (\mathbf{a}_1 - \mathbf{w}_3) = 0$ we get the statement required.

**13.56.** Let $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then $\mathbf{e}_1 \vee \mathbf{x} = x_2(\mathbf{e}_1 \vee \mathbf{e}_2)$ and $\mathbf{x} \vee \mathbf{e}_2 = x_1(\mathbf{e}_1 \vee \mathbf{e}_2)$, i.e.,

$$\mathbf{x} = \frac{(\mathbf{x} \vee \mathbf{e}_2)\mathbf{e}_1 + (\mathbf{e}_1 \vee \mathbf{x})\mathbf{e}_2}{\mathbf{e}_1 \vee \mathbf{e}_2}.$$

Multiplying this expression by $(\mathbf{e}_1 \vee \mathbf{e}_2)\mathbf{y}$ from the right we get

$$(1) \qquad (\mathbf{x} \vee \mathbf{e}_2)(\mathbf{e}_1 \vee \mathbf{y}) + (\mathbf{e}_1 \vee \mathbf{x})(\mathbf{e}_2 \vee \mathbf{y}) + (\mathbf{e}_2 \vee \mathbf{e}_1)(\mathbf{x} \vee \mathbf{y}) = 0.$$

Let $\mathbf{e}_1 = \overrightarrow{AB}$, $\mathbf{e}_2 = \overrightarrow{AC}$, $\mathbf{x} = \overrightarrow{AD}$ and $\mathbf{y} = \overrightarrow{AE}$. Then

$$S = a + \mathbf{x} \vee \mathbf{e}_2 + d = c + \mathbf{y} \vee \mathbf{e}_2 + a = d + \mathbf{x} \vee \mathbf{e}_1 + b,$$

i.e.,

$$\mathbf{x} \vee \mathbf{e}_2 = S - a - d, \mathbf{y} \vee \mathbf{e}_2 = S - c - a$$

and $\mathbf{x} \vee \mathbf{e}_1 = S - d - b$. Substituting these expressions into (1) we get the statement required.

# Chapter 14. THE CENTER OF MASS

## Background

1. Consider a system of mass points on a plane, i.e., there is a set of pairs $(X_i, m_i)$, where $X_i$ is a point on the plane and $m_i$ a positive number. The *center of mass* of the system of points $X_1, \ldots, X_n$ with masses $m_1, \ldots, m_n$, respectively, is a point, $O$, which satisfies

$$m_1 \overrightarrow{OX_1} + \cdots + m_n \overrightarrow{OX_n} = \overrightarrow{0}.$$

The center of mass of any system of points exists and is unique (Problem 14.1).

2. A careful study of the solution of Problem 14.1 reveals that the positivity of the numbers $m_i$ is not actually used; it is only important that their sum is nonzero. Sometimes it is convenient to consider systems of points for which certain masses are positive and certain are negative (but the sum of masses is nonzero).

3. The most important property of the center of mass which lies in the base of almost all its applications is the following

**Theorem on mass regroupping.** *The center of mass of a system of points does not change if part of the points are replaced by one point situated in their center of mass and whose mass is equal to the sum of their masses* (Problem 14.2).

4. The *moment of inertia* of a system of points $X_1, \ldots, X_n$ with masses $m_1, \ldots, m_n$ with respect to point $M$ is the number

$$I_M = m_1 MX_1^2 + \cdots + m_n MX_n^2.$$

The applications of this notion in geometry are based on the relation $I_M = I_O + mOM^2$, where $O$ is the center of mass of a system and $m = m_1 + \cdots + m_n$ (Problem 14.17).

## §1. Main properties of the center of mass

**14.1.** a) Prove that the center of mass exists and is unique for any system of points.

b) Prove that if $X$ is an arbitrary point and $O$ the center of mass of points $X_1, \ldots, X_n$ with masses $m_1, \ldots, m_n$, then

$$\overrightarrow{XO} = \frac{1}{m_1 + \cdots + m_n}(m_1 \overrightarrow{XX_1} + \cdots + m_n \overrightarrow{XX_n}).$$

**14.2.** Prove that the center of mass of the system of points $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ with masses $a_1, \ldots, a_n, b_1, \ldots, b_m$ coincides with the center of mass of two points — the center of mass $X$ of the first system with mass $a_1 + \cdots + a_n$ and the center of mass $Y$ of the second system with mass $b_1 + \cdots + b_m$.

**14.3.** Prove that the center of mass of points $A$ and $B$ with masses $a$ and $b$ belongs to segment $AB$ and divides it in the ratio of $b : a$.

## §2. A theorem on mass regrouping

**14.4.** Prove that the medians of triangle $ABC$ intersect at one point and are divided by it in the ratio of $2:1$ counting from the vertices.

**14.5.** Let $ABCD$ be a convex quadrilateral; let $K$, $L$, $M$ and $N$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$, respectively. Prove that the intersection point of segments $KM$ and $LN$ is the midpoint of these segments and also the midpoint of the segment that connects the midpoints of the diagonals.

**14.6.** Let $A_1$, $B_1$, ..., $F_1$ be the midpoints of sides $AB$, $BC$, ..., $FA$, respectively, of a hexagon. Prove that the intersection points of the medians of triangles $A_1C_1E_1$ and $B_1D_1F_1$ coincide.

**14.7.** Prove Ceva's theorem (Problem 4.48 b)) with the help of mass regrouping.

**14.8.** On sides $AB$, $BC$, $CD$ and $DA$ of convex quadrilateral $ABCD$ points $K$, $L$, $M$ and $N$, respectively, are taken so that $AK:KB = DM:MC = \alpha$ and $BL:LC = AN:ND = \beta$. Let $P$ be the intersection point of segments $KL$ and $LN$. Prove that $NP:PL = \alpha$ and $KP:PM = \beta$.

**14.9.** Inside triangle $ABC$ find point $O$ such that for any straight line through $O$, intersecting $AB$ at $K$ and intersecting $BC$ at $L$ the equality $p\frac{AK}{KB} + q\frac{CL}{LB} = 1$ holds, where $p$ and $q$ are given positive numbers.

**14.10.** Three flies of equal mass crawl along the sides of triangle $ABC$ so that the center of their mass is fixed. Prove that the center of their mass coincides with the intersection point of medians of $ABC$ if it is known that one fly had crawled along the whole boundary of the triangle.

**14.11.** On sides $AB$, $BC$ and $CA$ of triangle $ABC$, points $C_1$, $A_1$ and $B_1$, respectively, are taken so that straight lines $CC_1$, $AA_1$ and $BB_1$ intersect at point $O$. Prove that

a) $\frac{CO}{OC_1} = \frac{CA_1}{A_1B} + \frac{CB_1}{B_1A}$;

b) $\frac{AO}{OA_1} \cdot \frac{BO}{OB_1} \cdot \frac{CO}{OC_1} = \frac{AO}{OA_1} + \frac{BO}{OB_1} + \frac{CO}{OC_1} + 2 \geq 8$.

**14.12.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$, respectively, are taken so that $\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B}$. Prove that the centers of mass of triangles $ABC$ and $A_1B_1C_1$ coincide.

**14.13.** On a circle, $n$ points are given. Through the center of mass of $n-2$ points a straight line is drawn perpendicularly to the chord that connects the two remaining points. Prove that all such straight lines intersect at one point.

**14.14.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$, respectively, are taken so that segments $AA_1$, $BB_1$ and $CC_1$ intersect at point $P$. Let $l_a$, $l_b$, $l_c$ be the lines that connect the midpoints of segments $BC$ and $B_1C_1$, $CA$ and $C_1A_1$, $AB$ and $A_1B_1$, respectively. Prove that lines $l_a$, $l_b$ and $l_c$ intersect at one point and this point belongs to segment $PM$, where $M$ is the center of mass of triangle $ABC$.

**14.15.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$, respectively, are taken; straight lines $B_1C_1$, $BB_1$ and $CC_1$ intersect straight line $AA_1$ at points $M$, $P$ and $Q$, respectively. Prove that:

a) $\frac{A_1M}{MA} = \frac{A_1P}{PA} + \frac{A_1Q}{QA}$;

b) if $P = Q$, then $MC_1:MB_1 = \frac{BC_1}{AB} : \frac{CB_1}{AC}$.

**14.16.** On line $AB$ points $P$ and $P_1$ are taken and on line $AC$ points $Q$ and $Q_1$ are taken. The line that connects point $A$ with the intersection point of lines $PQ$ and $P_1Q_1$

intersects line $BC$ at point $D$. Prove that

$$\frac{\overline{BD}}{\overline{CD}} = \frac{\frac{\overline{BP}}{\overline{PA}} - \frac{\overline{BP_1}}{\overline{P_1A}}}{\frac{\overline{CQ}}{\overline{QA}} - \frac{\overline{CQ_1}}{\overline{Q_1A}}}.$$

## §3. The moment of inertia

For point $M$ and a system of mass points $X_1$, ..., $X_n$ with masses $m_1$, ..., $m_n$ the quantity $I_M = m_1 MX_1^2 + \cdots + m_n MX_n^2$ is called the *moment of inertia* with respect to $M$.

**14.17.** Let $O$ be the center of mass of a system of points whose sum of masses is equal to $m$. Prove that the moments of inertia of this system with respect to $O$ and with respect to an arbitrary point $X$ are related as follows: $I_X = I_O + mXO^2$.

**14.18.** a) Prove that the moment of inertia with respect to the center of mass of a system of points of unit masses is equal to $\frac{1}{n}\sum_{i<j} a_{ij}^2$, where $n$ is the number of points and $a_{ij}$ the distance between points whose indices are $i$ and $j$.

b) Prove that the moment of inertia with respect to the center of mass of a system of points whose masses are $m_1, \ldots, m_n$ is equal to $\frac{1}{m}\sum_{i<j} m_i m_j a_{ij}^2$, where $m = m_1 + \cdots + m_n$ and $a_{ij}$ is the distance between the points whose indices are $i$ and $j$.

**14.19.** a) Triangle $ABC$ is an equilateral one. Find the locus of points $X$ such that $AX^2 = BX^2 + CX^2$.

b) Prove that for the points of the locus described in heading a) the pedal triangle with respect to the triangle $ABC$ is a right one.

**14.20.** Let $O$ be the center of the circumscribed circle of triangle $ABC$ and $H$ the intersection point of the heights of triangle $ABC$. Prove that $a^2 + b^2 + c^2 = 9R^2 - OH^2$.

**14.21.** Chords $AA_1$, $BB_1$ and $CC_1$ in a disc with center $O$ intersect at point $X$. Prove that

$$\frac{AX}{XA_1} + \frac{BX}{XB_1} + \frac{CX}{XC_1} = 3$$

if and only if point $X$ belongs to the circle with diameter $OM$, where $M$ is the center of mass of triangle $ABC$.

**14.22.** On sides $AB$, $BC$, $CA$ of triangle $ABC$ pairs of points $A_1$ and $B_2$, $B_1$ and $C_2$, $C_1$ and $A_2$, respectively, are taken so that segments $A_1A_2$, $B_1B_2$ and $C_1C_2$ are parallel to the sides of triangle $ABC$ and intersect at point $P$. Prove that

$$PA_1 \cdot PA_2 + PB_1 \cdot PB_2 + PC_1 \cdot PC_2 = R^2 - OP^2,$$

where $O$ is the center of the circumscribed circle.

**14.23.** Inside a circle of radius $R$, consider $n$ points. Prove that the sum of squares of the pairwise distances between the points does not exceed $n^2R^2$.

**14.24.** Inside triangle $ABC$ point $P$ is taken. Let $d_a$, $d_b$ and $d_c$ be the distances from $P$ to the sides of the triangle; $R_a$, $R_b$ and $R_c$ the distances from $P$ to the vertices. Prove that

$$3(d_a^2 + d_b^2 + d_c^2) \geq (R_a \sin A)^2 + (R_b \sin B)^2 + (R_c \sin C)^2.$$

**14.25.** Points $A_1, \ldots, A_n$ belong to the same circle and $M$ is their center of mass. Lines $MA_1, \ldots, MA_n$ intersect this circle at points $B_1, \ldots, B_n$ (distinct from $A_1, \ldots, A_n$). Prove that

$$MA_1 + \cdots + MA_n \leq MB_1 + \cdots + MB_n.$$

## §4. Miscellaneous problems

**14.26.** Prove that if a polygon has several axes of symmetry, then all of them intersect at one point.

**14.27.** A centrally symmetric figure on a graph paper consists of $n$ "corners" and $k$ rectangles of size $1 \times 4$ depicted on Fig. 145. Prove that $n$ is even.
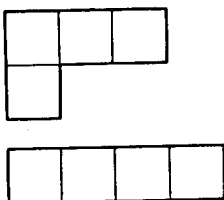


Figure 145 (14.27)

**14.28.** Solve Problem 13.44 making use the properties of the center of mass.

**14.29.** On sides $BC$ and $CD$ of parallelogram $ABCD$ points $K$ and $L$, respectively, are taken so that $BK : KC = CL : LD$. Prove that the center of mass of triangle $AKL$ belongs to diagonal $BD$.

## §5. The barycentric coordinates

Consider triangle $A_1A_2A_3$ whose vertices are mass points with masses $m_1$, $m_2$ and $m_3$, respectively. If point $X$ is the center of mass of the triangle's vertices, then the triple $(m_1 : m_2 : m_3)$ is called the *barycentric coordinates* of point $X$ with respect to triangle $A_1A_2A_3$.

**14.30.** Consider triangle $A_1A_2A_3$. Prove that
a) any point $X$ has some barycentric coordinates with respect to $\triangle A_1A_2A_3$;
b) provided $m_1 + m_2 + m_3 = 1$ the barycentric coordinates of $X$ are uniquely defined.

**14.31.** Prove that the barycentric coordinates with respect to $\triangle ABC$ of point $X$ which belongs to the interior of $ABC$ are equal to $(S_{BCX} : S_{CAX} : S_{ABX})$.

**14.32.** Point $X$ belongs to the interior of triangle $ABC$. The straight lines through $X$ parallel to $AC$ and $BC$ intersect $AB$ at points $K$ and $L$, respectively. Prove that the barycentric coordinates of $X$ with respect to $\triangle ABC$ are equal to $(BL : AK : LK)$.

**14.33.** Consider $\triangle ABC$. Find the barycentric coordinates with respect to $\triangle ABC$ of
a) the center of the circumscribed circle;
b) the center of the inscribed circle;
c) the orthocenter of the triangle.

**14.34.** The baricentric coordinates of point $X$ with respect to $\triangle ABC$ are $(\alpha : \beta : \gamma)$, where $\alpha + \beta + \gamma = 1$. Prove that $\overrightarrow{XA} = \beta\overrightarrow{BA} + \gamma\overrightarrow{CA}$.

**14.35.** Let $(\alpha : \beta : \gamma)$ be the barycentric coordinates of point $X$ with respect to $\triangle ABC$ and $\alpha + \beta + \gamma = 1$ and let $M$ be the center of mass of triangle $ABC$. Prove that

$$3\overrightarrow{XM} = (\alpha - \beta)\overrightarrow{AB} + (\beta - \gamma)\overrightarrow{BC} + (\gamma - \alpha)\overrightarrow{CA}.$$

**14.36.** Let $M$ be the center of mass of triangle $ABC$ and $X$ an arbitrary point. On lines $BC$, $CA$ and $AB$ points $A_1$, $B_1$ and $C_1$, respectively, are taken so that $A_1X \parallel AM$, $B_1X \parallel BM$ and $C_1X \parallel CM$. Prove that the center of mass $M_1$ of triangle $A_1B_1C_1$ coincides with the midpoint of segment $MX$.

**14.37.** Find an equation of the circumscribed circle of triangle $A_1A_2A_3$ (kto sut' indexy? iz 14.36?) in the barycentric coordinates.

**14.38.** a) Prove that the points whose barycentric coordinates with respect to $\triangle ABC$ are $(\alpha : \beta : \gamma)$ and $(\alpha^{-1} : \beta^{-1} : \gamma^{-1})$ are *isotomically conjugate* with respect to triangle $ABC$.

b) The lengths of the sides of triangle $ABC$ are equal to $a$, $b$ and $c$. Prove that the points whose barycentric coordinates with respect to $\triangle ABC$ are $(\alpha : \beta : \gamma)$ and $(\frac{a^2}{\alpha} : \frac{b^2}{\beta} : \frac{c^2}{\gamma})$ are *isogonally conjugate* with respect to $ABC$.

## Solutions

**14.1.** Let $X$ and $O$ be arbitrary points. Then

$$m_1 \overrightarrow{OX_1} + \cdots + m_n \overrightarrow{OX_n} =$$
$$(m_1 + \cdots + m_n)\overrightarrow{OX} + m_1 \overrightarrow{XX_1} + \cdots + m_n \overrightarrow{XX_n}$$

and, therefore, $O$ is the center of mass of the given system of points if and only if

$$(m_1 + \cdots + m_n)\overrightarrow{OX} + m_1 \overrightarrow{XX_1} + \cdots + M_n \overrightarrow{XX_n} = \overrightarrow{0},$$

i.e., $\overrightarrow{OX} = \frac{1}{m_1 + \cdots + m_n}(m_1 \overrightarrow{XX_1} + \cdots + m_n \overrightarrow{XX_n})$.

This argument gives a solution to the problems of both headings.

**14.2.** Let $Z$ be an arbitrary point; $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_m$. Then $\overrightarrow{ZX} = \frac{a_1 \overrightarrow{ZX_1} + \cdots + a_n \overrightarrow{ZX_n}}{a}$ and $\overrightarrow{ZY} = \frac{b_1 \overrightarrow{ZY_1} + \cdots + b_m \overrightarrow{ZY_m}}{b}$. If $O$ is the center of mass of point $X$ whose mass is $a$ and of point $Y$ whose mass is $b$, then

$$\overrightarrow{ZO} = \frac{a\overrightarrow{ZX} + b\overrightarrow{ZY}}{a + b} =$$
$$\frac{a_1 \overrightarrow{ZX_1} + \cdots + a_n \overrightarrow{ZX_n} + b_1 \overrightarrow{ZY_1} + \cdots + b_m \overrightarrow{ZY_m}}{a + b},$$

i.e., $O$ is the center of mass of the system of points $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ with masses $a_1, \ldots, a_n, b_1, \ldots, b_m$.

**14.3.** Let $O$ be the center of mass of the given system. Then $a\overrightarrow{OA} + b\overrightarrow{OB} = \overrightarrow{0}$ and, therefore, $O$ belongs to segment $AB$ and $aOA = bOB$, i.e., $AO : OB = b : a$.

**14.4.** Let us place unit masses at points $A$, $B$ and $C$. Let $O$ be the center of mass of this system of points. Point $O$ is also the center of mass of points $A$ of mass 1 and $A_1$ of mass 2, where $A_1$ is the center of mass of points $B$ and $C$ of unit mass, i.e., $A_1$ is the midpoint of segment $BC$. Therefore, $O$ belongs to median $AA_1$ and divides it in the ratio $AO : OA_1 = 2 : 1$. We similarly prove that the remaining medians pass through $O$ and are divided by it in the ratio of $2 : 1$.

**14.5.** Let us place unit masses in the vertices of quadrilateral $ABCD$. Let $O$ be the center of mass of this system of points. It suffices to prove that $O$ is the midpoint of segments $KM$ and $LN$ and the midpoint of the segment connecting the midpoints of the diagonals. Clearly, $K$ is the center of mass of points $A$ and $B$ while $M$ is the center of mass of points $C$ and $D$. Therefore, $O$ is the center of mass of points $K$ and $M$ of mass 2, i.e., $O$ is the center of mass of segment $KM$.

Similarly, $O$ is the midpoint of segment $LN$. Considering centers of mass of pairs of points $(A, C)$ and $(B, D)$ (i.e., the midpoints of diagonals) we see that $O$ is the midpoint of the segment connecting the midpoints of diagonals.

**14.6.** Let us place unit masses in the vertices of the hexagon; let $O$ be the center of mass of the obtained system of points. Since points $A_1$, $C_1$ and $E_1$ are the centers of mass of pairs of points $(A, B)$, $(C, D)$ and $(E, F)$, respectively, point $O$ is the center of mass of the system of points $A_1$, $C_1$ and $E_1$ of mass 2, i.e., $O$ is the intersection point of the medians of triangle $A_1C_1E_1$ (cf. the solution of Problem 14.4).

We similarly prove that $O$ is the intersection point of medians of triangle $B_1D_1F_1$.

**14.7.** Let lines $AA_1$ and $CC_1$ intersect at $O$ and let $AC_1 : C_1B = p$ and $BA_1 : A_1C = q$. We have to prove that line $BB_1$ passes through $O$ if and only if $CB_1 : B_1A = 1 : pq$.

Place masses 1, $p$ and $pq$ at points $A$, $B$ and $C$, respectively. Then point $C_1$ is the center of mass of points $A$ and $B$ and point $A_1$ is the center of mass of points $B$ and $C$. Therefore, the center of mass of points $A$, $B$ and $C$ with given masses is the intersection point $O$ of lines $CC_1$ and $AA_1$.

On the other hand, $O$ belongs to the segment which connects $B$ with the center of mass of points $A$ and $C$. If $B_1$ is the center of mass of points $A$ and $C$ of masses 1 and $pq$, respectively, then $AB_1 : B_1C = pq : 1$. It remains to notice that there is one point on segment $AC$ which divides it in the given ratio $AB_1 : B_1C$.

**14.8.** Let us place masses 1, $\alpha$, $\alpha\beta$ and $\beta$ at points $A$, $B$, $C$ and $D$, respectively. Then points $K$, $L$, $M$ and $N$ are the centers of mass of the pairs of points $(A, B)$, $(B, C)$, $(C, D)$ and $(D, A)$, respectively. Let $O$ be the center of mass of points $A$, $B$, $C$ and $D$ of indicated mass. Then $O$ belongs to segment $NL$ and $NO : OL = (\alpha\beta + \alpha) : (1 + \beta) = \alpha$. Point $O$ belongs to the segment $KM$ and $KO : OM = (\beta + \alpha\beta) : (1 + \alpha) = \beta$. Therefore, $O$ is the intersection point of segments $KM$ and $LN$, i.e., $O = P$ and $NP : PL = NO : OL = \alpha$, $KP : PM = \beta$.

**14.9.** Let us place masses $p$, 1 and $q$ in vertices $A$, $B$ and $C$, respectively. Let $O$ be the center of mass of this system of points. Let us consider a point of mass 1 as two coinciding points of mass $x_a$ and $x_c$, where $x_a + x_c = 1$. Let $K$ be the center of mass of points $A$ and $B$ of mass $p$ and $x_a$ and $L$ the center of mass of points $C$ and $B$ of mass $q$ and $x_c$, respectively. Then $AK : KB = x_a : p$ and $CL : LB = x_c : q$, whereas point $O$ which is the center of mass of points $K$ and $L$ of mass $p + x_a$ and $q + x_c$, respectively, belongs to line $KL$. By varying $x_a$ from 0 to 1 we get two straight lines passing through $O$ and intersecting sides $AB$ and $BC$. Therefore, for all these lines we have

$$\frac{pAK}{KB} + \frac{qCL}{LB} = x_a + x_c = 1.$$

**14.10.** Denote the center of mass of the flies by $O$. Let one fly be sited in vertex $A$ and let $A_1$ be the center of mass of the two other flies. Clearly, point $A_1$ lies inside triangle $ABC$ and point $O$ belongs to segment $AA_1$ and divides it in the ratio of $AO : OA_1 = 2 : 1$. Therefore, point $O$ belongs to the interior of the triangle obtained from triangle $ABC$ by a homothety with coefficient $\frac{2}{3}$ and center $A$.

Considering such triangles for all the three vertices of triangle $ABC$ we see that their unique common point is the intersection point of the medians of triangle $ABC$. Since one fly visited all the three vertices of the triangle $ABC$ and point $O$ was fixed during this, $O$ should belong to all these three small triangles, i.e., $O$ coincides with the intersection point of the medians of triangle $ABC$.

**14.11.** a) Let $AB_1 : B_1C = 1 : p$ and $BA_1 : A_1C = 1 : q$. Let us place masses $p$, $q$, 1 at points $A$, $B$, $C$, respectively. Then points $A_1$ and $B_1$ are the centers of mass of the pairs of points $(B, C)$ and $(A, C)$, respectively. Therefore, the center of mass of the system of points $A$, $B$ and $C$ belongs both to segment $AA_1$ and to segment $BB_1$, i.e., coincides with $O$. It

follows that $C_1$ is the center of mass of points $A$ and $B$. Therefore,

$$\frac{CO}{OC_1} = p + q = \frac{CB_1}{B_1A} + \frac{CA_1}{A_1B}.$$

b) By heading a) we have

$$\frac{AO}{OA_1} \cdot \frac{BO}{OB_1} \cdot \frac{CO}{OC_1} = \frac{1+q}{p} \cdot \frac{1+p}{q} \cdot \frac{p+q}{1} =$$

$$p + q + \frac{p}{q} + \frac{q}{p} + \frac{1}{p} + \frac{1}{q} + 2 = \frac{AO}{OA_1} + \frac{BO}{OB_1} + \frac{CO}{OC_1} + 2.$$

It is also clear that

$$p + \frac{1}{p} \geq 2, \ q + \frac{1}{q} \geq 2 \text{ and } \frac{p}{q} + \frac{q}{p} \geq 2.$$

**14.12.** Let $M$ be the center of mass of triangle $ABC$. Then

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = \overrightarrow{0}.$$

Moreover,

$$\overrightarrow{AB_1} + \overrightarrow{BC_1} + \overrightarrow{CA_1} = k(\overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{CB}) = \overrightarrow{0}.$$

Adding these identities we get $\overrightarrow{MB_1} + \overrightarrow{MC_1} + \overrightarrow{MA_1} = \overrightarrow{0}$, i.e., $M$ is the center of mass of triangle $A_1B_1C_1$.

REMARK. We similarly prove a similar statement for an arbitrary $n$-gon.

**14.13.** Let $M_1$ be the center of mass of $n - 2$ points; $K$ the midpoint of the chord connecting the two remaining points, $O$ the center of the circle, and $M$ the center of mass of all the given points. If line $OM$ intersects a(?) line drawn through $M_1$ at point $P$, then

$$\frac{\overline{OM}}{\overline{MP}} = \frac{\overline{KM}}{\overline{MM_1}} = \frac{n-2}{2}$$

and, therefore, the position of point $P$ is uniquely determined by the position of points $O$ and $M$ (if $M = O$, then $P = O$).

**14.14.** Let $P$ be the center of mass of points $A$, $B$ and $C$ of masses $a$, $b$ and $c$, respectively, $M$ the center of mass of points $A$, $B$ and $C$ (the mass of $M$ is $a + b + c$) and $Q$ the center of mass of the union of these two systems of points. The midpoint of segment $AB$ is the center of mass of points $A$, $B$ and $C$ of mass $a + b + c - \frac{ab}{c}$, $a + b + c - \frac{ab}{c}$ and $0$, respectively, and the midpoint of segment $A_1B_1$ is the center of mass of points $A$, $B$ and $C$ of mass $\frac{a(b+c)}{c}$, $\frac{b(a+c)}{c}$ and $(b + c) + (a + c)$, respectively. Point $O$ is the center of mass of the union of these systems of points.

**14.15.** a) Place masses $\beta$, $\gamma$ and $b + c$ in points $B$, $C$ and $A$ so that $CA_1 : BA_1 = \beta : \gamma$, $BC_1 : AC_1 = b : \beta$ and $AB_1 : CB_1 = \gamma : c$. Then $M$ is the center of mass of this system and, therefore, $\frac{A_1M}{AM} = \frac{b+c}{\beta+\gamma}$. Point $P$ is the center of mass of points $A$, $B$ and $C$ of masses $c$, $\beta$ and $\gamma$ and, therefore, $\frac{A_1P}{PA} = \frac{c}{\beta+\gamma}$. Similarly, $\frac{A_1Q}{AQ} = \frac{b}{b+\gamma}$.

b) As in heading a), we get $\frac{MC_1}{MB_1} = \frac{c+\gamma}{b+\beta}$, $\frac{BC_1}{AB} = \frac{b}{b+\beta}$ and $\frac{AC}{CB_1} = \frac{c+\gamma}{c}$. Moreover, $b = c$ because straight lines $AA_1$, $BB_1$ and $CC_1$ intersect at one point (cf. Problem 14.7).

**14.16.** The intersection point of lines $PQ$ and $P_1Q_1$ is the center of mass of points $A$, $B$ and $C$ of masses $a$, $b$ and $c$ and $P$ is the center of mass of points $A$ and $B$ of masses $a - x$

and $b$ while $Q$ is the center of mass of points $A$ and $C$ of masses $x$ and $c$. Let $p = \frac{BP}{PA} = \frac{a-x}{b}$ and $q = \frac{CQ}{QA} = \frac{x}{c}$. Then $pb + qc = a$. Similarly, $p_1 b + q_1 c = a$. It follows that

$$\frac{\overline{BD}}{\overline{CD}} = -\frac{c}{b} = \frac{(p - p_1)}{(q - q_1)}.$$

**14.17.** Let us enumerate the points of the given system. Let $\mathbf{x}_i$ be the vector with the beginning at $O$ and the end at the point of index $i$ and of mass $m_i$. Then $\sum m_i \mathbf{x}_i = \mathbf{0}$. Further, let $\mathbf{a} = \overrightarrow{OX}$. Then

$$I_O = \sum m_i x_i^2,$$
$$I_M = \sum m_i (\mathbf{x}_i + \mathbf{a})^2 = \sum m_i x_i^2 + 2(\sum m_i \mathbf{x}_i, \mathbf{a}) + \sum m_i a^2 = I_O + ma^2.$$

**14.18.** a) Let $\mathbf{x}_i$ be the vector with the beginning at the center of mass $O$ and the end at the point of index $i$. Then

$$\sum_{i,j} (\mathbf{x}_i - \mathbf{x}_j)^2 = \sum_{i,j} (x_i^2 + x_j^2) - 2\sum_{i,j} (\mathbf{x}_i, \mathbf{x}_j),$$

where the sum runs over all the possible pairs of indices. Clearly,

$$\sum_{i,j} (x_i^2 + x_j^2) = 2n \sum_i x_i^2 = 2nI_O; \quad \sum_{i,j} (\mathbf{x}_i, \mathbf{x}_j) = \sum_i (\mathbf{x}_i, \sum_j \mathbf{x}_j) = 0.$$

Therefore, $2nI_O = \sum_{i,j} (\mathbf{x}_i - \mathbf{x}_j)^2 = 2\sum_{i<j} a_{ij}^2$.

b) Let $\mathbf{x}_i$ be the vector with the beginning at the center of mass $O$ and the end at the point with index $i$. Then

$$\sum_{i,j} m_i m_j (\mathbf{x}_i - \mathbf{x}_j)^2 = \sum_{i,j} m_i m_j (x_i^2 + x_j^2) - 2\sum_{i,j} m_i m_j (\mathbf{x}_i, \mathbf{x}_j).$$

It is clear that

$$\sum_{i,j} m_i m_j (x_i^2 + x_j^2) = \sum_i m_i \sum_j (m_j x_i^2 + m_j x_j^2) = \sum_i m_i (mx_i^2 + I_O) = 2mI_O$$

and

$$\sum_{i,j} m_i m_j (\mathbf{x}_i, \mathbf{x}_j) = \sum_i m_i (\mathbf{x}_i, \sum_j m_j \mathbf{x}_j) = 0.$$

Therefore,

$$2mI_O = \sum_{i,j} m_i m_j (\mathbf{x}_i - \mathbf{x}_j)^2 = 2\sum_{i<j} m_i m_j a_{ij}^2.$$

**14.19.** a) Let $M$ be the point symmetric to $A$ through line $BC$. Then $M$ is the center of mass of points $A$, $B$ and $C$ whose masses are $-1$, $1$ and $1$, respectively, and, therefore,

$$-AX^2 + BX^2 + CX^2 = I_X = I_M + (-1 + 1 + 1)MX^2 = (-3 + 1 + 1)a^2 + MX^2,$$

where $a$ is the length of the side of triangle $ABC$. As a result we see that the locus to be found is the circle of radius $a$ with the center at $M$.

b) Let $A'$, $B'$ and $C'$ be the projections of point $X$ to lines $BC$, $CA$ and $AB$, respectively. Points $B'$ and $C'$ belong to the circle with diameter $AX$ and, therefore, $B'C' = AX \sin B'AC' = \frac{\sqrt{3}}{2}AX$. Similarly, $C'A' = \frac{\sqrt{3}}{2}BX$ and $A'B' = \frac{\sqrt{3}}{2}CX$. Therefore, if $AX^2 = BX^2 + CX^2$, then $\angle B'A'C' = 90°$.

**14.20.** Let $M$ be the center of mass of the vertices of triangle $ABC$ with unit masses in them. Then

$$I_O = I_M + 3MO^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3MO^2$$

(cf. Problems 14.17 and 14.18 a)). Since $OA = OB = OC = R$, it follows that $I_O = 3R^2$. It remains to notice that $OH = 3OM$ (Problem 5.105).

**14.21.** It is clear that

$$\frac{AX}{XA_1} = \frac{AX^2}{AX \cdot XA_1} = \frac{AX^2}{R^2 - OX^2}.$$

Therefore, we have to verify that $AX^2 + BX^2 + CX^2 = 3(R^2 - OX^2)$ if and only if $OM^2 = OX^2 + MX^2$. To this end it suffices to notice that

$$AX^2 + BX^2 + CX^2 = I_X = I_M + 3MX^2 =$$
$$I_O - 3MO^2 + 3MX^2 = 3(R^2 - MO^2 + MX^2).$$

**14.22.** Let $P$ be the center of mass of points $A$, $B$ and $C$ whose masses are $\alpha$, $\beta$ and $\gamma$, respectively. We may assume that $\alpha + \beta + \gamma = 1$. If $K$ is the intersection point of lines $CP$ and $AB$, then

$$\frac{BC}{PA_1} = \frac{CK}{PK} = \frac{CP + PK}{PK} = 1 + \frac{CP}{PK} = 1 + \frac{\alpha + \beta}{\gamma} = \frac{1}{\gamma}.$$

Similar arguments show that the considered quantity is equal to $\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2 = I_P$ (cf. Problem 14.18 b)). Since $I_O = \alpha R^2 + \beta R^2 + \gamma R^2 = R^2$, we have $I_P = I_O - OP^2 = R^2 - OP^2$.

**14.23.** Let us place unit masses in the given points. As follows from the result of Problem 14.18 a) the sum of squared distances between the given points is equal to $nI$, where $I$ is the moment of inertia of the system of points with respect to its center of mass. Now, consider the moment of inertia of the system with respect to the center $O$ of the circle. On the one hand, $I \leq I_O$ (see Problem 14.17). On the other hand, since the distance from $O$ to any of the given points does not exceed $R$, it follows that $I_O \leq nR^2$. Therefore, $nI \leq n^2R^2$ and the equality is attained only if $I = I_O$ (i.e., when the center of mass coincides with the center of the circle) and $I_O = nR^2$ (i.e., all the points lie on the given circle).

**14.24.** Let $A_1$, $B_1$ and $C_1$ be projections of point $P$ to sides $BC$, $CA$ and $AB$, respectively; let $M$ be the center of mass of triangle $A_1B_1C_1$. Then

$$3(d_a^2 + d_b^2 + d_c^2) = 3I_P \geq$$
$$3I_M = A_1B_1^2 + B_1C_1^2 + C_1A_1^2 = (R_c \sin C)^2 + (R_a \sin A)^2 + (R_b \sin B)^2$$

because, for example, segment $A_1B_1$ is a chord of the circle with diameter $CP$.

**14.25.** Let $O$ be the center of the given circle. If chord $AB$ passes through $M$, then $AM \cdot BM = R^2 - d^2$, where $d = MO$. Denote by $I_X$ the moment of inertia of the system of points $A_1$, ..., $A_n$ with respect to $X$. Then $I_O = I_M + nd^2$ (see Problem 14.17). On the other hand, since $OA_i = R$, we deduce that $I_O = nR^2$. Therefore,

$$A_iM \cdot B_iM = R^2 - d^2 = \frac{1}{n}(A_1M^2 + \cdots + A_nM^2).$$

Set $a_i = A_iM$. Then the inequality to be proved takes the form

$$a_1 + \cdots + a_n \leq \frac{1}{n}(a_1^2 + \cdots + a_n^2)(\frac{1}{a_1} + \cdots + \frac{1}{a_n}).$$

To prove this inequality we have to make use of the inequality

$$x + y \leq (\frac{x^2}{y}) + (\frac{y^2}{x})$$

which is obtained from the inequality $xy \leq x^2 - xy + y^2$ by multiplying both of its sides by $\frac{x+y}{xy}$.

**14.26.** Let us place unit masses in the vertices of the polygon. Under the symmetry through a line this system of points turns into itself and, therefore, its center of mass also turns into itself. It follows that all the axes of symmetry pass through the center of mass of the vertices.

**14.27.** Let us place unit masses in the centers of the cells which form "corners" and rectangles. Let us split each initial small cell of the graph paper into four smaller cells getting as a result a new graph paper. It is easy to verify that now the center of mass of a corner belongs to the center of a new small cell and the center of mass of a rectangle is a vertex of a new small cell, cf. Fig. 146.



Figure 146 (Sol. 14.27)

It is clear that the center of mass of a figure coincides with its center of symmetry and the center of symmetry of the figure consisting of the initial cells can only be situated in a vertex of a new cell. Since the masses of corners and bars (rectangles) are equal, the sum of vectors with the source in the center of mass of a figure and the targets in the centers of mass of all the corners and bars is equal to zero. If the number of corners had been odd, then the sum of the vectors would have had half integer coordinates and would have been nonzero. Therefore, the number of corners is an even one.

**14.28.** Let us place unit masses in the vertices of the polygon $A_1 \ldots A_n$. Then $O$ is the center of mass of the given system of points. Therefore, $\overrightarrow{A_iO} = \frac{1}{n}(\overrightarrow{A_iA_1} + \cdots + \overrightarrow{A_iA_n})$ and $A_iO \leq \frac{1}{n}(A_iA_1 + \cdots + A_iA_n)$; it follows that

$$d = A_1O + \cdots + A_nO \leq \frac{1}{n} \sum_{i,j=1}^{n} A_iA_j.$$

We can express the number $n$ either in the form $n = 2m$ or in the form $n = 2m + 1$. Let $P$ be the perimeter of the polygon. It is clear that

$$A_1A_2 + \cdots + A_nA_1 = P,$$
$$A_1A_3 + A_2A_4 + \cdots + A_nA_2 \leq 2P,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_1A_{m+1} + A_2A_{m+2} + \cdots + A_nA_m \leq mP$$

and in the left-hand sides of these inequalities all the sides and diagonals are encountered. Since they enter the sum $\sum_{i,j=1}^{n} A_i A_j$ twice, it is clear that

$$d \leq \frac{1}{n} \sum_{i,j=1}^{n} A_i A_j \leq \frac{2}{n}(P + 2P + \cdots + mP) = \frac{m(m+1)}{n} P.$$

For $n$ even this inequality can be strengthened due to the fact that in this case every diagonal occuring in the sum $A_1 A_{m+1} + \cdots + A_n A_{m+n}$ is counted twice, i.e., instead of $mP$ we can take $\frac{m}{2}P$. This means that for $n$ even we have

$$d \leq \frac{2}{n}(P + 2P + \cdots + (m-1)P + \frac{m}{2}P) = \frac{m^2}{n} P.$$

Thus, we have

$$d \leq \begin{cases} \frac{m^2}{n}P = \frac{n}{4}P & \text{if } n \text{ is even} \\ \frac{m(m+1)}{n}P = \frac{n^2-1}{4n}P & \text{if } n \text{ is odd.} \end{cases}$$

**14.29.** Let $k = \frac{BK}{BC} = 1 - \frac{DL}{DC}$. Under the projection to a line perpendicular to diagonal $BD$ points $A$, $B$, $K$ and $L$ pass into points $A'$, $B'$, $K'$ and $L'$, respectively, such that

$$B'K' + B'L' = kA'B' + (1-k)A'B' = A'B'.$$

It follows that the center of mass of points $A'$, $K'$ and $L'$ coincides with $B'$. It remains to notice that under the projection a center of mass turns into a center of mass.

**14.30.** Introduce the following notations: $\mathbf{e}_1 = \overrightarrow{A_3 A_1}$, $\mathbf{e}_2 = \overrightarrow{A_3 A_2}$ and $\mathbf{x} = \overrightarrow{XA_3}$. Point $X$ is the center of mass of the vertices of triangle $A_1 A_2 A_3$ with masses $m_1$, $m_2$, $m_3$ attached to them if and only if

$$m_1(\mathbf{x} + \mathbf{e}_1) + m_2(\mathbf{x} + \mathbf{e}_2) + m_3\mathbf{x} = \mathbf{0},$$

i.e., $m\mathbf{x} = -(m_1\mathbf{e}_1 + m_2\mathbf{e}_2)$, where $m = m_1 + m_2 + m_3$. Let us assume that $m = 1$. Any vector $\mathbf{x}$ on the plane can be represented in the form $\mathbf{x} = -m_1\mathbf{e}_1 - m_2\mathbf{e}_2$, where the numbers $m_1$ and $m_2$ are uniquely defined. The number $m_3$ is found from the relation $m_3 = 1 - m_1 - m_2$.

**14.31.** This problem is a reformulation of Problem 13.29.

REMARK. If we assume that the areas of triangles $BCX$, $CAX$ and $ABX$ are *oriented*, then the statement of the problem remains true for all the points situated outside the triangle as well.

**14.32.** Under the projection to line $AB$ parallel to line $BC$ vector $\mathbf{u} = \overrightarrow{XA} \cdot BL + \overrightarrow{XB} \cdot AK + \overrightarrow{XC} \cdot LK$ turns into vector $\overrightarrow{LA} \cdot BL + \overrightarrow{LB} \cdot AK + \overrightarrow{LB} \cdot LK$. The latter vector is the zero one since $\overrightarrow{LA} = \overrightarrow{LK} + \overrightarrow{KA}$. Considering the projection to line $AB$ parallel to line $AC$ we get $\mathbf{u} = \mathbf{0}$.

**14.33.** Making use of the result of Problem 14.31 it is easy to verify that the answer is as follows: a) $(\sin 2\alpha : \sin 2\beta : \sin 2\gamma)$; b) $(a : b : c)$; c) $(\tan \alpha : \tan \beta : \tan \gamma)$.

**14.34.** Adding vector $(\beta + \gamma)\overrightarrow{XA}$ to both sides of the equality $\alpha\overrightarrow{XA} + \beta\overrightarrow{XB} + \gamma\overrightarrow{XC} = \overrightarrow{0}$ we get

$$\overrightarrow{XA} = (\beta + \gamma)\overrightarrow{XA} + \beta\overrightarrow{BX} + \gamma\overrightarrow{CX} = \beta\overrightarrow{BA} + \gamma\overrightarrow{CA}.$$

**14.35.** By Problem 14.1 b) we have $3\overrightarrow{XM} = \overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC}$. Moreover, $\overrightarrow{XA} = \beta\overrightarrow{BA} + \gamma\overrightarrow{CA}$, $\overrightarrow{XB} = \alpha\overrightarrow{AB} + \gamma\overrightarrow{CB}$ and $\overrightarrow{XC} = \alpha\overrightarrow{AC} + \beta\overrightarrow{BC}$ (see Problem 14.34).

**14.36.** Let the lines through point $X$ parallel to $AC$ and $BC$ intersect the line $AB$ at points $K$ and $L$, respectively. If $(\alpha : \beta : \gamma)$ are the barycentric coordinates of $X$ and $\alpha + \beta + \gamma = 1$, then

$$2\overrightarrow{XC_1} = \overrightarrow{XK} + \overrightarrow{XL} = \gamma\overrightarrow{CA} + \gamma\overrightarrow{CB}$$

(see the solution of Problem 14.42). Therefore,

$$3\overrightarrow{XM_1} = \overrightarrow{XA_1} + \overrightarrow{XB_1} + \overrightarrow{XC_1} =$$
$$\tfrac{1}{2}(\alpha(\overrightarrow{AB} + \overrightarrow{AC}) + \beta(\overrightarrow{BA} + \overrightarrow{BC}) + \gamma\overrightarrow{CA} + \overrightarrow{CB}) = \tfrac{3}{2}\overrightarrow{XM}$$

(see Problem 14.35).

**14.37.** Let $X$ be an arbitrary point, $O$ the center of the circumscribed circle of the given triangle, $\mathbf{e}_i = \overrightarrow{OA_i}$ and $\mathbf{a} = \overrightarrow{XO}$. If the barycentric coordinates of $X$ are $(x_1 : x_2 : x_3)$, then $\sum x_i(\mathbf{a} + \mathbf{e}_i) = \sum x_i \overrightarrow{XA_i} = \mathbf{0}$ because $X$ is the center of mass of points $A_1$, $A_2$, $A_3$ with masses $x_1$, $x_2$, $x_3$. Therefore, $(\sum x_i)\mathbf{a} = -\sum x_i\mathbf{e}_i$.

Point $X$ belongs to the circumscribed circle of the triangle if and only if $|\mathbf{a}| = XO = R$, where $R$ is the radius of this circle. Thus, the circumscribed circle of the triangle is given in the barycentric coordinates by the equation

$$R^2\left(\sum x_i\right)^2 = \left(\sum x_i\mathbf{e}_i\right)^2,$$

i.e.,

$$R\sum^2 x_i^2 + 2R^2 \sum_{i<j} x_ix_j = R^2 \sum x_i^2 + 2\sum_{i<j} x_ix_j(\mathbf{e}_i, \mathbf{e}_j)$$

because $|\mathbf{e}_i| = R$. This equation can be rewritten in the form

$$\sum_{i<j} x_ix_j(R^2 - (\mathbf{e}_i, \mathbf{e}_j)) = 0.$$

Now notice that $2(R^2 - (\mathbf{e}_i, \mathbf{e}_j)) = a_{ij}^2$, where $a_{ij}$ is the length of side $A_iA_j$. Indeed,

$$a_{ij}^2 = |\mathbf{e}_i - \mathbf{e}_j|^2 = |\mathbf{e}_i|^2 + |\mathbf{e}_j|^2 - 2(\mathbf{e}_i, \mathbf{e}_j) = 2(R^2 - (\mathbf{e}_i, \mathbf{e}_j)).$$

As a result we see that the circumscribed circle of triangle $A_1A_2A_3$ is given in the barycentric coordinates by the equation $\sum_{i<j} x_ix_ja_{ij} = 0$, where $a_{ij}$ is the length of side $A_iA_j$.

**14.38.** a) Let $X$ and $Y$ be the points with barycentric coordinates $(\alpha : \beta : \gamma)$ and $(\alpha^{-1} : \beta^{-1} : \gamma^{-1})$ and let lines $CX$ and $CY$ intersect line $AB$ at points $X_1$ and $Y_1$, respectively. Then

$$\overline{AX_1} : \overline{BX_1} = \beta : \alpha = \alpha^{-1} : \beta^{-1} = \overline{BY_1} : \overline{AY_1}.$$

Similar arguments for lines $AX$ and $BX$ show that points $X$ and $Y$ are isotomically conjugate with respect to triangle $ABC$.

b) Let $X$ be the point with barycentric coordinates $(\alpha : \beta : \gamma)$. We may assume that $\alpha + \beta + \gamma = 1$. Then by Problem 14.34 we have

$$\overrightarrow{AX} = \beta\overrightarrow{AB} + \gamma\overrightarrow{AC} = \beta c\left(\frac{\overrightarrow{AB}}{c}\right) + \gamma b\left(\frac{\overrightarrow{AC}}{b}\right).$$

Let $Y$ be the point symmetric to $X$ through the bisector of angle $\angle A$ and $(\alpha' : \beta' : \gamma')$ the barycentric coordinates of $Y$. It suffices to verify that $\beta' : \gamma' = \frac{b^2}{\beta} : \frac{c^2}{\gamma}$. The symmetry through the bisector of angle $\angle A$ interchanges unit vectors $\frac{\overrightarrow{AB}}{c}$ and $\frac{\overrightarrow{AC}}{b}$, consequently, $\overrightarrow{AY} = \beta c\frac{\overrightarrow{AC}}{b} + \gamma b\frac{\overrightarrow{AB}}{c}$. It follows that

$$\beta' : \gamma' = \frac{\gamma b}{c} : \beta cb = \frac{b^2}{\beta} : \frac{c^2}{\gamma}.$$

# Chapter 15. PARALLEL TRANSLATIONS

## Background

1. The *parallel translation* by vector $\overrightarrow{AB}$ is the transformation which sends point $X$ into point $X'$ such that $\overrightarrow{XX'} = \overrightarrow{AB}$.

2. The composition (i.e., the consecutive execution) of two parallel translations is, clearly, a parallel translation.

## Introductory problems

**1.** Prove that every parallel translation turns any circle into a circle.

**2.** Two circles of radius $R$ are tangent at point $K$. On one of them we take point $A$,n on the other one we take point $B$ such that $\angle AKB = 90°$. Prove that $AB = 2R$.

**3.** Two circles of radius $R$ intersect at points $M$ and $N$. Let $A$ and $B$ be the intersection points of these circles with the perpendicular erected at the midpoint of segment $MN$. It so happens that the circles lie on one side of line $MN$. Prove that $MN^2 + AB^2 = 4R^2$.

**4.** Inside rectangle $ABCD$, point $M$ is taken. Prove that there exists a convex quadrilateral with perpendicular diagonals of the same length as $AB$ and $BC$ whose sides are equal to $AM$, $BM$, $CM$, $DM$.

## §1. Solving problems with the aid of parallel translations

**15.1.** Where should we construct bridge $MN$ through the river that separates villages $A$ and $B$ so that the path $AMNB$ from $A$ to $B$ was the shortest one? (The banks of the river are assumed to be parallel lines and the bridge perpendicular to the banks.)

**15.2.** Consider triangle $ABC$. Point $M$ inside the triangle moves parallel to side $BC$ to its intersection with side $CA$, then parallel to $AB$ to its intersection with $BC$, then parallel to $AC$ to its intersection with $AB$, and so on. Prove that after a number of steps the trajectory of the point becomes a closed one.

**15.3.** Let $K$, $L$, $M$ and $N$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$, respectively, of convex quadrilateral $ABCD$.

a) Prove that $KM \leq \frac{1}{2}(BC + AD)$ and the equality is attained only if $BC \parallel AD$.

b) For given lengths of the sides of quadrilateral $ABCD$ find the maximal value of the lengths of segments $KM$ and $LN$.

**15.4.** In trapezoid $ABCD$, sides $BC$ and $AD$ are parallel, $M$ the intersection point of the bisectors of angles $\angle A$ and $\angle B$, and $N$ the intersection point of the bisectors of angles $\angle C$ and $\angle D$. Prove that $2MN = |AB + CD - BC - AD|$.

**15.5.** From vertex $B$ of parallelogram $ABCD$ heights $BK$ and $BH$ are drawn. It is known that $KH = a$ and $BD = b$. Find the distance from $B$ to the intersection point of the heights of triangle $BKH$.

**15.6.** In the unit square a figure is placed such that the distance between any two of its points is not equal to 0.001. Prove that the area of this figure does not exceed a) 0.34; b) 0.287.

## §2. Problems on construction and loci

**15.7.** Consider angle $\angle ABC$ and straight line $l$. Construct a line parallel to $l$ on which the legs of angle $\angle ABC$ intercept a segment of given length $a$.

**15.8.** Consider two circles $S_1$, $S_2$ and line $l$. Draw line $l_1$ parallel to $l$ so that:

a) the distance between the intersection points of $l_1$ with circles $S_1$ and $S_2$ is of a given value $a$;

b) $S_1$ and $S_2$ intercept on $l_1$ equal chords;

c) $S_1$ and $S_2$ intercept on $l_1$ chords the sum (or difference) of whose lengths is equal to a given value.

**15.9.** Consider nonintersecting chords $AB$ and $CD$ on a circle. Construct a point $X$ on the circle so that chords $AX$ and $BX$ would intercept on chord $CD$ a segment, $EF$, of a given length $a$.

**15.10.** Construct quadrilateral $ABCD$ given the quadrilateral's angles and the lengths of sides $AB = a$ and $CD = b$.

**15.11.** Given point $A$ and circles $S_1$ and $S_2$. Through $A$ draw line $l$ so that $S_1$ and $S_2$ intercept on $l$ equal chords.

**15.12.** a) Given circles $S_1$ and $S_2$ intersect at points $A$ and $B$. Through point $A$ draw line $l$ so that the intercept of this line between circles $S_1$ and $S_2$ were of a given length.

b) Consider triangle $ABC$ and triangle $PQR$. In triangle $ABC$ inscribe a triangle equal to $PQR$.

**15.13.** Construct a quadrilateral given its angles and diagonals.

* * *

**15.14.** Find the loci of the points for which the following value is given: a) the sum, b) the difference of the distances from these points to the two given straight lines.

**15.15.** An angle made of a transparent material moves so that two nonintersecting circles are tangent to its legs from the inside. Prove that on the angle a point circumscribing an arc of a circle can be marked.

## Problems for independent study

**15.16.** Consider two pairs of parallel lines and point $P$. Through $P$ draw a line on which both pairs of parallel lines intercept equal segments.

**15.17.** Construct a parallelogram given its sides and an angle between the diagonals.

**15.18.** In convex quadrilateral $ABCD$, sides $AB$ and $CD$ are equal. Prove that

a) lines $AB$ and $CD$ form equal angles with the line that connects the midpoints of sides $AC$ and $BD$;

b) lines $AB$ and $CD$ form equal angles with the line that connects the midpoints of diagonals $BC$ and $AD$.

**15.19.** Among all the quadrilaterals with given lengths of the diagonals and an angle between them find the one of the least perimeter.

**15.20.** Given a circle and two neighbouring vertices of a parallelogram. Construct the parallelogram if it is known that its other two (not given) vertices belong to the given circle.

## Solutions

**15.1.** Let $A'$ be the image of point $A$ under the parallel translation by $\overrightarrow{MN}$. Then $A'N = AM$ and, therefore, the length of path $AMNB$ is equal to $A'N + NB + MN$. Since the length of segment $MN$ is a constant, we have to find point $N$ for which the sum

$A'N + NB$ is the least one. It is clear that the sum is minimal if $N$ belongs to segment $A'B$, i.e., $N$ is the closest to $B$ intersection point of the bank and segment $A'B$.
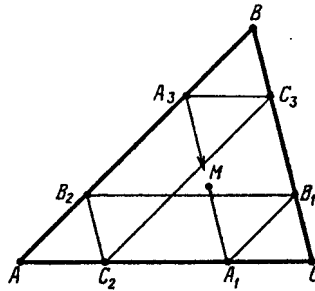


Figure 147 (Sol. 15.2)

**15.2.** Denote the consecutive points of the trajectory on the sides of the triangle as on Fig. 147:
$$A_1, \ B_1, \ B_2, \ C_2, \ C_3, \ A_3, \ A_4, \ B_4, \ldots$$
Since $A_1B_1 \parallel AB_2$, $B_1B_2 \parallel CA_1$ and $B_1C \parallel B_2C_2$, it is clear that triangle $AB_2C_2$ is the image of triangle $A_1B_1C$ under a parallel translation. Similarly, triangle $A_3BC_3$ is the image of triangle $AB_2C_2$ under a parallel translation and $A_4B_4C$ is obtained in the same way from $A_3BC_3$. But triangle $A_1B_1C$ is also the image of triangle $A_3BC_3$ under a parallel translation, hence, $A_1 = A_4$, i.e., after seven steps the trajectory becomes closed. (It is possible for the trajectory to become closed sooner. Under what conditions?)

**15.3.** a) Let us complement triangle $CBD$ to parallelogram $CBDE$. Then $2KM = AE \leq AD + DE = AD + BC$ and the equality is attained only if $AD \parallel BC$.

b) Let $a = AB$, $b = BC$, $c = CD$ and $d = DA$. If $|a - c| = |b - d| \neq 0$ then by heading a) the maximum is attained in the degenerate case when all points $A$, $B$, $C$ and $D$ belong to one line. Now suppose that, for example, $|a - c| < |b - d|$. Let us complement triangles $ABL$ and $LCD$ to parallelograms $ABLP$ and $LCDQ$, respectively; then $PQ \geq |b - d|$ and, therefore,
$$LN^2 = \frac{1}{4}(2LP^2 + 2LQ^2 - PQ^2) \leq \frac{1}{4}(2(a^2 + c^2) - (b - d)^2).$$
Moreover, by heading a) $KM \leq \frac{1}{2}(b + d)$. Both equalities are attained when $ABCD$ is a trapezoid with bases $AD$ and $BC$.

**15.4.** Let us construct circle $S$ tangent to side $AB$ and rays $BC$ and $AD$; translate triangle $CND$ parallelly (in the direction of bases $BC$ and $AD$) until $N'$ coincides with point $M$, i.e., side $C'D'$ becomes tangent to circle $S$ (Fig. 148).
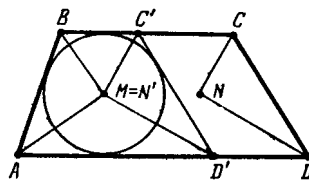


Figure 148 (Sol. 15.4)

For the circumscribed trapezoid $ABC'D'$ the equality $2MN' = |AB+C'D'-BC'-AD'|$ is obvious because $N' = M$. Under the passage from trapezoid $ABC'D'$ to trapezoid $ABCD$ the left-hand side of this equality accrues by $2N'N$ and the right-hand side accrues by $CC' + DD' = 2NN'$. Hence, the equality is preserved.

**15.5.** Denote the intersection point of heights of triangle $BKH$ by $H_1$. Since $HH_1 \perp BK$ and $KH_1 \perp BH$, it follows that $HH_1 \parallel AD$ and $KH_1 \parallel DC$, i.e., $H_1HDK$ is a parallelogram. Therefore, under the parallel translation by vector $\overrightarrow{H_1H}$ point $K$ passes to point $D$ and point $B$ passes to point $P$ (Fig. 149). Since $PD \parallel BK$, it follows that $BPDK$ is a rectangle and $PK = BD = b$. Since $BH_1 \perp KH$, it follows that $PH \perp KH$. It is also clear that $PH = BH_1$.
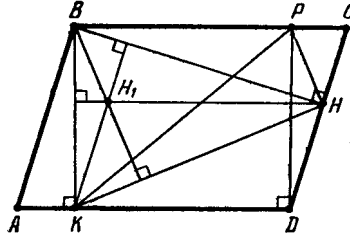


Figure 149 (Sol. 15.5)

In right triangle $PKH$, hypothenuse $KP = b$ and the leg $KH = a$ are known; therefore, $BH_1 = PH = \sqrt{b^2 - a^2}$.

**15.6.** a) Denote by $F$ the figure that lies inside the unit square $ABCD$; let $S$ be its area. Let us consider two vectors $\overrightarrow{AA_1}$ and $\overrightarrow{AA_2}$, where point $A_1$ belongs to side $AD$ and $AA_1 = 0.001$ and where point $A_2$ belongs to the interior of angle $\angle BAD$, $\angle A_2AA_1 = 60°$ and $AA_2 = 0.001$ (Fig. 150).
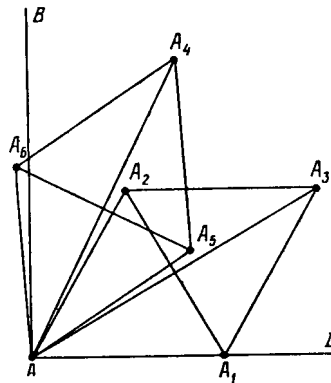


Figure 150 (Sol. 15.6 a))

Let $F_1$ and $F_2$ be the images of $F$ under the parallel translations by vectors $\overrightarrow{AA_1}$ and $\overrightarrow{AA_2}$, respectively. The figures $F$, $F_1$ and $F_2$ have no common points and belong to the interior of the square with side 1.001. Therefore, $2S < 1.001^2$, i.e., $S < 0.335 < 0.34$.

b) Consider vector $\overrightarrow{AA_3} = \overrightarrow{AA_1} + \overrightarrow{AA_2}$. Let us rotate $\overrightarrow{AA_3}$ about point $A$ through an acute angle counterclockwise so that point $A_3$ turns into point $A_4$ such that $A_3A_4 = 0.001$. Let us also consider vectors $\overrightarrow{AA_5}$ and $\overrightarrow{AA_6}$ of length 0.001 each constituting an angle of 30° with vector $\overrightarrow{AA_4}$ and situated on both sides of it (Fig. 151).

Denote by $F_i$ the image of figure $F$ under the parallel translation by the vector $\overrightarrow{AA_i}$. Denote the area of the union of figures $A$ and $B$ by $S(A \cup B)$ and by $S(A \cap B)$ the area of their intersection.

For definiteness, let us assume that $S(F_4 \cap F) \leq S(F_3 \cap F)$. Then $S(F_4 \cap F) \leq \frac{1}{2}S$ and, therefore, $S(F_4 \cup F) \geq \frac{3}{2}S$. The figures $F_5$ and $F_6$ do not intersect either each other
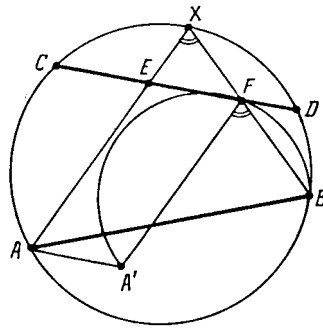
Figure 151 (Sol. 15.6 b))

or figures $F$ or $F_4$ and, therefore, $S(F \cup F_4 \cup F_5 \cup F_6) \geq \frac{7}{2}S$. (If it would have been that $S(F_3 \cap F) \leq S(F_4 \cap F)$, then instead of figures $F_5$ and $F_6$ we should have taken $F_1$ and $F_2$.) Since the lengths of vectors $\overrightarrow{AA_i}$ do not exceed $0.001\sqrt{3}$, all the figures considered lie inside a square with side $1 + 0.002\sqrt{3}$. Therefore, $7S/2 \leq (1 + 0.002\sqrt{3})^2$ and $S < 0.287$.

**15.7.** Given two vectors $\pm\mathbf{a}$ parallel to $l$ and of given length $a$. Consider the images of ray $BC$ under the parallel translations by these vectors. Their intersection point with ray $BA$ belongs to the line to be constructed (if they do not intersect, then the problem has no solutions).

**15.8.** a) Let $S_1'$ be the image of circle $S_1$ under the parallel translation by a vector of length $a$ parallel to $l$ (there are two such vectors). The desired line passes through the intersection point of circles $S_1'$ and $S_2$.

b) Let $O_1$ and $O_2$ be the projections of the centers of circles $S_1$ and $S_2$ to line $l$; let $S_1'$ be the image of the circle $S_1$ under the parallel translation by vector $\overrightarrow{O_1O_2}$. The desired line passes through the intersection point of circles $S_1'$ and $S_2$.

c) Let $S_1'$ be the image of circle $S_1$ under the parallel translation by a vector parallel to $l$. Then the lengths of chords cut by the line $l_1$ on circles $S_1$ and $S_1'$ are equal. If the distance between the projections of the centers of circles $S_1'$ and $S_2$ to line $l$ is equal to $\frac{1}{2}a$, then the sum of difference of the lengths of chords cut by the line parallel to $l$ and passing through the intersection point of circles $S_1'$ and $S_2$ is equal to $a$. Now it is easy to construct circle $S_1'$.

**15.9.** Suppose that point $X$ is constructed. Let us translate point $A$ by vector $\overrightarrow{EF}$, i.e., let us construct point $A'$ such that $\overrightarrow{EF} = \overrightarrow{AA'}$. This construction can be performed since we know vector $\overrightarrow{EF}$: its length is equal to $a$ and it is parallel to $CD$.
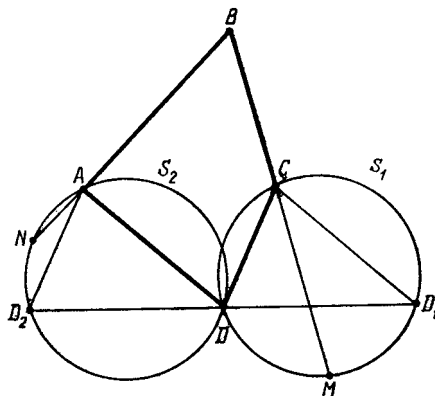


Figure 152 (Sol. 15.9)

Since $AX \parallel A'F$, it follows that $\angle A'FB = \angle AXB$ and, therefore, angle $\angle A'FB$ is known. Thus, point $F$ belongs to the intersection of two figures: segment $CD$ and an arc of the circle whose points are vertices of the angles equal to $\angle AXB$ that subtend segment $A'B$, see Fig. 152.

**15.10.** Suppose that quadrilateral $ABCD$ is constructed. Denote by $D_1$ the image of point $D$ under the parallel translation by vector $\overrightarrow{CB}$. In triangle $ABD_1$, sides $AB$, $BD_1$ and angle $\angle ABD_1$ are known. Hence, the following construction.

Let us arbitrarily construct ray $BC'$ and then draw rays $BD_1'$ and $BA'$ so that $\angle D_1'BC' = 180° - \angle C$, $\angle A'BC' = \angle B$ and these rays lie in the half plane on one side of ray $BC'$.

On rays $BA'$ and $BD_1'$, draw segments $BA = a$ and $BD_1 = b$, respectively. Let us draw ray $AD'$ so that $\angle BAD' = \angle A$ and rays $BC'$, $AD'$ lie on one side of line $AB$. Vertex $D$ is the intersection point of ray $AD'$ and the ray drawn from $D_1$ parallel to ray $BC'$. Vertex $C$ is the intersection point of $BC'$ and the ray drawn from $D$ parallel to ray $D_1B$.

**15.11.** Suppose that points $M$ and $N$ at which line $l$ intersects circle $S_2$ are constructed. Let $O_1$ and $O_2$ be the centers of circles $S_1$ and $S_2$; let $O_1'$ be the image of point $O_1$ under the parallel translation along $l$ such that $O_1'O_2 \perp MN$; let $S_1'$ be the image of circle $S_1$ under the same translation.

Let us draw tangents $AP$ and $AQ$ to circles $S_1'$ and $S_2$, respectively. Then $AQ^2 = AM \cdot AN = AP^2$ and, therefore, $O_1'A^2 = AP^2 + R^2$, where $R$ is the radius of circle $S_1'$. Since segment $AP$ can be constructed, we can also construct segment $AO_1'$. It remains to notice that point $O_1'$ belongs to both the circle of radius $AO_1'$ with the center at $A$ and to the circle with diameter $O_1O_2$.

**15.12.** a) Let us draw through point $A$ line $PQ$, where $P$ belongs to circle $S$ and $Q$ belongs to circle $S_2$. From the centers $O_1$ and $O_2$ of circles $S_1$ and $S_2$, respectively, draw perpendiculars $O_1M$ and $O_2N$ to line $PQ$. Let us parallelly translate segment $MN$ by a vector $\overrightarrow{MO_1}$. Let $C$ be the image of point $N$ under this translation.

Triangle $O_1CO_2$ is a right one and $O_1C = MN = \frac{1}{2}PQ$. It follows that in order to construct line $PQ$ for which $PQ = a$ we have to construct triangle $O_1CO_2$ of given hypothenuse $O_1O_2$ and leg $O_1C = \frac{1}{2}a$ and then draw through $A$ the line parallel to $O_1C$.

b) It suffices to solve the converse problem: around the given triangle $PQR$ circumscribe a triangle equal (?) to the given triangle $ABC$. Suppose that we have constructed triangle $ABC$ whose sides pass through given points $P$, $Q$ and $R$. Let us construct the arcs of circles whose points serve as vertices for angles $\angle A$ and $\angle B$ that subtend segments $RP$ and $QP$, respectively. Points $A$ and $B$ belong to these arcs and the length of segment $AB$ is known.

By heading a) we can construct line $AP$ through $P$ whose intercept between circles $S_1$ and $S_2$ is of given length. Draw lines $AR$ and $BQ$; we get triangle $ABC$ equal to the given triangle since these triangles have by construction equal sides and the angles adjacent to it.

**15.13.** Suppose that the desired quadrilateral $ABCD$ is constructed. Let $D_1$ and $D_2$ be the images of point $D$ under the translations by vectors $\overrightarrow{AC}$ and $\overrightarrow{CA}$, respectively. Let us circumscribe circles $S_1$ and $S_2$ around triangles $DCD_1$ and $DAD_2$, respectively. Denote the intersection points of lines $BC$ and $BA$ with circles $S_1$ and $S_2$ by $M$ and $N$, respectively, see Fig. 153. It is clear that $\angle DCD_1 = \angle DAD_2 = \angle D$, $\angle DCM = 180° - \angle C$ and $\angle DAN = 180° - \angle A$.

This implies the following construction. On an arbitrary line $l$, take a point, $D$, and construct points $D_1$ and $D_2$ on $l$ so that $DD_1 = DD_2 = AC$. Fix one of the half planes $\Pi$ determined by line $l$ and assume that point $B$ belongs to this half plane. Let us construct a circle $S_1$ whose points belonging to $\Pi$ serve as vertices of the angles equal to $\angle D$ that subtend segment $DD_1$.
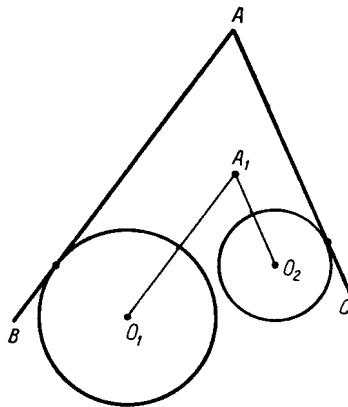
Figure 153 (Sol. 15.13)

We similarly construct circle $S_2$. Let us construct point $M$ on $S_1$ so that all the points of the part of the circle that belongs to $\Pi$ serve as vertices of the angles equal to $180° - \angle C$ that subtend segment $DM$.

Point $N$ is similarly constructed. Then segment $MN$ subtends angle $\angle B$, i.e., $B$ is the intersection point of the circle with center $D$ of radius $DB$ and the arc of the circle serve as vertices of the angles equal to $\angle B$ that subtend segment $MN$ (it also belongs to the half plane $\Pi$). Points $C$ and $A$ are the intersection points of lines $BM$ and $BN$ with circles $S_1$ and $S_2$, respectively.

**15.14.** From a point $X$ draw perpendiculars $XA_1$ and $XA_2$ to given lines $l_1$ and $l_2$, respectively. On ray $A_1X$, take point $B$ so that $A_1B = a$. Then if $XA_1 \pm XA_2 = a$, we have $XB = XA_2$. Let $l_1'$ be the image of line $l_1$ under the parallel translation by vector $\overrightarrow{A_1B}$ and $M$ the intersection point of lines $l_1'$ and $l_2$. Then in the indicated cases ray $MX$ is the bisector of angle $\angle A_2MB$. As a result we get the following answer.

Let the intersection points of lines $l_1$ and $l_2$ with the lines parallel to lines $l_1$ and $l_2$ and distant from them by $a$ form rectangle $M_1M_2M_3M_4$. The locus to be found is either a) the sides of this rectangle; or b) the extensions of these sides.

**15.15.** Let leg $AB$ of angle $\angle BAC$ be tangent to the circle of radius $r_1$ with center $O_1$ and leg $AC$ be tangent to the circle of radius $r_2$ with center $O_2$. Let us parallelly translate line $AB$ inside angle $\angle BAC$ by distance $r_1$ and let us parallelly translate line $AC$ inside angle $\angle BAC$ by distance $r_2$. Let $A_1$ be the intersection point of the translated lines (Fig. 154).
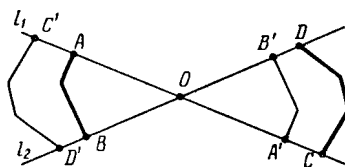


Figure 154 (Sol. 15.15)

Then $\angle O_1A_1O_2 = \angle BAC$. The constant(?) angle $O_1A_1O_2$ subtends fixed segment $O_1O_2$ and, therefore, point $A_1$ traverses an arc of a(?) circle.

# Chapter 16. CENTRAL SYMMETRY

## Background

1. The *symmetry through point A* is the transformation of the plane which sends point $X$ into point $X'$ such that $A$ is the midpoint of segment $XX'$. The other names of such a transformation: the *central symmetry with center A* or just the *symmetry with center A*.

Notice that the symmetry with center $A$ is a particular case of two other transformations: it is the rotation through an angle of $180°$ with center $A$ and also the homothety with center $A$ and coefficient $-1$.

2. If a figure turns into itself under the symmetry through point $A$, then $A$ is called the *center of symmetry* of this figure.

3. The following notations for transformations are used in this chapter:

$S_A$ — the symmetry with center $A$;

$T_{\mathbf{a}}$ — the translation by vector $\mathbf{a}$.

4. We will denote the composition of symmetries through points $A$ and $B$ by $S_B \circ S_A$; here we assume that we first perform symmetry $S_A$ and then symmetry $S_B$. This notation might look unnatural at first glance, but it is, however, justified by the identity $(S_B \circ S_A)(X) = S_B(S_A(X))$.

The composition of maps is associative: $F \circ (G \circ H) = (F \circ G) \circ H$. Therefore, the order of the compositions is inessential and we may simply write $F \circ G \circ H$.

5. The compositions of two central symmetries or of a symmetry with a parallel translation are calculated according to the following formulas (Problem 16.9):

a) $S_B \circ S_A = T_{2\overrightarrow{AB}}$;

b) $T_{\mathbf{a}} \circ S_A = S_B$ and $S_B \circ T_{\mathbf{a}} = S_A$, where $\mathbf{a} = 2\overrightarrow{AB}$.

## Introductory problems

**1.** Prove that under any central symmetry any circle turns into a circle.

**2.** Prove that a quadrilateral with a center of symmetry is a parallelogram.

**3.** The opposite sides of a convex hexagon are equal and parallel. Prove that the hexagon has a center of symmetry.

**4.** Consider parallelogram $ABCD$ and point $M$. The lines parallel to lines $MC$, $MD$, $MA$ and $MB$ are drawn through points $A$, $B$, $C$ and $D$, respectively. Prove that the lines drawn intersect at one point.

**5.** Prove that the opposite sides of a hexagon formed by the sides of a triangle and the tangents to its circumscribed circle parallel to the sides of the triangle are equal.

## §1. Solving problems with the help of a symmetry

**16.1.** Prove that if in a triangle a median and a bisector coincide, then the triangle is an isosceles one.

**16.2.** Two players lay out nickels on a rectangular table taking turns. It is only allowed to place a coin onto an unoccupied place. The *loser* is the one who can not make any move. Prove that the first player can always win in finitely many moves.

**16.3.** A circle intersects sides $BC$, $CA$, $AB$ of triangle $ABC$ at points $A_1$ and $A_2$, $B_1$ and $B_2$, $C_1$ and $C_2$, respectively. Prove that if the perpendiculars to the sides of the triangle drawn through points $A_1$, $B_1$ and $C_1$ intersect at one point, then the perpendiculars to the sides drawn through $A_2$, $B_2$ and $C_2$ also intersect at one point.

**16.4.** Prove that the lines drawn through the midpoints of the circumscribed quadrilateral perpendicularly to the opposite sides intersect at one point.

**16.5.** Let $P$ be the midpoint of side $AB$ of convex quadrilateral $ABCD$. Prove that if the area of triangle $PCD$ is equal to a half area of quadrilateral $ABCD$, then $BC \parallel AD$.

**16.6.** Unit circles $S_1$ and $S_2$ are tangent at point $A$; the center $O$ of circle $S$ of radius 2 belongs to $S_1$. Circle $S_1$ is tangent to circle $S$ at point $B$. Prove that line $AB$ passes through the intersection point of circles $S_2$ and $S$.

**16.7.** In triangle $ABC$ medians $AF$ and $CE$ are drawn. Prove that if $\angle BAF = \angle BCE = 30°$, then triangle $ABC$ is an equilateral one.

**16.8.** Consider a convex $n$-gon with pairwise nonparallel sides and point $O$ inside it. Prove that it is impossible to draw more than $n$ lines through $O$ so that each line divides the area of the $n$-gon in halves.

## §2. Properties of the symmetry

**16.9.** a) Prove that the composition of two central symmetries is a parallel translation.

b) Prove that the composition of a parallel translation with a central symmetry (in either order) is a central symmetry.

**16.10.** Prove that if a point is reflected symmetrically through points $O_1$, $O_2$ and $O_3$ and then reflected symmetrically once again through the same points, then it assumes the initial position.

**16.11.** a) Prove that a bounded figure cannot have more than one center of symmetry.

b) Prove that no figure can have precisely two centers of symmetry.

c) Let $M$ be a finite set of points on a plane. Point $O$ will be called an "almost center of symmetry" of the set $M$ if we can delete a point from $M$ so that $O$ becomes the center of symmetry of the remaining set. How many "almost centers of symmetry" can a set have?

**16.12.** On segment $AB$, consider $n$ pairs of points symmetric through the midpoint; $n$ of these $2n$ points are painted blue and the remaining are painted red. Prove that the sum of distances from $A$ to the blue points is equal to the sum of distances from $B$ to the red points.

## §3. Solving problems with the help of a symmetry. Constructions

**16.13.** Through a common point $A$ of circles $S_1$ and $S_2$ draw a straight line so that these circles would intercept on it equal chords.

**16.14.** Given point $A$, a line and a circle. Through $A$ draw a line so that $A$ divides the segment between the intersection points of the line drawn with the given line and the given circle in halves.

**16.15.** Given angle $ABC$ and point $D$ inside it. Construct a segment with the endpoints on the legs of the given angle and with the midpoint at $D$.

**16.16.** Consider an angle and points $A$ and $B$ inside it. Construct a parallelogram for which points $A$ and $B$ are opposite vertices and the two other vertices belong to the legs of the angle.

**16.17.** Given four pairwise nonparallel straight lines and point $O$ not belonging to these lines. Construct a parallelogram whose center is $O$ and the vertices lie on the given lines, one on each.

**16.18.** Consider two concentric circles $S_1$ and $S_2$. Draw a line on which these circles intercept three equal segments.

**16.19.** Consider nonintersecting chords $AB$ and $CD$ of a circle and point $J$ on chord $CD$. Construct point $X$ on the circle so that chords $AX$ and $BX$ would intercept on chord $CD$ segment $EF$ which $J$ divides in halves.

**16.20.** Through a common point $A$ of circles $S_1$ and $S_2$ draw line $l$ so that the difference of the lengths of the chords intercepted by circles $S_1$ and $S_2$ on $l$ were of given value $a$.

**16.21.** Given $m = 2n + 1$ points — the midpoints of the sides of an $m$-gon — construct the vertices of the $m$-gon.

## Problems for independent study

**16.22.** Construct triangle $ABC$ given medians $m_a$, $m_b$ and angle $\angle C$.

**16.23.** a) Given a point inside a parallelogram; the point does not belong to the segments that connect the midpoints of the opposite sides. How many segments divided in halves by the given point are there such that their endpoints are on the sides of the parallelogram?

b) A point inside the triangle formed by the midlines of a given triangle is given. How many segments divided in halves by the given point and with the endpoints on the sides of the given triangle are there?

**16.24.** a) Find the locus of vertices of convex quadrilaterals the midpoints of whose sides are the vertices of a given square.

b) Three points are given on a plane. Find the locus of vertices of convex quadrilaterals the midpoints of three sides of each of which are the given points.

**16.25.** Points $A$, $B$, $C$, $D$ lie in the indicated order on a line and $AB = CD$. Prove that for any point $P$ on the plane we have $AP + DP \geq BP + CP$.

## Solutions

**16.1.** Let median $BD$ of triangle $ABC$ be a bisector as well. Let us consider point $B_1$ symmetric to $B$ through point $D$. Since $D$ is the midpoint of segment $AC$, the quadrilateral $ABCB_1$ is a parallelogram. Since $\angle ABB_1 = \angle B_1BC = \angle AB_1B$, it follows that triangle $B_1AB$ is an isosceles one and $AB = AB_1 = BC$.

**16.2.** The first player places a nickel in the center of the table and then places nickels symmetrically to the nickels of the second player with respect to the center of the table. Using this strategy the first player has always a possibility to make the next move. It is also clear that the play will be terminated in a finite number of moves.

**16.3.** Let the perpendiculars to the sides drawn through points $A_1$, $B_1$ and $C_1$ intersect at point $M$. Denote the center of the circle by $O$. The perpendicular to side $BC$ drawn through point $A_1$ is symmetric through point $O$ to the perpendicular to side $BC$ drawn through $A_2$. It follows that the perpendiculars to the sides drawn through points $A_2$, $B_2$ and $C_2$ intersect at the point symmetric to $M$ through point $O$.

**16.4.** Let $P$, $Q$, $R$ and $S$ be the midpoints of sides $AB$, $BC$, $CD$ and $DA$, respectively, and $M$ the intersection point of segments $PR$ and $QS$ (i.e., the midpoint of both of these segments, see Problem 14.5); $O$ the center of the circumscribed circle and $O'$ the point symmetric to $O$ through $M$. Let us prove that the lines mentioned in the formulation of the problem pass through $O'$. Indeed, $O'POR$ is a parallelogram and, therefore, $O'P \parallel OR$. Since $R$ is the midpoint of chord $CD$, it follows that $OR \perp CD$, i.e., $O'P \perp CD$.

For lines $O'Q, O'R$ and $O'S$ the proof is similar.

**16.5.** Let point $D'$ be symmetric to $D$ through $P$. If the area of triangle $PCD$ is equal to a half area of quadrilateral $ABCD$, then it is equal to$S_{PBC} + S_{PAD}$, i.e., it is equal to $S_{PBC} + S_{PBD'}$. Since $P$ is the midpoint of segment $DD'$, it follows that $S_{PCD'} = S_{PCD} = S_{PBC} + S_{PBD'}$ and, therefore, point $B$ belongs to segment $D'C$. It remains to notice that $D'B \parallel AD$.

**16.6.** Circles $S_1$ and $S_2$ are symmetric through point $A$. Since $OB$ is the diameter of circle $S_1$, it follows that $\angle BAO = 90°$ and, therefore, under the symmetry through $A$ point $B$ becomes on the circle $S$ again. It follows that under the symmetry through $A$ point $B$ turns into the intersection point of circles $S_2$ and $S$.

**16.7.** Since $\angle EAF = \angle ECF = 30°$, we see that points $A$, $E$, $F$ and $C$ belong to one circle $S$ and if $O$ is its center, then $\angle EOF = 60°$. Point $B$ is symmetric to $A$ through $E$ and, therefore, $B$ belongs to circle $S_1$ symmetric to circle $S$ through $E$. Similarly, point $B$ belongs to circle $S_2$ symmetric to circle $S$ through point $F$. Since triangle $EOF$ is an equilateral one, the centers of circles $S$, $S_1$ and $S_2$ form an equilateral triangle with side $2R$, where $R$ is the radius of these circles. Therefore, circles $S_1$ and $S_2$ have a unique common point — $B$ — and triangle $BEF$ is an equilateral one. Thus, triangle $ABC$ is also an equilateral one.

**16.8.** Consider a polygon symmetric to the initial one through point $O$. Since the sides of the polygons are pairwise nonparallel, the contours of these polygons cannot have common segments but could only have common points. Since the polygons are convex ones, each side has not more than two intersection points; therefore, there are not more than $2n$ intersection points of the contours (more precisely, not more than $n$ pairs of points symmetric through $O$).

Let $l_1$ and $l_2$ be the lines passing through $O$ and dividing the area of the initial polygon in halves. Let us prove that inside each of the four parts into which these lines divide the plane there is an intersection point of the contours.

Suppose that one of the parts has no such points between lines $l_1$ and $l_2$. Denote the intersection points of lines $l_1$ and $l_2$ with the sides of the polygon as indicated on Fig. 12.
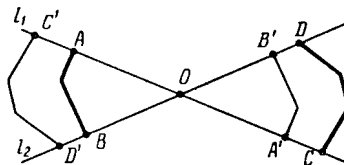


Figure 154 (Sol. 16.8)

Let points $A'$, $B'$, $C'$ and $D'$ be symmetric trough $O$ to points $A$, $B$, $C$ and $D$, respectively. For definiteness sake, assume that point $A$ is closer to $O$ than $C'$. Since segments $AB$ and $C'D'$ do not intersect, point $B$ is closer to $O$ than $D'$. It follows that $S_{ABO} < S_{C'D'O} = S_{CDO}$, where $ABO$ is a convex figure bounded by segments $AO$ and $BO$ and the part of the boundary of the $n$-gon between points $A$ and $B$.

On the other hand, $S_{ABO} = S_{CDO}$ because lines $l_1$ and $l_2$ divide the area of the polygon in halves. Contradiction.

Therefore, between every pair of lines which divide the area of the polygon in halves there is a pair of symmetric intersection points of contours; in other words, there are not more than $n$ such lines.

**16.9.** a) Let the central symmetry through $O_1$ send point $A$ into $A_1$; let the central symmetry through $O_2$ send point $A_1$ into $A_2$. Then $O_1O_2$ is the midline of triangle $AA_1A_2$ and, therefore, $\overrightarrow{AA_2} = 2\overrightarrow{O_1O_2}$.

b) Let $O_2$ be the image of point $O_1$ under the translation by vector $\frac{1}{2}\mathbf{a}$. By heading a) we have $S_{O_1} \circ S_{O_2} = T_{\mathbf{a}}$. Multiplying this equality by $S_{O_1}$ from the right or by $S_{O_2}$ from the left and taking into account that $S_X \circ S_X$ is the identity transformation we get $S_{O_1} = S_{O_2} \circ T_{\mathbf{a}}$ and $S_{O_2} = T_{\mathbf{a}} \circ S_{O_1}$.

**16.10.** By the preceding problem $S_B \circ S_A = T_{2\overrightarrow{AB}}$; therefore,

$$S_{O_3} \circ S_{O_2} \circ S_{O_1} \circ S_{O_3} \circ S_{O_2} \circ S_{O_1} = T_{2(\overrightarrow{O_2O_3}+\overrightarrow{O_3O_1}+\overrightarrow{O_1O_2})}$$

is the identity transformation.

**16.11.** a) Suppose that a bounded figure has two centers of symmetry: $O_1$ and $O_2$. Let us introduce a coordinate system whose absciss axis is directed along ray $O_1O_2$. Since $S_{O_2} \circ S_{O_1} = T_{2\overrightarrow{O_1O_2}}$, the figure turns into itself under the translation by vector $2\overrightarrow{O_1O_2}$. A bounded figure cannot possess such a property since the image of the point with the largest absciss does not belong to the figure.

b) Let $O_3 = S_{O_2}(O_1)$. It is easy to verify that $S_{O_3} = S_{O_2} \circ S_{O_1} \circ S_{O_2}$ and, therefore, if $O_1$ and $O_2$ are the centers of symmetry of a figure, then $O_3$ is also a center of symmetry, moreover, $O_3 \neq O_1$ and $O_3 \neq O_2$.

c) Let us demonstrate that a finite set can only have 0, 1, 2 or 3 "almost centers of symmetry". The corresponding examples are given on Fig. 13. It only remains to prove that a finite set cannot have more than three "almost centers of symmetry".
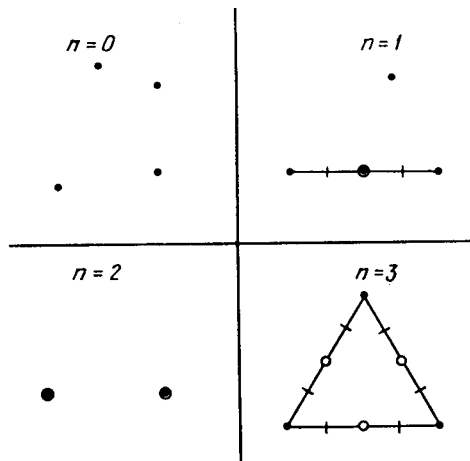


Figure 155 (Sol. 16.11)

There are finitely many "almost centers of symmetry" since they are the midpoints of the segments that connect the points of the set. Therefore, we can select a line such that the projections of "almost centers of symmetry" to the line are distinct. Therefore, it suffices to carry out the proof for the points which belong to one line.

Let $n$ points on a line be given and $x_1 < x_2 < \cdots < x_{n-1} < x_n$ be their coordinates. If we discard the point $x_1$, then only point $\frac{1}{2}(x_2 + x_n)$ can serve as the center of symmetry of the remaining set; if we discard $x_n$, then only point $\frac{1}{2}(x_1 + x_{n-1})$ can be the center of symmetry of the remaining set and if we discard any other point, then only point $\frac{1}{2}(x_1 + x_n)$ can be the center of symmetry of the remaining set. Therefore, there can not be more than 3 centers of symmetry.

**16.12.** A pair of symmetric points is painted different colours, therefore, it can be discarded from the consideration; let us discard all such pairs. In the remaining set of points the number of blue pairs is equal to the number of red pairs. Moreover, the sum of the distances from either of points $A$ or $B$ to any pair of symmetric points is equal to the length of segment $AB$.

**16.13.** Consider circle $S_1'$ symmetric to circle $S_1$ through point $A$. The line to be found passes through the intersection points of $S_1'$ and $S_2$.

**16.14.** Let $l'$ be the image of line $l$ under the symmetry through point $A$. The desired line passes through point $A$ and an intersection point of line $l'$ with the circle $S$.
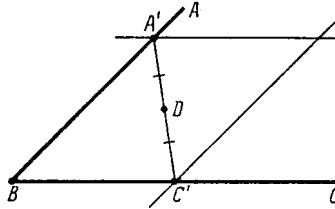


Figure 156 (Sol. 16.15)

**16.15.** Let us construct the intersection points $A'$ and $C'$ of the lines symmetric to the lines $BC$ and $AB$ through the point $D$ with lines $AB$ and $BC$, respectively, see Fig. 14. It is clear that point $D$ is the midpoint of segment $A'C'$ because points $A'$ and $C'$ are symmetric through $D$.

**16.16.** Let $O$ be the midpoint of segment $AB$. We have to construct points $C$ and $D$ that belong to the legs of the angle so that point $O$ is the midpoint of segment $CD$. This construction is described in the solution of the preceding problem.

**16.17.** Let us first separate the lines into pairs. This can be done in three ways. Let the opposite vertices $A$ and $C$ of parallelogram $ABCD$ belong to one pair of lines, $B$ and $D$ to the other pair. Consider the angle formed by the first pair of lines and construct points $A$ and $C$ as described in the solution of Problem 16.15. Construct points $B$ and $D$ in a similar way.

**16.18.** On the smaller circle, $S_1$, take an arbitrary point, $X$. Let $S_1'$ be the image of $S_1$ under the symmetry with respect to $X$, let $Y$ be the intersection point of circles $S_1'$ and $S_2$. Then $XY$ is the line to be found.
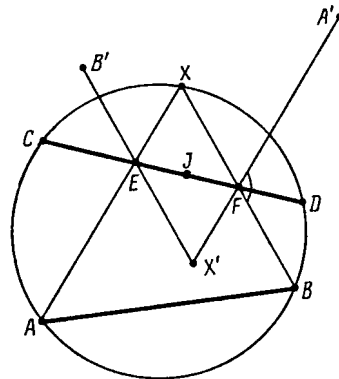


Figure 157 (Sol. 16.19)

**16.19.** Suppose $X$ is constructed. Denote the images of points $A$, $B$ and $X$ under the symmetry through point $J$ by $A'$, $B'$ and $X'$, respectively, see Fig. 15. Angle $\angle A'FB = 180° - \angle AXB$ is known and, therefore, point $F$ is the intersection point of segment $CD$ with the arc of the circle whose points serve as vertices of angles of value $180° - \angle AXB$ that subtend segment $BA'$. Point $X$ is the intersection point of line $BF$ with the given circle.

**16.20.** Suppose that line $l$ is constructed. Let us consider circle $S_1'$ symmetric to circle $S_1$ through point $A$. Let $O_1$, $O_1'$ and $O_2$ be the centers of circles $S_1$, $S_1'$ and $S_2$, as shown on Fig. 16.
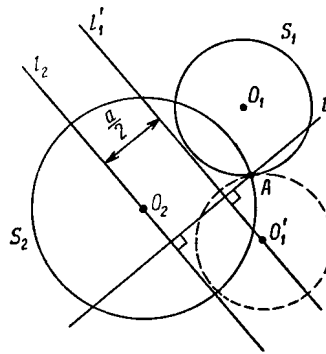
Figure 158 (Sol. 16.20)

Let us draw lines $l_1'$ and $l_2$ through $O_1'$ and $O_2$ perpendicularly to line $l$. The distance between lines $l_1'$ and $l_2$ is equal to a half difference of the lengths of chords intercepted by $l$ on circles $S_1$ and $S_2$. Therefore, in order to construct $l$, we have to construct the circle of radius $\frac{1}{2}a$ with center $O_1'$; line $l_2$ is tangent to this circle. Having constructed $l_2$, drop the perpendicular from point $A$ to $l_2$; this perpendicular is line $l$.

**16.21.** Let $B_1$, $B_2$, ..., $B_m$ be the midpoints of sides $A_1A_2$, $A_2A_3$, ..., $A_mA_1$ of polygon $A_1A_2\ldots A_m$. Then $S_{B_1}(A_1) = A_2$, $S_{B_2}(A_2) = A_3, \ldots, S_{B_m}(A_m) = A_1$. It follows that $S_{B_m} \circ \cdots \circ S_{B_1}(A_1) = A_1$, i.e., $A_1$ is a fixed point of the composition of symmetries $S_{B_m} \circ S_{B_{m-1}} \circ \cdots \circ S_{B_1}$. By Problem 16.9 the composition of an odd number of central symmetries is a central symmetry, i.e., has a unique fixed point. This point can be constructed as the midpoint of the segment that connects points $X$ and $S_{B_m} \circ S_{B_{m-1}} \circ \cdots \circ S_{B_1}(X)$, where $X$ is an arbitrary point.

# Chapter 17. THE SYMMETRY THROUGH A LINE

## Background

1. The *symmetry through a line l* (notation: $S_l$) is a transformation of the plane which sends point $X$ into point $X'$ such that $l$ is the midperpendicular to segment $XX'$. Such a transformation is also called *the axial symmetry* and $l$ is called *the axis of the symmetry*.

2. If a figure turns into itself under the symmetry through line $l$, then $l$ is called *the axis of symmetry* of this figure.

3. The composition of two symmetries through axes is a parallel translation, if the axes are parallel, and a rotation, if they are not parallel, cf. Problem 17.22.

Axial symmetries are a sort of "bricks" all the other motions of the plane are constructed from: any motion is a composition of not more than three axial symmetries (Problem 17.35). Therefore, the composition of axial symmetries give much more powerful method for solving problems than compositions of central symmetries. Moreover, it is often convenient to decompose a rotation into a composition of two symmetries with one of the axes of symmetry being a line passing through the center of the rotation.

## Introductory problems

**1.** Prove that any axial symmetry sends any circle into a circle.

**2.** A quadrilateral has an axis of symmetry. Prove that this quadrilateral is either an equilateral trapezoid or is symmetric through a diagonal.

**3.** An axis of symmetry of a polygon intersects its sides at points $A$ and $B$. Prove that either point $A$ is a vertex of the polygon or the midpoint of a side perpendicular to the axis of symmetry.

**4.** Prove that if a figure has two perpendicular axes of symmetry, it has a center of symmetry.

## §1. Solving problems with the help of a symmetry

**17.1.** Point $M$ belongs to a diameter $AB$ of a circle. Chord $CD$ passes through $M$ and intersects $AB$ at an angle of $45°$. Prove that the sum $CM^2 + DM^2$ does not depend on the choice of point $M$.

**17.2.** Equal circles $S_1$ and $S_2$ are tangent to circle $S$ from the inside at points $A_1$ and $A_2$, respectively. An arbitrary point $C$ of circle $S$ is connected by segments with points $A_1$ and $A_2$. These segments intersect $S_1$ and $S_2$ at points $B_1$ and $B_2$, respectively. Prove that $A_1A_2 \parallel B_1B_2$.

**17.3.** Through point $M$ on base $AB$ of an isosceles triangle $ABC$ a line is drawn. It intersects sides $CA$ and $CB$ (or their extensions) at points $A_1$ and $B_1$. Prove that $A_1A : A_1M = B_1B : B_1M$.

## §2. Constructions

**17.4.** Construct quadrilateral $ABCD$ whose diagonal $AC$ is the bisector of angle $\angle A$ knowing the lengths of its sides.

**17.5.** Construct quadrilateral $ABCD$ in which a circle can be inscribed knowing the lengths of two neighbouring sides $AB$ and $AD$ and the angles at vertices $B$ and $D$.

**17.6.** Construct triangle $ABC$ knowing $a$, $b$ and the difference of angles $\angle A - \angle B$.

**17.7.** Construct triangle $ABC$ given its side $c$, height $h_c$ and the difference of angles $\angle A - \angle B$.

**17.8.** Construct triangle $ABC$ given a) $c$, $a - b$ $(a > b)$ and angle $\angle C$; b) $c$, $a + b$ and angle $\angle C$.

**17.9.** Given line $l$ and points $A$ and $B$ on one side of it. Construct point $X$ on $l$ such that $AX + XB = a$, where $a$ is given.

**17.10.** Given acute angle $\angle MON$ and points $A$ and $B$ inside it. Find point $X$ on leg $OM$ such that triangle $XYZ$, where $Y$ and $Z$ are the intersection points of lines $XA$ and $XB$ with $ON$, were isosceles, i.e., $XY = XZ$.

**17.11.** Given line $MN$ and two points $A$ and $B$ on one side of it. Construct point $X$ on $MN$ such that $\angle AXM = 2\angle BXN$.

$$* * *$$

**17.12.** Given three lines $l_1$, $l_2$ and $l_3$ intersecting at one point and point $A_1$ on $l_1$. Construct triangle $ABC$ so that $A_1$ is the midpoint of its side $BC$ and lines $l_1$, $l_2$ and $l_3$ are the midperpendiculars to the sides.

**17.13.** Construct triangle $ABC$ given points $A$, $B$ and the line on which the bisector of angle $\angle C$ lies.

**17.14.** Given three lines $l_1$, $l_2$ and $l_3$ intersecting at one point and point $A$ on line $l_1$. Construct triangle $ABC$ so that $A$ is its vertex and the bisectors of the triangle lie on lines $l_1$, $l_2$ and $l_3$.

**17.15.** Construct a triangle given the midpoints of two of its sides and the line that contains the bisector drawn to one of these sides.

## §3. Inequalities and extremals

**17.16.** On the bisector of the exterior angle $\angle C$ of triangle $ABC$ point $M$ distinct from $C$ is taken. Prove that $MA + MB > CA + CB$.

**17.17.** In triangle $ABC$ median $AM$ is drawn. Prove that $2AM \geq (b + c)\cos(\frac{1}{2}\alpha)$.

**17.18.** The inscribed circle of triangle $ABC$ is tangent to sides $AC$ and $BC$ at points $B_1$ and $A_1$. Prove that if $AC > BC$, then $AA_1 > BB_1$.

**17.19.** Prove that the area of any convex quadrilateral does not exceed a half-sum of the products of opposite sides.

**17.20.** Given line $l$ and two points $A$ and $B$ on one side of it, find point $X$ on line $l$ such that the length of segment $AXB$ of the broken line was minimal.

**17.21.** Inscribe a triangle of the least perimeter in a given acute triangle.

## §4. Compositions of symmetries

**17.22.** a) Lines $l_1$ and $l_2$ are parallel. Prove that $S_{l_1} \circ S_{l_2} = T_{2\mathbf{a}}$, where $T_{\mathbf{a}}$ is the parallel translation that sends $l_1$ to $l_2$ and such that $\mathbf{a} \perp l_1$.

b) Lines $l_1$ and $l_2$ intersect at point $O$. Prove that $S_{l_2} \circ S_{l_1} = R_O^{2\alpha}$, where $R_O^{\alpha}$ is the rotation about $O$ through the angle of $\alpha$ that sends $l_1$ to $l_2$.

**17.23.** On the plane, there are given three lines $a$, $b$, $c$. Let $T = S_a \circ S_b \circ S_c$. Prove that $T \circ T$ is a parallel translation (or the identity map).

**17.24.** Let $l_3 = S_{l_1}(l_2)$. Prove that $S_{l_3} = S_{l_1} \circ S_{l_2} \circ S_{l_1}$.

**17.25.** The inscribed circle is tangent to the sides of triangle $ABC$ at points $A_1$, $B_1$ and $C_1$. Points $A_2$, $B_2$ and $C_2$ are symmetric to these points through the bisectors of the corresponding angles of the triangle. Prove that $A_2B_2 \parallel AB$ and lines $AA_2$, $BB_2$ and $CC_2$ intersect at one point.

**17.26.** Two lines intersect at an angle of $\gamma$. A grasshopper hops from one line to another one; the length of each jump is equal to 1 m and the grasshopper does not jump backwards whenever possible. Prove that the sequence of jumps is periodic if and only if $\gamma/\pi$ is a rational number.

**17.27.** a) Given a circle and $n$ lines. Inscribe into the circle an $n$-gon whose sides are parallel to given lines.

b) $n$ lines go through the center $O$ of a circle. Construct an $n$-gon circumscribed about this circle such that the vertices of the $n$-gon belong to these lines.

**17.28.** Given $n$ lines, construct an $n$-gon for which these lines are a) the midperpendiculars to the sides; b) the bisectors of the inner or outer angles at the vertices.

**17.29.** Given a circle, a point and $n$ lines. Into the circle inscribe an $n$-gon one of whose sides passes through the given point and the other sides are parallel to the given lines.

## §5. Properties of symmetries and axes of symmetries

**17.30.** Point $A$ lies at the distance of 50 cm from the center of the disk of radius 1 cm. It is allowed to reflect point $A$ symmetrically through any line intersecting the disk. Prove that a) after 25 reflections point $A$ can be driven inside the given circle; b) it is impossible to perform this in 24 reflections.

**17.31.** On a circle with center $O$ points $A_1$, ..., $A_n$ which divide the circle into equal archs and a point $X$ are given. Prove that the points symmetric to $X$ through lines $OA_1$, ..., $OA_n$ constitute a regular polygon.

**17.32.** Prove that if a planar figure has exactly two axes of symmetry, then these axes are perpendicular to each other.

**17.33.** Prove that if a polygon has several (more than 2) axes of symmetry, then all of them intersect at one point.

**17.34.** Prove that if a polygon has an even number of axes of symmetry, then it has a center of symmetry.

## §6. Chasles's theorem

A transformation which preserves distances between points (i.e., such that if $A'$ and $B'$ are the images of points $A$ and $B$, respectively, then $A'B' = AB$) is called a *movement*. A movement of the plane that preserves 3 points which do not belong to one line preserves all the other points.

**17.35.** Prove that any movement of the plane is a composition of not more than three symmetries through lines.

A movement which is the composition of an even number of symmetries through lines is called a *first type movement* or a *movement that preserves the orientation of the plane*.

A movement which is the composition of an odd number of symmetries through lines is called a *second type movement* or a *movement inversing the orientation of the plane*.

We will not prove that *the composition of an odd number of symmetries through lines is impossible to represent in the form of the composition of an odd number of symmetries through lines and the other way round* because this fact, though true, is beyond the scope of our book.

**17.36.** Prove that any first type movement is either a rotation or a parallel translation.

The composition of a symmetry through line $l$ and the translation by a vector parallel to $l$ (this vector might be the zero one) is called a *transvection*.

**17.37.** Prove that any second type movement is a transvection.

### Problems for independent study

**17.38.** Given a nonconvex quadrilateral of perimeter $P$. Prove that there exists a convex quadrilateral of the same perimeter but of greater area.

**17.39.** Can a bounded figure have a center of symmetry and exactly one axis of symmetry?

**17.40.** Point $M$ belongs to the circumscribed circle of triangle $ABC$. Prove that the lines symmetric to the lines $AM$, $BM$ and $CM$ through the bisectors of angles $\angle A$, $\angle B$ and $\angle C$ are parallel to each other.

**17.41.** The vertices of a convex quadrilateral belong to different sides of a square. Prove that the perimeter of this quadrilateral is not shorter than $2\sqrt{2}a$, where $a$ is the length of the square's side.

**17.42.** A ball lies on a rectangular billiard table. Construct a trajectory traversing along which the ball would return to the initial position after one reflexion from each side of the table.

### Solutions

**17.1.** Denote the points symmetric to points $C$ and $D$ through line $AB$ by $C'$ and $D'$, respectively. Since $\angle C'MD = 90°$, it follows that $CM^2 + MD^2 = C'M^2 + MD^2 = C'D^2$. Since $\angle C'CD = 45°$, chord $C'D$ is of constant length.

**17.2.** In circle $S$, draw the diameter which is at the same time the axis of symmetry of circles $S_1$ and $S_2$. Let points $C'$ and $B_2'$ be symmetric to points $C$ and $B_2$ through this diameter: see Fig. 17.
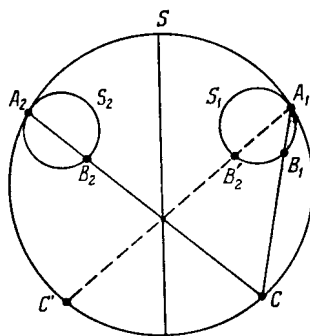


Figure 159 (Sol. 17.2)

Circles $S_1$ and $S$ are homothetic with the center of homothety at point $A_1$; let this homothety send line $B_1B_2'$ into line $CC'$. Therefore, these lines are parallel to each other. It is also clear that $B_2B_2' \parallel CC'$. Therefore, points $B_1$, $B_2'$ and $B_2$ belong to one line and this line is parallel to line $CC'$.

**17.3.** Let the line symmetric to line $A_1B_1$ through line $AB$ intersect sides $CA$ and $CB$ (or their extensions) at points $A_2$ and $B_2$, respectively. Since $\angle A_1AM = \angle B_2BM$ and $\angle A_1MA = \angle B_2MB$, it follows that $A_1AM \sim B_2BM$, i.e., $A_1A : A_1M = B_2B : B_2M$. Moreover, since $MB$ is a bisector in triangle $B_1MB_2$, it follows that $B_2B : B_2M = B_1B : B_1M$.

**17.4.** Suppose that quadrilateral $ABCD$ is constructed. Let, for definiteness sake, $AD > AB$. Denote by $B'$ the point symmetric to $B$ through diagonal $AC$. Point $B'$ belongs to side $AD$ and $B'D = AD - AB$. In triangle $B'CD$, the lengths of all the sides are known: $B'D = AD - AB$ and $B'C = BC$. Constructing triangle $B'CD$ on the extension of side $B'D$ beyond $B'$ let us construct point $A$.

Further construction is obvious.

**17.5.** Suppose that quadrilateral $ABCD$ is constructed. For definiteness sake, assume that $AD > AB$. Let $O$ be the center of the circumscribed circle; let point $D'$ be symmetric to $D$ through line $AO$; let $A'$ be the intersection point of lines $AO$ and $DC$; let $C'$ be the intersection point of lines $BC$ and $A'D'$ (Fig. 18).
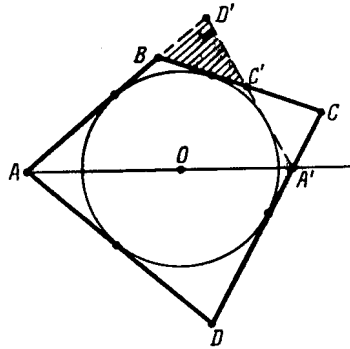


Figure 160 (Sol. 17.5)

In triangle $BC'D'$, side $BD'$ and adjacent angles are known: $\angle D'BC' = 180° - \angle B$ and $\angle BD'C' = \angle D$. Let us construct triangle $BC'D'$ given these elements. Since $AD' = AD$, we can construct point $A$. Further, let us construct $O$ — the intersection point of bisectors of angles $ABC'$ and $BD'C'$. Knowing the position of $O$ we can construct point $D$ and the inscribed circle. Point $C$ is the intersection point of line $BC'$ and the tangent to the circle drawn from $D$.

**17.6.** Suppose that triangle $ABC$ is constructed. Let $C'$ be the point symmetric to $C$ through the midperpendicular to segment $AB$. In triangle $ACC'$ there are known $AC = b$, $AC' = a$ and $\angle CAC' = \angle A - \angle B$. Therefore, the triangle can be constructed. Point $B$ is symmetric to $A$ through the midperpendicular to segment $CC'$.

**17.7.** Suppose that triangle $ABC$ is constructed. Denote by $C'$ the point symmetric to $C$ through the midperpendicular to side $AB$ and by $B'$ the point symmetric to $B$ through line $CC'$. For definiteness, let us assume that $AC < BC$. Then

$$\angle ACB' = \angle ACC' + \angle C'CB = 180° - \angle A + \angle C'CB = 180° - (\angle A - \angle B)$$

i.e., angle $\angle ACB'$ is known.

Triangle $ABB'$ can be constructed because $AB = c$, $BB' = 2h_c$ and $\angle ABB' = 90°$. Point $C$ is the intersection point of the midperpendicular to segment $BB'$ and the arc of the circle whose points serve as vertices of angles of value $180° - (\angle A - \angle B)$ that subtend segment $AB'$.

**17.8.** a) Suppose triangle $ABC$ is constructed. Let $C'$ be the point symmetric to $A$ through the bisector of angle $\angle C$. Then

$$\angle BC'A = 180° - \angle AC'C = 180° - \frac{1}{2}(180° - \angle C) = 90° + \frac{1}{2}\angle C$$

and $BC' = a - b$.

In triangle $ABC'$, there are known $AB = c$, $BC' = a - b$ and $\angle C' = 90° + \frac{1}{2}\angle C$. Since $\angle C' > 90°$, triangle $ABC'$ is uniquely constructed from these elements. Point $C$ is the intersection point of the midperpendicular to segment $AC'$ with line $BC'$.

b) The solution is similar to that of heading a). For $C'$ we should take the point symmetric to $A$ through the bisector of the outer angle $\angle C$ in triangle $ABC$.

Since $\angle AC'B = \frac{1}{2}\angle C < 90°$, the problem can have two solutions.

**17.9.** Let $S$ be the circle of radius $a$ centered at $B$, let $S'$ be the circle of radius $AX$ with center $X$ and $A'$ the point symmetric to $A$ through line $l$. Then circle $S'$ is tangent to circle $S$ and point $A'$ belongs to circle $S'$. It remains to draw circle $S'$ through the given points $A$ and $A'$ tangent to the given circle $S$ and find its center $X$, cf. Problem 8.56 b).
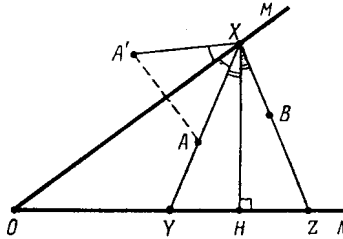


Figure 161 (Sol. 17.10)

**17.10.** Let the projection of point $A$ to line $ON$ be closer to point $O$ than the projection of point $B$. Suppose that the isosceles triangle $XYZ$ is constructed. Let us consider point $A'$ symmetric to point $A$ through line $OM$. Let us drop perpendicular $XH$ from point $X$ to line $ON$ (Fig. 19). Since

$$\angle A'XB = \angle A'XO + \angle OXA + \angle YXH + \angle HXZ =$$
$$2\angle OXY + 2\angle YXH = 2\angle OXH = 180° - 2\angle MON,$$

angle $\angle A'XB$ is known. Point $X$ is the intersection point of line $OM$ and the arc whose points serve as vertices of angles of $180° - 2\angle MON$ that subtend $A'B$. In addition, the projection of $X$ onto $ON$ must lie between the projections of $A$ and $B$.

Conversely, if $\angle A'XB = 180° - \angle MON$ and the projection of $X$ to line $ON$ lies between the projections of $A$ and $B$, then triangle $XYZ$ is an isosceles one.

**17.11.** Suppose that point $X$ is constructed. Let $B'$ be the point symmetric to point $B$ through line $MN$; the circle of radius $AB'$ with center $B'$ intersects line $MN$ at point $A'$. Then ray $B'X$ is the bisector of angle $\angle AB'A'$. It follows that $X$ is the intersection point of lines $B'O$ and $MN$, where $O$ is the midpoint of segment $AA'$.

**17.12.** Through point $A_1$ draw line $BC$ perpendicular to line $l_1$. Vertex $A$ of triangle $ABC$ to be found is the intersection point of lines symmetric to line $BC$ through lines $l_2$ and $l_3$.

**17.13.** Let point $A'$ be symmetric to $A$ through the bisector of angle $\angle C$. Then $C$ is the intersection point of line $A'B$ and the line on which the bisector of angle $\angle C$ lies.

**17.14.** Let $A_2$ and $A_3$ be points symmetric to $A$ through lines $l_2$ and $l_3$, respectively. Then points $A_2$ and $A_3$ belong to line $BC$. Therefore, points $B$ and $C$ are the intersection points of line $A_2A_3$ with lines $l_2$ and $l_3$, respectively.

**17.15.** Suppose that triangle $ABC$ is constructed and $N$ is the midpoint of $AC$, $M$ the midpoint of $BC$ and the bisector of angle $\angle A$ lies on the given line, $l$. Let us construct point $N'$ symmetric to $N$ through line $l$. Line $BA$ passes through point $N'$ and is parallel to $MN$. In this way we find vertex $A$ and line $BA$. Having drawn line $AN$, we get line $AC$. It remains to construct a segment whose endpoints belong to the legs of angle $\angle BAC$ and whose midpoint is $M$, cf. the solution of Problem 16.15.

**17.16.** Let points $A'$ and $B'$ be symmetric to $A$ and $B$, respectively, through line $CM$. Then $AM + MB = A'M + MB > A'B = A'C + CB = AC + CB$.

**17.17.** Let points $B'$, $C'$ and $M'$ be symmetric to points $B$, $C$ and $M$ through the bisector of the outer angle at vertex $A$. Then

$$AM + AM' = MM' = \frac{1}{2}(BB' + CC') = (b + c)\sin(90° - \frac{1}{2}\alpha) = (b + c)\cos(\frac{1}{2}\alpha).$$

**17.18.** Let point $B'$ be symmetric to $B$ through the bisector of angle $\angle ACB$. Then $B'A_1 = BB_1$, i.e., it remains to verify that $B'A_1 < AA_1$. To this end it suffices to notice that $\angle AB'A_1 > \angle AB'B > 90°$.

**17.19.** Let $D'$ be the point symmetric to $D$ through the midperpendicular to segment $AC$. Then

$$S_{ABCD} = S_{ABCD'} = S_{BAD'} + S_{BCD'} \leq \frac{1}{2}AB \cdot AD' + \frac{1}{2}BC \cdot CD' =$$

$$\frac{1}{2}(AB \cdot CD + BC \cdot AD).$$

**17.20.** Let point $A'$ be symmetric to $A$ through line $l$. Let $X$ be a point on line $l$. Then $AX + XB = A'X + XB \geq A'B$ and the equality is attained only if $X$ belongs to segment $A'B$. Therefore, the point to be found is the intersection point of line $l$ with segment $A'B$.

**17.21.** Let $PQR$ be the triangle determined by the bases of the heights of triangle $ABC$ and let $P'Q'R'$ be any other triangle inscribed in triangle $ABC$. Further, let points $P_1$ and $P_2$ (respectively $P_1'$ and $P_2'$) be symmetric to point $P$ (resp. $P'$) through lines $AB$ and $AC$, respectively (Fig. 20).
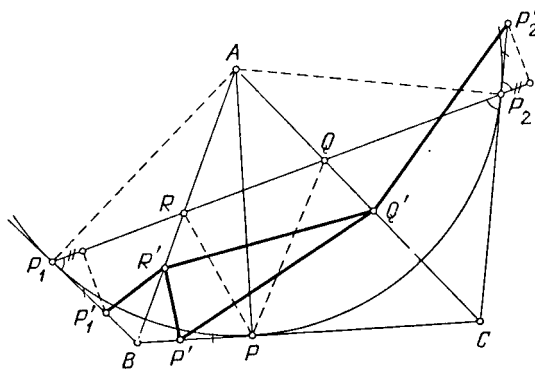


Figure 162 (Sol. 17.21)

Points $Q$ and $R$ belong to segment $P_1P_2$ (see Problem 1.57) and, therefore, the perimeter of triangle $PQR$ is equal to the length of segment $P_1P_2$. The perimeter of triangle $P'Q'R'$ is, however, equal to the length of the broken segment $P_1'R'Q'P_2'$, i.e., it is not shorter than

the length of segment $P_1'P_2'$. It remains to notice that $(P_1'P_2')^2 = P_1P_2^2 + 4d^2$, where $d$ is the distance from point $P_1'$ to line $P_1P_2$.

**17.22.** Let $X$ be an arbitrary point, $X_1 = S_{l_1}(X)$ and $X_2 = S_{l_2}(X_1)$.

a) On line $l_1$, select an arbitrary point $O$ and consider a coordinate system with $O$ as the origin and the absciss axis directed along line $l_1$. Line $l_2$ is given in this coordinate system by the equation $y = a$. Let $y$, $y_1$ and $y_2$ be ordinates of points $X$, $X_1$ and $X_2$, respectively. It is clear that $y_1 = -y$ and $y_2 = (a - y_1) + a = y + 2a$. Since points $X$, $X_1$ and $X_2$ have identical abscisses, it follows that $X_2 = T_{2\mathbf{a}}(X)$, where $T_\mathbf{a}$ is the translation that sends $l_1$ to $l_2$, and $\mathbf{a} \perp l_1$.

b) Consider a coordinate system with $O$ as the origin and the absciss axis directed along line $l_1$. Let the angle of rotation from line $l_1$ to $l_2$ in this coordinate system be equal to $\alpha$ and the angles of rotation from the absciss axes to rays $OX$, $OX_1$ and $OX_2$ be equal to $\varphi$, $\varphi_1$ and $\varphi_2$, respectively. Clearly, $\varphi_1 = -\varphi$ and $\varphi_2 = (\alpha - \varphi_1) + \alpha = \varphi + 2\alpha$. Since $OX = OX_1 = OX_2$, it follows that $X_2 = R_O^{2\alpha}(X)$, where $R_O^\alpha$ is the translation that sends $l_1$ to $l_2$.

**17.23.** Let us represent $T \circ T$ as the composition of three transformations:

$$T \circ T = (S_a \circ S_b \circ S_c) \circ (S_a \circ S_b \circ S_c) = (S_a \circ S_b) \circ (S_c \circ S_a) \circ (S_b \circ S_c).$$

Here $S_a \circ S_b$, $S_c \circ S_a$ and $S_b \circ S_c$ are rotations through the angles of $2\angle(b, a)$, $2\angle(a, c)$ and $2\angle(c, b)$, respectively. The sum of the angles of the rotations is equal to

$$2(\angle(b, a) + \angle(a, c) + \angle(c, b)) = 2\angle(b, b) = 0°$$

and this value is determined up to $2 \cdot 180° = 360°$. It follows that this composition of rotations is a parallel translation, cf. Problem 18.33.

**17.24.** If points $X$ and $Y$ are symmetric through line $l_3$, then points $S_{l_1}(X)$ and $S_{l_1}(Y)$ are symmetric through line $l_2$, i.e., $S_{l_1}(X) = S_{l_2} \circ S_{l_1}(Y)$. It follows that $S_{l_1} \circ S_{l_3} = S_{l_2} \circ S_{l_1}$ and $S_{l_3} = S_{l_1} \circ S_{l_2} \circ S_{l_1}$.

**17.25.** Let $O$ be the center of the inscribed circle; let $a$ and $b$ be lines $OA$ and $OB$. Then $S_a \circ S_b(C_1) = S_a(A_1) = A_2$ and $S_b \circ S_a(C_1) = S_b(B_1) = B_2$. Points $A_2$ and $B_2$ are obtained from point $C_1$ by rotations with center $O$ through opposite angles and, therefore, $A_2B_2 \parallel AB$.

Similar arguments show that the sides of triangles $ABC$ and $A_2B_2C_2$ are parallel and, therefore, these triangles are homothetic. Lines $AA_2$, $BB_2$ and $CC_2$ pass through the center of homothety which sends triangle $ABC$ to $A_2B_2C_2$. Notice that this homothety sends the circumscribed circle of triangle $ABC$ into the inscribed circle, i.e., the center of homothety belongs to the line that connects the centers of these circles.

**17.26.** For every jump vector there are precisely two positions of a grasshopper for which the jump is given by this vector. Therefore, a sequence of jumps is periodic if and only if there exists but a finite number of distinct jump vectors.

Let $\mathbf{a}_1$ be the jump vector of the grasshopper from line $l_2$ to line $l_1$; let $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$,... be vectors of the successive jumps. Then $\mathbf{a}_2 = S_{l_2}(\mathbf{a}_1)$, $\mathbf{a}_3 = S_{l_1}(\mathbf{a}_2)$, $\mathbf{a}_4 = S_{l_2}(\mathbf{a}_3)$ , ... Since the composition $S_{l_1} \circ S_{l_2}$ is a rotation through an angle of $2\gamma$ (or $2\pi - 2\gamma$), it follows that vectors $\mathbf{a}_3$, $\mathbf{a}_5$, $\mathbf{a}_7$, ... are obtained from $\mathbf{a}_1$ by rotations through angles of $2\gamma$, $4\gamma$, $6\gamma$, ... (or through angles of $2(\pi - \gamma)$, $4(\pi - \gamma)$, $6(\pi - \gamma)$, ... ). Therefore, the set $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, ... contains a finite number of distinct vectors if and only if $\gamma/\pi$ is a rational number. The set $\mathbf{a}_2$, $\mathbf{a}_4$, $\mathbf{a}_6$, ... is similarly considered.

**17.27.** a) Suppose polygon $A_1A_2 \ldots A_n$ is constructed. Let us draw through the center $O$ of the circle the midperpendiculars $l_1$, $l_2$, $\ldots$, $l_n$ to chords $A_1A_2$, $A_2A_3$, $\ldots$, $A_nA_1$, respectively. Lines $l_1$, $\ldots$, $l_n$ are known since they pass through $O$ and are perpendicular to

the given lines. Moreover, $A_2 = S_{l_1}(A_1)$, $A_3 = S_{l_2}(A_2)$, ..., $A_1 = S_{l_n}(A_n)$, i.e., point $A_1$ is a fixed point of the composition of symmetries $S_{l_n} \circ \cdots \circ S_{l_1}$. For $n$ odd there are precisely two fixed points on the circle; for $n$ even there are either no fixed points or all the points are fixed.

b) Suppose the desired polygon $A_1 \ldots A_n$ is constructed. Consider polygon $B_1 \ldots B_n$ formed by the tangent points of the circumscribed polygon with the circle. The sides of polygon $B_1 \ldots B_n$ are perpendicular to the given lines, i.e., they have prescribed directions and, therefore, the polygon can be constructed (see heading a)); it remains to draw the tangents to the circle at points $B_1$, ..., $B_n$.

**17.28.** Consider the composition of consecutive symmetries through given lines $l_1$, ..., $l_n$. In heading a) for vertex $A_1$ of the desired $n$-gon we have to take a fixed point of this composition, and in heading b) for line $A_1 A_n$ we have to take the(a) fixed line.

**17.29.** The consecutive symmetries through lines $l_1$, ..., $l_{n-1}$ perpendicular to given lines and passing through the center of the circle send vertex $A_1$ of the desired polygon to vertex $A_n$.

If $n$ is odd, then the composition of these symmetries is a rotation through a known angle and, therefore, we have to draw through point $M$ chord $A_1 A_n$ of known length.

If $n$ is even, then the considered composition is a symmetry through a line and, therefore, from $M$ we have to drop perpendicular to this line.

**17.30.** Let $O$ be the center of the given disk, $D_R$ the disk of radius $R$ with center $O$. Let us prove that the symmetries through the lines passing through $D_1$ send the set of images of points of $D_R$ into disk $D_{R+2}$. Indeed, the images of point $O$ under the indicated symmetries fill in disk $D_2$ and the disks of radius $R$ with centers in $D_2$ fill in disk $D_{R+2}$.

It follows that after $n$ reflections we can obtain from points of $D_1$ any point of $D_{2n+1}$ and only them. It remains to notice that point $A$ can be "herded" inside $D_R$ after $n$ reflections if and only if we can transform any point of $D_R$ into $A$ after $n$ reflections.

**17.31.** Denote symmetries through lines $OA_1$, ..., $OA_n$ by $S_1$, ..., $S_n$, respectively. Let $X_k = S_k(X)$ for $k = 1, \ldots, n$. We have to prove that under a rotation through point $O$ the system of points $X_1$, ..., $X_n$ turns into itself. Clearly,

$$S_{k+1} \circ S_k(X_k) = S_{k+1} \circ S_k \circ S_k(X) = X_{k+1}.$$

Transformations $S_{k+1} \circ S_k$ are rotations about $O$ through an angle of $\frac{4\pi}{n}$, see Problem 17.22 b).

REMARK. For $n$ even we get an $\frac{n}{2}$-gon.

**17.32.** Let lines $l_1$ and $l_2$ be axes of symmetry of a plane figure. This means that if point $X$ belongs to the figure, then points $S_{l_1}(X)$ and $S_{l_2}(X)$ also belong to the figure. Consider line $l_3 = S_{l_1}(l_2)$. Thanks to Problem 17.24 $S_{l_3}(X) = S_{l_1} \circ S_{l_2} \circ S_{l_1}(X)$ and, therefore, $l_3$ is also an axis of symmetry.

If the figure has precisely two axes of symmetry, then either $l_3 = l_1$ or $l_3 = l_2$. Clearly, $l_3 \neq l_1$ and, therefore, $l_3 = l_2$ i.e., line $l_2$ is perpendicular to line $l_1$.

**17.33.** Suppose that the polygon has three axes of symmetry which do not intersect at one point, i.e., they form a triangle. Let $X$ be the point of the polygon most distant from an inner point $M$ of this triangle. Points $X$ and $M$ lie on one side of one of the considered axes of symmetry, $l$. If $X'$ is the point symmetric to $X$ through $l$, then $M'X > MX$ and point $X'$ is distant from $M$ further than $X$. The obtained contradiction implies that all the axes of symmetry of a polygon intersect at one point.

**17.34.** All the axes of symmetry pass through one point $O$ (Problem 17.33). If $l_1$ and $l_2$ are axes of symmetry, then $l_3 = S_{l_1}(l_2)$ is also an axis of symmetry, see Problem 17.24.

Select one of the axes of symmetry $l$ of our polygon. The odd axes of symmetry are divided into pairs of lines symmetric through $l$. If line $l_1$ perpendicular to $l$ and passing through $O$ is not an axis of symmetry, then there is an odd number of axes of symmetry. Therefore, $l_1$ is an axis of symmetry. Clearly, $S_{l_1} \circ S_l = R_O^{180°}$ is a central symmetry i.e., $O$ is the center of symmetry.

**17.35.** Let $F$ be a movement sending point $A$ into $A'$ and such that $A$ and $A'$ are distinct; $S$ the symmetry through the midperpendicular $l$ to segment $AA'$. Then $S \circ F(A) = A$, i.e., $A$ is a fixed point of $S \circ F$. Moreover, if $X$ is a fixed point of transformation $F$, then $AX = A'X$, i.e., point $X$ belongs to line $l$; hence, $X$ is a fixed point of $S \circ F$. Thus, point $A$ and all the fixed points of $F$ are fixed points of the transformation $S \circ F$.

Take points $A$, $B$ and $C$ not on one line and consider their images under the given movement $G$. We can construct transformations $S_1$, $S_2$ and $S_3$ which are either symmetries through lines or identity transformations such that $S_3 \circ S_2 \circ S_1 \circ G$ preserves points $A$, $B$ and $C$, i.e., is the identity transformation $E$. Multiplying the equality $S_3 \circ S_2 \circ S_1 \circ G = E$ from the left consecutively by $S_3$, $S_2$ and $S_1$ and taking into account that $S_i \circ S_i = E$ we get $G = S_1 \circ S_2 \circ S_3$.

**17.36.** Thanks to Problem 17.35 any first type movement is a composition of two symmetries through lines. It remains to make use of the result of Problem 17.22.

**17.37.** By Problem 17.35 any second type movement can be represented in the form $S_3 \circ S_2 \circ S_1$, where $S_1$, $S_2$ and $S_3$ are symmetries through lines $l_1$, $l_2$ and $l_3$, respectively. First, suppose that the lines $l_2$ and $l_3$ are not parallel. Then under the rotation of the lines $l_2$ and $l_3$ about their intersection point through any angle the composition $S_3 \circ S_2$ does not change (see Problem 17.22 b)), consequently, we can assume that $l_2 \perp l_1$. It remains to rotate lines $l_1$ and $l_2$ about their intersection point so that line $l_2$ became parallel to line $l_3$.

Now, suppose that $l_2 \parallel l_3$. If line $l_1$ is not parallel to these lines, then it is possible to rotate $l_1$ and $l_2$ about their intersection point so that lines $l_2$ and $l_3$ become nonparallel. If $l_1 \parallel l_2$, then it is possible to perform a parallel transport of $l_1$ and $l_2$ so that lines $l_2$ and $l_3$ coincide.

# Chapter 18. ROTATIONS

## Background

1. We will not give a rigorous definition of a rotation. To solve the problems it suffices to have the following idea on the notion of the rotation: a *rotation with center $O$* (or *about the point $O$*) through an angle of $\varphi$ is the transformation of the plane which sends point $X$ into point $X'$ such that:

a) $OX' = OX$;

b) the angle from vector $\overrightarrow{OX}$ to vector $\overrightarrow{OX'}$ is equal to $\varphi$.

2. In this chapter we make use of the following notations for the transformations and their compositions:

$T_{\mathbf{a}}$ is a translation by vector $\mathbf{a}$;

$S_O$ is the symmetry through point $O$;

$S_l$ is the symmetry through line $l$;

$R_O^{\varphi}$ is the rotation with center $O$ through an angle of $\varphi$;

$F \circ G$ is the composition of transformations $F$ and $G$ defined as $(F \circ G)(X) = F(G(X))$.

3. The problems solvable with the help of rotations can be divided into two big classes: problems which do not use the properties of compositions of rotations and properties which make use of these properties. To solve the problems which make use of the properties of the compositions of rotations the following result of Problem 18.33 is handy: $R_B^{\beta} \circ R_A^{\alpha} = R_C^{\gamma}$, where $\gamma = \alpha + \beta$ and $\angle BAC = \frac{1}{2}\alpha$, $\angle ABC = \frac{1}{2}\beta$.

## Introductory problems

**1.** Prove that any rotation sends any circle into a circle.

**2.** Prove that a convex $n$-gon is a regular one if and only if it turns into itself under the rotation through an angle of $\frac{360°}{n}$ about a point.

**3.** Prove that triangle $ABC$ is an equilateral one if and only if under the rotation through $60°$ (either clockwise or counterclockwise) about point $A$ vertex $B$ turns into vertex $C$.

**4.** Prove that the midpoints of the sides of a regular polygon determine a regular polygon.

**5.** Through the center of a square two perpendicular lines are drawn. Prove that their intersection points with the sides of the square determine a square.

## §1. Rotation by $90°$

**18.1.** On sides $BC$ and $CD$ of square $ABCD$ points $M$ and $K$, respectively, are taken so that $\angle BAM = \angle MAK$. Prove that $BM + KD = AK$.

**18.2.** In triangle $ABC$ median $CM$ and height $CH$ are drawn. Through an arbitrary point $P$ of the plane in which $ABC$ lies the lines are drawn perpendicularly to $CA$, $CM$ and $CB$. They intersect $CH$ at points $A_1$, $M_1$ and $B_1$, respectively. Prove that $A_1 M_1 = B_1 M_1$.

**18.3.** Two squares $BCDA$ and $BKMN$ have a common vertex $B$. Prove that median $BE$ of triangle $ABK$ and height $BF$ of triangle $CBH$ belong to one line.

The vertices of each square are counted clockwise.

**18.4.** Inside square $A_1A_2A_3A_4$ point $P$ is taken. From vertex $A_1$ we drop the perpendicular on $A_2P$; from $A_2$ on $A_3P$; from $A_3$ on $A_4P$ and from $A_4$ on $A_1P$. Prove that all four perpendiculars (or their extensions) intersect at one point.

**18.5.** On sides $CB$ and $CD$ of square $ABCD$ points $M$ and $K$ are taken so that the perimeter of triangle $CMK$ is equal to the doubled length of the square's side. Find the value of angle $\angle MAK$.

**18.6.** On the plane three squares (with same orientation) are given: $ABCD$, $AB_1C_1D_1$ and $A_2B_2CD_2$; the first square has common vertices $A$ and $C$ with the two other squares. Prove that median $BM$ of triangle $BB_1B_2$ is perpendicular to segment $D_1D_2$.

**18.7.** Triangle $ABC$ is given. On its sides $AB$ and $BC$ squares $ABMN$ and $BCPQ$ are constructed outwards. Prove that the centers of these squares and the midpoints of segments $MQ$ and $AC$ form a square.

**18.8.** A parallelogram is circumscribed about a square. Prove that the perpendiculars dropped from the vertices of the parallelograms to the sides of the square form a square.

## §2. Rotation by $60°$

**18.9.** On segment $AE$, on one side of it, equilateral triangles $ABC$ and $CDE$ are constructed; $M$ and $P$ are the midpoints of segments $AD$ and $BE$. Prove that triangle $CPM$ is an equilateral one.

**18.10.** Given three parallel lines. Construct an equilateral triangle so that its vertices belong to the given lines.

**18.11.** Geven a square, consider all possible equilateral triangles $PKM$ with fixed vertex $P$ and vertex $K$ belonging to the square. Find the locus of vertices $M$.

**18.12.** On sides $BC$ and $CD$ of parallelogram $ABCD$, equilateral triangles $BCP$ and $CDQ$ are constructed outwards. Prove that triangle $APQ$ is an equilateral one.

**18.13.** Point $M$ belongs to arc $\smile AB$ of the circle circumscribed about an equilateral triangle $ABC$. Prove that $MC = MA + MB$.

**18.14.** Find the locus of points $M$ that lie inside equilateral triangle $ABC$ and such that $MA^2 = MB^2 + MC^2$.

**18.15.** Hexagon $ABCDEF$ is a regular one, $K$ and $M$ are the midpoints of segments $BD$ and $EF$, respectively. Prove that triangle $AMK$ is an equilateral one.

**18.16.** Let $M$ and $N$ be the midpoints of sides $CD$ and $DE$, respectively, of regular hexagon $ABCDEF$, let $P$ be the intersection point of segments $AM$ and $BN$.

a) Find the value of the angle between lines $AM$ and $BN$.

b) Prove that $S_{ABP} = S_{MDNP}$.

**18.17.** On sides $AB$ and $BC$ of an equilateral triangle $ABC$ points $M$ and $N$ are taken so that $MN \parallel AC$; let $E$ be the midpoint of segment $AN$ and $D$ the center of mass of triangle $BMN$. Find the values of the angles of triangle $CDE$.

**18.18.** On the sides of triangle $ABC$ equilateral triangles $ABC_1$, $AB_1C$ and $A_1BC$ are constructed outwards. Let $P$ and $Q$ be the midpoints of segments $A_1B_1$ and $A_1C_1$. Prove that triangle $APQ$ is an equilateral one.

**18.19.** On sides $AB$ and $AC$ of triangle $ABC$ equilateral triangles $ABC'$ and $AB'C$ are constructed outwards. Point $M$ divides side $BC$ in the ratio of $BM : MC = 3 : 1$; points $K$ and $L$ are the midpoints of sides $AC'$ and $B'C$, respectively. Prove that the angles of triangle $KLM$ are equal to $30°$, $60°$ and $90°$.

**18.20.** Equilateral triangles $ABC$, $CDE$, $EHK$ (vertices are circumvent counterclockwise) are placed on the plane so that $\overrightarrow{AD} = \overrightarrow{DK}$. Prove that triangle $BHD$ is also an equilateral one.

**18.21.** a) Inside an acute triangle find a point the sum of distances from which to the vertices is the least one.

b) Inside triangle $ABC$ all the angles of which are smaller than $120°$ a point $O$ is taken; it serves as vertex of the angles of $120°$ that subtend the sides. Prove that the sum of distances from $O$ to the vertices is equal to $\frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}S$.

**18.22.** Hexagon $ABCDEF$ is inscribed in a circle of radius $R$ and $AB = CD = EF = R$. Prove that the midpoints of sides $BC$, $DE$ and $FA$ determine an equilateral triangle.

**18.23.** On sides of a convex centrally symmetric hexagon $ABCDEF$ equilateral triangles are constructed outwards. Prove that the midpoints of the segments connecting the vertices of neighbouring triangles determine a regular hexagon.

## §3. Rotations through arbitrary angles

**18.24.** Given points $A$ and $B$ and circle $S$ construct points $C$ and $D$ on $S$ so that $AC \parallel BD$ and the value of arc $\smile CD$ is a given quantity $\alpha$.

**18.25.** A rotation with center $O$ transforms line $l_1$ into line $l_2$ and point $A_1$ on $l_1$ into point $A_2$. Prove that the intersection point of lines $l_1$ and $l_2$ belongs to the circle circumscribed about triangle $A_1OA_2$.

**18.26.** Two equal letters $\Gamma$ lie on the plane. Denote by $A$ and $A'$ the endpoints of the shorter segments of these letters. Points $A_1$, ..., $A_{n-1}$ and $A'_1$, ..., $A'_{n-1}$ divide the longer segments into $n$ equal parts (the division points are numbered starting from the outer endpoints of longer segments). Lines $AA_i$ and $A'A'_i$ intersect at point $X_i$. Prove that points $X_1$, ..., $X_{n-1}$ determine a convex polygon.

**18.27.** Along two lines that intersect at point $P$ two points are moving with the same speed: point $A$ along one line and point $B$ along the other one. They pass $P$ not simultaneously. Prove that at all times the circle circumscribed about triangle $ABP$ passes through a fixed point distinct from $P$.

**18.28.** Triangle $A_1B_1C_1$ is obtained from triangle $ABC$ by a rotation through an angle of $\alpha$ ($\alpha < 180°$) about the center of its circumscribed circle. Prove that the intersection points of sides $AB$ and $A_1B_1$, $BC$ and $B_1C_1$, $CA$ and $C_1A_1$ (or their extensions) are the vertices of a triangle similar to triangle $ABC$.

**18.29.** Given triangle $ABC$ construct a line which divides the area and perimeter of triangle $ABC$ in halves.

**18.30.** On vectors $\overrightarrow{A_iB_i}$, where $i = 1, \ldots, k$ similarly oriented regular $n$-gons $A_iB_iC_iD_i \ldots$ ($n \geq 4$) are constructed (a given vector serving as a side). Prove that $k$-gons $C_1 \ldots C_k$ and $D_1 \ldots D_k$ are regular and similarly oriented ones if and only if the $k$-gons $A_1 \ldots A_k$ and $B_1 \ldots B_k$ are regular and similarly oriented ones.

**18.31.** Consider a triangle. Consider three lines symmetric through the triangles sides to an arbitrary line passing through the intersection point of the triangle's heights. Prove that the three lines intersect at one point.

**18.32.** A lion runs over the arena of a circus which is a disk of radius 10 m. Moving along a broken line the lion covered 30 km. Prove that the sum of all the angles of his turns is not less than 2998 radian.

## §4. Compositions of rotations

**18.33.** Prove that the composition of two rotations through angles whose sum is not proportional to $360°$ is a rotation. In which point is its center and what is the angle of the rotation equal to? Investigate also the case when the sum of the angles of rotations is a multiple of $360°$.

**\* \* \***

**18.34.** On the sides of an arbitrary convex quadrilateral squares are constructed outwards. Prove that the segments that connect the centers of opposite squares have equal lengths and are perpendicular to each other.

**18.35.** On the sides of a parallelogram squares are constructed outwards. Prove that their centers form a square.

**18.36.** On sides of triangle $ABC$ squares with centers $P$, $Q$ and $R$ are constructed outwards. On the sides of triangle $PQR$ squares are constructed inwards. Prove that their centers are the midpoints of the sides of triangle $ABC$.

**18.37.** Inside a convex quadrilateral $ABCD$ isosceles right triangles $ABO_1$, $BCO_2$, $CDO_3$ and $DAO_4$ are constructed. Prove that if $O_1 = O_3$, then $O_2 = O_4$.

**\* \* \***

**18.38.** a) On the sides of an arbitrary triangle equilateral triangles are constructed outwards. Prove that their centers form an equilateral triangle.

b) Prove a similar statement for triangles constructed inwards.

c) Prove that the difference of the areas of equilateral triangles obtained in headings a) and b) is equal to the area of the initial triangle.

**18.39.** On sides of triangle $ABC$ equilateral triangles $A'BC$ and $B'AC$ are constructed outwards and $C'AB$ inwards; $M$ is the center of mass of triangle $C'AB$. Prove that $A'B'M$ is an isosceles triangle such that $\angle A'MB' = 120°$.

**18.40.** Let angles $\alpha$, $\beta$, $\gamma$ be such that $0 < \alpha$, $\beta$, $\gamma < \pi$ and $\alpha + \beta + \gamma = \pi$. Prove that if the composition of rotations $R_C^{2\gamma} \circ R_B^{2\beta} \circ R_A^{2\alpha}$ is the identity transformation, then the angles of triangle $ABC$ are equal to $\alpha$, $\beta$, $\gamma$.

**18.41.** Construct an $n$-gon given $n$ points which are the vertices of isosceles triangles constructed on the sides of this $n$-gon and such that the angles of these triangles at the vertices are equal to $\alpha_1$, $\ldots$, $\alpha_n$.

**18.42.** On the sides of an arbitrary triangle $ABC$ isosceles triangles $A'BC$, $AB'C$ and $ABC'$ are constructed outwards with angles $\alpha$, $\beta$ and $\gamma$ at vertices $A'$, $B'$ and $C'$, respectively, such that $\alpha + \beta + \gamma = 2\pi$. Prove that the angles of triangle $A'B'C'$ are equal to $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}\gamma$.

**18.43.** Let $AKL$ and $AMN$ be similar isosceles triangles with vertex $A$ and angle $\alpha$ at the vertex; $GNK$ and $G'LM$ similar isosceles triangles with angle $\pi - \alpha$ at the vertex. Prove that $G = G'$. (All the triangles are oriented ones.)

**18.44.** On sides $AB$, $BC$ and $CA$ of triangle $ABC$ points $P$, $Q$ and $R$, respectively, are taken. Prove that the centers of the circles circumscribed about triangles $APR$, $BPQ$ and $CQR$ constitute a triangle similar to triangle $ABC$.

## Problems for independent study

**18.45.** On the plane, the unit circle with center at $O$ is drawn. Two neighbouring vertices of a square belong to this circle. What is the maximal distance from point $O$ that the two other of the square's vertices can have?

**18.46.** On the sides of convex quadrilateral $ABCD$, equilateral triangles $ABM$, $CDP$ are constructed outwards and $BCN$, $ADK$ inwards. Prove that $MN = AC$.

**18.47.** On the sides of a convex quadrilateral $ABCD$, squares with centers $M$, $N$, $P$, $Q$ are constructed outwards. Prove that the midpoints of the diagonals of quadrilaterals $ABCD$ and $MNPQ$ form a square.

**18.48.** Inside an equilateral triangle $ABC$ lies point $O$. It is known that $\angle AOB = 113°$, $\angle BOC = 123°$. Find the angles of the triangle whose sides are equal to segments $OA$, $OB$, $OC$.

**18.49.** On the plane, there are drawn $n$ lines ($n > 2$) so that no two of them are parallel and no three intersect at one point. It is known that it is possible to rotate the plane about a point $O$ through an angle of $\alpha$ ($\alpha < 180°$) so that each of the drawn lines coincides with some other of the drawn lines. Indicate all $n$ for which this is possible.

**18.50.** Ten gears of distinct shapes are placed so that the first gear is meshed with the second one, the second one with the third one, etc., the tenth is meshed with the first one. Is it possible for such a system to rotate? Can a similar system of 11 gears rotate?

**18.51.** Given a circle and a point. a) Construct an equilateral triangle whose heights intersect at the given point and two vertices belong to the given circle.

b) Construct a square two vertices of which belong to the given circle and the diagonals intersect at the given point.

## Solutions

**18.1.** Let us rotate square $ABCD$ about point $A$ through $90°$ so that $B$ turns into $D$. This rotation sends point $M$ into point $M'$ and point $K$ into point $K'$. It is clear that $\angle BMA = \angle DM'A$. Since $\angle MAK = \angle MAB = \angle M'AD$, it follows that $\angle MAD = \angle M'AK$. Therefore,

$$\angle MA'K = \angle MAD = \angle BMA = \angle DM'A.$$

Hence, $AK = KM' = KD + DM' = KD + BM$.

**18.2.** Under the rotation through $90°$ about point $P$ lines $PA_1$, $PB_1$, $PM_1$ and $CH$ turn into lines parallel to $CA$, $CB$, $CM$ and $AB$, respectively. It follows that under such a rotation of triangle $PA_1B_1$ segment $PM_1$ turns into a median of the (rotated) triangle.

**18.3.** Consider a rotation through $90°$ about point $B$ which sends vertex $K$ into vertex $N$ and vertex $C$ into $A$. This rotation sends point $A$ into point $A'$ and point $E$ into point $E'$. Since $E'$ and $B$ are the midpoints of sides $A'N$ and $A'C$ of triangle $A'NC$, it follows that $BE' \parallel NC$. But $\angle EBE' = 90°$ and, therefore, $BE \perp NC$.

**18.4.** A rotation through an angle of $90°$ about the center of the square sends point $A_1$ to point $A_2$. This rotation sends the perpendiculars dropped from points $A_1$, $A_2$, $A_3$ and $A_4$ into lines $A_2P$, $A_3P$, $A_4P$ and $A_1P$, respectively. Therefore, the intersection point is the image of point $P$ under the inverse rotation.

**18.5.** Let us turn the given square through an angle of $90°$ about point $A$ so that vertex $B$ would coincide with $D$. Let $M'$ be the image of $M$ under this rotation. Since by the hypothesis

$$MK + MC + CK = (BM + MC) + (KD + CK),$$

it follows that $MK = BM + KD = DM' + KD = KM'$. Moreover, $AM = AM'$; hence, $\triangle AMK = \triangle AM'K$, consequently, $\angle MAK = \angle M'AK = \frac{1}{2}\angle MAM' = 45°$.

**18.6.** Let $R$ be the rotation through an angle of $90°$ that sends $\overrightarrow{BC}$ to $\overrightarrow{BA}$. Further, let $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CB_2} = \mathbf{b}$ and $\overrightarrow{AB_1} = \mathbf{c}$. Then $\overrightarrow{BA} = R\mathbf{a}$, $\overrightarrow{D_2C} = R\mathbf{b}$ and $\overrightarrow{AD_1} = R\mathbf{c}$. Hence, $\overrightarrow{D_2D_1} = R\mathbf{b} - \mathbf{a} + R\mathbf{a} + R\mathbf{c}$ and $2\overrightarrow{BM} = \mathbf{a} + \mathbf{b} + R\mathbf{a} + \mathbf{c}$. Therefore, $R(2\overrightarrow{BM}) = \overrightarrow{D_2D_1}$ because $R(R\mathbf{a}) = -\mathbf{a}$.

**18.7.** Let us introduce the following notations: $\mathbf{a} = \overrightarrow{BM}$, $\mathbf{b} = \overrightarrow{BC}$; let $R\mathbf{a}$ and $R\mathbf{b}$ be the vectors obtained from vectors $\mathbf{a}$ and $\mathbf{b}$ under a rotation through an angle of $90°$, i.e., $R\mathbf{a} = \overrightarrow{BA}$, $R\mathbf{b} = \overrightarrow{BQ}$. Let $O_1$, $O_2$, $O_3$ and $O_4$ be the midpoints of segments $AM$, $MQ$, $QC$

and $CA$, respectively. Then

$$\overrightarrow{BO_1} = \frac{(\mathbf{a} + R\mathbf{a})}{2}, \quad \overrightarrow{BO_2} = \frac{(\mathbf{a} + R\mathbf{b})}{2},$$

$$\overrightarrow{BO_3} = \frac{(\mathbf{b} - R\mathbf{b})}{2}, \quad \overrightarrow{BO_4} = \frac{(\mathbf{b} + R\mathbf{a})}{2}.$$

Therefore, $\overrightarrow{O_1O_2} = \frac{1}{2}(R\mathbf{b} - R\mathbf{a}) = -\overrightarrow{O_3O_4}$ and $\overrightarrow{O_2O_3} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = -\overrightarrow{O_4O_1}$. Moreover, $\overrightarrow{O_1O_2} = R(\overrightarrow{O_2O_3})$.

**18.8.** Parallelogram $A_1B_1C_1D_1$ is circumscribed around square $ABCD$ so that point $A$ belongs to side $A_1B_1$, $B$ to side $B_1C_1$, etc. Let us drop perpendiculars $l_1$, $l_2$, $l_3$ and $l_4$ from vertices $A_1$, $B_1$, $C_1$ and $D_1$, respectively to the sides of the square. To prove that these perpendiculars form a square, it suffices to verify that under a rotation through an angle of 90° about the center $O$ of square $ABCD$ lines $l_1$, $l_2$, $l_3$ and $l_4$ turn into each other. Under the rotation about $O$ through an angle of 90° points $A_1$, $B_1$, $C_1$ and $D_1$ turn into points $A_2$, $B_2$, $C_2$ and $D_2$ (Fig. 21).
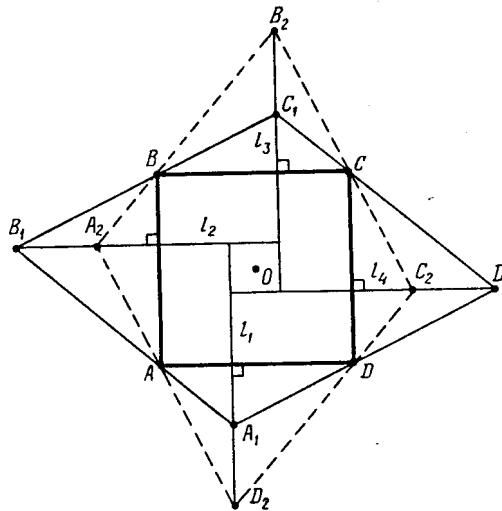


Figure 163 (Sol. 18.8)

Since $AA_2 \perp B_1B$ and $BA_2 \perp B_1A$, it follows that $B_1A_2 \perp AB$. This means that line $l_1$ turns under the rotation through an angle of 90° about $O$ into $l_2$. For the other lines the proof is similar.

**18.9.** Let us consider a rotation through an angle of 60° about point $C$ that turns $E$ into $D$. Under this rotation $B$ turns into $A$, i.e., segment $BE$ turns into $AD$. Therefore, the midpoint $P$ of segment $BE$ turns into the midpoint $M$ of segment $AD$, i.e., triangle $CPM$ is an equilateral one.

**18.10.** Suppose that we have constructed triangle $ABC$ so that its vertices $A$, $B$ and $C$ lie on lines $l_1$, $l_2$ and $l_3$, respectively. Under the rotation through an angle of 60° with center $A$ point $B$ turns into point $C$ and, therefore, $C$ is the intersection point of $l_3$ and the image of $l_2$ under the rotation through an angle of 60° about $A$.

**18.11.** The locus to be found consists of two squares obtained from the given one by rotations through angles of $\pm 60°$ about $P$.

**18.12.** Under the rotation through an angle of 60° vectors $\overrightarrow{QC}$ and $\overrightarrow{CP}$ turn into $\overrightarrow{QD}$ and $\overrightarrow{CB} = \overrightarrow{DA}$, respectively. Therefore, under this rotation vector $\overrightarrow{QP} = \overrightarrow{QC} + \overrightarrow{CP}$ turns into vector $\overrightarrow{QD} + \overrightarrow{DA} = \overrightarrow{QA}$.

**18.13.** Let $M'$ be the image of $M$ under the rotation through an angle of 60° about $B$ that turns $A$ into $C$. Then $\angle CM'B = \angle AMB = 120°$. Triangle $MM'B$ is an equilateral one and, therefore, $\angle BM'M = 60°$. Since $\angle CM'B + \angle BM'M = 180°$, point $M'$ belongs to segment $MC$. Therefore, $MC = MM' + M'C = MB + MA$.

**18.14.** Under the rotation through an angle of 60° about $A$ sending $B$ to $C$ point $M$ turns into point $M'$ and point $C$ into point $D$. The equality $MA^2 = MB^2 + MC^2$ is equivalent to the equality $M'M^2 = M'C^2 + MC^2$, i.e., $\angle MCM' = 90°$ and, therefore,

$$\angle MCB + \angle MBC = \angle MCB + \angle M'CD = 120° - 90° = 30°$$

that is $\angle BMC = 150°$. The locus to be found is the arc of the circle situated inside the triangle and such that the pionts of the arc serve as vertices of angles of 150° subtending segment $BC$.

**18.15.** Let $O$ be the center of a hexagon. Consider a rotation about $A$ through an angle of 60° sending $B$ to $O$. This rotation sends segment $OC$ into segment $FE$. Point $K$ is the midpoint of diagonal $BD$ of parallelogram $BCDO$ because it is the midpoint of diagonal $CO$. Therefore, point $K$ turns into $M$ under our rotation; in other words, triangle $AMK$ is an equilateral one.

**18.16.** There is a rotation through an angle of 60° about the center of the given hexagon that sends $A$ into $B$. It sends segment $CD$ into $DE$ and, therefore, sends $M$ into $N$. Therefore, this rotation sends $AM$ into $BN$, that is to say, the angle between these segments is equal to 60°. Moreover, this rotation turns pentagon $AMDEF$ into $BNEFA$; hence, the areas of the pentagons are equal. Cutting from these congruent pentagons the common part, pentagon $APNEF$, we get two figures of the same area: triangle $ABP$ and quadrilateral $MDNP$.

**18.17.** Consider the rotation through an angle of 60° about $C$ sending $B$ to $A$. It sends points $M$, $N$ and $D$ into $M'$, $N'$ and $D'$, respectively. Since $AMNN'$ is a parallelogram, the midpoint $E$ of diagonal $AN$ is its center of symmetry. Therefore, under the symmetry through point $E$ triangle $BMN$ turns into $M'AN'$ and, therefore, $D$ turns into $D'$. Hence, $E$ is the midpoint of segment $DD'$. Since triangle $CDD'$ is an equilateral one, the angles of triangle $CDE$ are equal to 30°, 60° and 90°.

**18.18.** Consider a rotation about $A$ sending point $C_1$ into $B$. Under this rotation equilateral triangle $A_1BC$ turns into triangle $A_2FB_1$ and segment $A_1C_1$ into segment $A_2B$. It remains to notice that $BA_1A_2B_1$ is a parallelogram, i.e., the midpoint of segment $A_2B$ coincides with the midpoint of segment $A_1B_1$.

**18.19.** Let $\overrightarrow{AB} = 4\mathbf{a}$, $\overrightarrow{CA} = 4\mathbf{b}$. Further, let $R$ be the rotation sending vector $\overrightarrow{AB}$ into $\overrightarrow{AC'}$ (and, therefore, sending $\overrightarrow{CA}$ into $\overrightarrow{CB'}$). Then $\overrightarrow{LM} = (\mathbf{a} + \mathbf{b}) - 2R\mathbf{b}$ and $\overrightarrow{LK} = -2R\mathbf{b} + 4\mathbf{b} + 2R\mathbf{a}$. It is easy to verify that $\mathbf{b} + R^2\mathbf{b} = R\mathbf{b}$. Hence, $2R(\overrightarrow{LM}) = \overrightarrow{LK}$ which implies the required statement.

**18.20.** Under the rotation about point $C$ through an angle of 60° counterclockwise point $A$ turns into $B$ and $D$ into $E$ and, therefore, vector $\overrightarrow{DK} = \overrightarrow{AD}$ turns into $\overrightarrow{BE}$. Since the rotation about point $H$ through an angle of 60° counterclockwise sends $K$ into $E$ and $\overrightarrow{DK}$ into $\overrightarrow{BE}$, it sends $D$ into $B$ which means that triangle $BHD$ is an equilateral one.

**18.21.** a) Let $O$ be a point inside triangle $ABC$. The rotation through an angle of 60° about $A$ sends $B$, $C$ and $O$ into some points $B'$, $C'$ and $O'$, respectively, see Fig. 22. Since
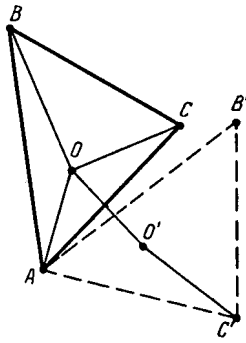
Figure 164 (Sol. 18.21)

$AO = OO'$ and $OC = O'C'$, we have:

$$BO + AO + CO = BO + OO' + O'C'.$$

The length of the broken line $BOO'C'$ is minimal if and only if this broken line is a segment, i.e., if $\angle AOB = \angle AO'C' = \angle AOC = 120°$. To construct the desired point, we can make use of the result of Problem 2.8.

b) The sum of distances from $O$ to the vertices is equal to the length of segment $BC'$ obtained in heading a). It is also clear that

$$(BC')^2 = b^2 + c^2 - 2bc\cos(\alpha + 60°) =$$
$$b^2 + c^2 - bc\cos\alpha + bc\sqrt{3}\sin\alpha =$$
$$\tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}S.$$

**18.22.** Let $P$, $Q$ and $R$ be the midpoints of sides $BC$, $DE$ and $FA$; let $O$ be the center of the circumscribed circle. Suppose that triangle $PQR$ is an equilateral one. Let us prove then that the midpoints of sides $BC$, $DE'$ and $F'A$ of hexagon $ABCDE'F'$ in which vertices $E'$ and $F'$ are obtained from vertices $E$ and $F$ after a rotation through an angle about point $O$ also form an equilateral triangle.

This will complete the proof since for a regular hexagon the midpoints of sides $BC$, $DE$ and $FA$ constitute an equilateral triangle and any of the considered hexagons can be obtained from a regular one with the help of rotations of triangles $OCD$ and $OEF$.
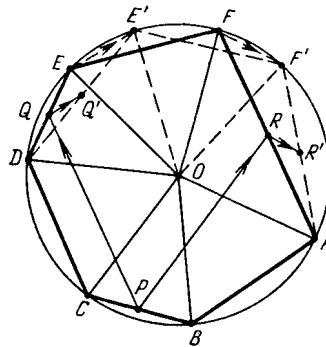


Figure 165 (Sol. 18.22)

Let $Q'$ and $R'$ be the midpoints of sides $DE'$ and $AF'$, see Fig. 23. Under the rotation through an angle of 60° vector $\overrightarrow{EE'}$ turns into $\overrightarrow{FF'}$. Since $\overrightarrow{QQ'} = \tfrac{1}{2}\overrightarrow{EE'}$ and $\overrightarrow{RR'} = \tfrac{1}{2}\overrightarrow{FF'}$, this rotation sends $\overrightarrow{QQ'}$ into $\overrightarrow{RR'}$. By hypothesis, triangle $PQR$ is an equilateral one, i.e., under the rotation through an angle of 60° vector $\overrightarrow{PQ}$ turns into $\overrightarrow{PR}$. Therefore, vector

$\overrightarrow{PQ'} = \overrightarrow{PQ} + \overrightarrow{QQ'}$ turns into vector $\overrightarrow{PR'} = \overrightarrow{PR} + \overrightarrow{RR'}$ under a rotation through an angle of $60°$. This means that triangle $PQ'R'$ is an equilateral one.

**18.23.** Let $K$, $L$, $M$ and $N$ be vertices of equilateral triangles constructed (wherewards?) on sides $BC$, $AB$, $AF$ and $FE$, respectively; let also $B_1$, $A_1$ and $F_1$ be the midpoints of segments $KL$, $LM$ and $MN$ (see Fig. 24).
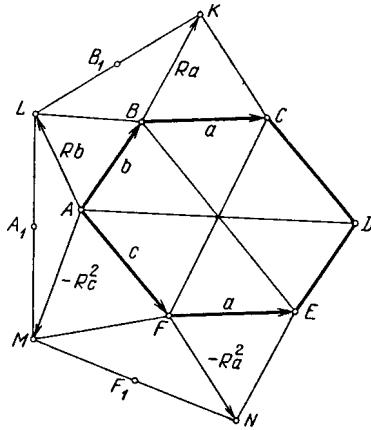


Figure 166 (Sol. 18.23)

Further, let $\mathbf{a} = \overrightarrow{BC} = \overrightarrow{FE}$, $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AF}$; let $R$ be the rotation through an angle of $60°$ that sends $\overrightarrow{BC}$ into $\overrightarrow{BK}$. Then $\overrightarrow{AM} = -R^2\mathbf{c}$ and $\overrightarrow{FN} = -R^2\mathbf{a}$. Therefore, $2\overrightarrow{A_1B_1} = R^2\mathbf{c} + R\mathbf{a} + \mathbf{b}$ and $2\overrightarrow{F_1A_1} = R^2\mathbf{a} - \mathbf{c} + R\mathbf{b}$, i.e., $\overrightarrow{F_1A_1} = R(\overrightarrow{A_1B_1})$.

**18.24.** Suppose a rotation through an angle of $\alpha$ about the center of circle $S$ sends $C$ into $D$. This rotation sends point $A$ into point $A'$. Then $\angle(BD, DA') = \alpha$, i.e., point $D$ belongs to the arc of the circle whose points serve as vertices of the angles of $\alpha$ that subtend segment $A'B$.

**18.25.** Let $P$ be the intersection point of lines $l_1$ and $l_2$. Then

$$\angle(OA_1, A_1P) = \angle(OA_1, l_1) = \angle(OA_2, l_2) = \angle(OA_2, A_2P).$$

Therefore, points $O$, $A_1$, $A_2$ and $P$ belong to one circle.

**18.26.** It is possible to identify similar letters $\Gamma$ after a rotation about $O$ (unless they can be identified by a parallel translation in which case $AA_i \parallel A'A_i'$). Thanks to Problem 18.25 point $X_i$ belongs to the circle circumscribed about triangle $A'OA$. It is clear that the points that belong to one circle constitute a convex polygon.

**18.27.** Let $O$ be the center of rotation $R$ that sends segment $A(t_1)A(t_2)$ into segment $B(t_1)B(t_2)$, where $t_1$ and $t_2$ are certain time moments. Then this rotation sends $A(t)$ into $B(t)$ at any moment $t$. Therefore, by Problem 18.25 point $O$ belongs to the circle circumscribed about triangle $APB$.

**18.28.** Let $A$ and $B$ be points on the circle with center $O$; let $A_1$ and $B_1$ be the images of these points under the rotation through an angle of $\alpha$ about $O$. Let $P$ and $P_1$ be the midpoints of segments $AB$ and $A_1B_1$; let $M$ be the intersection point of lines $AB$ and $A_1B_1$. The right triangles $POM$ and $P_1OM$ have a common hypothenuse and equal legs $PO = P_1O$, therefore, these triangles are equal and $\angle MOP = \angle MOP_1 = \frac{1}{2}\alpha$. Point $M$ is obtained from point $P$ under a rotation through an angle of $\frac{1}{2}\alpha$ and a subsequent homothety with coefficient $\frac{1}{\cos(\frac{1}{2}\alpha)}$ and center $O$.

The intersection points of lines $AB$ and $A_1B_1$, $AC$ and $A_1C_1$, $BC$ and $B_1C_1$ are the vertices of a triangle which is homothetic with coefficient $\frac{1}{\cos(\frac{1}{2}\alpha)}$ to the triangle determined

by the midpoints of the sides of triangle $ABC$. It is clear that the triangle determined by the midpoints of the sides of triangle $ABC$ is similar to triangle $ABC$.

**18.29.** By Problem 5.50 the line which divides in halves both the area and the perimeter of a triangle passes through the center of its inscribed circle. It is also clear that if the line passes through the center of the inscribed circle of a triangle and divides its perimeter in halves, then it divides in halves its area as well. Therefore, we have to draw a line passing through the center of the inscribed circle of the triangle and dividing its perimeter in halves.

Suppose we have constructed points $M$ and $N$ on sides $AB$ and $AC$ of triangle $ABC$ so that line $MN$ passes through the center $O$ of the inscribed circle and divides the perimeter of the triangle in halves. On ray $AC$ construct point $D$ so that $AD = p$, where $p$ is a semiperimeter of triangle $ABC$. Then $AM = ND$. Let $Q$ be the center of rotation $R$ that sends segment $AM$ into segment $DN$ (so that $A$ goes to $D$ and $M$ to $N$). Since the angle between lines $AM$ and $CN$ is known, it is possible to construct $Q$: it is the vertex of isosceles triangle $AQD$, where $\angle AQD = 180° - \angle A$ and points $B$ and $Q$ lie on one side of line $AD$. The rotation $R$ sends segment $OM$ into segment $O'N$. We can now construct point $O'$. Clearly, $\angle ONO' = \angle A$ because the angle between lines $OM$ and $O'N$ is equal to $\angle A$. Therefore, point $N$ is the intersection point of line $AC$ and the arc of the circle whose points serve as vertices for the angles equal to $\angle A$ that subtend segment $OO'$. Constructing point $N$, draw line $ON$ and find point $M$.

It is easy to verify that if the constructed points $M$ and $N$ belong to sides $AB$ and $AC$, then $MN$ is the desired line. The main point of the proof is the proof of the fact that the rotation about $Q$ through an angle of $180° - \angle A$ sends $M$ into $N$. To prove this fact, one has to make use of the fact that $\angle ONO' = \angle A$, i.e., this rotation sends line $OM$ into line $O'N$.

**18.30.** Suppose that the $k$-gons $C_1 \ldots C_k$ and $D_1 \ldots D_k$ are regular and similarly oriented. Let $C$ and $D$ be the centers of these $k$-gons; let $\mathbf{c}_i = \overrightarrow{CC_i}$ and $\mathbf{d}_i = \overrightarrow{DD_i}$. Then

$$\overrightarrow{C_iD_i} = \overrightarrow{C_iC} + \overrightarrow{CD} + \overrightarrow{DD_i} = -\mathbf{c}_i + \overrightarrow{CD} + \mathbf{d}_i.$$

The rotation $R^\varphi$, where $\varphi$ is the angle at a vertex of a regular $n$-gon, sends $\overrightarrow{C_iD_i}$ into $\overrightarrow{C_iB_i}$. Therefore,

$$\overrightarrow{XB_i} = \overrightarrow{XC} + \mathbf{c}_i + \overrightarrow{C_iB_i} = \overrightarrow{XC} + \mathbf{c}_i + R^\varphi(-\mathbf{c}_i + \overrightarrow{CD} + \mathbf{d}_i).$$

Let us select point $X$ so that $\overrightarrow{XC} + R^\varphi(\overrightarrow{CD}) = \overrightarrow{0}$. Then $\overrightarrow{XB_i} = \mathbf{c}_i + R^\varphi(\mathbf{d}_i - \mathbf{c}_i) = R^{i\psi}\mathbf{u}$, where $\mathbf{u} = \mathbf{c}_k + R^\varphi(\mathbf{d}_k - \mathbf{c}_k)$ and $R^\psi$ is the rotation sending $\mathbf{c}_k$ to $\mathbf{c}_1$. Hence, $B_1 \ldots B_k$ is a regular $k$-gon with center $X$.

We similarly prove that $A_1 \ldots A_k$ is a regular $k$-gon.

The converse statement is similarly proved.

**18.31.** Let $H$ be the intersection point of heights of triangle $ABC$; let $H_1$, $H_2$ and $H_3$ be points symmetric to $H$ through sides $BC$, $CA$ and $AB$, respectively. Points $H_1$, $H_2$ and $H_3$ belong to the circle circumscribed about triangle $ABC$ (Problem 5.9). Let $l$ be a line passing through $H$. The line symmetric to $l$ through $BC$ (resp. through $CA$ and $AB$) intersects the circumscribed circle at point $H_1$ (resp. $H_2$ and $H_3$) and at a point $P_1$ (resp. $P_2$ and $P_3$).

Consider another line $l'$ passing through $H$. Let $\varphi$ be the angle between $l$ and $l'$. Let us construct points $P_1'$, $P_2'$ and $P_3'$ for line $l'$ in the same way as points $P_1$, $P_2$ and $P_3$ were constructed for line $l$. Then $\angle P_iH_iP_i' = \varphi$, i.e., the value of arc $\smile P_iP_i'$ is equal to $2\varphi$ (the direction of the rotation from $P_i$ to $P_i'$ is opposite to that of the rotation from $l$ to $l'$). Therefore, points $P_1'$, $P_2'$ and $P_3'$ are the images of points $P_1$, $P_2$ and $P_3$ under a certain rotation. It is clear that if for $l'$ we take the height of the triangle dropped from vertex $A$, then $P_1' = P_2' = P_3' = A$, and, therefore, $P_1 = P_2 = P_3$.

**18.32.** Suppose that the lion ran along the broken line $A_1 A_2 \ldots A_n$. Let us rectify the lion's trajectory as follows. Let us rotate the arena of the circus and all(?) the further trajectory about point $A_2$ so that point $A_3$ would lie on ray $A_1 A_2$. Then let us rotate the arena and the further trajectory about point $A_3$ so that point $A_4$ were on ray $A_1 A_2$, and so on. The center $O$ of the arena turns consecutively into points $O_1 = O$, $O_2$, $\ldots$, $O_{n-1}$; and points $A_1$, $\ldots$, $A_n$ into points $A'_1$, $\ldots$, $A'_n$ all on one line (Fig. 25).
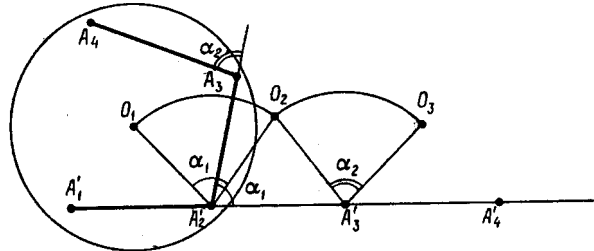


Figure 167 (Sol. 18.32)

Let $\alpha_{i-1}$ be the angle of through which the lion turned at point $A'_i$. Then $\angle O_{i-1} A'_i O_i = \alpha_{i-1}$ and $A'_i O_{i-1} = A'_i O_i \leq 10$; hence, $O_i O_{i-1} \leq 10 \alpha_{i-1}$. Hence,

$$30000 = A'_1 A'_n \leq A'_1 O_1 + O_1 O_2 + \cdots + O_{n-2} O_{n-1} + O_{n-1} A'_n \leq$$
$$10 + 10(\alpha_1 + \cdots + \alpha_{n-2}) + 10$$

i.e., $\alpha_1 + \cdots + \alpha_{n-2} \geq 2998$.

**18.33.** Consider the composition of the rotations $R^\beta_B \circ R^\alpha_A$. If $A = B$, then the statement of the problem is obvious and, therefore, let us assume that $A \neq B$. Let $l = AB$; let lines $a$ and $b$ pass through points $A$ and $B$, respectively, so that $\angle(a, l) = \frac{1}{2}\alpha$ and $\angle(l, b) = \frac{1}{2}\beta$. Then

$$R^\beta_B \circ R^\alpha_A = S_b \circ S_l \circ S_l \circ S_a = S_b \circ S_a.$$

If $a \parallel b$, then $S_a \circ S_b = T_{2\mathbf{u}}$, where $T_{\mathbf{u}}$ is a parallel translation sending $a$ into $b$ and such that $\mathbf{u} \perp a$. If lines $a$ and $b$ are not parallel and $O$ is their intersection point, then $S_a \circ S_b$ is the rotation through an angle of $\alpha + \beta$ with center $O$. It is also clear that $a \parallel b$ if and only if $\frac{1}{2}\alpha + \frac{1}{2}\beta = k\pi$, i.e., $\alpha + \beta = 2k\pi$.

**18.34.** Let $P$, $Q$, $R$ and $S$ be the centers of squares constructed outwards on sides $AB$, $BC$, $CD$ and $DA$, respectively. On segments $QR$ and $SP$, construct inwards isosceles right triangles with vertices $O_1$ and $O_2$. Then $D = R^{90°}_R \circ R^{90°}_Q(B) = R^{180°}_{O_1}(B)$ and $B = R^{90°}_P \circ R^{90°}_S(D) = R^{180°}_{O_2}(D)$, i.e., $O_1 = O_2$ is the midpoint of segment $BD$.

The rotation through an angle of 90° about point $O = O_1 = O_2$ that sends $Q$ into $R$ sends point $S$ into $P$, i.e., it sends segment $QS$ into $RP$ and, therefore, these segments are equal and perpendicular to each other.

**18.35.** Let $P$, $Q$, $R$ and $S$ be the centers of squares constructed outwards on the sides $AB$, $BC$, $CD$ and $DA$ of parallelogram $ABCD$. By the previous problem $PR = QS$ and $PR \perp QS$. Moreover, the center of symmetry of parallelogram $ABCD$ is the center of symmetry of quadrilateral $PQRS$. This means that $PQRS$ is a parallelogram with equal and perpendicular diagonals, hence, a square.

**18.36.** Let $P$, $Q$ and $R$ be the centers of squares constructed outwards on sides $AB$, $BC$ and $CA$. Let us consider a rotation through an angle of 90° with center $R$ that sends $C$ to $A$. Under the rotation about $P$ through an angle of 90° in the same direction point $A$ turns into $B$. The composition of these two rotations is a rotation through an angle of 180° and,

therefore, the center of this rotation is the midpoint of segment $BC$. On the other hand, the center of this rotation is a vertex of an isoscceles right triangle with base $PR$, i.e., it is the center of a square constructed on $PR$. This square is constructed inwards on a side of triangle $PQR$.

**18.37.** If $O_1 = O_3$, then $R_D^{90°} \circ R_C^{90°} \circ R_B^{90°} \circ R_A^{90°} = R_{O_3}^{180°} \circ R_{O_1}^{180°} = E$. Therefore,

$$E = R_A^{90°} \circ E \circ R_A^{-90°} = R_A^{90°} \circ R_D^{90°} R_C^{90°} \circ R_B^{90°} = R_{O_4}^{180°} \circ R_{O_2}^{180°},$$

where $E$ is the identity transformation, i.e., $O_4 = O_2$.

**18.38.** a) See solution of a more general Problem 18.42 (it suffices to set $\alpha = \beta = \gamma = 120°$). In case b) proof is analogous.

b) Let $Q$ and $R$ (resp. $Q_1$ and $R_1$) be the centers of equilateral triangles constructed outwards (resp. inwards) on sides $AC$ and $AB$. Since $AQ = \frac{1}{\sqrt{3}}b$, $AR = \frac{1}{\sqrt{3}}c$ and $\angle QAR = 60° + \alpha$, it follows that $3QR^2 = b^2 + c^2 - 2bc\cos(\alpha + 60°)$. Similarly, $3Q_1R_1^2 = b^2 + c^2 - 2bc\cos(\alpha - 60°)$. Therefore, the difference of areas of the obtained equilateral triangles is equal to

$$\frac{(QR^2 - Q_1R_1^2)\sqrt{3}}{4} = \frac{bc\sin\alpha\sin 60°}{\sqrt{3}} = S_{ABC}.$$

**18.39.** The combination of a rotation through an angle of $60°$ about $A'$ that sends $B$ to $C$, a rotation through an angle of $60°$ about $B'$ that sends $C$ to $A$ and a rotation through an angle of $120°$ about $M$ that sends $A$ to $B$ has $B$ as a fixed point. Since the first two rotations are performed in the direction opposite to the direction of the last rotation, it follows that the composition of these rotations is a parallel translation with a fixed point, i.e., the identity transformation:

$$R_M^{-120°} \circ R_{B'}^{60°} \circ R_{A'}^{60°} = E.$$

Therefore, $R_{B'}^{60°} \circ R_{A'}^{60°} = R_M^{120°}$, i.e., $M$ is the center of the rotation $R_{B'}^{60°} \circ R_{A'}^{60°}$. It follows that $\angle MA'B' = \angle MB'A' = 30°$, i.e., $A'B'M$ is an isosceles triangle and $\angle A'MB' = 120°$.

**18.40.** The conditions of the problem imply that $R_C^{-2\gamma} = R_B^{2\beta} \circ R_A^{2\alpha}$, i.e., point $C$ is the center of the composition of rotations $R_B^{2\beta} \circ R_A^{2\alpha}$. This means that $\angle BAC = \alpha$ and $\angle ABC = \beta$ (see Problem 18.33). Therefore, $\angle ACB = \pi - \alpha - \beta = \gamma$.

**18.41.** Denote the given points by $M_1, \ldots, M_n$. Suppose that we have constructed polygon $A_1 A_2 \ldots A_n$ so that triangles $A_1 M_1 A_2$, $A_2 M_2 A_3$, $\ldots$, $A_n M_n A_1$ are isosceles, where $\angle A_i M_i A_{i+1} = \alpha_i$ and the sides of the polygon are bases of these isosceles triangles. Clearly, $R_{M_n}^{\alpha_n} \circ \cdots \circ R_{M_1}^{\alpha_1}(A_1) = A_1$. If $\alpha_1 + \cdots + \alpha_n \neq k \cdot 360°$, then point $A_1$ is the center of the rotation $R_{M_n}^{\alpha_n} \circ \cdots \circ R_{M_1}^{\alpha_1}$.

We can construct the center of the composition of rotations. The construction of the other vertices of the polygon is done in an obvious way. If $\alpha_1 + \cdots + \alpha_n = k \cdot 360°$, then the problem is ill-posed: either an arbitrary point $A_1$ determines a polygon with the required property or there are no solutions.

**18.42.** Since $R_{C'}^{\gamma} \circ R_{B'}^{\beta} \circ R_{A'}^{\alpha}(B) = R_{C'}^{\gamma} \circ R_{B'}^{\beta}(C) = R_{C'}^{\gamma}(A) = B$, it follows that $B$ is a fixed point of the composition $R_{C'}^{\gamma} \circ R_{B'}^{\beta} \circ R_{A'}^{\alpha}$. Since $\alpha + \beta + \gamma = 2\pi$, it follows that this composition is a parallel translation with a fixed point, i.e., the identity transformation. It remains to make use of the result of Problem 18.40.

**18.43.** Since $R_{G'}^{\pi-\alpha} \circ R_A^{\alpha}(N) = L$ and $R_G^{\pi-\alpha} \circ R_A^{\alpha}(L) = N$, it follows that the transformations $R_{G'}^{\pi-\alpha} \circ R_A^{\alpha}$ and $R_G^{\pi-\alpha} \circ R_A^{\alpha}$ are central symmetries with respect to the midpoint of segment $LN$, i.e., $R_{G'}^{\pi-\alpha} \circ R_A^{\alpha} = R_G^{\pi-\alpha} \circ R_A^{\alpha}$. Therefore, $R_{G'}^{\pi-\alpha} = R_G^{\pi-\alpha}$ and $G' = G$.

**18.44.** Let $A_1$, $B_1$ and $C_1$ be the centers of the circumscribed circles of triangles $APR$, $BPQ$ and $CQR$. Under the successive rotations with centers $A_1$, $B_1$ and $C_1$ through angles

$2\alpha$, $2\beta$ and $2\gamma$ point $R$ turns first into $P$, then into $Q$, and then returns home. Since $2\alpha+2\beta+2\gamma = 360°$, the composition of the indicated rotations is the identity transformation. It follows that the angles of triangle $A_1B_1C_1$ are equal to $\alpha$, $\beta$ and $\gamma$ (see Problem 18.40).

# Chapter 19. HOMOTHETY AND ROTATIONAL HOMOTHETY

## Background

1. A *homothety* is a transformation of the plane sending point $X$ into point $X'$ such that $\overrightarrow{OX'} = k\overrightarrow{OX}$, where point $O$ and the number $k$ are fixed. Point $O$ is called *the center of homothety* and the number $k$ *the coefficient of homothety*.

We will denote the homothety with center $O$ and coefficient $k$ by $H_O^k$.

2. Two figures are called *homothetic* if one of them turns into the other one under a homothety.

3. A *rotational homothety* is the composition of a homothety and a rotation with a common center. The order of the composition is inessential since $R_O^\varphi \circ H_O^k = H_O^k \circ R_O^\varphi$.

We may assume that the coefficient of a rotational homothety is positive since $R_O^{180°} \circ H_O^k = H_O^{-k}$.

4. The composition of two homotheties with coefficients $k_1$ and $k_2$, where $k_1 k_2 \neq 1$, is a homothety with coefficient $k_1 k_2$ and its center belongs to the line that connects the centers of these homotheties (see Problem 19.23).

5. The center of a rotational homothety that sends segment $AB$ into segment $CD$ is the intersection point of the circles circumscribed about triangles $ACP$ and $BDP$, where $P$ is the intersection point of lines $AB$ and $CD$ (see Problem 19.41).

## Introductory problems

**1.** Prove that a homothety sends a circle into a circle.

**2.** Two circles are tangent at point $K$. A line passing through $K$ intersects these circles at points $A$ and $B$. Prove that the tangents to the circles through $A$ and $B$ are parallel to each other.

**3.** Two circles are tangent at point $K$. Through $K$ two lines are drawn that intersect the first circle at points $A$ and $B$ and the second one at points $C$ and $D$. Prove that $AB \parallel CD$.

**4.** Prove that points symmetric to an arbitrary point with respect to the midpoints of a square's sides are vertices of a square.

**5.** Two points $A$ and $B$ and a line $l$ on the plane are given. What is the trajectory of movement of the intersection point of medians of triangle $ABC$ when $C$ moves along $l$?

## §1. Homothetic polygons

**19.1.** A quadrilateral is cut by diagonals into four triangles. Prove that the intersection points of their medians form a parallelogram.

**19.2.** The extensions of the lateral sides $AB$ and $CD$ of trapezoid $ABCD$ intersect at point $K$ and its diagonals intersect at point $L$. Prove that points $K$, $L$, $M$ and $N$, where $M$ and $N$ are the midpoints of bases $BC$ and $AD$, respectively, belong to one line.

**19.3.** The intersection point of diagonals of a trapezoid is equidistant from the lines to which the sides of the trapezoid belong. Prove that the trapezoid is an isosceles one.

**19.4.** Medians $AA_1$, $BB_1$ and $CC_1$ of triangle $ABC$ meet at point $M$; let $P$ be an arbitrary point. Line $l_a$ passes through point $A$ parallel to line $PA_1$; lines $l_b$ and $l_c$ are similarly defined. Prove that:

a) lines $l_a$, $l_b$ and $l_c$ meet at one point, $Q$;

b) point $M$ belongs to segment $PQ$ and $PM : MQ = 1 : 2$.

**19.5.** Circle $S$ is tangent to equal sides $AB$ and $BC$ of an isosceles triangle $ABC$ at points $P$ and $K$, respectively, and is also tangent from the inside to the circle circumscribed about triangle $ABC$. Prove that the midpoint of segment $PK$ is the center of the circle inscribed into triangle $ABC$.

**19.6.** A convex polygon possesses the following property: if all its sides are pushed by distance 1 outwards and extended, then the obtained lines form a polygon similar to the initial one. Prove that this polygon is a circumscribed one.

**19.7.** Let $R$ and $r$ be the radii of the circumscribed and inscribed circles of a triangle. Prove that $R \geq 2r$ and the equality is only attained for an equilateral triangle.

**19.8.** Let $M$ be the center of mass of an $n$-gon $A_1 \ldots A_n$; let $M_1$, ..., $M_n$ be the centers of mass of the $(n-1)$-gons obtained from the given $n$-gon by discarding vertices $A_1$, ..., $A_n$, respectively. Prove that polygons $A_1 \ldots A_n$ and $M_1 \ldots M_n$ are homothetic to each other.

**19.9.** Prove that any convex polygon $\Phi$ contains two nonintersecting polygons $\Phi_1$ and $\Phi_2$ similar to $\Phi$ with coefficient $\frac{1}{2}$.

See also Problem 5.87.

## §2. Homothetic circles

**19.10.** On a circle, points $A$ and $B$ are fixed and point $C$ moves along this circle. Find the locus of the intersection points of the medians of triangles $ABC$.

**19.11.** a) A circle inscribed into triangle $ABC$ is tangent to side $AC$ at point $D$, and $DM$ is its diameter. Line $BM$ intersects side $AC$ at point $K$. Prove that $AK = DC$.

b) In the circle, perpendicular diameters $AB$ and $CD$ are drawn. From point $M$ outside the circle there are drawn tangents to the circle that intersect $AB$ at points $E$ and $H$ and also lines $MC$ and $MD$ that intersect $AB$ at points $F$ and $K$, respectively. Prove that $EF = KH$.

**19.12.** Let $O$ be the center of the circle inscribed into triangle $ABC$, let $D$ be the point where the circle is tangent to side $AC$ and $B_1$ the midpoint of $AC$. Prove that line $B_1O$ divides segment $BD$ in halves.

**19.13.** The circles $\alpha$, $\beta$ and $\gamma$ are of the same radius and are tangent to the sides of angles $A$, $B$ and $C$ of triangle $ABC$, respectively. Circle $\delta$ is tangent from the outside to all the three circles $\alpha$, $\beta$ and $\gamma$. Prove that the center of $\delta$ belongs to the line passing through the centers of the circles inscribed into and circumscribed about triangle $ABC$.

**19.14.** Consider triangle $ABC$. Four circles of the same radius $\rho$ are constructed so that one of them is tangent to the three other ones and each of those three is tangent to two sides of the triangle. Find $\rho$ given the radii $r$ and $R$ of the circles inscribed into and circumscribed about the triangle.

## §3. Costructions and loci

**19.15.** Consider angle $\angle ABC$ and point $M$ inside it. Construct a circle tangent to the legs of the angle and passing through $M$.

**19.16.** Inscribe two equal circles in a triangle so that each of the circles were tangent to two sides of the triangle and the other circle.

**19.17.** Consider acute triangle $ABC$. Construct points $X$ and $Y$ on sides $AB$ and $BC$, respectively, so that a) $AX = XY = YC$; b) $BX = XY = YC$.

**19.18.** Construct triangle $ABC$ given sides $AB$ and $AC$ and bisector $AD$.

**19.19.** Solve Problem 16.18 with the help of homothety.

**19.20.** On side $BC$ of given triangle $ABC$, construct a point such that the line that connects the bases of perpendiculars dropped from this point to sides $AB$ and $AC$ is parallel to $BC$.

$$* * *$$

**19.21.** Right triangle $ABC$ is modified so that vertex $A$ of the right angle is fixed whereas vertices $B$ and $C$ slide along fixed circles $S_1$ and $S_2$ tangent to each other at $A$ from the outside. Find the locus of bases $D$ of heights $AD$ of triangles $ABC$.

See also problems 7.26–7.29, 8.15, 8.16, 8.70.

## §4. Composition of homotheties

**19.22.** A transformation $f$ has the following property: if $A'$ and $B'$ are the images of points $A$ and $B$, then $\overrightarrow{A'B'} = k\overrightarrow{AB}$, where $k$ is a constant. Prove that:
    a) if $k = 1$, then $f$ is a parallel translation;
    b) if $k \neq 1$, then $f$ is a homothety.

**19.23.** Prove that the composition of two homotheties with coefficients $k_1$ and $k_2$, where $k_1 k_2 \neq 1$, is a homothety with coefficient $k_1 k_2$ and its center belongs to the line that connects the centers of these homotheties. Investigate the case $k_1 k_2 = 1$.

**19.24.** Common outer tangents to the pairs of circles $S_1$ and $S_2$, $S_2$ and $S_3$, $S_3$ and $S_1$ intersect at points $A$, $B$ and $C$, respectively. Prove that points $A$, $B$ and $C$ belong to one line.

**19.25.** Trapezoids $ABCD$ and $APQD$ have a common base $AD$ and the length of all their bases are distinct. Prove that the intersections points of the following pairs of lines belong to one line:
    a) $AB$ and $CD$, $AP$ and $DQ$, $BP$ and $CQ$;
    b) $AB$ and $CD$, $AQ$ and $DP$, $BQ$ and $CP$.

## §5. Rotational homothety

**19.26.** Circles $S_1$ and $S_2$ intersect at points $A$ and $B$. Lines $p$ and $q$ passing through point $A$ intersect circle $S_1$ at points $P_1$ and $Q_1$ and circle $S_2$ at points $P_2$ and $Q_2$. Prove that the angle between lines $P_1 Q_1$ and $P_2 Q_2$ is equal to the angle between circles $S_1$ and $S_2$.

**19.27.** Circles $S_1$ and $S_2$ intersect at points $A$ and $B$. Under the rotational homothety $P$ with center $A$ that sends $S_1$ into $S_2$ point $M_1$ from circle $S_1$ turns into $M_2$. Prove that line $M_1 M_2$ passes through $B$.

**19.28.** Circles $S_1, \ldots, S_n$ pass through point $O$. A grasshopper hops from point $X_i$ on circle $S_i$ to point $X_{i+1}$ on circle $S_{i+1}$ so that line $X_i X_{i+1}$ passes through the intersection point of circles $S_i$ and $S_{i+1}$ distinct from $O$. Prove that after $n$ hops (from $S_1$ to $S_2$ from $S_2$ to $S_3, \ldots$, from $S_n$ to $S_1$) the grasshopper returns to the initial position.

**19.29.** Two circles intersect at points $A$ and $B$ and chords $AM$ and $AN$ are tangent to these circles. Let us complete triangle $MAN$ to parallelogram $MANC$ and divide segments $BN$ and $MC$ by points $P$ and $Q$ in equal proportions. Prove then that $\angle APQ = \angle ANC$.

**19.30.** Consider two nonconcentric circles $S_1$ and $S_2$. Prove that there exist precisely two rotational homotheties with the angle of rotation of $90°$ that send $S_1$ into $S_2$.

**\* \* \***

**19.31.** Consider square $ABCD$ and points $P$ and $Q$ on sides $AB$ and $BC$, respectively, so that $BP = BQ$. Let $H$ be the base of the perpendicular dropped from $B$ on $PC$. Prove that $\angle DHQ = 90°$.

**19.32.** On the sides of triangle $ABC$ similar triangles are constructed outwards: $\triangle A_1BC \sim \triangle B_1CA \sim \triangle C_1AB$. Prove that the intersection points of medians of triangles $ABC$ and $A_1B_1C_1$ coincide.

**19.33.** The midpoints of sides $BC$ and $B_1C_1$ of equilateral triangles $ABC$ and $A_1B_1C_1$ coincide (the vertices of both triangles are listed clockwise). Find the value of the angle between lines $AA_1$ and $BB_1$ and also the ratio of the lengths of segments $AA_1$ and $BB_1$.

**19.34.** Triangle $ABC$ turns under a rotational homothety into triangle $A_1B_1C_1$; let $O$ be an arbitrary point. Let $A_2$ be the vertex of parallelogram $OAA_1A_2$; let points $B_2$ and $C_2$ be similarly defined. Prove that $\triangle A_2B_2C_2 \sim \triangle ABC$.

**19.35.** On top of a rectangular map lies a map of the same locality but of lesser scale. Prove that it is possible to pierce by a needle both maps so that the points where both maps are pierced depict the same point of the locality.

**19.36.** Rotational homotheties $P_1$ and $P_2$ with centers $A_1$ and $A_2$ have the same angle of rotation and the product of their coefficients is equal to 1. Prove that the composition $P_2 \circ P_1$ is a rotation and its center coincides with the center of another rotation that sends $A_1$ into $A_2$ and whose angle of rotation is equal to $2\angle(\overrightarrow{MA_1}, \overrightarrow{MN})$, where $M$ is an arbitrary point and $N = P_1(M)$.

**19.37.** Triangles $MAB$ and $MCD$ are similar but have opposite orientations. Let $O_1$ be the center of rotation through an angle of $2\angle(\overrightarrow{AB}, \overrightarrow{BM})$ that sends $A$ to $C$ and $O_2$ the center of rotation through an angle of $2\angle(\overrightarrow{AB}, \overrightarrow{AM})$ that sends $B$ to $D$. Prove that $O_1 = O_2$.

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**19.38.** Consider a half circle with diameter $AB$. For every point $X$ on this half circle a point $Y$ is placed on ray $XA$ so that $XY = kXB$. Find the locus of points $Y$.

**19.39.** Consider point $P$ on side $AB$ of (unknown?) triangle $ABC$ and triangle $LMN$. Inscribe triangle $PXY$ similar to $LMN$ into triangle $ABC$.

**19.40.** Construct quadrilateral $ABCD$ given $\angle B + \angle D$ and the lengths $a = AB, b = BC$, $c = CD$ and $d = DA$.

See also Problem 5.122.

## §6. The center of a rotational homothety

**19.41.** a) Let $P$ be the intersection point of lines $AB$ and $A_1B_1$. Prove that if no points among $A$, $B$, $A_1$, $B_1$ and $P$ coincide, then the common point of circles circumscribed about triangles $PAA_1$ and $PBB_1$ is the center of a rotational homothety that sends $A$ to $A_1$ and $B$ to $B_1$ and that such a rotational homothety is unique.

b) Prove that the center of a rotational homothety that sends segment $AB$ to segment $BC$ is the intersection point of circles passing through point $A$ and tangent to line $BC$ at point $B$ and the circle passing through $C$ and tangent to line $AB$ at point $B$.

**19.42.** Points $A$ and $B$ move along two intersecting lines with constant but distinct speeds. Prove that there exists a point, $P$, such that at any moment $AP : BP = k$, where $k$ is the ratio of the speeds.

**19.43.** Construct the center $O$ of a rotational homothety with a given coefficient $k \neq 1$ that sends line $l_1$ into line $l_2$ and point $A_1$ that belongs to $l_1$ into point $A_2$. (?)

**19.44.** Prove that the center of a rotational homothety that sends segment $AB$ into segment $A_1B_1$ coincides with the center of a rotational homothety that sends segment $AA_1$ into segment $BB_1$.

**19.45.** Four intersecting lines form four triangles. Prove that the four circles circumscribed about these triangles have one common point.

**19.46.** Parallelogram $ABCD$ is not a rhombus. Lines symmetric to lines $AB$ and $CD$ through diagonals $AC$ and $DB$, respectively, intersect at point $Q$. Prove that $Q$ is the center of a rotational homothety that sends segment $AO$ into segment $OD$, where $O$ is the center of the parallelogram.

**19.47.** Consider wo regular pentagons with a common vertex. The vertices of each pentagon are numbered 1 to 5 clockwise so that the common vertex has number 1. Vertices with equal numbers are connected by straight lines. Prove that the four lines thus obtained intersect at one point.

**19.48.** On sides $BC$, $CA$ and $AB$ of triangle $ABC$ points $A_1$, $B_1$ and $C_1$ are taken so that $\triangle ABC \sim \triangle A_1B_1C_1$. Pairs of segments $BB_1$ and $CC_1$, $CC_1$ and $AA_1$, $AA_1$ and $BB_1$ intersect at points $A_2$, $B_2$ and $C_2$, respectively. Prove that the circles circumscribed about triangles $ABC_2$, $BCA_2$, $CAB_2$, $A_1B_1C_2$, $B_1C_1A_2$ and $C_1A_1B_2$ intersect at one point.

## §7. The similarity circle of three figures

Let $F_1$, $F_2$ and $F_3$ be three similar figures, $O_1$ the center of a rotational homothety that sends $F_2$ to $F_3$. Let points $O_2$ and $O_3$ be similarly defined. If $O_1$, $O_2$ and $O_3$ do not belong to one line, then triangle $O_1O_2O_3$ is called *the similarity triangle* of figures $F_1$, $F_2$ and $F_3$ and its circumscribed circle is called *the similarity circle* of these figures. In case points $O_1, O_2$ and $O_3$ coincide the similarity circle degenerates into *the center of similarity* and in case when not all these points coincide but belong to one line the similarity circle degenerates into *the axis of similarity.*

In the problems of this section we assume that the similarity circle of the figures considered is not degenerate.

**19.49.** Lines $A_2B_2$ and $A_3B_3$, $A_3B_3$ and $A_1B_1$, $A_1B_1$ and $A_2B_2$ intersect at points $P_1$, $P_2$, $P_3$, respectively.

a) Prove that the circumscribed circles of triangles $A_1A_2P_3$, $A_1A_3P_2$ and $A_2A_3P_1$ intersect at one point that belongs to the similarity circle of segments $A_1B_1$, $A_2B_2$ and $A_3B_3$.

b) Let $O_1$ be the center of rotational homothety that sends segment $A_2B_2$ into segment $A_3B_3$; points $O_2$ and $O_3$ be similarly defined. Prove that lines $P_1O_1$, $P_2O_2$ and $P_3O_3$ intersect at one point that belongs to the similarity circle of segments $A_1B_1$, $A_2B_2$ and $A_3B_3$.

Points $A_1$ and $A_2$ are called *correspondent* points of similar figures $F_1$ and $F_2$ if the rotational symmetry that sends $F_1$ to $F_2$ transforms $A_1$ into $A_2$. *Correspondent lines* and *correspondent segments* are analogously defined.

**19.50.** Let $A_1B_1$, $A_2B_2$ and $A_3B_3$ and also $A_1C_1, A_2C_2$ and $A_3C_3$ be correspondent segments of similar figures $F_1$, $F_2$ and $F_3$. Prove that the triangle formed by lines $A_1B_1$, $A_2B_2$ and $A_3B_3$ is similar to the triangle formed by lines $A_1C_1$, $A_2C_2$ and $A_3C_3$ and the center of the rotational homothety that sends one of these triangles into another one belongs to the similarity circle of figures $F_1$, $F_2$ and $F_3$.

**19.51.** Let $l_1$, $l_2$ and $l_3$ be the correspondent lines of similar figures $F_1$, $F_2$ and $F_3$ and let the lines intersect at point $W$.

a) Prove that $W$ belongs to the similarity circle of $F_1$, $F_2$ and $F_3$.

b) Let $J_1$, $J_2$ and $J_3$ be distinct from $W$ intersection points of lines $l_1$, $l_2$ and $l_3$ with the similarity circle. Prove that these points only depend on figures $F_1$, $F_2$ and $F_3$ and do not depend on the choice of lines $l_1$, $l_2$ and $l_3$.

Points $J_1$, $J_2$ and $J_3$ are called *constant points of similar figures* $F_1$, $F_2$ and $F_3$ and triangle $J_1J_2J_3$ is called *the constant triangle of similar figures*.

**19.52.** Prove that the constant triangle of three similar figures is similar to the triangle formed by their correspondent lines and these triangles have opposite orientations.

**19.53.** Prove that constant points of three similar figures are their correspondent points.

The *similarity circle of triangle* $ABC$ is the similarity circle of segments $AB$, $BC$ and $CA$ (or of any three similar triangles constructed from these segments). *Constant points of a triangle* are the constant points of the three figures considered.

**19.54.** Prove that the similarity circle of triangle $ABC$ is the circle with diameter $KO$, where $K$ is Lemoin's point and $O$ is the center of the circumscribed circle.

**19.55.** Let $O$ be the center of the circumscribed circle of triangle $ABC$, $K$ Lemoin's point, $P$ and $Q$ Brokar's points, $\varphi$ Brokar's angle (see Problems 5.115 and 5.117). Prove that points $P$ and $Q$ belong to the circle of diameter $KO$ and $OP = OQ$ and $\angle POQ = 2\varphi$.

## Problems for independent study

**19.56.** Given triangles $ABC$ and $KLM$. Inscribe triangle $A_1B_1C_1$ into triangle $ABC$ so that the sides of $A_1B_1C_1$ wre parallel to the respective sides of triangle $KLM$.

**19.57.** On the plane, there are given points $A$ and $E$. Construct a rhombus $ABCD$ with a given height for which $E$ is the midpoint of $BC$.

**19.58.** Consider a quadrilateral. Inscribe a rombus in it so that the sides of the rombus are parallel to the diagonals of the quandrangle.

**19.59.** Consider acute angle $\angle AOB$ and point $C$ inside it. Find point $M$ on leg $OB$ equidistant from leg $OA$ and from point $C$.

**19.60.** Consider acute triangle $ABC$. Let $O$ be the intersection point of its heights; $\omega$ the circle with center $O$ situated inside the triangle. Construct triangle $A_1B_1C_1$ circumscribed about $\omega$ and inscribed in triangle $ABC$.

**19.61.** Consider three lines $a$, $b$, $c$ and three points $A$, $B$, $C$ each on the respective line. Construct points $X$, $Y$, $Z$ on lines $a$, $b$, $c$, respectively, so that $BY : AX = 2$, $CZ : AX = 3$ and so that $X$, $Y$, $Z$ are all on one line.

## Solutions

**19.1.** A homothety with the center at the intersection point of the diagonals of the quadrilateral and with coefficient $3/2$ sends the intersection points of the medians of the triangles in question into the midpoints of the sides of the quadrilateral. It remains to make use of the result of Problem 1.2.

**19.2.** The homothety with center $K$ that sends $\triangle KBC$ into $\triangle KAD$ sends point $M$ into $N$ and, therefore, $K$ belongs to line $MN$. The homothety with center $L$ that sends $\triangle LBC$ into $\triangle LDA$ sends $M$ into $N$. Therefore, $L$ belongs to line $MN$.

**19.3.** Suppose the continuations of the lateral sides $AB$ and $CD$ intersect at point $K$ and the diagonals of the trapezoid intersect at point $L$. By the preceding problem line $KL$ passes through the midpoint of segment $AD$ and by the hypothesis this line divides angle $\angle AKD$ in halves. Therefore, triangle $AKD$ is an isosceles one (see Problem 16.1); hence, so is trapezoid $ABCD$.

**19.4.** The homothety with center $M$ and coefficient $-2$ sends lines $PA_1$, $PB_1$ and $PC_1$ into lines $l_a$, $l_b$ and $l_c$, respectively, and, therefore, the point $Q$ to be found is the image of $P$ under this homothety.

**19.5.** Consider homothety $H_B^k$ with center $B$ that sends segment $AC$ into segment $A'C'$ tangent to the circumscribed circle of triangle $ABC$. Denote the midpoints of segments $PK$ and $A'C'$ by $O_1$ and $D$, respectively, and the center of $S$ by $O$.

Circle $S$ is the inscribed circle of triangle $A'BC'$ and, therefore, it suffices to show that homothety $H_B^k$ sends $O_1$ to $O$. To this end it suffices to verify that $BO_1 : BO = BA : BA'$. This equality follows from the fact that $PO_1$ and $DA$ are heights of similar right triangles $BPO$ and $BDA'$.

**19.6.** Let $k$ be the similarity coefficient of polygons and $k < 1$. Shifting the sides of the initial polygon inside consecutively by $k$, $k^2$, $k^3$, ... units of length we get a contracting system of embedded convex polygons similar to the initial one with coefficients $k$, $k^2$, $k^3$, .... The only common point of these polygons is the center of the inscribed circle of the initial polygon.

**19.7.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $AC$ and $AB$, respectively. The homothety with center at the intersection point of the medians of triangle $ABC$ and with coefficient $-\frac{1}{2}$ sends the circumscribed circle $S$ of triangle $ABC$ into the circumscribed circle $S_1$ of triangle $A_1B_1C_1$. Since $S_1$ passes through all the vertices of triangle $ABC$, we can construct triangle $A'B'C'$ whose sides are parallel to the respective sides of triangle $ABC$ and for which $S_1$ is the inscribed circle, see Fig. 26.
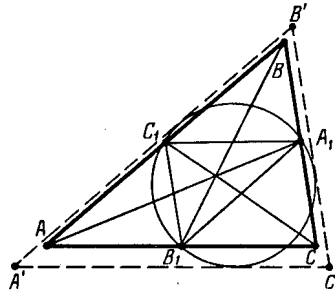

Figure 68 (Sol. 19.7)

Let $r$ and $r'$ be the radii of the inscribed circles of triangles $ABC$ and $A'B'C'$; let $R$ and $R_1$ be the radii of $S$ and $S_1$, respectively. Clearly, $r \leq r' = R_1 = R/2$. The equality is attained if triangles $A'B'C'$ and $ABC$ coincide, i.e., if $S_1$ is the inscribed circle of triangle $ABC$. In this case $AB_1 = AC_1$ and, therefore, $AB = AC$. Similarly, $AB = BC$.

**19.8.** Since
$$\overrightarrow{MM_i} = \frac{\overrightarrow{MA_1} + \cdots + \overrightarrow{MA_n} - \overrightarrow{MA_i}}{n - 1} = -\frac{\overrightarrow{MA_i}}{n - 1},$$
it follows that the homothety with center $M$ and coefficient $-\frac{1}{n-1}$ sends $A_i$ into $M_i$.

**19.9.** Let $A$ and $B$ be a pair of most distant from each other points of polygon $\Phi$. Then $\Phi_1 = H_A^{1/2}(\Phi)$ and $\Phi_2 = H_B^{1/2}(\Phi)$ are the required figures.

Indeed, $\Phi_1$ and $\Phi_2$ do not intersect because they lie on different sides of the midperpendicular to segment $AB$. Moreover, $\Phi_1$ and $\Phi_2$ are contained in $\Phi$ because $\Phi$ is a convex polygon.

**19.10.** Let $M$ be the intersection point of the medians of triangle $ABC$, $O$ the midpoints of segment $AB$. Clearly, $3\overrightarrow{OM} = \overrightarrow{OC}$ and, therefore, points $M$ fill in the circle obtained from the initial circle under the homothety with coefficient $\frac{1}{3}$ and center $O$.

**19.11.** a) The homothety with center $B$ that sends the inscribed circle into the escribed circle tangent to side $AC$ sends point $M$ into point $M'$. Point $M'$ is the endpoint of the diameter perpendicular to $AC$ and, therefore, $M'$ is the tangent point of the inscribed circle with $AC$, hence, it is the intersection point of $BM$ with $AC$. Therefore, $K = M'$ and $K$ is the tangent point of the escribed circle with side $AC$. Now it is easy to compute that $AK = \frac{1}{2}(a + b - c) = CD$, where $a$, $b$ and $c$ are the lengths of the sides of triangle $ABC$.

b) Consider a homothety with center $M$ that sends line $EH$ into a line tangent to the given circle. This homothety sends points $E$, $F$, $K$ and $H$ into points $E'$, $F'$, $K'$ and $H'$, respectively. By heading a) $E'F' = K'H'$; hence, $EF = KH$.

**19.12.** Let us make use of the solution and notations of Problem 19.11 a). Since $AK = DC$, then $B_1K = B_1D$ and, therefore, $B_1O$ is the midline of triangle $MKD$.

**19.13.** Let $O_\alpha$, $O_\beta$, $O_\gamma$ and $O_\delta$ be the centers of circles $\alpha$, $\beta$, $\gamma$ and $\delta$, respectively, $O_1$ and $O_2$ the centers of the inscribed and circumscribed circles, respectively, of triangle $ABC$. A homothety with center $O_1$ sends triangle $O_\alpha O_\beta O_\gamma$ into triangle $ABC$. This homothety sends point $O_2$ into the center of the circumscribed circle of triangle $O_\alpha O_\beta O_\gamma$; this latter center coincides with $O_\delta$. Therefore, points $O_1$, $O_2$ and $O_\delta$ belong to one line.

**19.14.** Let $A_1$, $B_1$ and $C_1$ be the centers of the given circles tangent to the sides of the triangle, $O$ the center of the circle tangent to these circles, $O_1$ and $O_2$ the centers of the inscribed and circumscribed circles of triangle $ABC$. Lines $AA_1$, $BB_1$ and $CC_1$ are the bisectors of triangle $ABC$ and, therefore, they intersect at point $O_1$. It follows that triangle $A_1B_1C_1$ turns into triangle $ABC$ under a homothety with center $O_1$ and the coefficient of the homothety is equal to the ratio of distances from $O_1$ to the sides of triangles $ABC$ and $A_1B_1C_1$, i.e., is equal to $\frac{r-\rho}{r}$.

Under this homothety the circumscribed circle of triangle $ABC$ turns into the circumscribed circle of triangle $A_1B_1C_1$. Since $OA_1 = OB_1 = OC_1 = 2\rho$, the radius of the circumscribed circle of triangle $A_1B_1C_1$ is equal to $2\rho$. Hence, $R\frac{r-\rho}{r} = 2\rho$, i.e., $\rho = \frac{rR}{2r+R}$.

**19.15.** On the bisector of angle $\angle ABC$ take an arbitrary point $O$ and construct a circle $S$ with center $O$ tangent to the legs of the angle. Line $BM$ intersects circle $S$ at points $M_1$ and $M_2$. The problem has two solutions: circle $S$ turns into the circles passing through $M$ and tangent to the legs of the angle under the homothety with center $B$ that sends $M_1$ into $M$ and under the homothety with center $B$ that sends $M_2$ into $M$.

**19.16.** Clearly, both circles are tangent to one of the triangle's sides. Let us show how to construct circles tangent to side $AB$. Let us take line $c'$ parallel to line $AB$. Let us construct circles $S_1'$ and $S_2'$ of the same radius tangent to each other and to line $c'$. Let us construct tangents $a'$ and $b'$ to these circles parallel to lines $BC$ and $AC$, respectively. The sides of triangle $A'B'C'$ formed by lines $a'$, $b'$ and $c'$ are parallel to respective sides of triangle $ABC$. Therefore, there exists a homothety sending triangle $A'B'C'$ into triangle $ABC$. The desired circles are the images of circles $S_1'$ and $S_2'$ with respect to this homothety.

**19.17.** a) On sides $AB$ and $BC$ of triangle $ABC$ fix segments $AX_1$ and $CY_1$ of equal length $a$. Through point $Y_1$ draw a line $l$ parallel to side $AC$. Let $Y_2$ be the intersection point of $l$ and the circle of radius $a$ with center $X_1$ situated(who?) inside the triangle. Then point $Y$ to be found is the intersection point of line $AY_2$ with side $BC$ and $X$ is a point on ray $AB$ such that $AX = CY$.

b) On side $AB$, take an arbitrary point $X_1$ distinct from $B$. The circle of radius $BX_1$ with center $X_1$ intersects ray $BC$ at points $B$ and $Y_1$. Construct point $C_1$ on line $BC$ such that $Y_1C_1 = BX_1$ and such that $Y_1$ lies between $B$ and $C_1$. The homothety with center $B$ that sends point $C_1$ into $C$ sends $X_1$ and $Y_1$ into points $X$ and $Y$ to be found.

**19.18.** Take segment $AD$ and draw circles $S_1$ and $S_2$ with center $A$ and radii $AB$ and $AC$, respectively. Vertex $B$ is the intersection point of $S_1$ with the image of $S_2$ under the homothety with center $D$ and coefficient $-\frac{DB}{DC} = -\frac{AB}{AC}$.

**19.19.** On the great circle $S_2$ take an arbitrary point $X$. Let $S_2'$ be the image of $S_2$ under the homothety with center $X$ and coefficient $\frac{1}{3}$, let $Y$ be the intersection point of $S_2'$ and $S_1$. Then $XY$ is the line to be found.

**19.20.** From points $B$ and $C$ draw perpendiculars to lines $AB$ and $AC$ and let $P$ be their intersection point. Then the intersection point of lines $AP$ and $BC$ is the desired one.

**19.21.** Let us draw common exterior tangents $l_1$ and $l_2$ to circles $S_1$ and $S_2$, respectively. Lines $l_1$ and $l_2$ intersect at a point $K$ which is the center of a homothety $H$ that sends $S_1$ to $S_2$. Let $A_1 = H(A)$. Points $A$ and $K$ lie on a line that connects the centers of the circles and, therefore, $AA_1$ is a diameter of $S_2$, i.e., $\angle ACA_1 = 90°$ and $A_1C \parallel AB$. It follows that segment $AB$ goes into $A_1C$ under $H$. Therefore, line $BC$ passes through $K$ and $\angle ADK = 90°$. Point $D$ belongs to circle $S$ with diameter $AK$. It is also clear that point $D$ lies inside the angle formed by lines $l_1$ and $l_2$. Therefore, the locus of points $D$ is the arc of $S$ cut off by $l_1$ and $l_2$.

**19.22.** The hypothesis of the problem implies that the map $f$ is one-to-one.

a) Suppose $f$ sends point $A$ to point $A'$ and $B$ to $B'$. Then

$$\overrightarrow{BB'} = \overrightarrow{BA} + \overrightarrow{AA'} + \overrightarrow{A'B'} = -\overrightarrow{AB} + \overrightarrow{AA'} + \overrightarrow{AB} = \overrightarrow{AA'},$$

i.e., $f$ is a parallel translation.

b) Consider three points $A$, $B$ and $C$ not on one line. Let $A'$, $B'$ and $C'$ be their images under $f$. Lines $AB$, $BC$ and $CA$ cannot coincide with lines $A'B'$, $B'C'$ and $C'A'$, respectively, since in this case $A = A'$, $B = B'$ and $C = C'$. Let $AB \neq A'B'$. Lines $AA'$ and $BB'$ are not parallel because otherwise quandrilateral $ABB'A'$ would have been a parallelogram and $\overrightarrow{AB} = \overrightarrow{A'B'}$. Let $O$ be the intersection point of $AA'$ and $BB'$. Triangles $AOB$ and $A'OB'$ are similar with similarity coefficient $k$ and, therefore, $\overrightarrow{OA'} = k\overrightarrow{OA}$, i.e., $O$ is a fixed point of the transformation $f$. Therefore,

$$\overrightarrow{Of(X)} = \overrightarrow{f(O)f(X)} = k\overrightarrow{OX}$$

for any $X$ which means that $f$ is a homothety with coefficient $k$ and center $O$.

**19.23.** Let $H = H_2 \circ H_1$, where $H_1$ and $H_2$ are homotheties with centers $O_1$ and $O_2$ and coefficients $k_1$ and $k_2$, respectively. Denote:

$$A' = H_1(A), \ B' = H_1(B), \ A'' = H_2(A'), \ B'' = H_2(B').$$

Then $\overrightarrow{A'B'} = k_1\overrightarrow{AB}$ and $\overrightarrow{A''B''} = k_2\overrightarrow{A'B'}$, i.e., $\overrightarrow{A''B''} = k_1k_2\overrightarrow{AB}$. With the help of the preceding problem this implies that for $k_1k_2 \neq 1$ the transformation $H$ is a homothety with coefficient $k_1k_2$ and if $k_1k_2 = 1$, then $H$ is a parallel translation.

It remains to verify that the fixed point of $H$ belongs to the line that connects the centers of homotheties $H_1$ and $H_2$. Since $\overrightarrow{O_1A'} = k_1\overrightarrow{O_1A}$ and $\overrightarrow{O_2A''} = k_2\overrightarrow{O_2A'}$, it follows that

$$\overrightarrow{O_2A''} = k_2(\overrightarrow{O_2O_1} + \overrightarrow{O_1A'}) = k_2(\overrightarrow{O_2O_1} + k_1\overrightarrow{O_1A}) =$$

$$k_2\overrightarrow{O_2O_1} + k_1k_2\overrightarrow{O_1O_2} + k_1k_2\overrightarrow{O_2A}.$$

For a fixed point $X$ we get the equation

$$\overrightarrow{O_2X} = (k_1k_2 - k_2)\overrightarrow{O_1O_2} + k_1k_2\overrightarrow{O_2X}$$

and, therefore, $\overrightarrow{O_2X} = \lambda\overrightarrow{O_1O_2}$, where $\lambda = \frac{k_1k_2-k_2}{1-k_1k_2}$.

**19.24.** Point $A$ is the center of homothety that sends $S_1$ to $S_2$ and $B$ is the center of homothety that sends $S_2$ to $S_3$. The composition of these homotheties sends $S_1$ to $S_3$ and its center belongs to line $AB$. On the other hand, the center of homothety that sends $S_1$ to $S_3$ is point $C$. Indeed, to the intersection point of the outer tangents there corresponds a homothety with any positive coefficient and a composition of homotheties with positive coefficients is a homothety with a positive coefficient.

**19.25.** a) Let $K$, $L$, $M$ be the intersection points of lines $AB$ and $CD$, $AP$ and $DQ$, $BP$ and $CQ$, respectively. These points are the centers of homotheties $H_K$, $H_L$ and $H_M$ with positive coefficients that consequtively send segments $BC$ to $AD$, $AD$ to $PQ$ and $BC$ to $PQ$. Clearly, $H_L \circ H_K = H_M$. Therefore, points $K$, $L$ and $M$ belong to one line.

b) Let $K$, $L$, $M$ be the intersection points of lines $AB$ and $CD$, $AQ$ and $DP$, $BQ$ and $CP$, respectively. These points are the centers of homotheties, $H_K$, $H_L$ and $H_M$ that consequtively send segments $BC$ to $AD$, $AD$ to $QP$, $BC$ to $QP$; the coefficient of the first homothety is a positive one those of two other homotheties are negative ones. Clearly, $H_L \circ H_K = H_M$. Therefore, points $K$, $L$ and $M$ belong to one line.

**19.26.** Since $\angle(P_1A, AB) = \angle(P_2A, AB)$, the oriented angle values of arcs $\smile BP_1$ and $\smile BP_2$ are equal. Therefore, the rotational homothety with center $B$ that sends $S_1$ to $S_2$ sends point $P_1$ to $P_2$ and line $P_1Q_1$ into line $P_2Q_2$.

**19.27.** Oriented angle values of arcs $\smile AM_1$ and $\smile AM_2$ are equal, consequently, $\angle(M_1B, BA) = \angle(M_2B, BA)$ and, therefore, points $M_1$, $M_2$ and $B$ belong to one line.

**19.28.** Let $P_i$ be a rotational homothety with center $O$ that sends circle $S_i$ to $S_{i+1}$. Then $X_{i+1} = P_i(X_i)$ (see Problem 19.27). It remains to observe that the composition $P_n \circ \cdots \circ P_2 \circ P_1$ is a rotational homothety with center $O$ that sends $S_1$ to $S_1$, i.e., is an identity transformation.

**19.29.** Since $\angle AMB = \angle NAB$ and $\angle BAM = \angle BNA$, we have $\triangle AMB \sim \triangle NAB$ and, therefore, $AN : AB = MA : MB = CN : MB$. Moreover, $\angle ABM = 180° - \angle MAN = \angle ANC$. It follows that $\triangle AMB \sim \triangle ACN$, i.e., the rotational homothety with center $A$ sending $M$ to $B$ sends $C$ to $N$ and, therefore, it maps $Q$ to $P$.

**19.30.** Let $O_1$ and $O_2$ be the centers of given circles, $r_1$ and $r_2$ be their radii. The coefficient $k$ of the rotational homothety which maps $S_1$ to $S_2$ is equal to $r_1/r_2$ and its center $O$ belongs to the circle with diameter $O_1O_2$. Moreover, $OO_1 : OO_2 = k = r_1/r_2$. It remains to verify that the circle with diameter $O_1O_2$ and the locus of points $O$ such that $OO_1 : OO_2 = k$ have precisely two common points. For $k = 1$ it is obvious and for $k \neq 1$ the locus in question is described in the solution of Problem 7.14: it is the(A?) circle and one of its intersection points with line $O_1O_2$ is an inner point of segment $O_1O_2$ whereas the other intersection point lies outside the segment.

**19.31.** Consider a transformation which sends triangle $BHC$ to triangle $PHB$, i.e., the composition of the rotation through an angle of $90°$ about point $H$ and the homothety with coefficient $BP : CB$ and center $H$. Since this transformation maps the vertices of any square into vertices of a square, it maps points $C$ and $B$ to points $B$ and $P$, respectively. Then it maps point $D$ to $Q$, i.e., $\angle DHQ = 90°$.

**19.32.** Let $P$ be a rotational homothety that sends $\overrightarrow{CB}$ to $\overrightarrow{CA_1}$. Then

$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} =$$

$$\overrightarrow{AC} + P(\overrightarrow{CB}) + \overrightarrow{CB} + P(\overrightarrow{BA}) + \overrightarrow{BA} + P(\overrightarrow{AC}) = \overrightarrow{0}.$$

Hence, if $M$ is the center of mass of triangle $ABC$, then

$$\overrightarrow{MA_1} + \overrightarrow{MB_1} + \overrightarrow{MC_1} =$$

$$(\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}) + (\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1}) = \overrightarrow{0}.$$

**19.33.** Let $M$ be the common midpoint of sides $BC$ and $B_1C_1$, $\mathbf{x} = \overrightarrow{MB}$ and $\mathbf{y} = \overrightarrow{MB_1}$. Further, let $P$ be the rotational homothety with center $M$, the angle of rotation $90°$ and coefficient $\sqrt{3}$ that sends $B$ to $A$ and $B_1$ to $A_1$. Then $\overrightarrow{BB_1} = \mathbf{y} - \mathbf{x}$ and $\overrightarrow{AA_1} = P(\mathbf{y}) - P(\mathbf{x}) = P(\overrightarrow{BB_1})$. Therefore, the angle between vectors $\overrightarrow{AA_1}$ and $\overrightarrow{BB_1}$ is equal to $90°$ and $AA_1 : BB_1 = \sqrt{3}$.

**19.34.** Let $P$ be the rotational homothety that sends triangle $ABC$ to triangle $A_1B_1C_1$. Then

$$\overrightarrow{A_2B_2} = \overrightarrow{A_2O} + \overrightarrow{OB_2} = \overrightarrow{A_1A} + \overrightarrow{BB_1} =$$

$$\overrightarrow{BA} + \overrightarrow{A_1B_1} =$$

$$- \overrightarrow{AB} + P(\overrightarrow{AB}).$$

Similarly, the transformation $f(\mathbf{a}) = -\mathbf{a} + P(\mathbf{a})$ sends the other vectors of the sides of triangle $ABC$ to the vectors of the sides of triangle $A_2B_2C_2$.

**19.35.** Let the initial map be rectangle $K_0$ on the plane, the smaller map rectangle $K_1$ contained in $K_0$. Let us consider a rotational homothety $f$ that maps $K_0$ to $K_1$. Let $K_{i+1} = f(K_i)$ for $i > 1$. Since the sequence $K_i$ for $i = 1, 2, \ldots$ is a contracting sequence of embedded polygons, there exists (by Helly's theorem) a unique fixed point $X$ that belongs to all the rectangles $K_i$.

Let us prove that $X$ is the required point, i.e., $f(X) = X$. Indeed, since $X$ belongs to $K_i$, point $f(X)$ belongs to $K_{i+1}$, i.e., point $f(X)$ belongs also to all rectangles $K_i$. Since there is just one point that belongs to all rectangles, we deduce that $f(X) = X$.

**19.36.** Since the product of coefficients of rotational homotheties $P_1$ and $P_2$ is equal to 1, their composition is a rotation (cf. Problem 17.36). Let $O$ be the center of rotation $P_2 \circ P_1$ and $R = P_1(O)$. Since $P_2 \circ P_1(O) = O$, it follows that $P_2(R) = O$. Therefore, by hypothesis $A_1O : A_1R = A_2O : A_2R$ and $\angle OA_1R = \angle OA_2R$, i.e., $\triangle OA_1R \sim \triangle OA_2R$. Moreover, $OR$ is a common side of these similar triangles; hence, $\triangle OA_1R = \triangle OA_2R$. Therefore, $OA_1 = OA_2$ and

$$\angle(\overrightarrow{OA_1}, \overrightarrow{OA_2}) = 2\angle(\overrightarrow{OA_1}, \overrightarrow{OR}) = 2\angle(\overrightarrow{MA_1}, \overrightarrow{MN}),$$

i.e., $O$ is the center of rotation through an angle of $2\angle(\overrightarrow{MA_1}, \overrightarrow{MN})$ that maps $A_1$ to $A_2$.

**19.37.** Let $P_1$ be the rotational homothety with center $B$ sending $A$ to $M$ and $P_2$ be rotational homothety with center $D$ sending $M$ to $C$. Since the product of coefficients of these rotational homotheties is equal to $(BM : BA) \cdot (DC : DM) = 1$, their composition $P_2 \circ P_1$ is a rotation (sending $A$ to $C$) through an angle of

$$\angle(\overrightarrow{AB}, \overrightarrow{BM}) + \angle(\overrightarrow{DM}, \overrightarrow{DC}) = 2\angle(\overrightarrow{AB}, \overrightarrow{BM}).$$

On the other hand, the center of the rotation $P_2 \circ P_1$ coincides with the center of the rotation through an angle of $2\angle(\overrightarrow{AB}, \overrightarrow{AM})$ that sends $B$ to $D$ (cf. Problem 19.36).

**19.38.** It is easy to verify that $\tan \angle XBY = k$ and $BY : BX = \sqrt{k^2 + 1}$, i.e., $Y$ is obtained from $X$ under the rotational homothety with center $B$ and coefficient $\sqrt{k^2 + 1}$, the

angle of rotation being of value $\arctan k$. The locus to be found is the image of the given half circle under this rotational homothety.

**19.39.** Suppose that triangle $PXY$ is constructed and points $X$ and $Y$ belong to sides $AC$ and $CB$, respectively. We know a transformation that maps $X$ to $Y$, namely, the rotational homothety with center $P$, the angle of rotation $\varphi = \angle XPY = \angle MLN$ and the homothety coefficient $k = PY : PX = LN \cdot LM$. Point $Y$ to be found is the intersection point of segment $BC$ and the image of segment $AC$ under this transformation.

**19.40.** Suppose that rectangle $ABCD$ is constructed. Consider the rotational homothety with center $A$ that sends $B$ to $D$. Let $C'$ be the image of point $C$ under this homothety. Then $\angle CDC' = \angle B + \angle D$ and $DC' = \frac{BC \cdot AD}{AB} = \frac{bd}{a}$.

We can recover triangle $CDC'$ from $CD$, $DC'$ and $\angle CDC'$. Point $A$ is the intersection point of the circle of radius $d$ with center $D$ and the locus of points $X$ such that $C'X : CX = d : a$ (this locus is a circle, see Problem 7.14). The further construction is obvious.

**19.41.** a) If $O$ is the center of a rotational homothety that sends segment $AB$ to segment $A_1B_1$, then

$$\angle(PA, AO) = \angle(PA_1, A_1O) \quad \text{and} \quad \angle(PB, BO) = \angle(PB_1, B_1O) \qquad (1)$$

and, therefore, point $O$ is the intersection point of the inscribed circles of triangles $PAA_1$ and $PBB_1$.

The case when these circles have only one common point $P$ is clear: this is when segment $AB$ turns into segment $A_1B_1$ under a homothety with center $P$.

If $P$ and $O$ are two intersection points of the circles considered, then equalities (1) imply that $\triangle OAB \sim \triangle OA_1B_1$ and, therefore, $O$ is the center of a rotational homothety that maps segment $AB$ into segment $A_1B_1$.

b) It suffices to notice that point $O$ is the center of a rotational homothety that maps segment $AB$ to segment $BC$ if and only if $\angle(BA, AO) = \angle(CB, BO)$ and $\angle(AB, BO) = \angle(BC, CO)$.

**19.42.** Let $A_1$ and $B_1$ be the positions of the points at one moment, $A_2$ and $B_2$ the position of the points at another moment. Then for point $P$ we can take the center of a rotational homothety that maps segment $A_1A_2$ to segment $B_1B_2$.

**19.43.** Let $P$ be the intersection point of lines $l_1$ and $l_2$. By Problem 19.41 point $O$ belongs to the circumscribed circle $S_1$ of triangle $A_1A_2P$. On the other hand, $OA_2 : OA_1 = k$. The locus of points $X$ such that $XA_2 : XA_1 = k$ is circle $S_2$ (by Problem 7.14). Point $O$ is the intersection point of circles $S_1$ and $S_2$ (there are two such points).

**19.44.** Let $O$ be the center of a rotational homothety that maps segment $AB$ to segment $A_1B_1$. Then $\triangle ABO \sim \triangle A_1B_1O$, i.e., $\angle AOB = \angle A_1OB_1$ and $AO : BO = A_1O : B_1O$. Therefore, $\angle AOA_1 = \angle BOB_1$ and $AO : A_1O = BO : B_1O$, i.e., $\triangle AA_1O \sim \triangle BB_1O$. Hence, point $O$ is the center of the rotational homothety that maps segment $AA_1$ to segment $BB_1$.

**19.45.** Let lines $AB$ and $DE$ intersect at point $C$ and lines $BD$ and $AE$ intersect at point $F$. The center of rotational homothety that maps segment $AB$ to segment $ED$ is the distinct from $C$ intersection point of the circumscribed circles of triangles $AEC$ and $BDC$ (see Problem 19.41) and the center of rotational homothety sending $AE$ to $BD$ is the intersection point of circles circumscribed about triangles $ABF$ and $EDF$. By Problem 19.44 the centers of these rotational homotheties coincide, i.e., all the four circumscribed circles have a common point.

**19.46.** The center $O$ of parallelogram $ABCD$ is equidistant from the following pairs of lines: $AQ$ and $AB$, $AB$ and $CD$, $CD$ and $DQ$ and, therefore, $QO$ is the bisector of angle $\angle AQD$. Let $\alpha = \angle BAO$, $\beta = \angle CDO$ and $\varphi = \angle AQO = \angle DQO$. Then $\alpha + \beta = \angle AOD = 360° - \alpha - \beta - 2\varphi$, i.e., $\alpha + \beta + \varphi = 180°$ and, therefore, $\triangle QAO \sim \triangle QOD$.

**19.47.** Let us solve a slightly more general problem. Suppose point $O$ is taken on circle $S$ and $H$ is a rotational homothety with center $O$. Let us prove that then all lines $XX'$, where $X$ is a point from $S$ and $X' = H(X)$, intersect at one point.

Let $P$ be the intersection point of lines $X_1X_1'$ and $X_2X_2'$. By Problem 19.41 points $O$, $P$, $X_1$ and $X_2$ lie on one circle and points $O$, $P$, $X_1'$ and $X_2'$ also belong to one circle. Therefore, $P$ is an intersection point of circles $S$ and $H(S)$, i.e., all lines $XX'$ pass through the distinct from $O$ intersection point of circles $S$ and $H(S)$.

**19.48.** Let $O$ be the center of a rotational homothety sending triangle $A_1B_1C_1$ to triangle $ABC$. Let us prove that, for instance, the circumscribed circles of triangles $ABC_2$ and $A_1B_1C_2$ pass through point $O$. Under the considered homothety segment $AB$ goes into segment $A_1B_1$; therefore, point $O$ coincides with the center of the rotational homothety that maps segment $AA_1$ to segment $BB_1$ (see Problem 19.44). By problem 19.41 the center of the latter homothety is the second intersection point of the circles circumscribed about triangles $ABC_2$ and $A_1B_1C_2$ (or is their tangent point).
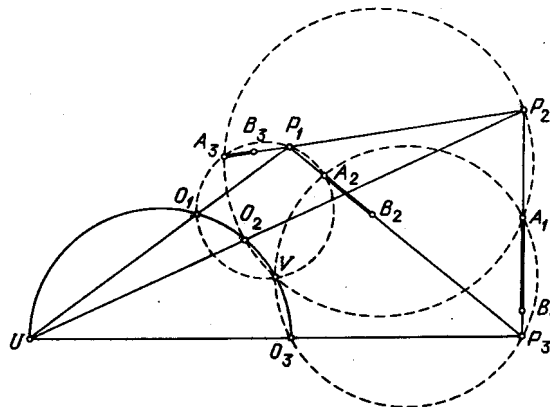

Figure 169 (Sol. 19.48)

**19.49.** Points $A_1$, $A_2$ and $A_3$ belong to lines $P_2P_3$, $P_3P_1$ and $P_1P_2$ (Fig. 27). Therefore, the circles circumscribed about triangles $A_1A_2P_3$, $A_1A_3P_2$ and $A_2A_3P_1$ have a common point $V$ (see Problem 2.80 a)), and points $O_3$, $O_2$ and $O_1$ lie on these circles (see Problem 19.41). Similarly, the circles circumscribed about triangles $B_1B_2P_3$, $B_1B_3P_2$ and $B_2B_3P_1$ have a common point $V'$. Let $U$ be the intersection point of lines $P_2O_2$ and $P_3O_3$. Let us prove that point $V$ belongs to the circle circumscribed about triangle $O_2O_3U$. Indeed,

$$\angle(O_2V, VO_3) = \angle(VO_2, O_2P_2) + \angle(O_2P_2, P_3O_3) + \angle(P_3O_3, O_3V) =$$
$$\angle(VA_1, A_1P_2) + \angle(O_2U, UO_3) + \angle(P_3A_1, A_1V) = \angle(O_2U, UO_3).$$

Analogous arguments show that point $V'$ belongs to the circle circumscribed about triangle $O_2O_3U$. In particular, points $O_2$, $O_3$, $V$ and $V'$ belong to one circle. Similarly, points $O_1$, $O_2$, $V$ and $V'$ belong to one circle and, therefore, points $V$ and $V'$ belong to the circle circumscribed about triangle $O_1O_2O_3$; point $U$ also belongs to this circle.

We can similarly prove that lines $P_1O_1$ and $P_2O_2$ intersect at one point that belongs to the similarity circle. Line $P_2O_2$ intersects the similarity circle at points $U$ and $O_2$ and, therefore, line $P_1O_1$ passes through point $U$.

**19.50.** Let $P_1$ be the intersection point of lines $A_2B_2$ and $A_3B_3$, let $P_1'$ be the intersection point of lines $A_2C_2$ and $A_3C_3$; let points $P_2$, $P_3$, $P_2'$ and $P_3'$ be similarly defined. The rotational homothety that sends $F_1$ to $F_2$ sends lines $A_1B_1$ and $A_1C_1$ to lines $A_2B_2$ and $A_2C_2$, respectively, and, therefore, $\angle(A_1B_1, A_2B_2) = \angle(A_1C_1, A_2C_2)$. Similar arguments show that $\triangle P_1P_2P_3 \sim \triangle P_1'P_2'P_3'$.

The center of the rotational homothety that maps segment $P_2P_3$ to $P_2'P_3'$ belongs to the circle circumscribed about triangle $A_1P_3P_3'$ (see Problem 19.41). Since

$$\angle(P_3A_1, A_1P_3') = \angle(A_1B_1, A_1C_1) = \angle(A_2B_2, A_2C_2) = \angle(P_3A_2, A_2P_3'),$$

the circle circumscribed about triangle $A_1P_3P_3'$ coincides with the circle circumscribed about triangle $A_1A_2P_3$. Similar arguments show that the center of the considered rotational homothety is the intersection point of the circles circumscribed about triangles $A_1A_2P_3$, $A_1A_3P_2$ and $A_2A_3P_1$; this point belongs to the similarity circle of figures $F_1$, $F_2$ and $F_3$ (see Problem 19.49 a)).

**19.51.**   a) Let $l_1'$, $l_2'$ and $l_3'$ be the corresponding lines of figures $F_1$, $F_2$ and $F_3$ such that $l_i' \parallel l_i$. These lines form triangle $P_1P_2P_3$. The rotational homothety with center $O_3$ that maps $F_1$ to $F_2$ sends lines $l_1$ and $l_1'$ to lines $l_2$ and $l_2'$, respectively, and, therefore, the homothety with center $O_3$ that maps $l_1$ to $l_1'$ sends line $l_2$ to $l_2'$. Therefore, line $P_3O_3$ passes through point $W$.

Similarly, lines $P_1O_1$ and $P_2O_2$ pass through point $W$; hence, $W$ belongs to the similarity circle of figures $F_1$, $F_2$ and $F_3$ (see Problem 19.49 b)).
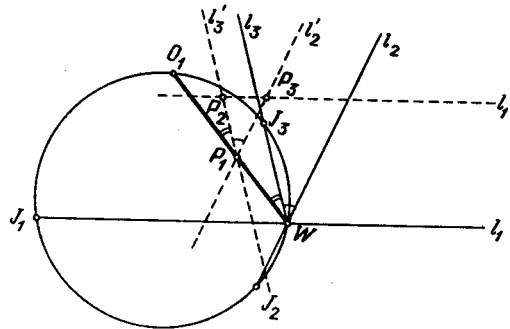


Figure 170 (Sol. 19.51 a))

b) The ratio of the distances from point $O_1$ to lines $l_2'$ and $l_3'$ is equal to the coefficient of the rotational homothety that maps $F_2$ to $F_3$ and the angle $\angle P_1$ of triangle $P_1P_2P_3$ is equal to the angle of the rotation. Therefore, $\angle(O_1P_1, P_1P_2)$ only depends on figures $F_2$ and $F_3$. Since $\angle(O_1W, WJ_3) = \angle(O_1P_1, P_1P_2)$, arc $\smile O_1J_3$ is fixed (see Fig. 28) and, therefore, point $J_3$ is fixed. We similarly prove that points $J_1$ and $J_2$ are fixed.

**19.52.** Let us make use of notations from Problem 19.51. Clearly,

$$\angle(J_1J_2, J_2J_3) = \angle(J_1W, WJ_3) = \angle(P_3P_2, P_2P_1).$$

For the other angles of the triangle the proof is similar.

**19.53.** Let us prove, for instance, that under the rotational homothety with center $O_1$ that maps $F_2$ to $F_3$ point $J_2$ goes to $J_3$. Indeed, $\angle(J_2O_1, O_1J_3) = \angle(J_2W, WJ_3)$. Moreover, lines $J_2W$ and $J_3W$ are the corresponding lines of figures $F_2$ and $F_3$ and, therefore, the distance from lines $J_2W$ and $J_3W$ to point $O_1$ is equal to the similarity coefficient $k_1$; hence, $\frac{O_1J_2}{O_1J_3} = k_1$.

**19.54.** Let $O_a$ be the intersection point of the circle passing through point $B$ and tangent to line $AC$ at point $A$ and the circle passing through point $C$ and tangent to line $AB$ at point $A$.

By Problem 19.41 b) point $O_a$ is the center of rotational homothety that sends segment $BA$ to segment $AC$. Having similarly defined points $O_b$ and $O_c$ and making use of the result of Problem 19.49 b) we see that lines $AO_a$, $BO_b$ and $CO_c$ intersect at a point that belongs

to the similarity circle $S$. On the other hand, these lines intersect at Lemoin's point $K$ (see Problem 5.128).

The midperpendiculars to the sides of the triangle are the corresponding lines of the considered similar figures. The midperpendiculars intersect at point $O$; hence, $O$ belongs to the similarity circle $S$ (see Problem 19.51 a)). Moreover, the midperpendiculars intersect $S$ at fixed points $A_1$, $B_1$ and $C_1$ of triangle $ABC$ (see Problem 19.51 b)). On the other hand, the lines passing through point $K$ parallel to $BC$, $CA$ and $AB$ are also corresponding lines of the considered figures (see solution to Problem 5.132), therefore, they also intersect circle $S$ at points $A_1$, $B_1$ and $C_1$. Hence, $OA_1 \perp A_1K$, i.e., $OK$ is a diameter of $S$.

**19.55.** If $P$ is the first of Brokar's points of triangle $ABC$, then $CP$, $AP$ and $BP$ are the corresponding lines for similar figures constructed on segments $BC$, $CA$ and $AB$. Therefore, point $P$ belongs to the similarity circle $S$ (see Problem 19.51 a)). Similarly, point $Q$ belongs to $S$. Moreover, lines $CP$, $AP$ and $BP$ intersect $S$ at fixed points $A_1$, $B_1$ and $C_1$ of triangle $ABC$ (cf. Problem 19.51 b)). Since $KA_1 \parallel BC$ (see the solution of Problem 19.54), it follows that $\angle(PA_1, A_1K) = \angle(PC, CB) = \varphi$, i.e., $\smallsmile PK = 2\varphi$. Similarly, $\smallsmile KQ = 2\varphi$. Therefore, $PQ \perp KO$; hence, $OP = OQ$ and $\angle POQ = \frac{1}{2} \smallsmile PQ = 2\varphi$.

# Chapter 20. THE PRINCIPLE OF AN EXTREMAL ELEMENT

## Background

1. Solving various problems it is often convenient to consider a certain extremal or "boundary" element, i.e., an element at which a certain function takes its maximal or minimal value. For instance, the longest or the shortest side a triangle, the greatest or the smallest angle, etc. This method for solving problems is sometimes called *the principle* (or the *rule*) *of an extremal element*; this term, however, is not conventional.
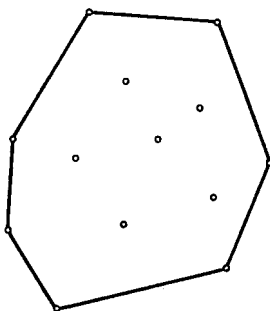


Figure 171

2. Let $O$ be the intersection point of the diagonals of a convex quadrilateral. Its vertices can be denoted so that $CO \leq AO$ and $BO \leq DO$ (see Fig. *). Then under symmetries with respect to point $O$ triangle $BOC$ is mapped inside triangle $AOD$, i.e., in a certain sense triangle $BOC$ is the smallest and triangle $AOD$ is the greatest (see §4).

3. The vertices of the convex hull and the basic lines are also extremal elements; to an extent these notions are used in §5 where they are defined and where their main properties are listed.

## §1. The least and the greatest angles

**20.1.** Prove that if the lengths of all the sides of a triangle are smaller than 1, then its area is smaller than $\frac{1}{4}\sqrt{3}$.

**20.2.** Prove that the disks constructed on the sides of a convex quadrilateral as on diameters completely cover this quadrilateral.

**20.3.** In a country, there are 100 airports such that all the pairwise distances between them are distinct. From each airport a plane lifts up and flies to the nearest airport. Prove that there is no airport to which more than five planes can arrive.

**20.4.** Inside a disk of radius 1, eight points are placed. Prove that the distance between some two of them is smaller than 1.

**20.5.** Six disks are placed on the plane so that point $O$ is inside each of them. Prove that one of these disks contains the center of some other disk.

**20.6.** Inside an acute triangle point $P$ is taken. Prove that the greatest distance from $P$ to the vertices of this triangle is smaller than twice the shortest of the distances from $P$ to the sides of the triangle.

**20.7.** The lengths of a triangle's bisectors do not exceed 1. Prove that the area of the triangle does not exceed $\frac{1}{\sqrt{3}}$.

## §2. The least and the greatest distances

**20.8.** Given $n \geq 3$ points on the plane not all of them on one line. Prove that there is a circle passing through three of the given points such that none of the remaining points lies inside the circle.

**20.9.** Several points are placed on the plane so that all the pairwise distances between them are distinct. Each of these points is connected with the nearest one by a line segment. Do some of these segments constitute a closed broken line?

**20.10.** Prove that at least one of the bases of perpendiculars dropped from an interior point of a convex polygon to its sides is on the side itself and not on its extension.

**20.11.** Prove that in any convex pentagon there are three diagonals from which one can construct a triangle.

**20.12.** Prove that it is impossible to cover a polygon with two polygons which are homothetic to the given one with coefficient $k$ for $0 < k < 1$.

**20.13.** Given finitely many points on the plane such that any line passing through two of the given points contains one more of the given points. Prove that all the given points belong to one line.

**20.14.** In plane, there are given finitely many pairwise non-parallel lines such that through the intersection point of any two of them one more of the given lines passes. Prove that all these lines pass through one point.

**20.15.** In plane, there are given $n$ points. The midpoints of all the segments with both endpoints in these points are marked, the given points are also marked. Prove that there are not less than $2n - 3$ marked points.

See also Problems 9.17, 9.19.

## §3. The least and the greatest areas

**20.16.** In plane, there are $n$ points. The area of any triangle with vertices in these points does not exceed 1. Prove that all these points can be placed in a triangle whose area is equal to 4.

**20.17.** Polygon $M'$ is homothetic to a polygon $M$ with homothety coefficient equal to $-\frac{1}{2}$. Prove that there exists a parallel translation that sends $M'$ inside $M$.

## §4. The greatest triangle

**20.18.** Let $O$ be the intersection point of diagonals of convex quadrilateral $ABCD$. Prove that if the perimeters of triangles $ABO$, $BCO$, $CDO$ and $DAO$ are equal, then $ABCD$ is a rhombus.

**20.19.** Prove that if the center of the inscribed circle of a quadrilateral coincides with the intersection point of the diagonals, then this quadrilateral is a rhombus.

**20.20.** Let $O$ be the intersection point of the diagonals of convex quadrilateral $ABCD$. Prove that if the radii of inscribed circles of triangles $ABO$, $BCO$, $CDO$ and $DAO$ are equal, then $ABCD$ is a rhombus.

## §5. The convex hull and the base lines

While solving problems of this section we will consider convex hulls of systems of points and base lines of convex polygons.

The *convex hull* of a finite set of points is the least convex polygon which contains all these points. The word "least" means that the polygon is not contained in any other such polygon. Any finite system of points possesses a unique convex hull (Fig. 29).
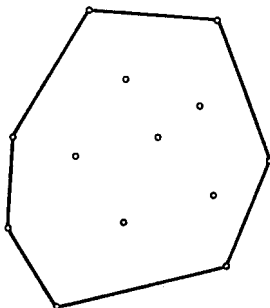
Figure 171

A *base line* of a convex polygon is a line passing through its vertex and with the property that the polygon is situated on one side of it. It is easy to verify that for any convex polygon there exist precisely two base lines parallel to a given line (Fig. 30).
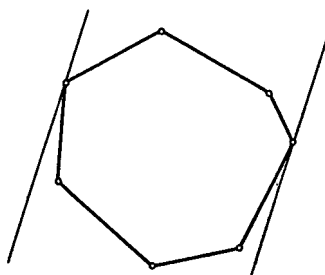
Figure 172

**20.21.** Solve Problem 20.8 making use of the notion of the convex hull.

**20.22.** Given $2n + 3$ points on a plane no three of which belong to one line and no four of which belong to one circle. Prove that one can select three points among these so that $n$ of the remaining points lie inside the circle drawn through the selected points and $n$ of the points lie outside the circle.

**20.23.** Prove that any convex polygon of area 1 can be placed inside a rectangle of area 2.

**20.24.** Given a finite set of points in plane prove that there always exists a point among them for which not more than three of the given points are the nearest to it.

**20.25.** On the table lie $n$ cardboard and $n$ plastic squares so that no two cardboard and no two plastic squares have common points, the boundary points included. It turned out that the set of vertices of the cardboard squares coincides with that of the plastic squares. Is it necessarily true that every cardboard square coincides with a plastic one?

**20.26.** Given $n \geq 4$ points in plane so that no three of them belong to one line. Prove that if for any 3 of them there exists a fourth (among the given ones) together with which they form vertices of a parallelogram, then $n = 4$.

## §6. Miscellaneous problems

**20.27.** In plane, there are given a finite set of (not necessarily convex) polygons each two of which have a common point. Prove that there exists a line having a common point with all these polygons.

**20.28.** Is it possible to place 1000 segments on the plane so that the endpoints of every segment are interior points of certain other of these segments?

**20.29.** Given four points in plane not on one line. Prove that at least one of the triangles with vertices in these points is not an acute one.

**20.30.** Given an infinite set of rectangles in plane. The vertices of each of the rectangles lie in points with coordinates $(0, 0)$, $(0, m)$, $(n, 0)$, $(n, m)$, where $n$ and $m$ are positive integers (each rectangle has its own numbers). Prove that among these rectangles one can select such a pair that one is contained inside the other one.

**20.31.** Given a convex polygon $A_1 \ldots A_n$, prove that the circumscribed circle of triangle $A_i A_{i+1} A_{i+2}$ contains the whole polygon.

## Solutions

**20.1.** Let $\alpha$ be the least angle of the triangle. Then $\alpha \le 60°$. Therefore, $S = \frac{bc \sin \alpha}{2} \le \frac{\sin 60°}{2} = \frac{\sqrt{3}}{4}$.

**20.2.** Let $X$ be an arbitrary point inside a convex quadrilateral. Since

$$\angle AXB + \angle BXC + \angle CXD + \angle AXD = 360°,$$

the maximal of these angles is not less than $90°$. Let, for definiteness sake, $\angle AXB \ge 90°$. Then point $X$ is inside the circle with diameter $AB$.

**20.3.** If airplanes from points $A$ and $B$ arrived to point $O$, then $AB$ is the longest side of triangle $AOB$, i.e., $\angle AOB > 60°$. Suppose that airplanes from points $A_1$, ..., $A_n$ arrived to point $O$. Then one of the angles $\angle A_i O A_j$ does not exceed $\frac{360°}{n}$. Therefore, $\frac{360°}{n} > 60°$, i.e., $n < 6$.

**20.4.** At least seven points are distinct from the center $O$ of the circle. Therefore, the least of the angles $\angle A_i O A_j$, where $A_i$ and $A_j$ are given points, does not exceed $\frac{360°}{7} < 60°$. If $A$ and $B$ are points corresponding to the least angle, then $AB < 1$ because $AO \le 1$, $BO \le 1$ and angle $\angle AOB$ cannot be the largest angle of triangle $AOB$.

**20.5.** One of the angles between the six segments that connect point $O$ with the centers of the disks does not exceed $\frac{360°}{6} = 60°$. Let $\angle O_1 O O_2 \le 60°$, where $O_1$ and $O_2$ are the centers of the disks of radius $r_1$ and $r_2$, respectively. Since $\angle O_1 O O_2 \le 60°$, this angle is not the largest angle in triangle $O_1 O O_2$ and, therefore, either $O_1 O_2 \le O_1 O$ or $O_1 O_2 \le O_2 O$. Let, for definiteness, $O_1 O_2 \le O_1 O$. Since point $O$ is inside the circles, $O_1 O < r_1$. Therefore, $O_1 O_2 \le O_1 O < r_1$, i.e., point $O_2$ is inside the disk of radius $r_1$ with center $O_1$.

**20.6.** Let us drop perpendiculars $PA_1$, $PB_1$ and $PC_1$ from point $P$ to sides $BC$, $CA$ and $AB$, respectively, and select the greatest of the angles formed by these perpendiculars and rays $PA$, $PB$ and $PC$. Let, for definiteness sake, this be angle $\angle APC_1$. Then $\angle APC_1 \ge 60°$; hence, $PC_1 : AP = \cos APC_1 \le \cos 60° = \frac{1}{2}$, i.e., $AP \ge 2PC_1$. Clearly, the inequality still holds if $AP$ is replaced with the greatest of the numbers $AP$, $BP$ and $CP$ and $PC_1$ is replaced with the smallest of the numbers $PA_1$, $PB_1$ and $PC_1$.

**20.7.** Let, for definiteness, $\alpha$ be the smallest angle of triangle $ABC$; let $AD$ be the bisector. One of sides $AB$ and $AC$ does not exceed $AD/\cos(\alpha/2)$ since otherwise segment $BC$ does not pass through point $D$. Let, for definiteness,

$$AB \le \frac{AD}{\cos(\alpha/2)} \le \frac{AD}{\cos 30°} \le \frac{2}{\sqrt{3}}.$$

Then $S_{ABC} = \frac{1}{2}h_c AB \leq \frac{1}{2}l_c AB \leq \frac{1}{\sqrt{3}}$.

**20.8.** Let $A$ and $B$ be those of the given points for which the distance between them is minimal. Then inside the circle with diameter $AB$ there are no given points. Let $C$ be the remaining point — the vertex of the greatest angle that subtends segment $AB$. Then inside the circle passing through points $A$, $B$ and $C$ there are no given points.

**20.9.** Suppose that we have obtained a closed broken line. Then $AB$ is the longest link of this broken line and $AC$ and $BD$ are the links neighbouring to $AB$. Then $AC < AB$, i.e., $B$ is not the point closest to $A$ and $BD < AB$, i.e., $A$ is not the point closest to $B$. Therefore, points $A$ and $B$ cannot be connected. Contradiction.

**20.10.** Let $O$ be the given point. Let us draw lines containing the sides of the polygon and select among them the one which is the least distant from point $O$. Let this line contain side $AB$. Let us prove that the base of the perpendicular dropped from $O$ to $AB$ belongs to side $AB$ itself. Suppose that the base of the perpendicular dropped from $O$ to line $AB$ is point $P$ lying outside segment $AB$. Since $O$ belongs to the interior of the convex polygon, segment $OP$ intersects side $CD$ at point $Q$. Clearly, $OQ < OP$ and the distance from $O$ to line $CD$ is smaller than $OQ$. Therefore, line $CD$ is less distant from point $O$ than line $AB$. This contradicts the choice of line $AB$.

**20.11.** Let $BE$ be the longest diagonal of pentagon $ABCDE$. Let us prove then that from segments $BE$, $EC$ and $BD$ one can construct a triangle. To this end, it suffices to verify that $BE < EC + BD$. Let $O$ be the intersection point of diagonals $BD$ and $EC$. Then

$$BE < BO + OE < BD + EC.$$

**20.12.** Let $O_1$ and $O_2$ be the centers of homotheties, each with coefficient $k$, sending polygon $M$ to polygons $M_1$ and $M_2$, respectively. Then a point from $M$ the most distant from line $O_1O_2$ is not covered by polygons $M_1$ and $M_2$.

**20.13.** Suppose that not all of the given points lie on one line. Through every pair of given points draw a line (there are finitely many of such lines) and select the least nonzero distance from the given points to these lines. Let the least distance be the one from point $A$ to line $BC$, where points $B$ and $C$ are among given ones.

On line $BC$, there lies one more of the given points, $D$. Drop perpendicular $AQ$ from point $A$ to line $BC$. Two of the points $B$, $C$ and $D$ lie to one side of point $Q$, let these be $C$ and $D$. Let, for definiteness, $CQ < DQ$ (Fig. 31).
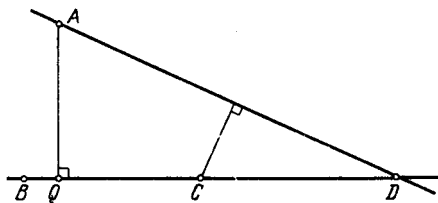


Figure 173 (Sol. 20.13)

Then the distance from point $C$ to line $AD$ is smaller than that from $A$ to line $BC$ which contradicts to the choice of point $A$ and line $BC$.

**20.14.** Suppose that not all lines pass through one point. Consider the intersection points of lines and select the least nonzero distance from these points to the given lines. Let the least distance be the one from point $A$ to line $l$. Through point $A$ at least three of given lines pass. Let them intersect line $l$ at points $B$, $C$ and $D$. From point $A$ drop perpendicular $AQ$ to line $l$.
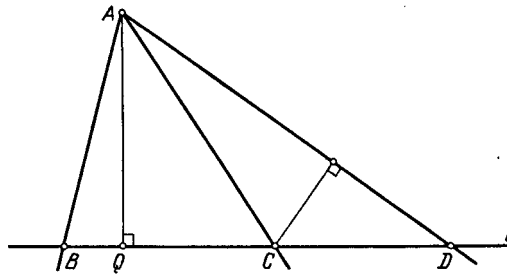
Figure 174 (Sol. 20.14)

Two of the points $B$, $C$ and $D$ lie on one side of point $Q$, let them be $C$ and $D$. Let, for definiteness, $CQ < DQ$ (Fig. 32). Then the distance from point $C$ to line $AD$ is smaller than the distance from point $A$ to line $l$ which contradicts the choice of $A$ and $l$.

**20.15.** Let $A$ and $B$ be the most distant from each other given points. The midpoints of the segments that connect point $A$ (resp. $B$) with the other points are all distinct and lie inside the circle of radius $\frac{1}{2}AB$ with center $A$ (resp. $B$). The two disks obtained have only one common point and, therefore, there are no less than $2(n-1)-1 = 2n-3$ distinct fixed points.

**20.16.** Among all the triangles with vertices in the given points select a triangle of the greatest area. Let this be triangle $ABC$. Let us draw through vertex $C$ line $l_c$ so that $l_c \parallel AB$. If points $X$ and $A$ lie on different sides of line $l_c$, then $S_{ABX} > S_{ABC}$. Therefore, all the given points lie on one side of $l_c$.

Similarly, drawing lines $l_b$ and $l_a$ through points $B$ and $A$ so that $l_b \parallel AC$ and $l_a \parallel BC$ we see that all given points lie inside (or on the boundary of) the triangle formed by lines $l_a$, $l_b$ and $l_c$. The area of this triangle is exactly four times that of triangle $ABC$ and, therefore, it does not exceed 4.

**20.17.** Let $ABC$ be the triangle of the greatest area among these with vertices in the vertices of polygon $M$. Then $M$ is contained inside triangle $A_1B_1C_1$ the midpoints of whose sides are points $A$, $B$ and $C$. The homothety with center in the center of mass of triangle $ABC$ and with coefficient $-\frac{1}{2}$ sends triangle $A_1B_1C_1$ to triangle $ABC$ and, therefore, sends polygon $M$ inside triangle $ABC$.

**20.18.** For definiteness, we may assume that $AO \geq CO$ and $DO \geq BO$. Let points $B_1$ and $C_1$ be symmetric to points $B$ and $C$ through point $O$ (Fig. 33).
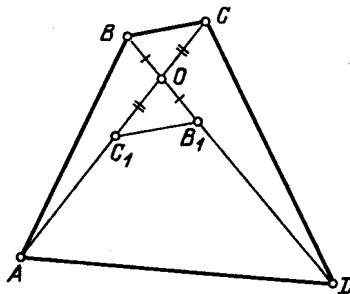


Figure 175 (Sol. 20.18)

Since triangle $B_1OC_1$ lies inside triangle $AOD$, it follows that $P_{AOD} \geq P_{B_1OC_1} = P_{BOC}$ and the equality is attained only if $B_1 = D$ and $C_1 = A$ (see Problem 9.27 b)). Therefore, $ABCD$ is a parallelogram. Therefore, $AB - BC = P_{ABO} - P_{BCO} = 0$, i.e., $ABCD$ is a rhombus.

**20.19.** Let $O$ be the intersection point of the diagonals of quadrilateral $ABCD$. For definiteness, we may assume that $AO \geq CO$ and $DO \geq BO$. Let points $B_1$ and $C_1$ be symmetric to points $B$ and $C$, respectively, through point $O$. Since $O$ is the center of the circle inscribed into the quadrilateral, we see that segment $B_1C_1$ is tangent to this circle. Therefore, segment $AD$ can be tangent to this circle only if $B_1 = D$ and $C_1 = A$, i.e., if $ABCD$ is a parallelogram. One can inscribe a circle into this parallelogram since this parallelogram is a rhombus.

**20.20.** For definiteness, we may assume that $AO \geq CO$ and $DO \geq BO$. Let points $B_1$ and $C_1$ be symmetric to points $B$ and $C$ through point $O$. Then triangle $C_1OB_1$ is contained inside triangle $AOD$ and, therefore, the inscribed circle $S$ of triangle $C_1OB_1$ is contained inside triangle $AOD$. Suppose that segment $AD$ does not coincide with segment $C_1B_1$. Then circle $S$ turns into the inscribed circle of triangle $AOD$ under the homothety with center $O$ and coefficient greater than 1, i.e., $r_{AOD} > r_{C_1OB_1} = r_{COB}$. We have got a contradiction; hence, $A = C_1$ and $D = B_1$, i.e., $ABCD$ is a parallelogram.

In parallelogram $ABCD$, the areas of triangles $AOB$ and $BOC$ are equal and, therefore, if the inscribed circles have equal radii, then they have equal perimeters since $S = pr$. It follows that $AB = BC$, i.e., $ABCD$ is a rhombus.

**20.21.** Let $AB$ be the side of the convex hull of the given points, $B_1$ be the nearest to $A$ of all the given points that lie on $AB$. Select the one of the remaining points that is the vertex of the greatest angle that subtends segment $AB_1$. Let this be point $C$. Then the circumscribed circle of triangle $AB_1C$ is the one to be found.

**20.22.** Let $AB$ be one of the sides of the convex hull of the set of given points. Let us enumerate the remaining points in the order of increase of the angles with vertex in these points that subtend segment $AB$, i.e., denote them by $C_1, C_2, \ldots, C_{2n+1}$ so that

$$\angle AC_1B < \angle AC_2B \leq \cdots < \angle AC_{2n+1}B.$$

Then points $C_1, \ldots, C_n$ lie outside the circle circumscribed about triangle $ABC_{n+1}$ and points $C_{n+2}, \ldots, C_{2n+1}$ lie inside it, i.e., this is the circle to be constructed.

**20.23.** Let $AB$ be the greatest diagonal (or side) of the polygon. Through points $A$ and $B$ draw lines $a$ and $b$ perpendicular to line $AB$. If $X$ is a vertex of the polygon, then $AX \leq AB$ and $XB \leq AB$, therefore, the polygon lies inside the band formed by lines $a$ and $b$.
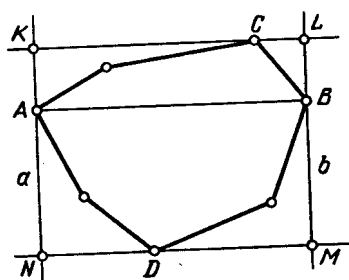


Figure 176 (Sol. 20.23)

Draw the base lines of the polygon parallel to $AB$. Let these lines pass through vertices $C$ and $D$ and together with $a$ and $b$ form rectangle $KLMN$ (see Fig. 34). Then

$$S_{KLMN} = 2S_{ABC} + 2S_{ABD} = 2S_{ACBD}.$$

Since quadrilateral $ACBD$ is contained in the initial polygon whose area is equal to 1, $S_{KLMN} \leq 2$.

**20.24.** Select the least of all the distances between the given points and consider points which have neighbours at this distance. Clearly, it suffices to prove the required statement for these points. Let $P$ be the vertex of the convex hull of these points. If $A_i$ and $A_j$ are the points nearest to $P$, then $A_iA_j \geq A_iP$ and $A_iA_j \geq A_jP$ and, therefore, $\angle A_iPA_j \geq 60°$. It follows that $P$ cannot have four nearest neighbours since otherwise one of the angles $\angle A_iPA_j$ would have been smaller than $\frac{180°}{3} = 60°$. Therefore, $P$ is the point to be found.

**20.25.** Suppose that there are cardboard squares that do not coincide with the plastic ones. Let us discard all the coinciding squares and consider the convex hull of the vertices of the remaining squares. Let $A$ be a vertex of this convex hull. Then $A$ is a vertex of two distinct squares, a cardboard one and a plastic one. It is easy to verify that one of the vertices of the smaller of these squares lies inside the larger one (Fig. 35).

Let, for definiteness, vertex $B$ of the cardboard square lie inside the plastic one. Then point $B$ lies inside a plastic square and is a vertex of another plastic square, which is impossible. This is a contradiction, hence, every cardboard square coincides with a plastic one.

**20.26.** Let us consider the convex hull of the given points. The two cases are possible:

1) The convex hull is a parallelogram, $ABCD$. If point $M$ lies inside parallelogram $ABCD$, then the vertices of all three parallelograms with vertices at $A$, $B$, and $M$ lie outside $ABCD$ (Fig. 36). Hence, in this case there can be no other points except $A$, $B$, $C$ and $D$.

2) The convex hull is not a parallelogram. Let $AB$ and $BC$ be edges of the convex hull. Let us draw base lines parallel to $AB$ and $BC$. Let these base lines pass through vertices $P$ and $Q$. Then the vertices of all the three parallelograms with vertices at $B$, $P$ and $Q$ lie outside the convex hull (Fig. 37).

They even lie outside the parallelogram formed by the base lines except for the case when $P$ and $Q$ are vertices of this parallelogram. In this last case the fourth vertex of the parallelogram does not belong to the convex hull since the convex hull is not a parallelogram.

**20.27.** In plane, take an arbitrary straight line $l$ and project all the polygons to it. We will get several segments any two of which have a common point. Let us order line $l$; consider left endpoints of the segments-projections and select the right-most left endpoint. The point belongs to all the segments and, therefore, the perpendicular drawn through it to $l$ intersects all the given polygons.

**20.28.** Let 1000 segments lie in plane. Take an arbitrary line $l$ not perpendicular to any of them and consider the projections of the endpoints of all these segments on $l$. It is clear that the endpoint of the segment whose projection is the left-most of the obtained points cannot belong to the interior of another segment.

**20.29.** Two variants of disposition of these four points are possible:

(1) The points are vertices of a convex quadrilateral, $ABCD$. Take the largest of the angles of its vertices. Let this be angle $\angle ABC$. Then $\angle ABC \geq 90°$, i.e., triangle $ABC$ is not an acute one.

(2) Point $D$ lies inside triangle $ABC$. Select the greatest of the angles $\angle ADB$, $\angle BDC$ and $\angle ADC$. Let this be angle $\angle ADB$. Then $\angle ADB \geq 120°$, i.e., triangle $ADB$ is an obtuse one.

We can prove in the following way that there are no other positions of the four points. The lines that pass through three of given points divide the plane into seven parts (Fig. 38). If the fourth given point belongs to the 2nd, 4th or 6th part, then we are in situation (1); if it belongs to the 1st, 3rd, 5th or 7th part, then we are in situation (2).

**20.30.** The rectangle with vertices at points $(0,0)$, $(0,m)$, $(n,0)$ and $(n,m)$ the horizontal side is equal to $n$ and vertical side is equal to $m$. From the given set select a rectangle with

Figure 177 (Sol. 20.25)
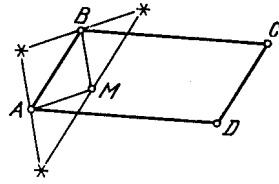
Figure 178 (Sol. 20.26)
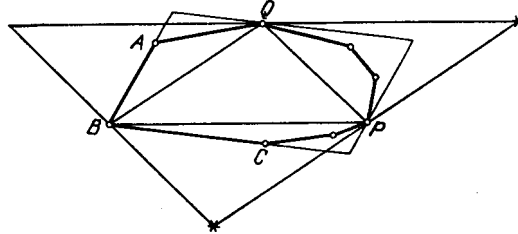


Figure 179 (Sol. 20.26)
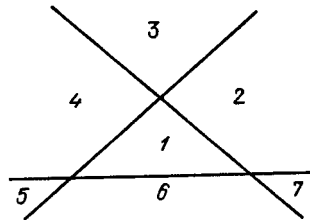


Figure 180 (Sol. 20.29)

the least horizontal side. Let the length of its vertical side be equal to $m_1$. Consider any side $m_1$ of the remaining rectangles. The two cases are possible:

1) The vertical sides of two of these $m_1$-rectangles are equal. Then one of them is contained in another one.

2) The vertical sides of all these rectangles are distinct. Then the vertical side of one of them is greater than $m_1$ and, therefore, it contains the rectangle with the least horizontal side.

**20.31.** Consider all the circles passing through two neighbouring vertices $A_i$ and $A_{i+1}$ and a vertex $A_j$ such that $\angle A_i A_j A_{i+1} < 90°$. At least one such circle exists. Indeed, one of the angles $\angle A_i A_{i+2} A_{i+1}$ and $\angle A_{i+1} A_i A_{i+2}$ is smaller than $90°$; in the first case set $A_j = A_{i+2}$ and in the second case set $A_j = A_i$. Among all such circles (for all $i$ and $j$) select a circle $S$ of the largest radius; let, for definiteness, it pass through points $A_1$, $A_2$ and $A_k$.

Suppose that vertex $A_p$ lies outside $S$. Then points $A_p$ and $A_k$ lie on one side of line $A_1 A_2$ and $\angle A_1 A_p A_2 < \angle A_1 A_k A_2 \leq 90°$. The law of sines implies that the radius of the circumscribed circle of triangle $A_1 A_p A_2$ is greater than that of $A_1 A_k A_2$. This is a contradiction and, therefore, $S$ contains the whole polygon $A_1 \ldots A_n$.

Let, for definiteness sake, $\angle A_2 A_1 A_k \leq \angle A_1 A_2 A_k$. Let us prove then that $A_2$ and $A_k$ are neighbouring vertices. If $A_k \neq A_3$, then

$$180° - \angle A_2 A_3 A_k \leq \angle A_2 A_1 A_k \leq 90°$$

and, therefore, the radius of the circumscribed circle of triangle $A_2 A_3 A_k$ is greater than the radius of the circumscribed circle of triangle $A_1 A_2 A_k$. Contradiction implies that $S$ passes through neighbouring vertices $A_1$, $A_2$ and $A_3$.

# Chapter 21. DIRICHLET'S PRINCIPLE

## Background

1. The most popular (Russian) formulation of *Dirichlet's* or *pigeonhole principle* is the following one: "If $m$ rabbits sit in $n$ hatches and $m > n$, then at least one hatch contains at least two rabbits."

It is even unclear at first glance why this absolutely transparent remark is a quite effective method for solving problems. The point is that in every concrete problem it is sometimes difficult to see what should we designate as the rabbits and the hatches and why there are more rabbits than the hatches. The choice of rabbits and hatches is often obscured; and from the formulation of the problem it is not often clear how to immediately deduce that one should apply Dirichlet's principle. What is very important is that this method gives a *nonconstructive* proof (naturally, we cannot say which precisely hatch contains two rabbits and only know that such a hatch exists) and an attempt to give a constructive proof, i.e., the proof by explicitly constructing or indicating the desired object can lead to far greater difficulties (and more profound results).

2. Certain problems are also solved by methods in a way similar to Dirichlet's principle. Let us formulate the corresponding statements (all of them are easily proved by the rule of contraries).

a) If several segments the sum of whose lengths is greater than 1 lie on a segment of length 1, then at least two of them have a common point.

b) If several arcs the sum of whose lengths is greater than $2\pi$ lie on the circle of radius 1, then at least two of them have a common point.

c) If several figures the sum of whose areas is greater than 1 are inside a figure of area 1, then at least two of them have a common point.

## §1. The case when there are finitely many points, lines, etc.

**21.1.** The nodes of an infinite graph paper are painted two colours. Prove that there exist two horizontal and two vertical lines on whose intersection lie points of the same colour.

**21.2.** Inside an equilateral triangle with side 1 five points are placed. Prove that the distance between certain two of them is shorter than 0.5.

**21.3.** In a $3 \times 4$ rectangle there are placed 6 points. Prove that among them there are two points the distance between which does not exceed $\sqrt{5}$.

**21.4.** On an $8 \times 8$ checkboard the centers of all the cells are marked. Is it possible to divide the board by 13 straight lines so that in each part there are not more than 1 of marked points?

**21.5.** Given 25 points in plane so that among any three of them there are two the distance between which is smaller than 1, prove that there exists a circle of radius 1 that contains not less than 13 of the given points.

**21.6.** In a unit square, there are 51 points. Prove that certain three of them can be covered by a disk of radius $\frac{1}{7}$.

**21.7.** Each of two equal disks is divided into 1985 equal sectors and on each of the disks some 200 sectors are painted (one colour). One of the disks was placed upon the other one and they began rotating one of the disks through multiples of $\frac{360°}{1985}$. Prove that there exists at least 80 positions for which not more than 20 of the painted sectors of the disks coincide.

**21.8.** Each of 9 straight lines divides a square into two quadrilaterals the ratio of whose areas is $2 : 3$. Prove that at least three of those nine straight lines pass through one point.

**21.9.** In a park, there grow 10, 000 trees planted by a so-called square-cluster method (100 rows of 100 trees each). What is the largest number of trees one has to cut down in order to satisfy the following condition: if one stands on any stump, then no other stump is seen (one may assume the trees to be sufficiently thin).

**21.10.** What is the least number of points one has to mark inside a convex $n$-gon in order for the interior of any triangle with the vertices at vertices of the $n$-gon to contain at least one of the marked points?

**21.11.** Point $P$ is taken inside a convex $2n$-gon. Through every vertex of the polygon and $P$ a line is drawn. Prove that there exists a side of the polygon which has no common interior points with neither of the drawn straight lines.

**21.12.** Prove that any convex $2n$-gon has a diagonal non-parallel to either of its sides.

**21.13.** The nodes of an infinite graph paper are painted three colours. Prove that there exists an isosceles right triangle with vertices of one colour.

## §2. Angles and lengths

**21.14.** Given $n$ pairwise nonparallel lines in plane. Prove that the angle between certain two of them does not exceed $\frac{180°}{n}$.

**21.15.** In a circle of radius 1 several chords are drawn. Prove that if every diameter intersects not more than $k$ chords, then the sum of the length of the chords is shorter than $k\pi$.

**21.16.** In plane, point $O$ is marked. Is it possible to place in plane a) five disks; b) four disks that do not cover $O$ and so that any ray with the beginning in $O$ would intersect not less than two disks? ("Intersect" means has a common point.)

**21.17.** Given a line $l$ and a circle of radius $n$. Inside the circle lie $4n$ segments of length 1. Prove that it is possible to draw a line which is either parallel or perpendicular to the given line and intersects at least two of the given segments.

**21.18.** Inside a unit square there lie several circles the sum of their lengths being equal to 10. Prove that there exists a straight line intersecting at least four of these circles.

**21.19.** On a segment of length 1 several segments are marked so that the distance between any two marked points is not equal to 0.1. Prove that the sum of the lengths of the marked segments does not exceed 0.5.

**21.20.** Given two circles the length of each of which is equal to 100 cm. On one of them 100 points are marked, on the other one there are marked several arcs with the sum of their lengths less than 1 cm. Prove that these circles can be identified so that no one of the marked points would be on a marked arc.

**21.21.** Given are two identical circles; on each of them $k$ arcs are marked, the angle value of each of the arcs is $> \frac{1}{k^2-k+1} \cdot 180°$. The circles can be identified(?) so that the marked arcs of one circle would coincide with the marked arcs of the other one. Prove that these circles can be identified so that all the marked arcs would lie on unmarked arcs.

## §3. Area

**21.22.** In square of side 15 there lie 20 pairwise nonintersecting unit squares. Prove that it is possible to place in the large square a unit disk so that it would not intersect any of the small squares.

**21.23.** Given an infinite graph paper and a figure whose area is smaller than the area of a small cell prove that it is possible to place this figure on the paper without covering any of the nodes of the mesh.

**21.24.** Let us call the figure formed by the diagonals of a unit square (Fig. 39) a *cross*. Prove that it is possible to place only a finite number of nonintersecting crosses in a disk of radius 100.
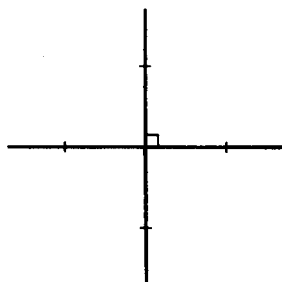


Figure 181 (21.24)

**21.25.** Pairwise distances between points $A_1$, ..., $A_n$ is greater than 2. Prove that any figure whose area is smaller than $\pi$ can be shifted by a vector not longer than 1 so that it would not contain points $A_1$, ..., $A_n$.

**21.26.** In a circle of radius 16 there are placed 650 points. Prove that there exists a ring (annulus) of inner radius 2 and outer radius 3 which contains not less than 10 of the given points.

**21.27.** There are given $n$ figures in plane. Let $S_{i_1 \ldots i_k}$ be the area of the intersection of figures indexed by $i_1$, ..., $i_k$ and $S$ be the area of the part of the plane covered by the given figures; $M_k$ the sum of all the $S_{i_1 \ldots i_k}$. Prove that:

a) $S = M_1 - M_2 + M_3 - \cdots + (-1)^{n+1} M_n$;

b) $S \geq M_1 - M_2 + M_3 - \cdots + (-1)^{m+1} M_m$ for $m$ even and $S \leq M_1 - M_2 + M_3 - \cdots + (-1)^{m+1} M_m$ for $m$ odd.

**21.28.** a) In a square of area 6 there are three polygons of total area 3. Prove that among them there are two polygons such that the area of their intersection is not less than 1.

b) In a square of area 5 there are nine polygons of total area 1. Prove that among them there are two polygons the area of whose intersection is not less than $\frac{1}{9}$.

**21.29.** On a rug of area 1 there are 5 patches the area of each of them being not less than 0.5. Prove that there are two patches such that the area of their intersection is not less than 0.2.

## Solutions

**21.1.** Let us take three vertical lines and nine horizontal lines. Let us consider only intersection points of these lines. Since there are only $2^3 = 8$ variants to paint three points two colours, there are two horizontal lines on which lie similarly coloured triples of points. Among three points painted two colours there are, by Dirichlet's principle, two similarly

coloured points. The vertical lines passing through these points together with the two horizontal lines selected earlier are the ones to be found.

**21.2.** The midlines of an equilateral triangle with side 1 separate it into four equilateral triangles with side 0.5. Therefore, one of the triangles contains at least two of the given points and these points cannot be vertices of the triangle. The distance between these points is less than 0.5.

**21.3.** Let us cut the rectangle into five figures as indicated on Fig. 40. One of the figures contains at least two points and the distance between any two points of each of the figures does not exceed $\sqrt{5}$.
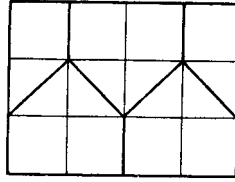


Figure 182 (Sol. 21.3)

**21.4.** 28 fields are adjacent to a side of an $8 \times 8$ chessboard. Let us draw 28 segments that connect the centers of neighbouring end(?) fields. Every line can intersect not more than 2 such segments and, therefore, 13 lines can intersect not more than 26 segments, i.e., there are at least 2 segments that do not intersect any of 13 drawn lines. Therefore, it is impossible to split the chessboard by 13 lines so that in each part there would be not more than 1 marked point since both endpoints of the segment that does not intersect with the lines belongs to one of the parts.

**21.5.** Let $A$ be one of the given points. If all the remaining points lie in disk $S_1$ of radius 1 with center $A$, then we have nothing more to prove.

Now, let $B$ be a given point that lies outside $S_1$, i.e., $AB > 1$. Consider disk $S_2$ of radius 1 with center $B$. Among points $A$, $B$ and $C$, where $C$ is any of the given points, there are two at a distance less than 1 and these cannot be points $A$ and $B$. Therefore, disks $S_1$ and $S_2$ contain all the given points, i.e., one of them contains not less than 13 points.

**21.6.** Let us divide a given square into 25 similar small squares with side 0.2. By Dirichlet's principle one of them contains no less than 3 points. The radius of the circumscribed circle of the square with side 0.2 is equal to $\frac{1}{5}\sqrt{2} < \frac{1}{7}$ and, therefore, it can be covered by a disk of radius $\frac{1}{7}$.

**21.7.** Let us take 1985 disks painted as the second of our disks and place them upon the first disk so that they would take all possible positions. Then over every painted sector of the first disk there lie 200 painted sectors, i.e., there are altogether $200^2$ pairs of coinciding painted sectors. Let there be $n$ positions of the second disk when not less 21 pairs of painted sectors coincide. Then the number of coincidences of painted sectors is not less than $21n$. Therefore, $21n \leq 200^2$, i.e., $n \leq 1904.8$. Since $n$ is an integer, $n \leq 1904$. Therefore, at least for $1985 - 1904 = 81$ positions not more than 20 pairs of painted sectors coincide.

**21.8.** The given lines cannot intersect neighbouring sides of square $ABCD$ since otherwise we would have not two quadrilaterals but a triangle and a pentagon. Let a line intersect sides $BC$ and $AD$ at points $M$ and $N$, respectively. Trapezoids $ABMN$ and $CDNM$ have equal heights, and, therefore, the ratio of their areas is equal to that of their midlines, i.e., $MN$ divides the segment that connects the midpoints of sides $AB$ and $CD$ in the ratio of $2:3$. There are precisely 4 points that divide the midlines of the square in the ratio of $2:3$. Since the given nine lines pass through these four points, then through one of the points at least three lines pass.
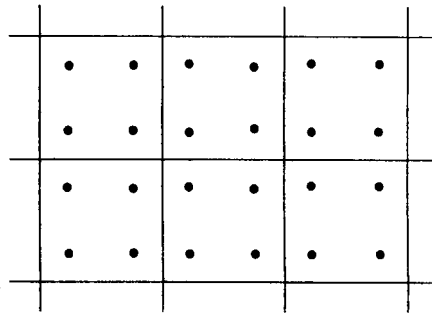
Figure 183 (Sol. 21.8)

**21.9.** Let us divide the trees into 2500 quadruples as shown in Fig. 41. In each such quadruple it is impossible to chop off more than 1 tree. On the other hand, one can chop off all the trees that grow in the left upper corners of the squares formed by our quadruples. Therefore, the largest number of trees that can be chopped off is equal to 2500.

**21.10.** Since any diagonal that goes out of one vertex divides an $n$-gon into $n - 2$ triangles, then $n - 2$ points are necessary.
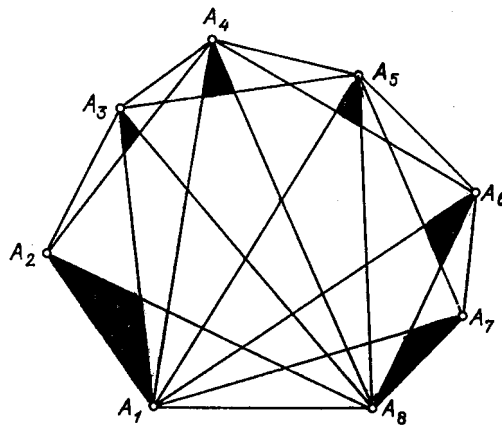


Figure 184 (Sol. 21.10)

From Fig. 42 one can deduce that $n - 2$ points are sufficient: it suffices to mark one points in each shaded triangle. Indeed, inside triangle $A_p A_q A_r$, where $p < q < r$, there is always contained a shaded triangle adjacent to vertex $A_q$.

**21.11.** The two cases are possible.

(1) Point $P$ lies on diagonal $AB$. Then lines $PA$ and $PB$ coincide and do not intersect the sides. There remain $2n - 2$ lines; they intersect not more than $2n - 2$ sides.

(2) Point $P$ does not belong to a diagonal of polygon $A_1 A_2 \ldots A_{2n}$. Let us draw diagonal $A_1 A_{n+1}$. On both sides of it there lie $n$ sides. Let, for definiteness, point $P$ be inside polygon $A_1 \ldots A_{n+1}$ (Fig. 43).

Then lines $PA_{n+1}$, $PA_{n+2}$, $\ldots$, $PA_{2n}$, $PA_1$ (there are $n + 1$ such lines) cannot intersect sides $A_{n+1}A_{n+2}$, $A_{n+2}A_{n+3}$, $\ldots$, $A_{2n}A_1$, respectively. Therefore, the remaining straight lines can intersect not more than $n - 1$ of these $n$ sides.

**21.12.** The number of diagonals of a $2n$-gon is equal to $\frac{2n(2n-3)}{2} = n(2n - 3)$. It is easy to verify that there are not more than $n - 2$ diagonals parallel to the given one. Therefore, there are not more than $2n(n-2)$ diagonals parallel to the sides. Since $2n(n-2) < n(2n-3)$, there exists a diagonal which is not parallel to any side.
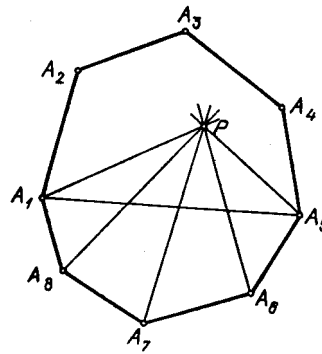
Figure 185 (Sol. 21.11)

**21.13.** Suppose that there does not exist an equilateral right triangle whose legs are parallel to the sides of the cells and with vertices of the same colour. For convenience we may assume that it is the cells which are painted, not the nodes.

Let us divide the paper into squares of side 4; then on the diagonal of each such square there are two cells of the same colour. Let $n$ be greater than the number of distinct colorings of the square of side 4. Consider a square consisting of $n^2$ squares of side 4. On its diagonal we can find two similarly painted squares of side 4. Finally, take square $K$ on whose diagonal we can find two similarly painted squares of side $4n$.
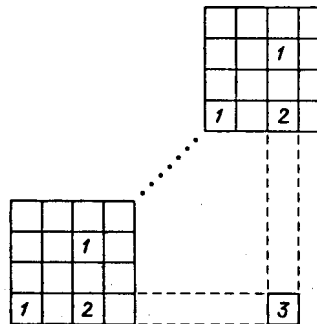


Figure 186 (Sol. 21.13)

Considering the square with side $4n$ and in it two similarly painted squares with side 4 we get four cells of the first colour, two cells of the second colour and one cell of the third colour, see Fig. 44. Similarly, considering square $K$ we get a cell which cannot be of the first, or second, or third colour.

**21.14.** In plane, take an arbitrary point and draw through it lines parallel to the given ones. They divide the plane into $2n$ angles whose sum is equal to $360°$. Therefore, one of these angles does not exceed $\frac{180°}{n}$.

**21.15.** Suppose the sum of the length of the chords is not shorter than $\pi k$. Let us prove that then there exists a diameter which intersects with at least $k+1$ chords. Since the length of the arc corresponding to the chord is greater than the length of this chord, the sum of the lengths of the archs corresponding to given chords is longer than $\pi k$. If we add to these arcs the arcs symmetric to them through the center of the circle, then the sum of the lengths of all these arcs becomes longer than $2\pi k$. Therefore, there exists a point covered by at least $k + 1$ of these arcs. The diameter drawn through this point intersects with at least $k + 1$ chord.

**21.16.** a) It is possible. Let $O$ be the center of regular pentagon $ABCDE$. Then the disks inscribed in angles $\angle AOC$, $\angle BOD$, $\angle COD$, $\angle DOA$ and $\angle EOB$ possess the required property.

b) It is impossible. For each of the four disks consider the angle formed by the tangents to the disk drawn through point $O$. Since each of these four angles is smaller than $180°$, their sum is less than $2 \cdot 360°$. Therefore, there exists a point on the plane covered by not more than 1 of these angles. The ray drawn through this point intersects with not more than one disk.

**21.17.** Let $l_1$ be an arbitrary line perpendicular to $l$. Denote the lengths of the projections of the $i$-th segment to $l$ and $l_1$ by $a_i$ and $b_i$, respectively. Since the length of each segment is equal to 1, we have $a_i + b_i \geq 1$. Therefore,

$$(a_1 + \cdots + a_{4n}) + (b_1 + \cdots + b_{4n}) \geq 4n.$$

Let, for definiteness,

$$a_1 + \cdots + a_{4n} \geq b_1 + \cdots + b_{4n}.$$

Then $a_1 + \cdots + a_{4n} \geq 2n$. The projection of any of the given segment is of length $2n$ because all of them lie inside the circle of radius $n$. If the projections of the given segments to $l$ would have had no common points, then we would had $a_1 + \cdots + a_{4n} < 2n$. Therefore, on $l$ there exists a point which is the image under the projection of at least two of the given segments. The perpendicular to $l$ drawn through this point intersects with at least two of given segments.

**21.18.** Let us project all the given circles on side $AB$ of square $ABCD$. The projection of the circle of length $l$ is a segment of length $\frac{l}{\pi}$. Therefore, the sum of the lengths of the projections of all the given circles is equal to $\frac{10}{\pi}$. Since $\frac{10}{\pi} > 3 = 3AB$, on segment $AB$ there is a point which belongs to projections of at least four circles. The perpendicular to $AB$ drawn through this point intersects at least four circles.

**21.19.** Let us cut the segment into ten segments of length 0.1, stack them in a pile and consider their projection to a similar segment as shown on Fig. 45.
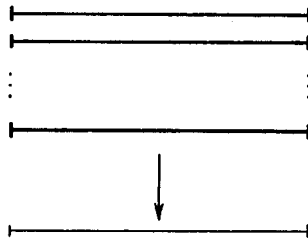


Figure 187 (Sol. 21.19)

Since the distance between any two painted points is not equal to 0.1, the painted points of neighbouring segments cannot be projected into one point. Therefore, neither of the points can be the image under the projection of painted points of more than 5 segments. It follows that the sum of the lengths of the projections of the painted segments (equal to the sum of their lengths) does not exceed $5 \cdot 0.1 = 0.5$.

**21.20.** Let us identify the given circles and let us place a painter in a fixed point of one of them. Let us rotate this circle and let the painter paint a point of the other circle each time when it is a marked point that belongs to a marked arc. We have to prove that after a complete revolution a part of the circle would remain unpainted.

The final result of the painter's job will be the same as if he were rotated 100 times and (s)he was asked to paint the other circle on the $i$-th revolution so that (s)he would have to

paint the $i$-th marked point that belongs to one of the marked arcs. Since in this case at each revolution less than 1 cm is being painted, it follows that after 100 revolutions there will be painted less than 100 cm. Therefore, a part of the circle will be unpainted.

**21.21.** Let us identify(?) our circles and place a painter into a fixed point of one of them. Let us rotate this circle and let the painter paint the point of the other circle against which he moves each time when some of the marked arcs intersect. We have to prove that after a full revolution a part of the circle will be unpainted.

The final result of the painter's job would be the same as if (s)he were rotated $k$ times and was asked to paint the circle on the $i$-th revolution when the $i$-th marked arc on which the painter resides would intersect with a marked arc of the other circle.

Let $\varphi_1$, ..., $\varphi_n$ be the angle parameters of the marked arcs. By the hypothesis $\varphi_1 < \alpha$, ..., $\varphi_n < \alpha$, where $\alpha = \frac{180°}{k^2-k+1}$. During the time when the marked arcs with counters $i$ and $j$ intersect the painter paints an arc of length $\varphi_i + \varphi_j$.

Therefore, the sum of the angle values of the arcs painted during the $i$-th revolution does not exceed $k\varphi_i(\varphi_1 + \cdots + \varphi_k)$ and the sum of the angle values of the arcs painted during all $k$ revolutions does not exceed $2k(\varphi_1 + \cdots + \varphi_k)$. Observe that during all this we have actually counted the intersection of arcs with similar(?) counters $k$ times.

In particular, point $A$ across which the painter moves at the moment when the marked arcs coincide has, definitely, $k$ coats of paint. Therefore, it is desirable to disregard the arcs that the painter paints at the moment when some of the marked arcs with similar counters intersect. Since all these arcs contain point $A$, we actually disregard only one arc and the angle value of this arc does not exceed $2\alpha$.

The sum of the angle values of the remaining part of the arcs painted during the $i$-th revolution does not exceed $(k-1)\varphi_1 + (\varphi_1 + \cdots + \varphi_k - \varphi_i)$ and the sum of the angle values of the remaining part of the arcs painted through all $k$ revolutions does not exceed

$$(2k-2) \cdot (\varphi_1 + \cdots + \varphi_k) < (2k^2 - 2k)\alpha.$$

A part of the circle will be unpainted if $(2k^2 - 2k)\alpha \le 360° - 2\alpha$, i.e., $\alpha \le \frac{180°}{k^2-k+1}$.

**21.22.** Let us consider a figure consisting of all the points whose distance from the small unit square is not greater than 1 (Fig. 46).
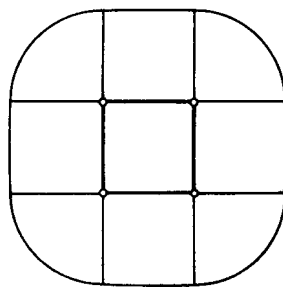


Figure 188 (Sol. 21.22)

It is clear that no unit disk whose center is outside this figure intersects the small square. The area of such a figure is equal to $\pi + 5$. The center of the needed disk should also lie at a distance greater than 1 from the sides of the large square, i.e., inside the square of side 13. Obviously, 20 figures of total area $\pi + 5$ cannot cover a square of side 13 because $20(\pi + 5) < 13^2$. The disk with the center in an uncovered point possesses the desired property.

**21.23.** Let us paint the figure to(?) the graph paper arbitrarily, cut the paper along the cells of the mesh and stack them in a pile moving them parallelly with themselves and

without turning. Let us consider the projection of this stack on a cell. The projections of parts of the figure cannot cover the whole cell since their area is smaller. Now, let us recall how the figure was placed on the graph paper and move the graph paper parallelly with itself so that its vertices would be in the points whose projection is an uncovered point. As a result we get the desired position of the figure.

**21.24.** For every cross consider a disk of radius $\frac{1}{2}\sqrt{2}$ with center in the center of the cross. Let us prove that if two such disks intersect, then the crosses themselves also intersect. The distance between the centers of equal intersecting disks does not exceed the doubled radius of any of them and, therefore, the distance between the centers of the corresponding crosses does not exceed $\frac{1}{\sqrt{2}}$. Let us consider a rectangle given by bars of the first cross and the center of the second one (Fig. 47).
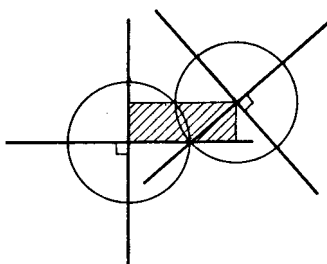


Figure 189 (Sol. 21.24)

One of the bars of the second cross passes through this rectangle and, therefore, it intersects the first cross since the length of the bar is equal to $\frac{1}{\sqrt{2}}$ and the length of the diagonal of the rectangle does not exceed $\frac{1}{\sqrt{2}}$. In the disk of a finite radius one can only place finitely many non-intersecting disks of radius $\frac{1}{2}\sqrt{2}$.

**21.25.** Let $\Phi$ be a given figure, $S_1, \ldots, S_n$ unit disks with centers at points $A_1, \ldots, A_n$. Since disks $S_1, \ldots, S_n$ do not intersect, then neither do figures $V_i = \Phi \cap S_i$, consequently, the sum of their areas does not exceed the area of figure $\Phi$, i.e., it is smaller than $\pi$. Let $O$ be an arbitrary point and $W_i$ the image of $V_i$ under the translation by vector $\overrightarrow{A_iO}$. The figures $W_i$ lie inside the unit disk $S$ centered at $O$ and the sum of their areas is smaller than the area of this disk. Therefore, point $B$ of disk $S$ does not belong to any of the figures $W_i$. It is clear that the translation by vector $\overrightarrow{BO}$ is the desired one.

**21.26.** First, notice that point $X$ belongs to the ring with center $O$ if and only if point $O$ belongs to a similar ring centered at $X$. Therefore, it suffices to show that if we construct rings with centers at given points, then not less than 10 rings will cover one of the points of the considered disk. The considered rings lie inside a disk of radius $16 + 3 = 19$ whose area is equal to $361\pi$. It remains to notice that $9 \cdot 361\pi = 3249\pi$ and the total area of the rings is equal to $650 \cdot 5\pi = 3250\pi$.

**21.27.** a) Let $\binom{n}{k}$ be the number of ways to choose $k$ elements from $n$ indistinguishable ones. One can verify the following Newton binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Denote by $W_m$ the area of the part of the plane covered by exactly $m$ figures. This part consists of pieces each of which is covered by certain $m$ figures. The area of each such piece has been counted $\binom{n}{k}$ times in calculation of $M_k$ because from $m$ figures we can form $\binom{n}{k}$

intersections of $k$ figures. Therefore,

$$M_k = \binom{n}{k} W_k + \binom{k+1}{k} W_{k+1} + \cdots + \binom{n}{k} W_n.$$

It follows that

$$M_1 - M_2 + M_3 - \cdots =$$
$$\binom{1}{1} W_1 + (\binom{2}{1} - \binom{2}{2}) W_2 + \cdots + (\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \ldots) W_n =$$
$$W_1 + \cdots + W_n$$

since

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots - (-1)^m \binom{m}{m} =$$
$$(-1 + \binom{m}{1} - \binom{m}{2} + \ldots) + 1) = -(1-1)^m + 1 = 1.$$

It remains to observe that $S = W_1 + \cdots + W_n$.

b) According to heading a)

$$S - (M_1 - M_2 + \cdots + (-1)^{m+1} M_m) =$$
$$(-1)^{m+2} M_{m+1} + (-1)^{m+3} M_{m+2} + \ldots \cdots + (-1)^{n+1} M_n =$$
$$\sum_{i=1}^{n} ((-1)^{m+2} \binom{i}{m+1} + \cdots + (-1)^{n+1} \binom{i}{n}) W_i$$

(it is convenient to assume that $\binom{n}{k}$ is defined for $k > n$ so that $\binom{n}{k} = 0$). Therefore, it suffices to verify that

$$\binom{i}{m+1} - \binom{i}{m+2} + \binom{i}{m+3} - \cdots + (-1)^{m+n+1} \binom{i}{n} \geq 0 \text{ for } i \leq n.$$

The identity

$$(x+y)^i = (x+y)^{i-1}(x+y)$$

implies that $\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j}$. Hence,

$$\binom{i}{m+1} - \binom{i}{m+2} + \cdots + (-1)^{m+n+1} \binom{i}{n} = \binom{i-1}{m} \pm \binom{i-1}{n}.$$

It remains to notice that $\binom{i-1}{n} = 0$ for $i \leq n$.

**21.28.** a) By Problem 21.27 a) we have

$$6 = 9 - (S_{12} + S_{23} + S_{13}) + S_{123},$$

i.e.,

$$S_{12} + S_{23} + S_{13} = 3 + S_{123} \geq 3.$$

Hence, one of the numbers $S_{12}$, $S_{23}$, $S_{13}$ is not less than 1.

b) By Problem 21.27 b) $5 \geq 9 - M_2$, i.e., $M_2 \geq 4$. Since from 9 polygons one can form $9 \cdot \frac{8}{2} = 36$ pairs, the area of the common part of one of such pairs is not less than $\frac{M_2}{36} \geq \frac{1}{9}$.

**21.29.** Let the area of the rug be equal to $M$, the area of the intersection of the patches indexed by $i_1$, ..., $i_k$ is equal to $S_{i_1...i_k}$ and $M_k = \sum S_{i_1...i_k}$. By Problem 21.27 a)

$$M - M_1 + M_2 - M_3 + M_4 - M_5 \geq 0$$

since $M \geq S$. One can write similar inequalities not only for the whole rug but also for every patch: if we consider the patch $S_1$ as the rug with patches $S_{12}$, $S_{13}$, $S_{14}$, $S_{15}$ we get

$$S_1 - \sum_i S_{1i} + \sum_{i<j} S_{1ij} - \sum_{i<j<k} S_{1ijk} + S_{12345} \geq 0.$$

Adding such inequalities for all five patches we get

$$M_1 - 2M_2 + 3M_3 - 4M_4 + 5M_5 \geq 0$$

(the summand $S_{i_1...i_k}$ enters the inequality for patches $i_1$, ..., $i_k$ and, therefore, it enters the sum of all inequalities with coefficient $k$). Adding the inequalities

$$3(M - M_1 + M_2 - M_3 + M_4 - M_5) \geq 0 \text{ and } M_1 - 2M_2 + 3M_3 - 4M_4 + 5M_5 \geq 0$$

we get

$$3M - 2M_1 + M_2 - M_4 + 2M_5 \geq 0.$$

Adding to this the inequality $M_4 - 2M_5 \geq 0$ (which follows from the fact that $S_{12345}$ enters every $S_{i_1 i_2 i_3 i_4}$, i.e., $M_4 \geq 5M_5 \geq 2M_5$) we get $3M - 2M_1 + M_2 \geq 0$, i.e., $M_2 \geq 2M_1 - 3M \geq 5 - 3 = 2$.

Since from five patches we can form ten pairs, the area of the intersection of patches from one of these pairs is not less than $\frac{1}{10}M_2 \geq 0.2$.

# Chapter 22. CONVEX AND NONCONVEX POLYGONS

## Background

1. There are several different (nonequivalent) definitions of a convex polygon. Let us give the most known and most often encountered definitions. A polygon is called *convex* if one of the following conditions is satisfied:

a) the polygon lies on one side of any of its sides (i.e., the intersections of the sides of the polygon do not intersect its other sides);

b) the polygon is the intersection (i.e., the common part) of several half planes;

c) any segment whose endpoints belong to the polygon wholly belongs to the polygon.

2. A figure is called a *convex* one if any segment with the endpoints in the points of a figure belongs to the figure.

3. In solutions of several problems of this chapter we make use of the notion of the *convex hull* and the *basic line*.

## §1. Convex polygons

**22.1.** Given $n$ points in plane such that any four of them are the vertices of a convex quadrilateral, prove that these points are the vertices of a convex $n$-gon.

**22.2.** Given five points in plane no three of which belong to one line, prove that four of these points are placed in the vertices of a convex quadrilateral.

**22.3.** Given several regular $n$-gons in plane prove that the convex hull of their vertices has not less than $n$ angles.

**22.4.** Among all numbers $n$ such that any convex 100-gon can be represented as an intersection (i.e., the common part) of $n$ triangles find the least number.

**22.5.** A convex heptagon will be called *singular* if three of its diagonals intersect at one point. Prove that by a slight movement of one of the vertices of a singular heptagon one can obtain a nonsingular heptagon.

**22.6.** In plane lie two convex polygons, $F$ and $G$. Denote by $H$ the set of midpoints of the segments one endpoint of each of which belongs to $F$ and the other one to $G$. Prove that $H$ is a convex polygon.

a) How many sides can $H$ have if $F$ and $G$ have $n_1$ and $n_2$ sides, respectively?

b) What value can the perimeter of $H$ have if the perimeters of $F$ and $G$ are equal to $P_1$ and $P_2$, respectively?

**22.7.** Prove that there exists a number $N$ such that among any $N$ points no three of which lie on one line one can select 100 points which are vertices of a convex polygon.

* * *

**22.8.** Prove that in any convex polygon except parallelogram one can select three sides whose extensions form a triangle which is ambient(?) with respect to the given polygon.

**22.9.** Given a convex $n$-gon no two sides of which are parallel, prove that there are not less than $n - 2$ distinct triangles such as discussed in Problem 22.8.

**22.10.** A point $O$ is inside a convex $n$-gon, $A_1 \ldots A_n$. Prove that among the angles $\angle A_i O A_j$ there are not fewer than $n - 1$ acute ones.

**22.11.** Convex $n$-gon $A_1 \ldots A_n$ is inscribed in a circle and among the vertices of the polygon there are no diametrically opposite points. Prove that among the triangles $A_p A_q A_r$ there is at least one acute triangle, then there are not fewer than $n - 2$ such acute triangles.

## §2. Helly's theorem

**22.12.** a) Given four convex figures in plane such that any three of them have a common point, prove that all of them have a common point.

b) (*Helly's theorem.*) Given $n$ convex figures in plane such that any three of them have a common point, prove that all $n$ figures have a common point.

**22.13.** Given $n$ points in plane such that any three of them can be covered by a unit disk, prove that all $n$ points can be covered by a unit disk.

**22.14.** Prove that inside any convex heptagon there is a point that does not belong to any of quadrilaterals formed by quadruples of its neighbouring vertices.

**22.15.** Given several parallel segments such that for any three of them there is a line that intersects them, prove that there exists a line that intersects all the points.

## §3. Non-convex polygons

In this section all polygons considered are non-convex unless otherwise mentioned.

**22.16.** Is it true that any pentagon lies on one side of not fewer than two of its sides?

**22.17.** a) Draw a polygon and point $O$ inside it so that the polygon's angle with vertex in $O$ would not subtend any side without intersecting some of the other of the polygon's sides.

b) Draw a polygon and point $O$ outside it so that the polygon's angle with vertex in $O$ would not subtend any side without intersecting some of the other of the polygon's sides.

**22.18.** Prove that if a polygon is such that point $O$ is the vertex of an angle that subtends its entire contour, then any point of the plane is the vertex of an angle that entirely subtends at least one of its sides.

**22.19.** Prove that for any polygon the sum of the outer angles adjacent to the inner ones that are smaller than $180°$ is $\geq 360°$.

**22.20.** a) Prove that any $n$-gon ($n \geq 4$) has at least one diagonal that completely lies inside it.

b) Find out what is the least number of such diagonals for an $n$-gon.

**22.21.** What is the maximal number of vertices of an $n$-gon from which one cannot draw a diagonal?

**22.22.** Prove that any $n$-gon can be cut into triangles by nonintersecting diagonals.

**22.23.** Prove that the sum of the inner angles of any $n$-gon is equal to $(n - 2)180°$.

**22.24.** Prove that the number of triangles into which an $n$-gon is cut by nonintersecting diagonals is equal to $n - 2$.

**22.25.** A polygon is cut by nonintersecting diagonals into triangles. Prove that at least two of these diagonals cut triangles off it.

**22.26.** Prove that for any 13-gon there exists a line containing exactly one of its sides; however, for any $n > 13$ there exists an $n$-gon for which the similar statement is false.

**22.27.** What is the largest number of acute angles in a nonconvex $n$-gon?

**22.28.** The following operations are done over a nonconvex non-selfintersecting polygon. If it lies on one side of line $AB$, where $A$ and $B$ are non-neighbouring vertices, then we reflect one of the parts into which points $A$ and $B$ divide the contour of the polygon through the

midpoint of segment $AB$. Prove that after several such operations the polygon becomes a convex one.

**22.29.** The numbers $\alpha_1$, ..., $\alpha_n$ whose sum is equal to $(n-2)\pi$ satisfy inequalities $0 < \alpha_i \leq 2\pi$. Prove that there exists an $n$-gon $A_1 \ldots A_n$ with anagles $\alpha_1$, ..., $\alpha_n$ at vertices $A_1$, ..., $A_n$, respectively.

## Solutions

**22.1.** Consider the convex hull of given points. It is a convex polygon. We have to prove that all the given points are its vertices. Suppose one of the given points (point $A$) is not a vertex, i.e., it lies inside or on the side of the polygon. The diagonals that go out of this vertex cut the convex hull into triangles; point $A$ belongs to one of the triangles. The vertices of this triangle and point $A$ cannot be vertices of a convex quadrilateral. Contradiction.

**22.2.** Consider the convex hull of given points. If it is a quadrilateral or a pentagon, then all is clear. Now, suppose that the convex hull is triangle $ABC$ and points $D$ and $E$ lie inside it. Point $E$ lies inside one of the triangles $ABD$, $BCD$, $CAD$; let for definiteness sake it belong to the interior of triangle $ABC$. Let $H$ be the intersection point of lines $CD$ and $AB$. Point $E$ lies inside one of the triangles $ADH$ and $BDH$. If, for example, $E$ lies inside triangle $ADH$, then $AEDC$ is a convex quadrilateral (Fig. 48).
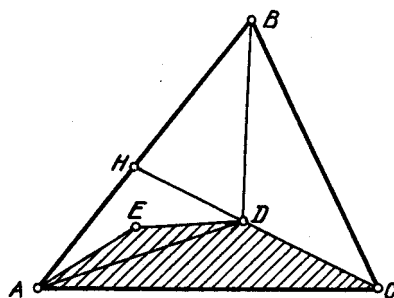


Figure 190 (Sol. 22.2)

**22.3.** Let the convex hull of the vertices of the given $n$-gons be an $m$-gon and $\varphi_1$, ..., $\varphi_m$ its angles. Since to every angle of the convex hull an angle of a regular $n$-gon is adjacent, $\varphi_i \geq (1-(\frac{2}{n}))\pi$ (in the right-hand side there stands the value of an angle of a regular $n$-gon). Therefore,

$$\varphi_1 + \cdots + \varphi_m \geq m(1-(\frac{2}{n}))\pi = (m-(\frac{2m}{n}))\pi.$$

On the other hand, $\varphi_1 + \cdots + \varphi_m = (m-2)\pi$; hence, $(m-2)\pi \geq (m-(\frac{2m}{n}))\pi$, i.e., $m \geq n$.

**22.4.** First, notice that it suffices to take 50 triangles. Indeed, let $\Delta_k$ be the triangle whose sides lie on rays $A_kA_{k-1}$ and $A_kA_{k+1}$ and which contains convex polygon $A_1 \ldots A_{100}$. Then this polygon is the intersection of the triangles $\Delta_2$, $\Delta_4$, ..., $\Delta_{100}$.
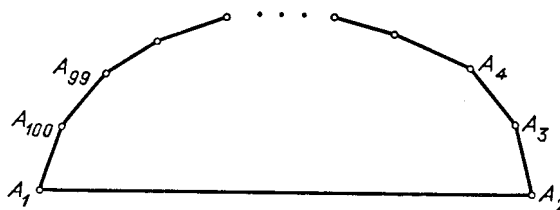


Figure 191 (Sol. 22.4)

On the other hand, the 100-gon depicted on Fig. 49 cannot be represented as the intersection of less than 50 triangles. Indeed, if three of its sides lie on the sides of one triangle, then one of these sides is side $A_1A_2$. All the sides of this polygon lie on the sides of $n$ triangles and, therefore, $2n + 1 \geq 100$, i.e., $n \geq 50$.

**22.5.** Let $P$ be the intersection point of diagonals $A_1A_4$ and $A_2A_5$ of convex heptagon $A_1 \ldots A_7$. One of the diagonals $A_3A_7$ or $A_3A_6$, let, for definiteness, this be $A_3A_6$, does not pass through point $P$. There are finitely many intersection points of the diagonals of hexagon $A_1 \ldots A_6$ and, therefore, in a vicinity of point $A_7$ one can select a point $A_7'$ such that lines $A_1A_7'$, $\ldots$, $A_6A_7'$ do not pass through these points, i.e., heptagon $A_1 \ldots A_7'$ is a nonsingular one.

**22.6.** First, let us prove that $H$ is a convex figure. Let points $A$ and $B$ belong to $H$, i.e., $A$ and $B$ be the midpoints of segments $C_1D_1$ and $C_2D_2$, where $C_1$ and $C_2$ belong to $F$ and $D_1$, respectively, and $D_2$ belong to $G$. We have to prove that the whole segment $AB$ belongs to $H$. It is clear that segments $C_1C_2$ and $D_1D_2$ belong to $F$ and $G$, respectively. The locus of the midpoints of segments with the endpoints on segments $C_1C_2$ and $D_1D_2$ is the parallelogram with diagonal $AB$ (Fig. 50); this follows from the fact that the locus of the midpoints of segments $CD$, where $C$ is fixed and $D$ moves along segment $D_1D_2$, is the midline of triangle $CD_1D_2$.
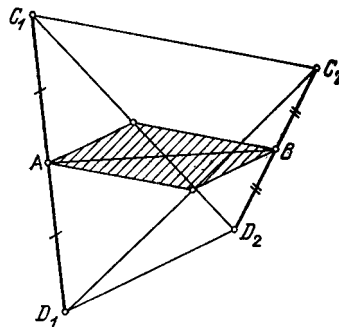


Figure 192 (Sol. 22.6)

In plane, take an arbitrary coordinate axis $Ox$. The set of all the points of the polygon whose projections to the axis have the largest value (Fig. 51) will be called the *basic set* of the polygon with respect to axis $Ox$.
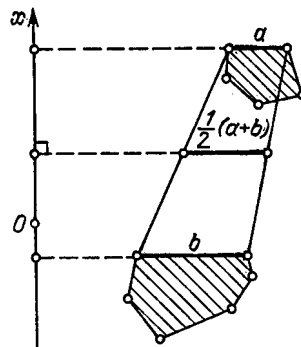


Figure 193 (Sol. 22.6)

The convex polygon is given by its basic sets for all possible axes $Ox$. If basic sets $F$ and $G$ with respect to an axis are segments of length $a$ and $b$, then the basic set of $H$ with

respect to the same axis is a segment of length $\frac{a+b}{2}$ (here we assume that a point is *segment of zero length*). Therefore, the perimeter of $H$ is equal to $\frac{P_1+P_2}{2}$ and the number of $H$'s sides can take any value from the largest — $n_1$ or $n_2$ — to $n_1 + n_2$ depending on for how many axes both basic sets of $F$ and $G$ are sides and not vertices simultaneously.

**22.7.** We will prove a more general statement. Recall that *cardinality* of a set is (for a finite set) the number of its element.

(*Ramsey's theorem.*) *Let $p$, $q$ and $r$ be positive integers such that $p$, $q \geq r$. Then there exists a number $N = N(p, q, r)$ with the following property: if $r$-tuples from a set $S$ of cardinality $N$ are divided at random into two nonintersecting families $\alpha$ and $\beta$, then either there exists a $p$-tuple of elements from $S$ all subsets of cardinality $r$ of which are contained in $\alpha$ or there exists a $q$-tuple all subsets of cardinality $r$ of which are contained in $\beta$.*

The desired statement follows easily from Ramsey's theorem. Indeed, let $N = N(p, 5, 4)$ and family $\alpha$ consist of quadruples of elements of an $N$-element set of points whose convex hulls are quadrilaterals. Then there exists a subset of $n$ elements of the given set of points the convex hulls of any its four-elements subset being quadrilaterals because there is no five-element subset such that the convex hulls of any four-element subsets of which are triangles (see Problem 22.2). It remains to make use of the result of Problem 22.1.

Now, let us prove Ramsey's theorem. It is easy to verify that for $N(p, q, 1)$, $N(r, q, r)$ and $N(p, r, r)$ one can take numbers $p + q - 1$, $q$ and $p$, respectively.

Now, let us prove that if $p > r$ and $q > r$, then for $N(p, q, r)$ one can take numbers $N(p_1, q_1, r - 1) + 1$, where $p_1 = N(p - 1, q, r)$ and $q_1 = N(p, q - 1, r)$. Indeed, let us delete from the $N(p, q, r)$-element set $S$ one element and divide the $(r - 1)$-element subsets of the obtained set $S'$ into two families: family $\alpha'$ (resp. $\beta'$) consists of subsets whose union with the deleted element enters $\alpha$ (resp. $\beta$). Then either (1) there exists a $p_1$-element subset of $S'$ all $(r - 1)$-element subsets of which are contained in $\alpha'$ or (2) there exists a $q_1$-element subset all whose $(r - 1)$ element subsets are contained in family $\beta'$.

Consider case (1). Since $p_1 = N(p - 1, q, r)$, it follows that either there exists a $q$-element subset of $S'$ all $r$-element subsets of which belong to $\beta$ (then these $q$ elements are the desired one) or there exists a $(p - 1)$-element subset of $S'$ all the $r$-element subsets of which are contained in $\alpha$ (then these $p - 1$ elements together with the deleted element are the desired ones).

Case (2) is treated similarly.

Thus, the proof of Ramsey's theorem can be carried out by induction on $r$, where in the proof of the inductive step we make use of induction on $p + q$.

**22.8.** If the polygon is not a triangle or parallelogram, then it has two nonparallel non-neighbouring sides. Extending them until they intersect, we get a new polygon which contains the initial one and has fewer number of sides. After several such operations we get a triangle or a parallelogram.

If we have got a triangle, then everything is proved; therefore, let us assume that we have got a parallelogram, $ABCD$. On each of its sides there lies a side of the initial polygon and one of its vertices, say $A$, does not belong to the initial polygon (Fig. 52). Let $K$ be a vertex of the polygon nearest to $A$ and lying on $AD$; let $KL$ be the side that does not lie on $AD$. Then the polygon is confined inside the triangle formed by lines $KL$, $BC$ and $CD$.

**22.9.** The proof will be carried out by induction on $n$. For $n = 3$ the statement is obvious. Let $n \geq 4$. By Problem 22.8 there exist lines $a$, $b$ and $c$ which are extensions of the sides of the given $n$-gon that constitute triangle $T$ which contains the given $n$-gon. Let line $l$ be the extension of some other side of the given $n$-gon. The extensions of all the sides
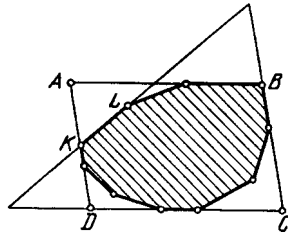
Figure 194 (Sol. 22.8)

of the $n$-gon except the side which lies on line $l$ form a convex $(n-1)$-gon that lies inside triangle $T$.

By the inductive hypothesis for this $(n-1)$-gon there exist $n-3$ required triangles. Moreover, line $l$ and two of the lines $a$, $b$ and $c$ also form a required triangle.

REMARK. If points $A_2$, ..., $A_n$ belong to a circle with center at $A_1$, where $\angle A_2A_1A_n < 90°$ and the $n$-gon $A_1 \ldots A_n$ is a convex one, then for this $n$-gon there exist precisely $n-2$ triangles required.

**22.10.** Proof will be carried out by induction on $n$. For $n = 3$ the proof is obvious. Now, let us consider $n$-gons $A_1 \ldots A_n$, where $n \geq 4$. Point $O$ lies inside triangle $A_pA_qA_r$. Let $A_k$ be a vertex of the given $n$-gon distinct from points $A_p$, $A_q$ and $A_r$. Selecting vertex $A_k$ in $n$-gon $A_1 \ldots A_n$ we get a $(n-1)$-gon to which the inductive hypothesis is applicable. Moreover, the angles $\angle A_kOA_p$, $\angle A_kOA_q$ and $\angle A_kOA_r$ cannot all be acute ones because the sum of certain two of them is greater than $180°$.

**22.11.** Proof will be carried out by induction on $n$. For $n = 3$ the statement is obvious. Let $n \geq 4$. Fix one acute triangle $A_pA_qA_r$ and let us discard vertex $A_k$ distinct from the vertices of this triangle. The inductive hypothesis is applicable to the obtained $(n-1)$-gon. Moreover, if, for instance, point $A_k$ lies on arc $A_pA_q$ and $\angle A_kA_pA_r \leq \angle A_kA_qA_r$, then triangle $A_kA_pA_r$ is an acute one.

Indeed, $\angle A_pA_kA_r = \angle A_pA_qA_r$, $\angle A_pA_rA_k < \angle A_pA_rA_q$ and $\angle A_kA_pA_r \leq 90°$; hence, $\angle A_kA_pA_r < 90°$.

**22.12.** a) Denote the given figures by $M_1$, $M_2$, $M_3$ and $M_4$. Let $A_i$ be the intersection point of all the figures except $M_i$. Two variants of arrangements of points $A_i$ are possible.

1) One of the points, for example, $A_4$ lies inside the triangle formed by the remaining points. Since points $A_1$, $A_2$, $A_3$ belong to the convex figure $M_4$, all points of $A_1A_2A_3$ also belong to $M_4$. Therefore, point $A_4$ belongs to $M_4$ and it belongs to the other figures by its definition.

2) $A_1A_2A_3A_4$ is a convex quadrilateral. Let $C$ be the intersection point of diagonals $A_1A_3$ and $A_2A_4$. Let us prove that $C$ belongs to all the given figures. Both points $A_1$ and $A_3$ belong to figures $M_2$ and $M_4$, therefore, segment $A_1A_3$ belongs to these figures. Similarly, segment $A_2A_4$ belongs to figures $M_1$ and $M_3$. It follows that the intersection point of segments $A_1A_3$ and $A_2A_4$ belongs to all the given figures.

b) Proof will be carried out by induction on the number of figures. For $n = 4$ the statement is proved in the preceding problem. Let us prove that if the statement holds for $n \geq 4$ figures, then it holds also for $n + 1$ figures. Given convex figures $\Phi_1$, ..., $\Phi_n$, $\Phi_{n+1}$ every three of which have a common point, consider instead of them figures $\Phi_1$, ..., $\Phi_{n-1}$, $\Phi_n'$, where $\Phi_n'$ is the intersection of $\Phi_n$ and $\Phi_{n+1}$. It is clear that $\Phi_n'$ is also a convex figure.

Let us prove that any three of the new figures have a common point. One can only doubt this for the triple of figures that contain $\Phi_n'$ but the preceding problem implies that figures

$\Phi_i, \Phi_j$, $\Phi_n$ and $\Phi_{n+1}$ always have a common point. Therefore, by the inductive hypothesis $\Phi_1, \ldots, \Phi_{n-1}, \Phi'_n$ have a common point; hence, $\Phi_1, \ldots, \Phi_n, \Phi_{n+1}$ have a common point.

**22.13.** A unit disk centered at $O$ covers certain points if and only if unit disks centered at these points contain point $O$. Therefore, our problem admits the following reformulation:

*Given $n$ points in plane such that any three unit disks centered at these points have a common point, prove that all these disks have a common point.*

This statement clearly follows from Helley's theorem.

**22.14.** Consider pentagons that remain after deleting pairs of neighbouring vertices of a heptagon. It suffices to verify that any three of the pentagons have a common point. For three pentagons we delete not more than 6 distinct vertices, i.e., one vertex remains. If vertex $A$ is not deleted, then the triangle shaded in Fig. 53 belongs to all three pentagons.
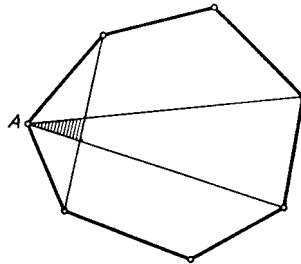


Figure 195 (Sol. 22.14)

**22.15.** Let us introduce the coordinate system with $Oy$-axis parallel to the given segments. For every segment consider the set of all points $(a, b)$ such that the line $y = ax + b$ intersects it. It suffices to verify that these sets are convex ones and apply to them Helley's theorem. For the segment with endpoints $(x_0, y_1)$ and $(x_0, y_2)$ the considered set is a band between parallel lines $ax_0 + b = y_1$ and $ax_0 + b = y_2$.

**22.16.** Wrong. A counterexample is given on Fig. 54.
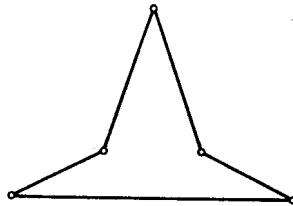


Figure 196 (Sol. 22.16)

**22.17.** The required polygons and points are drawn on Fig. 55.
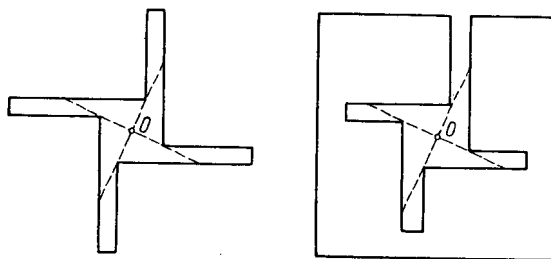


Figure 197 (Sol. 22.17)

**22.18.** Let the whole contour of polygon $A_1 \ldots A_n$ subtend an angle with vertex $O$. Then no other side of the polygon except $A_i A_{i+1}$ lies inside angle $\angle A_i O A_{i+1}$; hence, point $O$ lies inside the polygon (Fig. 56). Any point $X$ in plane belongs to one of the angles $\angle A_i O A_{i+1}$ and, therefore, side $A_i A_{i+1}$ subtends an angle with vertex in $X$.
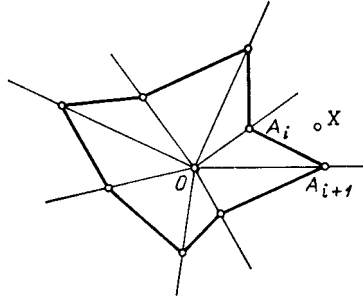


Figure 198 (Sol. 22.18)

**22.19.** Since all the inner angles of a convex $n$-gon are smaller than $180°$ and their sum is equal to $(n-2) \cdot 180°$, the sum of the exterior angles is equal to $360°$, i.e., for a convex polygon we attain the equality.
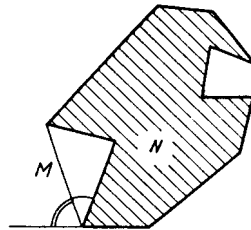


Figure 199 (Sol. 22.19)

Now, let $M$ be the convex hull of polygon $N$. Each angle of $M$ contains an angle of $N$ smaller than $180°$ and the angle of $M$ can be only greater than the angle of $N$, i.e., the exterior angle of $N$ is not less than the exterior angle of $M$ (Fig. 57). Therefore, even restricting to the angles of $N$ adjacent to the angles of $M$ we will get not less than $360°$.

**22.20.** a) If the polygon is a convex one, then the statement is proved. Now, suppose that the exterior angle of the polygon at vertex $A$ is greater than $180°$. The visible part of the side subtends an angle smaller than $180°$ with vertex at point $A$, therefore, parts of at least two sides subtend an angle with vertex at $A$. Therefore, there exist rays exiting point $A$ and such that on these rays the change of (parts of) sides visible from $A$ occurs (on Fig. 58 all such rays are depicted). Each of such rays determines a diagonal that lies entirely inside the polygon.

b) On Fig. 59 it is plotted how to construct an $n$-gon with exactly $n-3$ diagonals inside it. It remains to demonstrate that any $n$-gon has at least $n-3$ diagonals. For $n=3$ this statement is obvious.

Suppose the statement holds for all $k$-gons, where $k < n$ and let us prove it for an $n$-gon. By heading a) it is possible to divide an $n$-gon by its diagonal into two polygons: a $(k+1)$-gon and an $(n-k+1)$-gon, where $k+1 < n$ and $n-k+1 < n$. These parts have at least $(k+1)-3$ and $(n-k+1)-3$ diagonals, respectively, that lie inside these parts. Therefore, the $n$-gon has at least $1 + (k-2) + (n-k-2) = n-3$ diagonals that lie inside it.

**22.21.** First, let us prove that if $A$ and $B$ are neighbouring vertices of the $n$-gon, then either from $A$ or from $B$ it is possible to draw a diagonal. The case when the inner angle
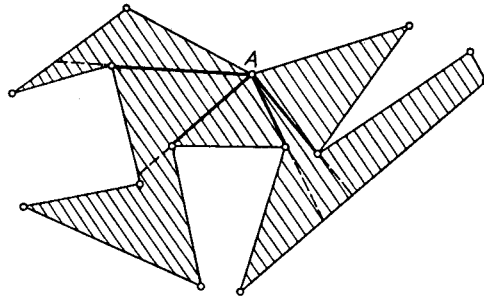
Figure 200 (Sol. 22.20 a))
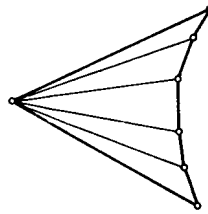


Figure 201 (Sol. 22.20 b))

of the polygon at $A$ is greater than $180°$ is considered in the solution of Problem 22.20 a). Now, suppose that the angle at vertex $A$ is smaller than $180°$. Let $B$ and $C$ be vertices neighbouring $A$.

If inside triangle $ABC$ there are no other vertices of the polygon, then $BC$ is the diagonal and if $P$ is the nearest to $A$ vertex of the polygon lying inside triangle $ABC$, then $AP$ is the diagonal. Hence, the number of vertices from which it is impossible to draw the diagonal does not exceed $\left[\frac{n}{2}\right]$ (the integer part of $\frac{n}{2}$). On the other hand, there exist $n$-gons for which this estimate is attained, see Fig. 60.
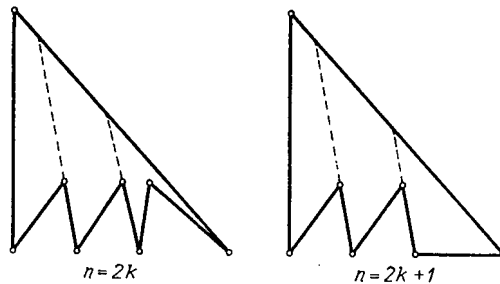


Figure 202 (Sol. 22.21)

**22.22.** Let us prove the statement by induction on $n$. For $n = 3$ it is obvious. Let $n \geq 4$. Suppose the statement is proved for all $k$-gons, where $k < n$; let us prove it for an $n$-gon. Any $n$-gon can be divided by a diagonal into two polygons (see Problem 22.20 a)) and the number of vertices of every of the smaller polygons is strictly less than $n$, i.e., they can be divided into triangles by the inductive hypothesis.

**22.23.** Let us prove the statement by induction. For $n = 3$ it is obvious. Let $n \geq 4$. Suppose it is proved for all $k$-gons, where $k < n$, and let us prove it for an $n$-gon. Any $n$-gon can be divided by a diagonal into two polygons (see Problem 22.20 a)). If the number of sides of one of the smaller polygons is equal to $k + 1$, then the number of sides of the other one is equal to $n - k + 1$ and both numbers are smaller than $n$. Therefore, the sum of the

angles of these polygons are equal to $(k-1)\cdot 180°$ and $(n-k-1)\cdot 180°$, respectively. It is also clear that the sum of the angles of a $n$-gon is equal to the sum of the angles of these polygons, i.e., it is equal to

$$(k-1+n-k-1)\cdot 180° = (n-2)\cdot 180°.$$

**22.24.** The sum of all the angles of the obtained triangles is equal to the sum of the angles of the polygon, i.e., it is equal to $(n-2)\cdot 180°$, see Problem 22.23. Therefore, the number of triangles is equal to $n-2$.

**22.25.** Let $k_i$ be the number of triangles in the given partition for which precisely $i$ sides are the sides of the polygon. We have to prove that $k_2 \geq 2$. The number of sides of the $n$-gon is equal to $n$ and the number of the triangles of the partition is equal to $n-2$, see Problem 22.24. Therefore, $2k_2 + k_1 = n$ and $k_2 + k_1 + k_0 = n-2$. Subtracting the second equality from the first one we get $k_2 = k_0 + 2 \geq 2$.

**22.26.** Suppose that there exists a 13-gon for which on any line that contains its side there lies at least one side. Let us draw lines through all the sides of this 13-gon. Since the number of sides is equal to 13, it is clear that one of the lines contains an odd number of sides, i.e., one of the lines has at least 3 sides. On these sides lie 6 vertices and through each vertex a line passes on which there lie at least 2 sides. Therefore, this 13-gon has not less than $3 + 2\cdot 6 = 15$ sides but this is impossible.
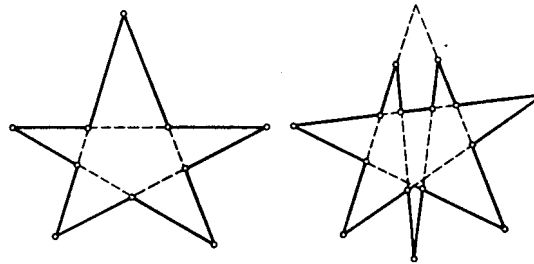


Figure 203 (Sol. 22.26)

For $n$ even, $n \geq 10$, the required example is the contour of a "star" (Fig. 61 a)) and an idea of how to construct an example for $n$ odd is illustrated on Fig. 61 b).

**22.27.** Let $k$ be the number of acute angles of the $n$-gon. Then the number of its angles is smaller than $k\cdot 90° + (n-k)\cdot 360°$. On the other hand, the sum of the angles of an $n$-gon is equal to $(n-2)\cdot 180°$ (see Problem 22.23) and, therefore, $k\cdot 90° + (n-k)\cdot 360° > (n-2)\cdot 180°$, i.e., $3k < 2n + 4$. It follows that $k \leq [\frac{2n}{3}] + 1$, where $[x]$ denotes the largest integer not exceeding $x$.
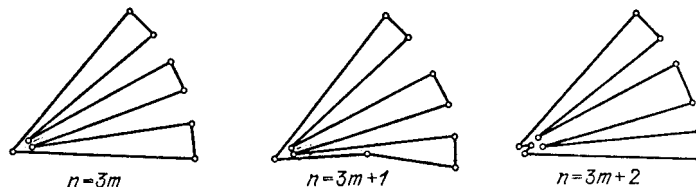


Figure 204 (Sol. 22.27)

Examples of $n$-gons with $[\frac{2n}{3}] + 1$ acute angles are given on Fig. 62.

**22.28.** Under these operations the vectors of the sides of a polygon remain the same only their order changes (Fig. 63). Therefore, there exists only a finite number of polygons that
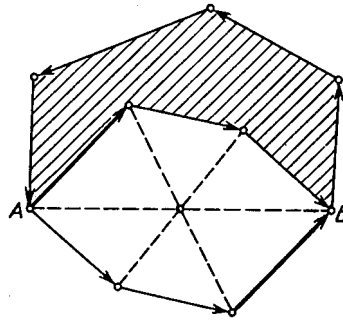
Figure 205 (Sol. 22.28)

may be obtained. Moreover, after each operation the area of the polygon strictly increases. Hence, the process terminates.

**22.29.** Let us carry out the proof by induction on $n$. For $n = 3$ the statement is obvious. Let $n \geq 4$. If one of the numbers $\alpha_i$ is equal to $\pi$, then the inductive step is obvious and, therefore, we may assume that all the numbers $\alpha_i$ are distinct from $\pi$. If $n \geq 4$, then

$$\frac{1}{n} \sum_{i=1}^{n} (\alpha_i + \alpha_{i+1}) = 2(n-2)\frac{\pi}{n} \geq \pi,$$

where the equality is only attained for a quadrilateral. Hence, in any case except for a parallelogram ($\alpha_1 = \pi - \alpha_2 = \alpha_3 = \pi - \alpha_4$), and (?) there exist two neighbouring numbers whose sum is greater than $\pi$. Moreover, there exist numbers $\alpha_i$ and $\alpha_{i+1}$ such that $\pi < \alpha_i + \alpha_{i+1} < 3\pi$. Indeed, if all the given numbers are smaller than $\pi$, then we can take the above-mentioned pair of numbers; if $\alpha_j > \pi$, then we can take numbers $\alpha_i$ and $\alpha_{i+1}$ such that $\alpha_i < \pi$ and $\alpha_{i+1} > \pi$. Let $\alpha_i^* = \alpha_i + \alpha_{i+1} - 1$. Then $0 < \alpha_i^* < 2\pi$ and, therefore, by the inductive hypothesis there exists an $(n-1)$-gon $M$ with angles $\alpha_1$, ..., $\alpha_{i-1}$, $\alpha_i^*$, $\alpha_{i+2}$, ..., $\alpha_n$.

Three cases might occur: 1) $\alpha_i^* < \pi$, 2) $\alpha_i^* = \pi$, 3) $\pi < \alpha_i^* < 2\pi$.

In the first case $\alpha_i + \alpha_{i+1} < 2\pi$ and, therefore, one of these numbers, say $\alpha_i$, is smaller than $\pi$. If $\alpha_{i+1} < \pi$, then let us cut from $M$ a triangle with angles $\pi - \alpha_i$, $\pi - \alpha_{i+1}$, $\alpha_i^*$ (Fig. 64 a)). If $\alpha_{i+1} > \pi$, then let us juxtapose to $M$ a triangle with angles $\alpha_i$, $\alpha_{i+1} - \pi$, $\pi - \alpha_i^*$ (Fig. 64 b)).

In the second case let us cut from $M$ a trapezoid with the base that belongs to side $A_{i-1}A_i^*A_{i+2}$ (Fig. 64 c)).

In the third case $\alpha_i + \alpha_{i+1} > \pi$ and, therefore, one of these numbers, say $\alpha_i$, is greater than $\pi$. If $\alpha_{i+1} > \pi$, then let us juxtapose to $M$ a triangle with angles $\alpha_i - \pi$, $\alpha_{i+1} - \pi$, $2\pi - \alpha_i^*$ (Fig. 64 d)), and if $\alpha_{i+1} < \pi$ let us cut off $M$ a triangle with angles $2\pi - \alpha_i$, $\pi - \alpha_{i+1}$ and $\alpha_i^* - \pi$ (Fig. 64 e)).
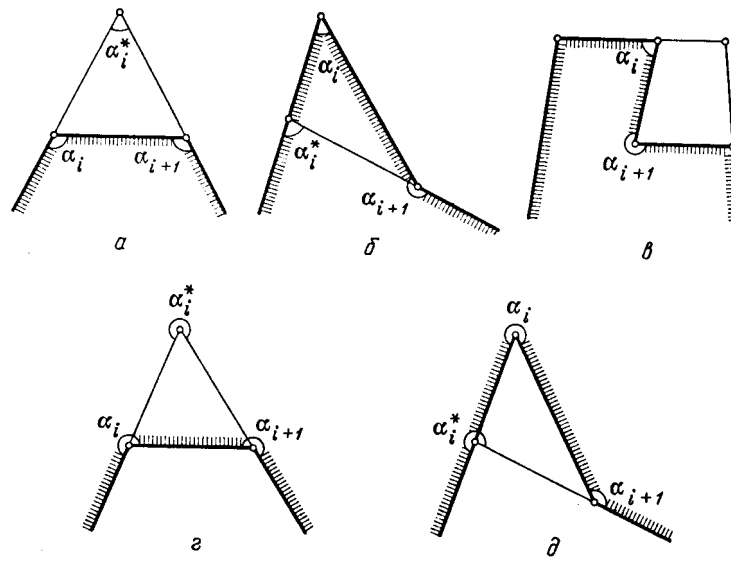
Figure 206 (Sol. 22.29)

# Chapter 23. DIVISIBILITY, INVARIANTS, COLORINGS

## Background

1. In a number of problems we encounter the following situation. A certain system consecutively changes its state and we have to find out something at its final state. It might be difficult or impossible to trace the whole intermediate processes but sometimes it is possible to answer the question with the help of a quantity that characterizes the state of the system and is preserved during all the transitions (such a quantity is sometimes called an *invariant* of the system considered). Clearly, in the final state the value of the invariant is the same as in the initial one, i.e., the system cannot occur in any state with another value of the invariant.

2. In practice this method reduces to the following. A quantity is calculated in two ways: first, it is simply calculated in the initial and final states and then its variation is studied under consecutive elementary transitions.

3. The simplest and most often encountered invariant is the parity of a number; the residue after a division not only by 2 but some other number can also be an invariant.

In the construction of invariants certain auxiliary colorings are sometimes convenient, i.e., partitions of considered objects into several groups, where each group consists of the objects of the same colour.

## §1. Even and odd

**23.1.** Can a line intersect (in inner points) all the sides of a nonconvex a) $(2n + 1)$-gon; b) $2n$-gon?

**23.2.** Given a closed broken plane line with a finite number of links and a line $l$ that intersects it at 1985 points, prove that there exists a line that intersects this broken line in more than 1985 points.

**23.3.** In plane, there lie three pucks $A$, $B$ and $C$. A hockey player hits one of the pucks so that it passes (along the straight line) between the other two and stands at some point. Is it possible that after 25 hits all the pucks return to the original places?

**23.4.** Is it possible to paint 25 small cells of the graph paper so that each of them has an odd number of painted neighbours? (Riddled cells are called *neighbouring* if they have a common side).

**23.5.** A circle is divided by points into $3k$ arcs so that there are $k$ arcs of length 1, 2, and 3. Prove that there are 2 diametrically opposite division points.

**23.6.** In plane, there is given a non-selfintersecting closed broken line no three vertices of which lie on one line. A pair of non-neighbouring links of the broken will be called a singular one if the extension of one of them intersects the other one. Prove that the number of singular pairs is always even.

**23.7.** (*Sperner's lemma.*) The vertices of a triangle are labeled by figures 0, 1 and 2. This triangle is divided into several triangles so that no vertex of one triangle lies on a side of the other one. The vertices of the initial triangle retain their old labels and the additional vertices get labels 0, 1, 2 so that any vertex on a side of the initial triangle should

be labelled by one of the vertices of this side, see Fig. 65. Prove that there exists a triangle in the partition labelled by 0, 1, 2.
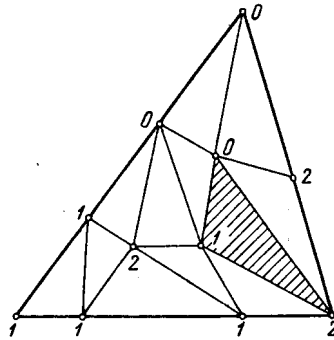


Figure 207 (23.6)

**23.7.** The vertices of a regular $2n$-gon $A_1 \ldots A_{2n}$ are divided into $n$ pairs. Prove that if $n = 4m+2$ or $n = 4m+3$, then the two pairs of vertices are the endpoints of equal segments.

## §2. Divisibility

**23.9.** On Fig. 66 there is depicted a hexagon divided into black and white triangles so that any two triangles have either a common side (and then they are painted different colours) or a common vertex, or they have no common points and every side of the hexagon is a side of one of the black triangles. Prove that it is impossible to find a similar partition for a 10-gon.
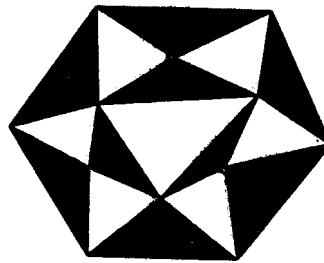


Figure 208 (23.9)

**23.10.** A square sheet of graph paper is divided into smaller squares by segments that follow the sides of the small cells. Prove that the sum of the lengths of these segments is divisible by 4. (The length of a side of a small cell is equal to 1).

## §3. Invariants

**23.11.** Given a chess board, it is allowed to simultaneously repaint into the opposite colour either all the cells of one row or those of a column. Can we obtain in this way a board with precisely one black small cell?

**23.12.** Given a chess board, it is allowed to simultaneously repaint into the opposite colour all the small cells situated inside a $2 \times 2$ square. Is it possible that after such repaintings there will be exactly one small black cell left?

**23.13.** Given a convex $2m$-gon $A_1 \ldots A_{2m}$ and point $P$ inside it not belonging to any of the diagonals, prove that $P$ belongs to an even number of triangles with vertices at points $A_1, \ldots, A_{2m}$.

**23.14.** In the center of every cell of a chess board stands a chip. Chips were interchanged so that the pairwise distances between them did not diminish. Prove that the pairwise distances did not actually alter at all.

**23.15.** A polygon is cut into several polygons so that the vertices of the obtained polygons do not belong to the sides of the initial polygon nor to the sides of the obtained polygons. Let $p$ be the number of the obtained smaller polygons, $q$ the number of segments which serve as the sides of the smaller polygons, $r$ the number of points which are their vertices. Prove that

$$p - q + r = 1. \qquad (Euler's\ formula)$$

**23.16.** A square field is divided into 100 equal square patches 9 of which are overgrown with weeds. It is known that during a year the weeds spread to those patches that have not less than two neighbouring (i.e., having a common side) patches that are already overgrown with weeds and only to them. Prove that the field will never overgrow completely with weeds.

**23.17.** Prove that there exist polygons of equal size and impossible to divide into polygons (perhaps, nonconvex ones) which can be translated into each other by a parallel translation.

**23.18.** Prove that it is impossible to cut a convex polygon into finitely many nonconvex quadrilaterals.

**23.18.** Given points $A_1$, ..., $A_n$. We considered a circle of radius $R$ encircling some of them. Next, we constructed a circle of radius $R$ with center in the center of mass of points that lie inside the first circle, etc. Prove that this process eventually terminates, i.e., the circles will start to coincide.

## §4. Auxiliary colorings

**23.20.** In every small cell of a $5 \times 5$ chess board sits a bug. At certain moment all the bugs crawl to neighbouring (via a horizontal or a vertical) cells. Is it necessary that some cell to become empty at the next moment?

**23.21.** Is it possible to tile by $1 \times 2$ domino chips a $8 \times 8$ chess board from which two opposite corner cells are cut out?

**23.22.** Prove that it is impossible to cut a $10 \times 10$ chess board into $T$-shaped figures consisting of four cells.

**23.23.** The parts of a toy railroad's line are of the form of a quarter of a circle of radius $R$. Prove that joining them consecutively so that they would smoothly turn into each other it is impossible to construct a closed path whose first and last links form the dead end depicted on Fig. 67.
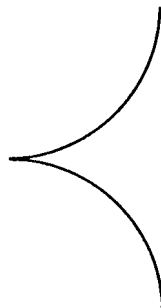


Figure 209 (23.23)

**23.24.** At three vertices of a square sit three grasshoppers playing the leap frog as follows. If a grasshopper $A$ jumps over a grasshopper $B$, then after the jump it lands at the same distance from $B$ but, naturally, on the other side and on the same line. Is it possible that after several jumps one of the grasshoppers gets to the fourth vertex of the square?

**23.25.** Given a square sheet of graph paper of size $100 \times 100$ cells. Several nonselfintersecting broken lines passing along the sides of the small cells and without common points are drawn. These broken lines are all strictly inside the square but their endpoints are invariably on the boundary. Prove that apart from the vertices of the square there will be one more node (of the graph paper inside the square or on the boundary) that does not belong to any of the broken lines.

## §5. More auxiliary colorings

**23.26.** An equilateral triangle is divided into $n^2$ equal equilateral triangles (Fig. 68). Some of them are numbered by numbers 1, 2, ..., $m$ and consecutively numbered triangles have adjacent sides. Prove that $m \le n^2 - n + 1$.
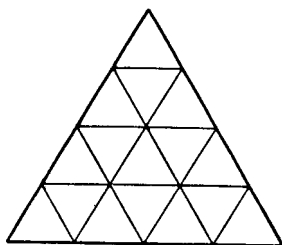


Figure 210 (23.26)

**23.27.** The bottom of a parallelepipedal box is tiled with tiles of size $2 \times 2$ and $1 \times 4$. The tiles had been removed from the box and in the process one tile of size $2 \times 2$ was lost. We replaced it with a tile of size $1 \times 4$. Prove that it will be impossible to tile now the bottom of the box.

**23.28.** Of a piece of graph paper of size $29 \times 29$ (of unit cells) 99 squares of size $2 \times 2$ were cut. Prove that it is still possible to cut off one more such square.

**23.29.** Nonintersecting diagonals divide a convex $n$-gon into triangles and at each of the $n$-gon's vertex an odd number of triangles meet. Prove that $n$ is divisible by 3.

\* \* \*

**23.30.** Is it possible to tile a $10 \times 10$ graph board by tiles of size $2 \times 4$?

**23.31.** On a graph paper some arbitrary $n$ cells are fixed. Prove that from them it is possible to select not less than $\frac{n}{4}$ cells without common points.

**23.32.** Prove that if the vertices of a convex $n$-gon lie in the nodes of graph paper and there are no other nodes inside or on the sides of the $n$-gon, then $n \le 4$.

**23.33.** From 16 tiles of size $1 \times 3$ and one tile of size $1 \times 1$ one constructed a $7 \times 7$ square. Prove that the $1 \times 1$ tile either sits in the center of the square or is adjacent to its boundary.

**23.34.** A picture gallery is of the form of a nonconvex $n$-gon. Prove that in order to overview the whole gallery $\left[\frac{n}{3}\right]$ guards suffices.

## §6. Problems on colorings

**23.35.** A plane is painted two colours. Prove that there exist two points of the same colour the distance between which is equal to 1.

**23.36.** A plane is painted three colours. Prove that there are two points of the same colour the distance between which is equal to 1.

**23.37.** The plane is painted seven colours. Are there necessarily two points of the same colour the distance between which is equal to 1?

(?)**23.38.** The points on sides of an equilateral triangle are painted two colours. Prove that there exists a right triangle with vertices of the same colour.

**\* \* \***

A *triangulation* of a polygon is its partition into triangles with the property that these triangles have either a common side or a common vertex or have no common points (i.e., the vertex of one triangle cannot belong to a side of the other one).

**23.39.** Prove that it is possible to paint the triangles of a triangulation three colours so that the triangles with a common side would be of different colours.

**23.40.** A polygon is cut by nonintersecting diagonals into triangles. Prove that the vertices of the polygon can be painted three colours so that all the vertices of each of the obtained triangles would be of different colours.

**23.41.** Several disks of the same radius were put on the table so that no two of them overlap. Prove that it is possible to paint disks four colours so that any two tangent disks would be of different colours.

## Solutions

**23.1.** a) Let a line intersect all the sides of the polygon. Consider all the vertices on one side of the line. To each of these vertices we can assign a pair of sides that intersect at it. Thus we get a partition of all the sides of the polygon into pairs. Therefore, if a line intersects all the sides of an $m$-gon, then $m$ is even.
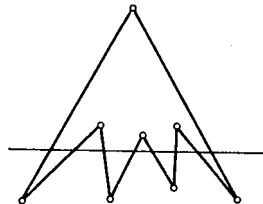


Figure 211 (Sol. 23.1)

b) It is clear from Fig. 69 how to construct $2n$-gon and a line that intersects all its sides for any $n$.

**23.2.** A line $l$ determines two half planes; one of them will be called *upper* the other one *lower*. Let $n_1$ (resp. $n_2$) be the number of the vertices of the broken line that lie on $l$ for which both links that intersect at this point belong to the upper (resp. lower) half plane and $m$ the number of all the remaining intersection points of $l$ and the broken line. Let us circumvent the broken line starting from a point that does not lie on $l$ (and returning to the same point). In the process we pass from one half plane to the other one only passing through any of $m$ intersection points. Since we will have returned to the same point from which we have started, $m$ is even.

By the hypothesis $n_1 + n_2 + m = 1985$ and, therefore, $n_1 + n_2$ is odd, i.e., $n_1 \neq n_2$.

Let for definiteness $n_1 > n_2$. Then let us draw in the upper halfplane a line $l_1$ parallel to $l$ and distant from it by a distance smaller than any nonzero distance from $l$ to any of the vertices of the broken line (Fig. 70). The number of intersection points of the broken line with $l_1$ is equal to $2n_1 + m > n_1 + n_2 + m = 1985$, i.e., $l_1$ is the desired line.
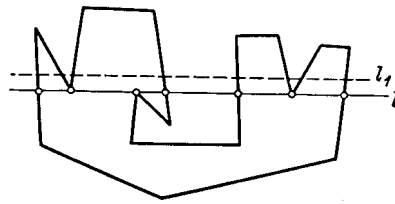
Figure 212 (Sol. 23.2)

**23.3.** No, they cannot. After each hit the orientation (i.e., the direction of the circumventing pass) of triangle $ABC$ changes.

**23.4.** Let on a graph paper several cells be painted and $n_k$ be the number of painted cells with exactly $k$ painted neighbours. Let $N$ be the number of common sides of painted cells. Since each of them belongs to exactly two painted cells,

$$N = \frac{n_1 + 2n_2 + 3n_3 + 4n_4}{2} = \frac{n_1 + n_3}{2} + n_2 + n_3 + 2n_4.$$

Since $N$ is an integer, $n_1 + n_3$ is even.

(?) We have proved that the number of painted cells with an odd number of painted cells is always even. Therefore, it is impossible to paint 25 cells so that each of them would have had an odd number of painted neighbours.

**23.5.** Suppose that the circle is divided into arcs as indicated and there are no diametrically opposite division points. Then against the endpoints of any arc of length 1 there are no division points and, therefore, against it there lies an arc of length 3. Let us delete one of the arcs of length 1 and the opposite arc of length 3. Then the circle is divided into two arcs.

If on one of them there lie $m$ arcs of length 1 and $n$ arcs of length 3, then on the other one there lie $m$ arcs of length 3 and $n$ arcs of length 1. The total number of arcs of length 1 and 3 lying on these two "great" arcs is equal to $2(k-1)$ and, therefore, $n + m = k - 1$.

Since beside arcs of length 1 and 3 there are only arcs of even length, the parity of the length of each of the considered arcs coincides with the parity of $k - 1$. On the other hand, the length of each of them is equal to $\frac{6k-1-3}{2} = 3k - 2$. We have obtained a contradiction since numbers $k - 1$ and $3k - 2$ are of opposite parities.

**23.6.** Take neighbouring links $AB$ and $BC$ and call the angle symmetric to angle $\angle ABC$ through point $B$ a *little angle* (on Fig. 71 the little angle is shaded).
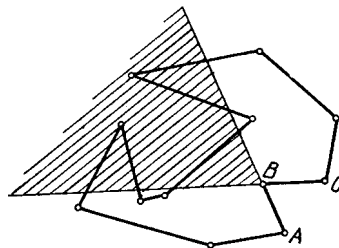


Figure 213 (Sol. 23.6)

We can consider similar little angles for all vertices of the broken line. It is clear that the number of singular pairs is equal to the number of intersection points of links with little angles. It remains to notice that the number of links of the broken line which intersect one angle is even because during the passage from $A$ to $C$ the broken line goes into the little angle as many times as it goes out of it.

**23.7.** Let us consider segments into which side 01 is divided. Let $a$ be the number of segments of the form 00 and $b$ the number of segments of the form 01. For every segment consider the number of zeros at its ends and add all these numbers. We get $2a + b$. On the other hand, all the "inner" zeros enter this sum twice and there is one more zero at a vertex of the initial triangle. Consequently, the number $2a + b$ is odd, i.e., $b$ is odd.

Let us now divide the triangle. Let $a_1$ be the total number of triangles of the form 001 and 011 and $b_1$ the total number of triangles of the form 012. For every triangle consider the number of its sides of the form 01 and add all these numbers. We get $2a_1 + b_1$. On the other hand all "inner" sides enter twice the sum and all the "boundary" sides lie on the side 01 of the initial triangle and their number is odd by above arguments. Therefore, the number $2a_1 + b_1$ is odd in particular $b_1 \neq 0$.

**23.8.** Suppose that all the pairs of vertices determine segments of distinct lengths. Let us assign to segment $A_p A_q$ the least of the numbers $|p - q|$ and $2n - |p - q|$. As a result, for the given $n$ pairs of vertices we get numbers $1, 2, \ldots, n$; let among these numbers there be $k$ even and $n - k$ odd ones. To odd numbers segments $A_p A_q$, where numbers $p$ and $q$ are of opposite parity, correspond. Therefore, among vertices of the other segments there are $k$ vertices with even numbers and $k$ vertices with odd numbers and the segments connect vertices with numbers of the same parity. Therefore, $k$ is even. For numbers $n$ of the form $4m$, $4m + 1$, $4m + 2$ and $4m + 3$ the number $k$ of even numbers is equal to $2m$, $2m$, $2m + 1$ and $2m + 1$, respectively, and therefore, either $n = 4m$ or $n = 4m + 1$.

**23.9.** Suppose we have succeeded to cut the decagon as required. Let $n$ be the number of sides of black triangles, $m$ the number of sides of white triangles. Since every side of an odd triangle (except the sides of a polygon) is also a side of a white triangle, then $n - m = 10$. On the other hand, both $n$ and $m$ are divisible by 3. Contradiction.

**23.10.** Let $Q$ be a square sheet of paper, $L(Q)$ the sum of lengths of the sides of the small cells that lie inside it. Then $L(Q)$ is divisible by 4 since all the considered sides split into quadruples of sides obtained from each other by rotations through angles of $\pm 90°$ and $180°$ about the center of the square.

If $Q$ is divided into squares $Q_1, \ldots, Q_n$, then the sum of the lengths of the segments of the partition is equal to $L(Q) - L(Q_1) - \cdots - L(Q_n)$. Clearly, this number is divisible by 4 since the numbers $L(Q), L(Q_1), \ldots, L(Q_n)$ are divisible by 4.

**23.11.** Repainting the horizontal or vertical containing $k$ black and $8 - k$ white cells we get $8 - k$ black and $k$ white cells. Therefore, the number of black cells changes by $(8 - k) - k = 8 - 2k$, i.e., by an even number. Since the parity of the number of black cells is preserved, we cannot get one black cell from the initial 32 black cells.

**23.12.** After repainting the $2 \times 2$ square containing $k$ black and $4 - k$ white cells we get $4 - k$ black and $k$ white cells. Therefore, the number of black cells changes by $(4 - k) - k = 4 - 2k$, i.e., by an even number. Since the parity of the number of black cells is preserved, we cannot get one black cell from the initial 32 black cells.

**23.13.** The diagonals divide a polygon into several parts. Parts that have a common side are called *neighbouring*. Clearly, from any inner point of the polygon we can get into any other point passing each time only from a neighbouring part to a neighbouring part. A part of the plane that lies outside the polygon can also be considered as one of these parts. The number of the considered triangles for the points of this part is equal to zero and, therefore, it suffices to prove that under the passage from a neighbouring part to a neighbouring one the parity of the number of triangles is preserved.

Let the common side of two neighbouring parts lie on diagonal (or side) $PQ$. Then for all the triangles considered, except the triangles with $PQ$ as a side, both these parts either simultaneously belong to or do not belong to. Therefore, under the passage from one part to

the other one the number of triangles changes by $k_1 - k_2$, where $k_1$ is the number of vertices of the polygon situated on one side of $PQ$ and $k_2$ is the number of vertices situated on the other side of $PQ$. Since $k_1 + k_2 = 2m - 2$, it follows that $k_1 - k_2$ is even.

**23.14.** If at least one of the distances between chips would increase, then the sum of the pairwise distances between chips would have also increased but the sum of all pairwise distances between chips does not vary under any permutation.

**23.15.** Let $n$ be the number of vertices of the initial polygon, $n_1$, ..., $n_p$ the number of vertices of the obtained polygons. On the one hand, the sum of angles of all the obtained polygons is equal to

$$\sum_{i=1}^{p}(n_i - 2)\pi = \sum_{i=1}^{p} n_i\pi - 2p\pi.$$

On the other hand, it is equal to

$$2(r - n)\pi + (n - 2)\pi.$$

It remains to observe that

$$\sum_{i=1}^{p} n_i = 2(q - n) + n.$$

**23.16.** It is easy to verify that the length of the boundary of the whole patch (of several patches) overgrown with weeds does not increase. Since in the initial moment it did not surpass $9 \cdot 4 = 36$, then at the final moment it cannot be equal to 40.

**23.17.** In plane, fix ray $AB$. To any polygon $M$ assign a number $F(M)$ (depending on $AB$) as follows. Consider all the sides of $M$ perpendicular to $AB$ and to each of them assign the number $\pm l$, where $l$ is the length of this side and the sine "plus" is taken if following this side in the direction of ray $AB$ we get inside $M$ and "minus" if we get outside $M$, see Fig. 72.
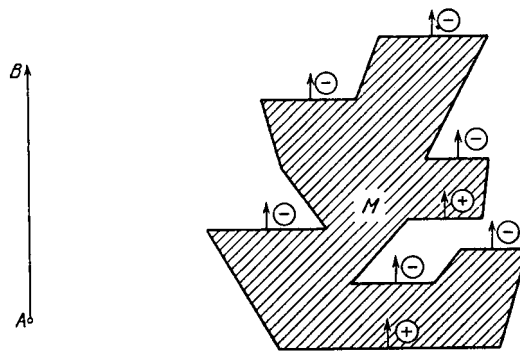


Figure 214 (Sol. 23.17)

Let us denote the sum of all the obtained numbers by $F(M)$; if $M$ has no sides perpendicular to $AB$, then $F(M) = 0$.

It is easy to see that if polygon $M$ is divided into the union of polygons $M_1$ and $M_2$, then $F(M) = F(M_1) + F(M_2)$ and if $M'$ is obtained from $M$ by a parallel translation, then $F(M') = F(M)$. Therefore, if $M_1$ and $M_2$ can be cut into parts that can be transformed into each other by a parallel translation, then $F(M_1) = F(M_2)$.

On Fig. 73 there are depicted congruent equilateral triangles $PQR$ and $PQS$ and ray $AB$ perpendicular to side $PQ$. It is easy to see that $F(PQR) = a$ and $F(PQS) = -a$, where $a$ is the length of the side of these equilateral triangles. Therefore, it is impossible to divide
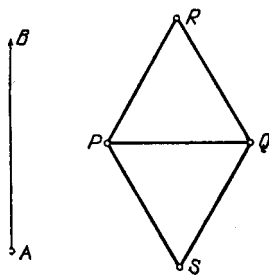
Figure 215 (Sol. 23.17)

congruent triangles $PQR$ and $PQS$ into parts that can be translated into each other by a parallel translation.

**23.18.** Suppose that a convex polygon $M$ is divided into nonconvex quadrilaterals $M_1$, ..., $M_n$. To every polygon $N$ assign the number $f(N)$ equal to the difference between the sum of its inner angles smaller than $180°$ and the sum of the angles that complements its angles greater than $180°$ to $360°$. Let us compare the numbers $A = f(M)$ and $B = f(M_1) + \cdots + f(M_n)$. To this end consider all the points that are vertices of triangles $M_1$, ..., $M_n$. These points can be divided into four types:

1) The (inner?) points of $M$. These points contribute equally to $A$ and to $B$.

2) The points on sides of $M$ or $M_i$. The contribution of each such point to $B$ exceeds the contribution to $A$ by $180°$.

(?)3) The inner points of the polygon in which the angles of the quadrilateral smaller than $180°$ in it. The contribution of every such point to $B$ is smaller than that to $A$ by $360°$.

4) The inner points of polygon $M$ in which the angles of the quadrilaterals meet and one of the angles is greater than $180°$. Such points give zero contribution to both $A$ and $B$.

As a result we see that $A \leq B$. On the other hand, $A > 0$ and $B = 0$. The inequality $A > 0$ is obvious and to prove that $B = 0$ it suffices to verify that if $N$ is a nonconvex quadrilateral, then $f(N) = 0$. Let the angles of $N$ be equal to $\alpha$, $\beta$, $\gamma$ and $\delta$, where $\alpha \geq \beta \geq \gamma \geq \delta$. Any nonconvex quadrilateral has exactly one angle greater than $180°$ and, therefore,
$$f(N) = \beta + \gamma + \delta - (360° - \alpha) = \alpha + \beta + \gamma + \delta - 360° = 0°.$$

We have obtained a contradiction and, therefore, it is impossible to cut a convex polygon into a finite number of nonconvex quadrilaterals.

**23.19.** Let $S_n$ be the circle constructed at the $n$-th step; $O_n$ its center. Consider the quantity $F_n = \sum(R^2 - O_nA_i^2)$, where the sum runs over points that are inside $S_n$ only. Let us denote the points lying inside circles $S_n$ and $S_{n+1}$ by letters $B$ with an index; the points that lie inside $S_n$ but outside $S_{n+1}$ by letters $C$ with an index and points lying inside $S_{n+1}$ but outside $S_n$ by letters $D$ with an index. Then
$$F_n = \sum(R^2 - O_nB_i^2) + \sum(R^2 - O_nC_i^2)$$
and
$$F_{n+1} = \sum(R^2 - O_{n+1}B_i^2) + \sum(R^2 - O_{n+1}D_i^2).$$
Since $O_{n+1}$ is the center of mass of the system of points $B$ and $C$, it follows that
$$\sum O_nB_i^2 + \sum O_nC_i^2 = qO_nO_{n+1}^2 + \sum O_{n+1}B_i^2 + \sum O_{n+1}C_i^2,$$
where $q$ is the total number of points of type $B$ and $C$. It follows that
$$F_{n+1} - F_n = qO_nO_{n+1}^2 + \sum(R^2 - O_{n+1}D_i^2) - \sum(R^2 - O_{n+1}C_i^2).$$

All the three summands are nonnegative and, therefore, $F_{n+1} \geq F_n$. In particular, $F_n \geq F_1 > 0$, i.e., $q > 0$.

There is a finite number of centers of mass of distinct subsets of given points and, therefore, there is also only finitely many distinct positions of circles $S_i$. Hence, $F_{n+1} = F_n$ for some $n$ and, therefore, $qO_nO_{n+1}^2 = 0$, i.e., $O_n = O_{n+1}$.

**23.20.** Since the total number of cells of a $5 \times 5$ chessboard is odd, the number of black fields cannot be equal to the number of white fields. Let, for definiteness, there be more black fields than white fields. Then there are less bugs that sit on white fields than there are black fields. Therefore, at least one of black fields will be empty since only bugs that sit on white fields crawl to black fields.

**23.21.** Since the fields are cut of one colour only, say, of black colour, there remain 32 white and 30 black fields. Since a domino piece always covers one white and one black field, it is impossible to tile with domino chips a $8 \times 8$ chessboard without two opposite corner fields.

**23.22.** Suppose that a $10 \times 10$ chessboard is divided into such tiles. Every tile contains either 1 or 3 black fields, i.e., always an odd number of them. The total number of figures themselves should be equal to $\frac{100}{4} = 25$. Therefore, they contain an odd number of black fields and the total of black fields is $\frac{100}{2} = 50$ copies. Contradiction.

(?)**23.23.** Let us divide the plane into equal squares with side $2R$ and paint them in a staggered order. Let us inscribe a circle into each of them. Then the details of the railway can be considered placed on these circles and the movement of the train that follows from the beginning to the end is performed clockwise on white fields and counterclockwise on black fields (or the other way round, see Fig. 74).
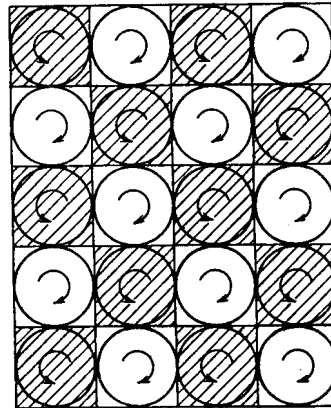


Figure 216 (Sol. 23.23)

Therefore, a deadend cannot arise since along both links of the deadend the movement is performed in the same fashion (clockwise or counterclockwise).

**23.24.** Let us consider the lattice depicted on Fig. 75 and paint it two colours as indicated in Fig. (white nodes are not painted on this Fig. and the initial square is shaded so that the grasshoppers sit in its white vertices). Let us prove that the grasshoppers can only reach white nodes, i.e., under the symmetry through a white node any white node turns into a white one. To prove this, it suffices to prove that under a symmetry through a white node a black node turns into a black one.

Let $A$ be a black node, $B$ a white one and $A_1$ the image of $A$ under the symmetry through $B$. Point $A_1$ is a black node if and only if $\overrightarrow{AA_1} = 2m\mathbf{e}_1 + 2n\mathbf{e}_2$, where $m$ and $n$ are integers.
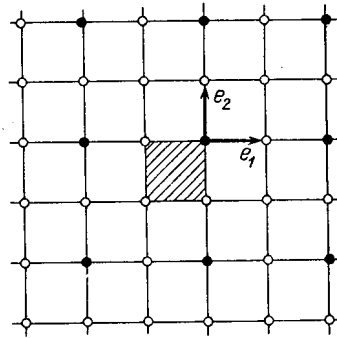
Figure 217 (Sol. 23.24)

It is clear that

$$\overrightarrow{AA_1} = 2\overrightarrow{AB} = 2(m\mathbf{e}_1 + n\mathbf{e}_2)$$

and, therefore, $A_1$ is a black node. Therefore, a grasshopper cannot reach the fourth vertex of the square.
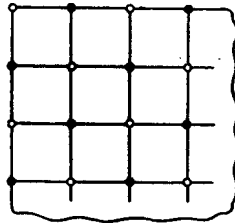


Figure 218 (Sol. 23.25)

**23.25.** Let us paint the nodes of the graph paper in a (?)chess order (Fig. 76). Since the endpoints of any unit segment are of different colours, the broken line with the endpoints of the same colour contains an odd number of nodes and an even number of nodes if its endpoints are of the same colour. Suppose that broken lines go out of all the nodes of the boundary (except for the vertices of the square). Let us prove then that all the broken lines together contain an even number of nodes. To this end it suffices to show that the number of broken lines with the endpoints of the same colour is even.

Let $4m$ white and $4n$ black nodes (the vertices of the square are not counted) are placed on the boundary of the square. Let $k$ be the number of broken lines with both endpoints white. Then there are $4m - 2k$ broken lines with endpoints of different colours and $\frac{4n - (4m - 2k)}{2} = 2(n - m) + k$ broken lines with black endpoints. It follows that there are $k + 2(n - m) + k = 2(n - m + k)$ — an even number — of broken lines with the endpoints of the same colour. It remains to notice that a $100 \times 100$ piece of paper contains an odd number of nodes. Therefore, the broken lines with an even number of nodes cannot pass through all the nodes.
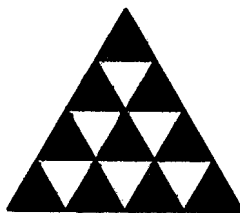


Figure 219 (Sol. 23.25)

**23.26.** Let us paint the triangles as shown on Fig. 77. Then there are $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ black triangles and $1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1)$ white triangles. It is clear that two triangles with consecutive indices are of distinct colours. Hence, among the numbered triangles the number of black triangles is only by 1 greater than that of white ones.

Therefore, the total number of numbered triangles does not exceed $n(n-1) + 1$.
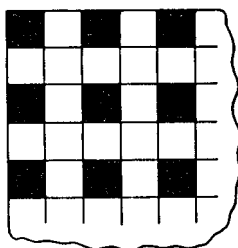


Figure 220 (Sol. 23.27)

**23.27.** Let us paint the bottom of the box two colours as shown on Fig. 78. Then every $2 \times 2$ tile covers exactly one black cell and a $1 \times 4$ tile covers 2 or 0 of them. Hence, the parity of the number of odd cells on the bottom of the box coincides with the parity of the number of $2 \times 2$ tiles. Since under the change of a $2 \times 2$ tile by a $1 \times 4$ tile the parity of the number of $2 \times 2$ tiles changes, we will not be able to tile the bottom of the box.
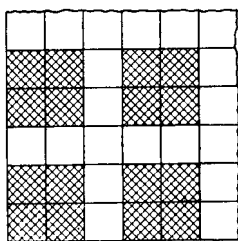


Figure 221 (Sol. 23.28)

**23.28.** In the given square piece of graph paper, let us shade $2 \times 2$ squares as shown on Fig. 79. We thus get 100 shaded squares. Every cut off square touches precisely one shaded square and therefore, at least one shaded square remains intact and can be cut off(?).

**23.29.** If a polygon is divided into parts by several diagonals, then these parts can be painted two colours so that parts with a common side were of distinct colours. This can be done as follows.
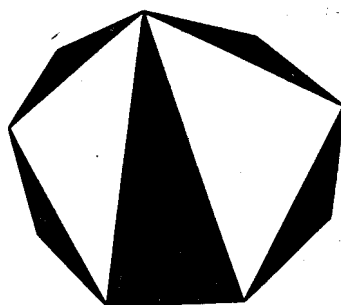


Figure 222 (Sol. 23.29)

Let us consecutively draw diagonals. Every diagonal splits the polygon into two parts. In one of them retain its painting and repaint the other one changing everywhere the white

colour to black and black to white. Performing this operation under all the needed diagonals, we get the desired coloring.

Since in the other case at every vertex an odd number of triangles meet, then under such a coloring all the sides of the polygon would belong to triangles of the same colour, for example, black, Fig. 80.

Denote the number of sides of white triangles by $m$. It is clear that $m$ is divisible by 3. Since every side of a white triangle is also a side of a black triangle and all the sides of the polygon are sides of the black triangles, it follows that the number of sides of black triangles is equal to $n + m$. Hence, $n + m$ is divisible by 3 and since $m$ is divisible by 3, then $n$ is divisible by 3.

**23.30.** Let us paint the chessboard four colours as shown on Fig. 81. It is easy to count the number of cells of the second colour: it is 26; that of the fourth is 24.
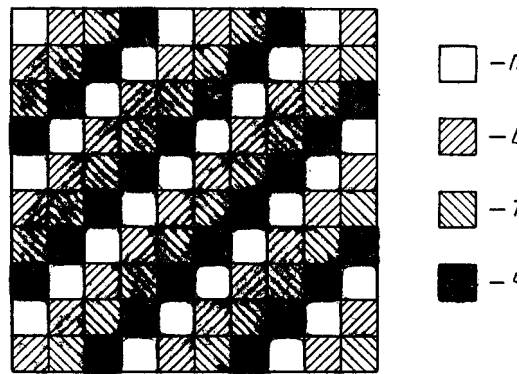


Figure 223 (Sol. 23.30)

Every $1 \times 4$ tile covers one cell of each colour. Therefore, it is impossible to tile a $10 \times 10$ chessboard with tiles of size $1 \times 4$ since otherwise there would have been an equal number of cells of every colour.

**23.31.** Let us paint the graph paper four colours as shown on Fig. 82. Among the given $n$ cells there are not less than $\frac{n}{4}$ cells of the same colour and such cells do not have common points.
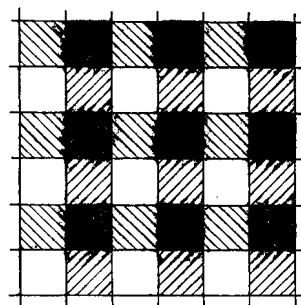


Figure 224 (Sol. 23.32)

**23.32.** Let us paint the nodes of graph paper four colours in the same order as the cells on Fig. 82 are painted. If $n \geq 5$, then there exist two vertices of an $n$-gon of the same colour. The midpoint of the segment with the endpoints in the nodes of the same colour is a node itself. Since the $n$-gon is a convex one, then the midpoint of the segment with the endpoints at its nodes lies either inside it or on its side.

**23.33.** Let us divide the obtained square into cells of size $1 \times 1$ and paint them three colours as shown on Fig. 83. It is easy to verify that it is possible to divide tiles of size $1 \times 3$ into two types: a tile of the first type covers one cell of the first colour and two cells of the second colour and a tile of the second type covers one cell of the second colour and two cells of the third colour.
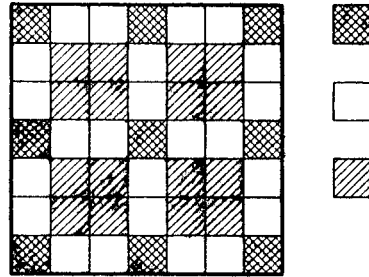


Figure 225 (Sol. 23.33)

Suppose that all the cells of the first colour are covered by tiles $1 \times 3$. Then there are 9 tiles of the first type and 7 tiles of the second type. Hence, they cover $9 \cdot 2 + 7 = 25$ cells of the second colour and $7 \cdot 2 = 14$ cells of the third colour. We have reached a contradiction and, therefore, one of the cells of the first colour is covered by the tile of size $1 \times 1$.

**23.34.** Let us cut the given $n$-gon by nonintersecting diagonals into triangles (cf. Problem 22.22). The vertices of the $n$-gon can be painted 3 colours so that all the vertices of each of the obtained triangles are of distinct colours (see Problem 23.40). There are not more than $[\frac{n}{3}]$ vertices of any colour; and it suffices to place guards at these points.

**23.35.** Let us consider an equilateral triangle with side 1. All of its three vertices cannot be of distinct colours and, therefore, two of the vertices are of the same colour; the distance between them is equal to 1.

**23.36.** Suppose that any two points situated at distance 1 are painted distinct colours. Consider an equilateral triangle $ABC$ with side 1; all its vertices are of distinct colours. Let point $A_1$ be symmetric to $A$ through line $BC$. Since $A_1B = A_1C = 1$, the colour of $A_1$ is distinct from that of $B$ and $C$ and $A_1$ is painted the same colour as $A$.

These arguments show that if $AA_1 = \sqrt{3}$, then points $A$ and $A_1$ are of the same colour. Therefore, all the points on the circle of radius $\sqrt{3}$ with center $A$ are of the same colour. It is clear that on this circle there are two points the distance between which is equal to 1. Contradiction.

**23.37.** Let us give an example of a seven-colour coloring of the plane for which the distance between any two points of the same colour is not equal to 1. Let us divide the plane into equal hexagons with side $a$ and paint them as shown on Fig. 84 (the points belonging to two or three hexagons can be painted any of the colours of these hexagons).

The greatest distance between points of the same colour that belong to one hexagon does not exceed $2a$ and the least distance between points of the same colour lying in distinct hexagons is not less than the length of segment $AB$ (see Fig. 84). It is clear that

$$AB^2 = AC^2 + BC^2 = 4a^2 + 3a^2 = 7a^2 > (2a)^2.$$

Therefore, if $2a < 1 < \sqrt{7}a$, i.e., $\frac{1}{\sqrt{7}} < a < \frac{1}{2}$, then the distance between points of the same colour cannot be equal to 1.

**23.38.** Suppose there does not exist a right triangle with vertices of the same colour. Let us divide every side of an equilateral triangle into three parts by two points. These points form a right hexagon. If two of its opposite vertices are of the same colour, then all the other
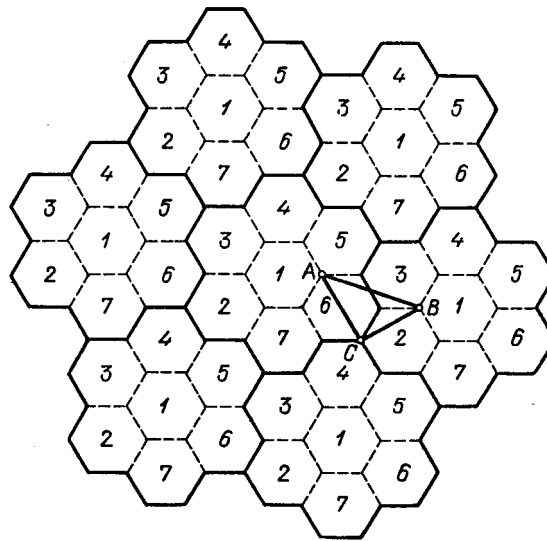
Figure 226 (Sol. 23.37)

vertices are of the second colour and therefore, there exists a right triangle with vertices of the second colour. Hence, the opposite vertices of the hexagon must be of distinct colours.

Therefore, there exist two neighbouring vertices of distinct colours; the vertices opposite to them are also of distinct colours. One of these pairs of vertices of distinct colours lies on a side of the triangle. The points of this side distinct from the vertices of the hexagon cannot be of either first or second colour. Contradiction.

**23.39.** Let us prove this statement by induction on the number of triangles of the triangulation. For one triangle the needed coloring exists. Now, let us suppose that it is possible to paint in the required way any triangulation consisting of less than $n$ triangles; let us prove that then we can paint any triangulation consisting of $n$ triangles.

Let us delete a triangle one of the sides of which lies on a side of the triangulated figure. The remaining part can be painted by the inductive hypothesis. (It is clear that this part can consist of several disjoint pieces but this does not matter.) Only two sides of the deleted triangle can be neighbouring with the other triangles. Therefore, it can be coloured the colour distinct from the colours of its two neighbouring triangles.

**23.40.** Proof is similar to that of Problem 23.39. The main difference is in that one must delete a triangle with two sides of the boundary of the polygon (cf. Problem 22.25).

**23.41.** Proof will be carried out by induction on the number of disks $n$. For $n = 1$ the statement is obvious. Let $M$ be any point, $O$ the most distant from $M$ center of a(?) given disk. Then the disk centered at $O$ is tangent to not more than 3 other given disks. Let us delete it and paint the other disks; this is possible thanks to the inductive hypothesis. Now, let us paint the deleted disk the colour distinct from the colours of the disks tangent to it.

# Chapter 24. INTEGER LATTICES

In plane, consider a system of lines given by equations $x = m$ and $y = n$, where $m$ and $n$ are integers. These lines form a lattice of squares or an *integer lattice*. The vertices of these squares, i.e., the points with integer coordinates, are called *the nodes* of the integer lattice.

## §1. Polygons with vertices in the nodes of a lattice

**24.1.** Is there an equilateral triangle with vertices in the nodes of an integer lattice?

**24.2.** Prove that for $n \neq 4$ a regular $n$-gon is impossible to place so that its vertices would lie in the nodes of an integer lattice.

**24.3.** Is it possible to place a right triangle with integer sides (i.e., with sides of integer length) so that its vertices would be in nodes of an integer lattice but none of its sides would pass along the lines of the lattice?

**24.4.** Is there a closed broken line with an odd number of links of equal length all vertices of which lie in the nodes of an integer lattice?

**24.5.** The vertices of a polygon (not necessarily convex one) are in nodes of an integer lattice. Inside the polygon lie $n$ nodes of the lattice and $m$ nodes lie on the polygon's boundary. Prove that the polygon's area is equal to $n + \frac{m}{2} - 1$. (*Pick's formula.*)

**24.6.** The vertices of triangle $ABC$ lie in nodes of an integer lattice and there are no other nodes on its sides whereas inside it there is precisely one node, $O$. Prove that $O$ is the intersection point of the medians of triangle $ABC$.

See also Problem 23.32.

## §2. Miscellaneous problems

**24.7.** On an infinite sheet of graph paper $N$, cells are painted black. Prove that it is possible to cut off a finite number of squares from this sheet so that the following two conditions are satisfied:

1) all black cells belong to the cut-off squares;

2) in any cut-off square $K$, the area of black cells constitutes not less than 0.2 and not more than 0.8 of the area of $K$.

**24.8.** The origin is the center of symmetry of a convex figure whose area is greater than 4. Prove that this figure contains at least one distinct from the origin point with integer coordinates. (*Minkowski's theorem.*)

**24.9.** In all the nodes of an integer lattice except one, in which a hunter stands, trees are growing and the trunks of these trees are of radius $r$ each. Prove that the hunter will not be able to see a hare that sits further than $\frac{1}{r}$ of the unit length from it.

**24.10.** Inside a convex figure of area $S$ and semiperimeter $p$ there are $n$ nodes of a lattice. Prove that $n > S - p$.

**24.11.** Prove that for any $n$ there exists a circle inside which there are exactly (not more nor less) $n$ integer points.

**24.12.** Prove that for any $n$ there exists a circle on which lies exactly (not more nor less) $n$ integer points.

## Solutions

**24.1.** Suppose that the vertices of an equilateral triangle $ABC$ are in nodes of an integer lattice. Then the tangents of all the angles formed by sides $AB$ and $AC$ with the lines of the lattice are rational. For any position of triangle $ABC$ either the sum or the difference of certain two of such angles $\alpha$ and $\beta$ is equal to $60°$. Hence,

$$\sqrt{3} = \tan 60° = \tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

is a rational number. Contradiction.

**24.2.** For $n = 3$ and $n = 6$ the statement follows from the preceding problem and, therefore, in what follows we will assume that $n \neq 3$, 4, 6. Suppose that there exist regular $n$-gons with vertices in nodes of an integer lattice ($n \neq 3$, 4, 6). Among all such $n$-gons we can select one with the shortest side. (To prove that we can do it, it suffices to observe that if $a$ is the length of a segment with the endpoints in nodes of the lattice, then $a = \sqrt{n^2 + m^2}$, where $n$ and $m$ are integers, i.e., there is only a finite number of distinct length of segments with the endpoints in nodes of the lattice shorter than the given length.) Let $\overrightarrow{A_i B_i} = \overrightarrow{A_{i+1} A_{i+2}}$. Then $B_1 \ldots B_n$ is a regular $n$-gon whose vertices lie in nodes of the integer lattice and its side is shorter than any side of the $n$-gon $A_1 \ldots A_n$. For $n = 5$ this is clear from Fig. 85 and for $n \geq 7$ look at Fig. 86. We have arrived to a contradiction with the choice of the $n$-gon $A_1 \ldots A_n$.
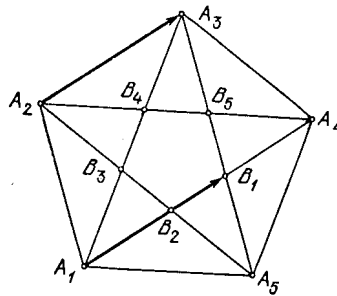


Figure 227 (Sol. 24.2)
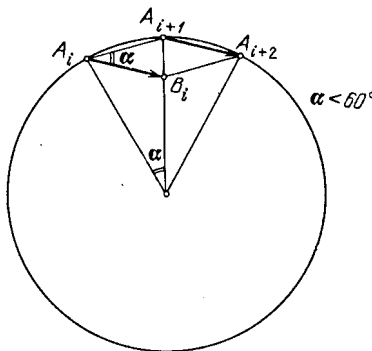


Figure 228 (Sol. 24.2)

**24.3.** It is easy to verify that the triangle with the vertices at points with coordinates $(0,0)$, $(12,16)$ and $(-12,9)$ possesses the required properties.

**24.4.** Suppose that there exists a closed broken line $A_1 \ldots A_n$ with an odd number of links of equal length all the vertices of which lie in nodes of an integer lattice. Let $a_i$ and

$b_i$ be coordinates of the projections of vector $\overrightarrow{A_iA_{i+1}}$ to the horizontal and vertical axes, respectively. Let $c$ be the length of the link of the broken line. Then $c^2 = a_i^2 + b_i^2$.

Hence, the residue after the division of $c^2$ by 4 is equal to 0, 1 or 2. If $c^2$ is divisible by 4, then $a_i$ and $b_i$ are divisible by 4 (this is proved by a simple case-by-case checking of all possible residues after the division of $a_i$ and $b_i$ by 4). Therefore, the homothety centered at $A_1$ with coefficient 0.5 sends our broken line into a broken line with a shorter links but whose vertices are also in the nodes of the lattice. After several such operations we get a broken line for which $c^2$ is not divisible by 4, i.e., the corresponding residue is equal to either 1 or 2.

Let us consider these variants, but first observe that

$$a_1 + \cdots + a_m = b_1 + \cdots + b_m = 0.$$

1) The residue after division of $c^2$ by 4 is equal to 1. Then one of the numbers $a_i$ and $b_i$ is odd and the other one is even; hence, the number $a_1 + \cdots + a_m$ is odd and cannot equal to zero.

2) The residue after division of $c^2$ by 4 is equal to 2. Then the numbers $a_i$ and $b_i$ are both odd; hence, $a_1 + \cdots + a_m + b_1 + \cdots + b_m$ is odd and cannot equal to zero.

**24.5.** To every polygon $N$ with vertices in nodes of an integer lattice assign the number $f(N) = n + \frac{m}{2} - 1$. Let polygon $M$ be cut into polygons $M_1$ and $M_2$ with vertices in nodes of the lattice. Let us prove that if Pick's formula holds for two of the polygons $M$, $M_1$ and $M_2$, then it is true for the third one as well.

To this end it suffices to prove that $f(M) = f(M_1) + f(M_2)$. The nodes which lie outside the line of cut contribute equally to $f(M)$ and $f(M_1) + f(M_2)$. "Nonterminal" nodes of the cut contribute 1 to $f(M)$ and 0.5 to $f(M_1)$ and $f(M_2)$. Each of the two terminal nodes of the cut contributes 0.5 to each of $f(M), f(M_1)$ and $f(M_2)$ and, therefore, the contribution of the terminal nodes to $f(M)$ is by 1 less than to $f(M_1) + f(M_2)$. Since we have to deduct 1 from each contribution to $f(M)$ and two from each contribution to $f(M_1) + f(M_2)$, it follows that $f(M) = f(M_1) + f(M_2)$.

Now, let us prove the validity of Pick's formula for an arbitrary triangle. If $M$ is a rectangle with sides of length $p$ and $q$ directed along the lines of the lattice, then

$$f(M) = (p-1)(q-1) + \frac{2(p+q)}{2} - 1 = pq,$$

i.e., Pick's formula holds for $M$. Cutting triangle $M$ into triangles $M_1$ and $M_2$ by a diagonal and making use of the fact that $f(M) = f(M_1) + f(M_2)$ and $f(M_1) = f(M_2)$ it is easy to prove the validity of Pick's formula for any right triangle with legs directed along the lines of the lattice. Cutting several such triangles from the rectangle we can get any triangle (Fig. 87).
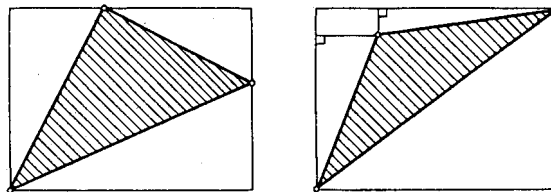


Figure 229 (Sol. 24.5)

To complete the proof of Pick's formula, it remains to notice that any polygon can be cut by diagonals into triangles.

**24.6.** Thanks to Pick's formula $S_{AOB} = S_{BOC} = S_{COA} = \frac{1}{2}$; hence, $O$ is the intersection point of medians of triangle $ABC$ (cf. Problem 4.2).

**24.7.** Take a sufficiently large square with side $2^n$ so that all the black cells are inside it and constitute less than 0.2 of its area. Let us divide this square into four identical squares. The painted area of each of them is less than 0.8 of the total. Let us leave those of them whose painted part constitutes more than 0.2 of the total and cut the remaining ones in the same way.

The painted area of the obtained $2 \times 2$ squares will be $\frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$ of the total or they will not be painted at all. Now, we have to cut off those of the obtained squares which contain painted cells.

**24.8.** Consider all the convex figures obtained from the given one by translations by vectors with both coordinates even. Let us prove that at least two of these figures intersect. The initial figure can be squeezed in the disk of radius $R$ centered in the origin, where for $R$ we can take an integer. Take those of the considered figures the coordinates of whose centers are nonnegative integers not greater than $2n$.

There are precisely $(n+1)^2$ of such figures and all of them lie inside a square with side $2(n+R)$. If they do not intersect, then for any $n$ we would have had $(n+1)^2 S < 4(n+R)^2$, where $S$ is the area of the given figure. Since $S > 4$, we can select $n$ so that the inequality $\frac{n+R}{n+1} < \sqrt{\frac{S}{4}}$ holds.
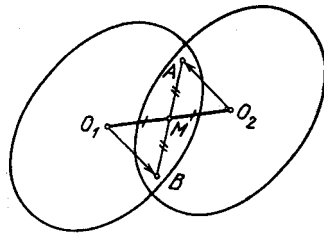


Figure 230 (Sol. 24.8)

Let now figures with centers $O_1$ and $O_2$ have a common point $A$ (Fig. 88). Let us prove that then the midpoint $M$ of segment $O_1 O_2$ belongs to both figures (it is clear that the coordinates of $M$ are integers). Let $\overrightarrow{O_1 B} = -\overrightarrow{O_2 A}$.

Since the given figure is centrally symmetric, point $B$ belongs to the figure with center $O_1$. This figure is convex and points $A$ and $B$ belong to it and, therefore, the midpoint of segment $AB$ also belongs to it. Clearly, the midpoint of segment $AB$ coincides with the midpoint of segment $O_1 O_2$.

**24.9.** Let the hunter sit at point $O$ and the hare at point $A$; let $A_1$ be the point symmetric to $A$ with respect to $O$. Consider figure $\Phi$ that contains all the points the distance from which to segment $AA_1$ does not exceed $r$ (Fig. 89).
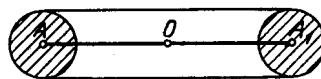


Figure 231 (Sol. 24.9)

It suffices to prove that $\Phi$ contains at least one node of the lattice (if the node gets into the shaded part, then point $A$ belongs to the trunk).

The area of $\Phi$ is equal to $4rh + \pi r^2$, where $h$ is the distance from the hunter to the hare. If $h > \frac{1}{r}$, then $4rh + \pi r^2 > 4$. By Minkowski's theorem $\Phi$ contains an integer point.

**24.10.** Consider the integer lattice given by equations $x = k + \frac{1}{2}$ and $y = l + \frac{1}{2}$, where $k$ and $l$ are integers. Let us prove that each small square of this lattice gives a nonnegative contribution to $n - S + p$. Consider two cases:

1) The figure contains the center of the square. Then $n' = 1$ and $S' \le 1$; hence, $n' - S' + p' \ge 0$.

2) The figure intersects the square but does not contain its center. Let us prove that in this case $S' \le p'$ and we can confine ourselves with the study of the cases depicted on Fig. 90 (i.e., we may assume that the center $O$ of the square lies on the boundary of the figure). Since the distances from the center of the square to its sides are equal to $\frac{1}{2}$, it follows that $p' \ge \frac{1}{2}$. Draw the base line through $O$ to this figure; we get $S' \le \frac{1}{2}$.
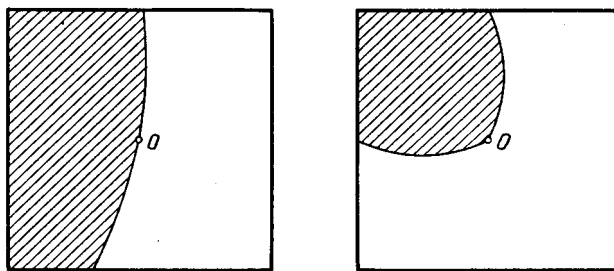


Figure 232 (Sol. 24.10)

It is also clear that all the contributions of the squares cannot be zero simultaneously.

**24.11.** First, let us prove that on the circle with center $A = (\sqrt{2}, \frac{1}{3})$ there cannot lie more than one integer point. If $m$ and $n$ are integers, then

$$(m - \sqrt{2})^2 + (n - \frac{1}{3})^2 = q - 2m\sqrt{2},$$

where $q$ is a rational number. Therefore, the equality

$$(m_1 - \sqrt{2})^2 + (n_1 - \frac{1}{3})^2 = (m_2 - \sqrt{2})^2 + (n_2 - \frac{1}{3})^2$$

implies that $m_1 = m_2$. By Vièta's theorem the sum of roots of equation $(n - \frac{1}{3})^2 = d$ is equal to $\frac{2}{3}$; hence, at least one root can be integer.

Now, let us arrange the radii of the circles with center $A$ passing through integer points in the increasing order: $R_1 < R_2 < R_3 < \ldots$. If $R_n < R < R_{n+1}$, then inside the circle of radius $R$ with center $A$ there are $n$ integer points.

**24.12.** First. let us prove that the equation $x^2 + y^2 = 5^k$ has $4(k+1)$ integer solutions. For $k = 0$ and $k = 1$ this statement is obvious. Let us prove that the equation $x^2 + y^2 = 5^k$ has exactly 8 solutions $(x, y)$ such that $x$ and $y$ are not divisible by 5. Together with $4(k-1)$ solutions of the form $(5a, 5b)$, where $(a, b)$ is a solution of the equation $a^2 + b^2 = 5^{k-2}$, they give the needed number of solutions.

These solutions are obtained from each other by permutations of $x$ and $y$ and changes of signs; we will call them *nontrivial solutions*.

Let $x^2 + y^2$ be divisible by 5. Then $(x + 2y)(x - 2y) = x^2 + y^2 - 5y^2$ is also divisible by 5. Hence, one of the numbers $x + 2y$ and $x - 2y$ is divisible by 5. It is also easy to verify that if $x + 2y$ and $x - 2y$ are divisible by 5, then both $x$ and $y$ are divisible by 5.

If $(x, y)$ is a nontrivial solution of equation $x^2 + y^2 = 5^k$, then $(x + 2y, 2x - y)$ and $(x - 2y, 2x + y)$ are solutions of equation $\xi^2 + \eta^2 = 5^{k+1}$ and precisely one of them is nontrivial. It remains to prove that a nontrivial solution is unique up to permutations of $x$ and $y$ and changes of signs.

Let $(x, y)$ be a nontrivial solution of the equation $x^2 + y^2 = 5^k$. Then the pairs

$$\left(\pm\frac{2x - y}{5}, \pm\frac{x + 2y}{5}\right) \quad \text{and} \quad \left(\pm\frac{x + 2y}{5}, \pm\frac{2x - y}{5}\right) \tag{1}$$

together with the pairs

$$\left(\pm\frac{2x + y}{5}, \pm\frac{x - 2y}{5}\right) \quad \text{and} \quad \left(\pm\frac{x - 2y}{5}, \pm\frac{2x + y}{5}\right) \tag{2}$$

are solutions of the equation $\xi^2 + \eta^2 = 5^{k-1}$ but the pairs of exactly one of these types will be integer since exactly one of the numbers $x + 2y$ and $x - 2y$ is divisible by 5. Thus, we will get a nontrivial solution because

$$(x + 2y)(x - 2y) = (x^2 + y^2) - 5y^2$$

for $k \geq 2$ is divisible by 5 but is not divisible by 25.

Therefore, each of the 8 nontrivial solutions of the equation $x^2 + y^2 = 5^k$ yields 8 nontrivial solutions of the equation $\xi^2 + \eta^2 = 5^{k-1}$ where for one half of the solutions we have to make use of formulas (1) and for the other half of the formulas (2).

Now, let us pass directly to the solution of the problem. Let $n = 2k + 1$. Let us prove that on the circle of radius $\frac{5^k}{3}$ with center $(\frac{1}{3}, 0)$ there lie exactly (not more nor less) $n$ integer points. The equation $x^2 + y^2 = 5^{2k}$ has $4(2k + 1)$ integer solutions. Moreover, after the division of $5^{2k}$ by 3 we have residue 1; hence, one of the numbers $x$ and $y$ is divisible by 3 and the residue after the division of the other one by 3 is equal to $\pm 1$. Therefore, in precisely one of the pairs $(x, y)$, $(x, -y)$, $(y, x)$ and $(-y, x)$ the residues after the divisionof the first and the second number by 3 are equal to $-1$ and 0, respectively. Hence, the equation $(3z - 1)^2 + (3t)^2 = 5^{2k}$ has precisely $2k + 1$ integer solutions.

Let $n = 2k$. Let us prove that on the circle of radius $\frac{5^{(k-1)/2}}{2}$ with center $(\frac{1}{2}, 0)$ there lie $n$ integer points. The equation $x^2 + y^2 = 5^{k-1}$ has $4k$ integer solutions; for them one of the numbers $x$ and $y$ is even and the other one is odd. Hence, the equation $(2z-1)^2 + (2t)^2 = 5^{k-1}$ has $2k$ integer solutions.

# Chapter 25. CUTTINGS

## §1. Cuttings into parallelograms

**25.1.** Prove that the following properties of convex polygon $F$ are equivalent:
1) $F$ has a center of symmetry;
2) $F$ can be cut into parallelograms.

**25.2.** Prove that if a convex polygon can be cut into centrally symmetric polygons, then it has a center of symmetry.

**25.3.** Prove that any regular $2n$-gon can be cut into rhombuss.

**25.4.** A regular octagon with side 1 is cut into parallelograms. Prove that among the parallelograms there is at least two rectangles and the sum of areas of all the rectangles is equal to 2.

## §2. How lines cut the plane

In plane, let there be drawn $n$ pairwise nonparallel lines no three of which intersect at one point. In Problems 25.5–25.9 we will consider properties of figures into which these lines cut the plane. A figure is called an *n-linked* one if it is bounded by $n$ links (i.e., a link is a line segment or a ray).

**25.5.** Prove that for $n = 4$ among the obtained parts of the plane there is a quadrilateral.

**25.6.** a) Find the total number of all the obtained figures.

b) Find the total number of *bounded* figures, i.e., of polygons.

**25.7.** a) Prove that for $n = 2k$ there are not more than $2k-1$ angles among the obtained figures.

b) Is it possible that for $n = 100$ there are only three angles among the obtained figures?

**25.8.** Prove that if among the obtained figures there is a $p$-linked and a $q$-linked ones, then $p + q \leq n + 4$.

**25.9.** Prove that for $n \geq 3$ there are not less than $\frac{2n-2}{3}$ triangles among the obtained parts.

Now, let us abandon the assumption that no three of the considered lines intersect at one point. If $P$ is the intersection point of two or several lines, then the number of lines of the given system passing through point $P$ will be denoted by $\lambda(P)$.

**25.10.** Prove that the number of segments into which the given lines are divided by their intersection points is equal to $n + \sum \lambda(P)$.

**25.11.** Prove that the number of parts into which given lines divide the plane is equal to $1 + n + \sum(\lambda(P) - 1)$ and among these parts there are $2n$ unbounded ones.

**25.12.** The parts into which the plane is cut by lines are painted red and blue so that the neighbouring parts are of distinct colours (cf. Problem 27.1). Let $r$ be the number of red parts, $b$ the number of blue parts. Prove that

$$r \leq 2b - 2 - \sum(\lambda(P) - 2)$$

where the equality is attained if and only if the red parts are triangles and angles.

## Solutions

**25.1.** Consider a convex polygon $A_1 \ldots A_n$. Prove that each of the properties 1) and 2) is equivalent to the following property:

3) For any vector $\overrightarrow{A_i A_{i+1}}$ there exists a vector $\overrightarrow{A_j A_{j+1}} = -\overrightarrow{A_i A_{i+1}}$.

Property 1) clearly implies property 3). Let us prove that property 3) implies property 1). If a convex polygon $A_1 \ldots A_n$ possesses property 3), then $n = 2m$ and $\overrightarrow{A_i A_{i+1}} = -\overrightarrow{A_{m+i} A_{m+i+1}}$. Let $O_i$ be the midpoint of segment $A_i A_{m+i}$. Since $A_i A_{i+1} A_{m+i} A_{m+i+1}$ is a parallelogram, we have $O_i = O_{i+1}$. Hence, all the points $O_i$ coincide and this point is the center of symmetry of the polygon.

Let us prove that property 2) implies property 3). Let a convex polygon $F$ be divided into parallelograms. We have to prove that for any side of $F$ there exists another side parallel and equal to it. From every side of $F$ a chain of parallelograms departs, i.e., this side sort of moves along them parallelly so that it can be split into several parts (Fig. 91).
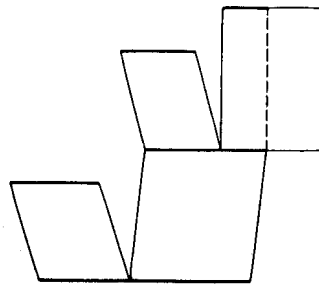


Figure 233 (Sol. 25.1)

Since a convex polygon can have only one more side parallel to the given one, all the bifurcations of the chain terminate in the same side which is not shorter than the side from which the chain starts. We can equally well begin the chain of parallelograms from the first side to the second one or from the second one to the first one; hence, the lengths of these sides are equal.

It remains to prove that property 3) implies property 2). A way of cutting a polygon with equal and parallel opposite sides is indicated on Fig. 92.
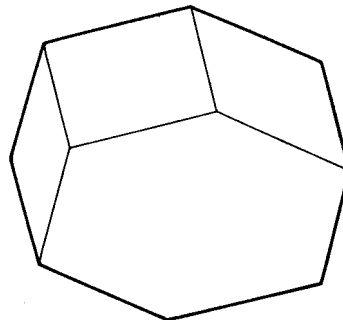


Figure 234 (Sol. 25.1)

After each such operation we get a polygon with a lesser number of sides which still possesses property 3) and by applying the same process to this polygon we eventually get a parallelogram.

**25.2.** Let us make use of the result of the preceding problem. If a convex polygon $M$ is cut into convex centrally symmetric polygons, then they can be cut into parallelograms. Therefore, $M$ can be cut into parallelograms, i.e., $M$ has a center of symmetry.

**25.3.** Let us prove by induction on $n$ that any $2n$-gon whose sides have the same length and opposite sides are parallel can be cut into rhombs. For $n = 2$ this is obvious and from Fig. 92 it is clear how to perform the inductive step.

**25.4.** Let us single out two perpendicular pairs of opposite sides in a regular octagon and, as in Problem 25.1, consider chains of parallelograms that connect the opposite sides. On the intersection of these chains rectangles stand. By considering two other pairs of opposite sides we will get at least one more rectangle.

It is possible to additionally cut parallelograms from every chain so that the chain would split into several "passes" and in each pass the neighbouring parallelograms are neighboring to each other along the whole sides, not a part of a side. The union of rectangles of a new partition coincides with the union of rectangles of the initial partition and, therefore, it suffices to carry out the proof for the new partition.

Every pass has a constant width; hence, the length of one side of each rectangle that enters a path is equal to the width of the path, and the sum of length of all the other sides is equal to the sum of the widths of the passes corresponding to the other pair of sides.

Therefore, the area of all the rectangles that constitute one path is equal to the product of the width of the path by the length of the side of the polygon, i.e., its value is equal to the width of the path. Hence, the area of all the rectangles corresponding to two perpendicular pairs of opposite sides is equal to 1 and the area of the union of the rectangles is equal to 2.

**25.5.** Denote the intersections points of one of the given lines with the other ones by $A$, $B$ and $C$. For definiteness, let us assume that point $B$ lies between $A$ and $C$. Let $D$ be the intersection point of lines through $A$ and $C$. Then any line passing through point $B$ and not passing through $D$ cuts triangle $ACD$ into a triangle and a quadrilateral.

**25.6.** a) Let $n$ lines divide the plane into $a_n$ parts. Let us draw one more line. This will increase the number of parts by $n + 1$ since the new line has $n$ intersection points with the already drawn lines. Therefore, $a_{n+1} = a_n + n + 1$. Since $a_1 = 2$, it follows that $a_n = 2 + 2 + 3 + \cdots + n = \frac{n^2 + n + 2}{2}$.

b) Encircle all the intersection points of the given lines. It is easy to verify that the number of unbounded figures is equal to $2n$. Therefore, the number of bounded figures is equal to

$$\frac{n^2 + n + 2}{2} - 2n = \frac{n^2 - 3n + 2}{2}.$$

**25.7.** a) All intersection points of given lines can be encircled in a circle. Lines divide this circle into $4k$ arcs. Clearly, two neighbouring arcs cannot simultaneously belong to angles; hence, the number of angles does not exceed $2k$, where the equality can only be attained if the arcs belonging to the angles alternate. It remains to prove that the equality cannot be attained. Suppose that the arcs belonging to angles alternate. Since on both sides from any of the given lines lie $2k$ arcs, the opposite arcs (i.e., the arcs determined by two lines) must belong to angles (Fig. 93) which is impossible.

b) For any $n$ there can be three angles among the obtained figures. On Fig. 94 it is shown how to construct the corresponding division of the plane.

**25.8.** Let us call a line which is the continuation of a segment or a ray that bounds a figure a (border?)*bounding* line of the figure. It suffices to show that two considered figures cannot have more than 4 common bounding lines. If two figures have 4 common bounding lines, then one of the figures lies in domain 1 and the other one lies in domain 2 (Fig. 95).

The fifth bounding line of the figure that lies in domain 1 must intersect two neighbouring sides of the quadrilateral 1; but then it cannot be bounding line for the figure that belongs to domain 2.
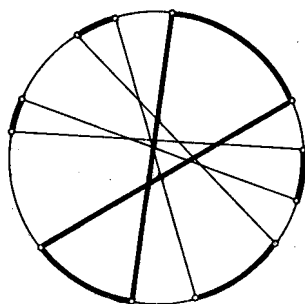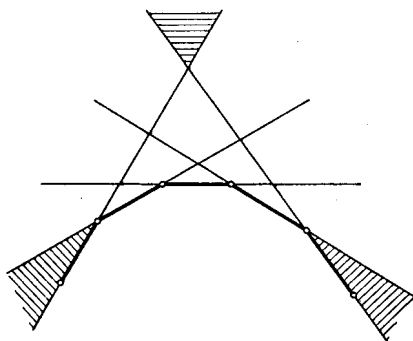
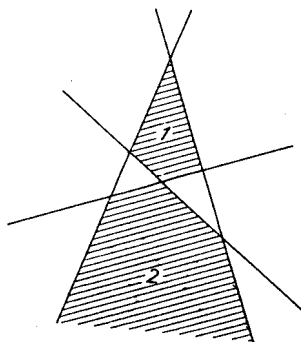Figure 235 (Sol. 25.7 a))



Figure 236 (Sol. 25.7 b))



Figure 237 (Sol. 25.8)

**25.9.** Consider all the intersection points of the given lines. Let us prove that these points may lie on one side of not more than two given lines. Suppose that all the intersection points lie on one side of three given lines. These lines constitute triangle $ABC$. The fourth line cannot intersect the sides of this triangle only, i.e., it intersects at least one extension of a side. Let, for definiteness, it intersect the continuation of side $AB$ beyond point $B$; let the intersection point be $M$. Then points $A$ and $M$ lie on distinct sides of line $BC$. Contradiction. Hence, there exist at least $n - 2$ lines on both sides of which there are intersection points.

If in the half plane given by line $l$ we select the nearest $l$ intersection point, then this point is a vertex of a triangle adjacent to $l$. Thus, there exists not less than $n - 2$ lines to each of which at least two triangles are adjacent and there are two lines to each of which at least one triangle is adjacent. Since every triangle is adjacent to exactly 3 lines, there are not less than $\frac{2(n-2)+2}{3}$ triangles.

**25.10.** If $P$ is the intersection point of given lines, then $2\lambda(P)$ segments or rays go out of $P$. Moreover, each of $x$ segments have two boundary points and each of $2n$ rays has one boundary point. Hence, $2x + 2n = 2\sum \lambda(P)$, i.e., $x = -n + \sum \lambda(P)$.

**25.11.** Let us carry out the proof by induction on $n$. For two lines the statement is obvious. Suppose that the statement holds for $n - 1$ line and consider a system consisting of $n$ lines. Let $f$ be the number of parts into which the given $n$ lines divide the plane; $g = 1 + n + \sum(\lambda(P) - 1)$. Let us delete one line from the given system and define similarly numbers $f'$ and $g'$ for the system obtained. If on the deleted line there lie $k$ intersection points of lines, then $f' = f - k - 1$ and $g' = 1 + (n-1) + \sum(\lambda'(P) - 1)$. It is easy to verify that $\sum(\lambda(P) - 1) = -k + \sum(\lambda'(P) - 1)$. By inductive hypothesis $f' = g'$.

Therefore, $f = f' + k + 1 = g' + k + 1 = g$. It is also clear that the number of unbounded parts is equal to $2n$.

**25.12.** Let $r'_k$ be the number of red $k$-gons, $r'$ the number of bounded red domains and the number of segments into which the given lines are divided by their intersection points be equal to $\sum \lambda(P) - n$, cf. Problem 25.10. Each segment is a side of not more than 1 red polygon, hence, $3r' \leq \sum_{k \geq 3} kr'_k \leq \sum \lambda(P) - n$, where the left inequality is only attained if and only if there are no red $k$-gons for $k > 3$, and the right inequality is only attained if and only if any segment is a side of a red $k$-gon, i.e., any unbounded red domain is an angle.

The number of bounded domains is equal to $1 - n + \sum(\lambda(P) - 1) = c$ (see Problem 25.11), hence, the number $b'$ of bounded blue domaions is equal to

$$c - r' \geq 1 - n + \sum(\lambda(P) - 1) - \frac{\sum \lambda(P) - n}{3} = 1 - \frac{2n}{3} + \sum\left(\frac{2\lambda(P)}{3} - 1\right).$$

The colours of $2n$ unbounded domains alternate; hence,

$$b = b' + n \geq 1 + \frac{n}{3} + \sum\left(\frac{2\lambda(P)}{3} - 1\right)$$

and

$$r = r' + n \leq \frac{2n + \sum \lambda(P)}{3}$$

and, therefore, $2b - r \geq 2 + \sum(\lambda(P) - 2)$.

# Chapter 26. SYSTEMS OF POINTS AND SEGMENTS. EXAMPLES AND COUNTEREXAMPLES

## §1. Systems of points

**26.1.** a) An architect wants to place four sky-scrapers so that any sightseer can see their spires in an arbitrary order. In other words, if the sky-scrapers are numbered, then for any ordered set $(i, j, k, l)$ of sky-scrapers one can stand at an arbitrary place in the town and by turning either clockwise or counterclockwise see first the spire of the sky-scraper $i$, next, that of $j$, $k$ and, lastly, $l$. Is it possible for the architect to perform this?

b) The same question for five sky-scrapers.

**26.2.** In plane, there are given $n$ points so that from any foresome of these points one can delete one point so that the remaining three points lie on one line. Prove that it is possible to delete one of the given points so that all the remaining points lie on one line.

**26.3.** Given 400 points in plane, prove that there are not fewer than 15 distinct distances between them.

**26.4.** In plane, there are given $n \geq 3$ points. Let $d$ be the greatest distance between any two of these points. Prove that there are not more than $n$ pairs of points with the distance between the points of any pair equal to $d$.

**26.5.** In plane, there are given 4000 points no three of which lie on one line. Prove that there are 1000 nonintersecting quadrilaterals (perhaps, nonconvex ones) with vertices at these points.

**26.6.** In plane, there are given 22 points no three of which lie on one line. Prove that it is possible to divide them into pairs so that the segments determined by pairs intersect at least at 5 points.

**26.7.** Prove that for any positive integer $N$ there exist $N$ points no three of which lie on one line and all the pairwise distances between them are integers.

See also Problems 20.13-20.15, 22.7.

## §2. Systems of segments, lines and circles

**26.8.** Construct a closed broken line of six links that intersects each of its links precisely once.

**26.9.** Is it possible to draw six points in plane and to connect them with nonintersecting segments so that each point is connected with precisely four other ones?

**26.10.** Point $O$ inside convex polygon $A_1 \ldots A_n$ possesses a property that any line $OA_i$ contains one more vertex $A_j$. Prove that no point except $O$ possesses such a property.

**26.11.** On a circle, $4n$ points are marked and painted alternatedly red and blue. Points of the same colour are divided into pairs and points from each pair are connected by segments of the same colour. Prove that if no three segments intersect at one point, then there exist at least $n$ intersection points of red segments with blue segments.

**26.12.** In plane, $n \geq 5$ circles are placed so that any three of them have a common point. Prove that then all the circles have a common point.

## §3. Examples and counterexamples

There are many wrong statements that at first glance seem to be true. To refute such statements we have to construct the corresponding example; such examples are called *counterexamples.*

**26.13.** Is there a triangle all the hights of which are shorter than 1 cm and the area is greater than 1 m $^2$?

**26.14.** In a convex quadrilateral $ABCD$ sides $AB$ and $CD$ are equal and angles $A$ and $C$ are equal. Must this quadrilateral be a parallelogram?

**26.15.** The list of sides and diagonals of a convex quandrilateral ordered with respect to length coincides with a similar list for another quandrilateral. Must these quandrilaterals be equal?

**26.16.** Let $n \geq 3$. Do there exist $n$ points that do not belong to one line and such that pairwise distances between which are irrational while the areas of all the triangles with vertices in these points are rational?

**26.17.** Do there exist three points $A$, $B$ and $C$ in plane such that for any point $X$ the length of at least one of the segments $XA$, $XB$ and $XC$ is irrational?

**26.18.** In an acute triangle $ABC$, median $AM$, bisector $BK$ and hight $CH$ are drawn. Can the area of the triangle formed by the intersection points of these segments be greater than $0.499 \cdot S_{ABC}$?

**26.19.** On an infinite list of graph paper (with small cells of size $1 \times 1$) the domino chips of size $1 \times 2$ are placed so that they cover all the cells. Is it possible to make it so that any line that follows the lines of the mash cuts only a finite number of chips?

**26.20.** Is it possible for a finite set of points to contain for every of its points precisely 100 points whose distance from the point is equal to 1?

**26.21.** In plane, there are several nonintersecting segments. Is it always possible to connect the endpoints of some of them by segments so that we get a closed nonselfintersecting broken line?

**26.22.** Consider a triangle. Must the triangle be an isosceles one if the center of its inscribed circle is equidistant from the midpoints of two of its sides?

**26.23.** The arena of a circus is illuminated by $n$ distinct spotlights. Each spotlight illuminates a convex figure. It is known that if any of the spotlights is turned off, then the arena is still completely illuminated, but if any two spotlights are turned off, then the arena is not completely illuminated. For which $n$ this is possible?

See also problems 22.16–22.18, 22.26, 22.27, 22.29, 23.37, 24.11, 24.12.

## Solutions

**26.1.** a) It is easy to verify that constructing the fourth building inside the triangle formed by the three other buildings we get the desired position of the buildings.

b) It is impossible to place in the desired way five buildings. Indeed, if we consecutively see buildings $A_1$, $A_2$, $\ldots$, $A_n$, then $A_1 A_2 \ldots A_n$ is a nonselfintersecting broken line. Therefore, if $ABCD$ is a convex quadrilateral, then it is impossible to see its vertices in the following order: $A$, $C$, $D$, $B$. It remains to notice that of five points no three of which lie on one line it is always possible to select four points which are vertices of a convex quandrilateral (Problem 22.2).

**26.2.** It is possible to assume that $n \geq 4$ and not all the points lie on one line. Then we can select four points $A$, $B$, $C$ and $D$ not on one line. By the hypothesis, three of them lie on one line. Let, for definiteness, points $A$, $B$ and $C$ lie on line $l$ and $D$ does not lie on $l$.

We have to prove that all the points except for $D$ lie on $l$. Suppose that a point $E$ does not belong to $l$. Let us consider points $A$, $B$, $D$, $E$. Both triples $A$, $B$, $D$ and $A$, $B$, $E$ do not lie on one line. Therefore, on one line there lies either triple $(A, D, E)$ or triple $(B, D, E)$. Let, for definiteness, points $A$, $D$ and $E$ lie on one line. Then no three of the points $B$, $C$, $D$, $E$ lie on one line. Contradiction.

**26.3.** Let the number of distinct distances between points be equal to $k$. Fix two points. Then all the other points are intersection points of two families of concentric circles containing $k$ circles each. Hence, the total number of points does not exceed $2k^2 + 2$. It remains to notice that $2 \cdot 14^2 + 2 = 394 < 400$.

**26.4.** A segment of length $d$ connecting a pair of given points will be called a *diameter*. The endpoints of all the diameters that begin at point $A$ lie on the circle centered in $A$ and of radius $d$. Since the distance between any two points does not exceed $d$, the endpoints of all the diameters beginning in $A$ belong to an arc whose angle value does not exceed $60°$. Therefore, if three diameters $AB$, $AC$ and $AD$ begin in point $A$, then one of the endpoints of these diameters lies inside the angle formed by the other two endpoints.

Let, for definiteness, point $C$ lie inside angle $\angle BAD$. Let us prove that then not more than one diameter begins in point $C$. Suppose that there is another diameter, $CP$, and points $B$ and $P$ lie on different sides of line $AC$ (Fig. 96). Then $ABCP$ is a convex quadrilateral; hence, $AB + CP < AC + BP$ (see Problem 9.14), i.e., $d + d < d + BP$ and, therefore, $BP > d$ which is impossible.
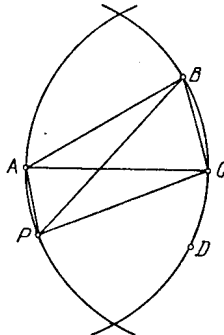


Figure 238 (Sol. 26.4)

As a result we see that either from each point there goes not more than two diameters or there exists a point from which there goes not more than one diameter. Now, the required statement can be proved by induction on the number of points. For $n = 3$ it is obvious.

Suppose the statement is proved for any system of $n$ points; let us prove it for a system of $n + 1$ points. In this system either there is a point from which there goes not more than one diameter or from each point there goes not more than two diameters. In the first case we delete this point and, making use of the fact that in the remaining system there are not more than $n$ diameters, get the desired.

The second case is obvious.

**26.5.** Let us draw all the lines that connect pairs of given points and select a line, $l$, not parallel to either of them. It is possible to divide the given points into quadruples with the help of lines parallel to $l$. The quadrilaterals with vertices in these quadruples of points are the desired ones (Fig. 97).

**26.6.** Let us divide the given points in an arbitrary way into six groups: four groups of four points each, a group of five points and a group of one point. Let us consider the group of five points. From these points we can select four points which are vertices of a convex
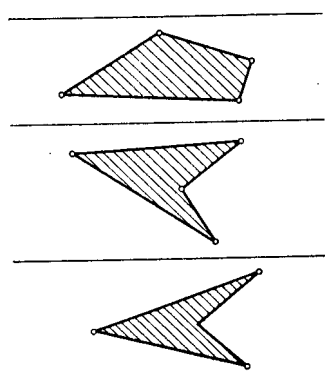
Figure 239 (Sol. 26.5)

quadrilateral $ABCD$ (see Problem 22.2). Let us unite points $A$, $C$ and $B$, $D$ into pairs. Then segments $AC$ and $BD$ given by pairs intersect. One of the five points is free. Let us adjoin it to the foursome of points and perform the same with the obtained 5-tuple of points, etc. After five of such operations there remain two points and we can unite them in a pair.

**26.7.** Since $\left(\frac{2n}{n^2+1}\right)^2 + \left(\frac{n^2-1}{n^2+1}\right)^2 = 1$, there exists an angle $\varphi$ with the property that $\sin\varphi = \frac{2n}{n^2+1}$ and $\cos\varphi = \frac{n^2-1}{n^2+1}$, where $0 < 2N\varphi < \frac{\pi}{2}$ for a sufficiently large $n$. Let us consider the circle of radius $R$ centered at $O$ and points $A_0$, $A_1$, ..., $A_{N-1}$ on it such that $\angle A_0 O A_k = 2k\varphi$. Then $A_i A_j = 2R\sin(|i-j|\varphi)$. Making use of the formulas

$$\sin(m+1)\varphi = \sin m\varphi \cos\varphi + \sin\varphi \cos m\varphi,$$

$$\cos(m+1)\varphi = \cos m\varphi \cos\varphi - \sin m\varphi \sin\varphi$$

it is easy to prove that the numbers $\sin m\varphi$ and $\cos m\varphi$ are rational for all positive integers $m$. Let us take for $R$ the greatest common divisor of all the denominators of the rational numbers $\sin\varphi$, ..., $\sin(N-1)\varphi$. Then $A_0$, ..., $A_{N-1}$ is the required system of points.

**26.8.** An example is depicted on Fig. 98.
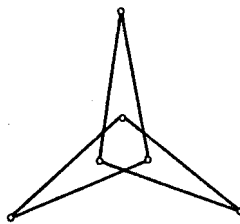


Figure 240 (Sol. 26.8)

**26.9.** It is possible. An example is plotted on Fig. 99.

**26.10.** The hypothesis implies that all the vertices of the polygon are divided into pairs that determine diagonals $A_i A_j$ which pass through point $O$. Therefore, the number of vertices is even and on both parts of each of such diagonals $A_i A_j$ there are an equal number of vertices. Hence, $j = i + m$, where $m$ is a half of the total number of vertices. Therefore, point $O$ is the intersection point of diagonals that connect opposite vertices. It is clear that the intersection point of these diagonals is unique.

**26.11.** If $AC$ and $BD$ are intersecting red segments, then the number of intersection points of any line with segments $AB$ and $CD$ does not exceed the number of intersection
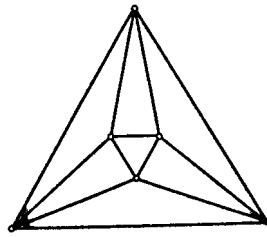
Figure 241 (Sol. 26.9)

points of this line with segments $AC$ and $BD$. Therefore, by replacing red segments $AC$ and $BD$ with segments $AB$ and $CD$ we do not increase the number of intersection points of red segments with blue ones and diminish the number of intersection points of red segments with red ones because the intersection point of $AC$ and $BD$ vanishes. After several such operations all red segments become nonintersecting ones and it remains to prove that in this case the number of intersection points of red segments with blue ones is not smaller than $n$.

Let us consider an arbitrary red segment. Since the other red segments do not intersect it, we deduce that on both sides of it there lies an even number of red points or, equivalently, an odd number of blue points. Therefore, there exists a blue segment that intersects the given red segment. Therefore, the number of intersection points of red segments with blue ones is not fewer than the number of red segments i.e., is not less than $n$.

**26.12.** Let $A$ be a common point of the first three circles $S_1$, $S_2$ and $S_3$. Denote the intersection points of $S_1$ and $S_2$, $S_2$ and $S_3$, $S_3$ and $S_1$ by $B$, $C$, $D$, respectively. Suppose there exists a circle $S$ not passing through point $A$. Then $S$ passes through points $B$, $C$ and $D$. Let $S'$ be the fifth circle. Each pair of points from the collection $A$, $B$, $C$, $D$ is a pair of intersection points of two of the circles $S_1$, $S_2$, $S_3$, $S$. Therefore, $S'$ passes through one point from each pair of points $A$, $B$, $C$, $D$. On the other hand, $S'$ cannot pass through three points from the set $A$, $B$, $C$, $D$ because each triple of these points determines one of the circles $S_1$, $S_2$, $S_3$, $S$. Hence, $S'$ does not pass through certain two of these points. Contradiction.

**26.13.** Let us consider rectangle $ABCD$ with sides $AB = 1$ cm and $BC = 500$ m. Let $O$ be the intersection point of the rectangle's diagonals. It is easy to verify that the area of $AOD$ is greater than $1$ m$^2$ and all its hights are shorter than $1$ cm.

**26.14.** No, not necessarily. On Fig. 100 it is shown how to get the required quadrilateral $ABCD$.
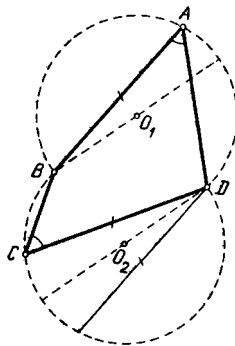


Figure 242 (Sol. 26.14)

**26.15.** Not necessarily. It is easy to verify that the list of the lengths of sides and diagonals for an isosceles trapezoid with height 1 and bases 2 and 4 coincides with the similar list for the quadrilateral with perpendicular diagonals of length 2 and 4 that are

divided by their intersection point into segments of length 1 and 1 and 1 and 3, respectively (Fig. 101).
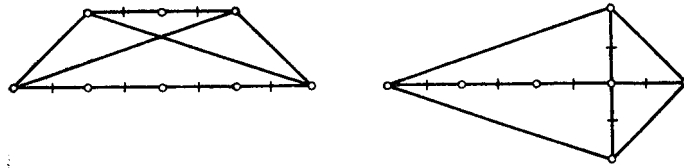


Figure 243 (Sol. 26.15)

**26.16.** Yes, there exist. Let us consider points $P_i = (i, i^2)$, where $i = 1, \ldots, n$. The areas of all the triangles with vertices in the nodes of an integer lattice are rational (see Problem 24.5) and the numbers $P_i P_j = |i - j|\sqrt{1 + (i + j)^2}$ are irrational.

**26.17.** Yes, there exist. Let $C$ be the midpoint of segment $AB$. Then

$$XC^2 = \frac{2XA^2 + 2XB^2 - AB^2}{2}.$$

If the number $AB^2$ is irrational, then the numbers $XA$, $XB$ and $XC$ cannot simultaneously be rational.

**26.18.** It can. Consider right triangle $ABC_1$ with legs $AB = 1$ and $BC_1 = 2n$. In this triangle draw median $AM_1$, bisector $BK_1$ and hight $C_1H_1$. The area of the triangle formed by these segments is greater than $S_{ABM_1} - S_{ABK_1}$. Clearly, $S_{ABK_1} < \frac{1}{2}$ and $S_{ABM_1} = \frac{n}{2}$, i.e., $S_{ABM_1} - S_{ABK_1} > (\frac{S}{2}) - (\frac{S}{2n})$, where $S = S_{ABC_1}$. Hence, for a sufficiently large $n$ the area of the triangle formed by segments $AM_1$, $BK_1$ and $C_1H_1$ will be greater than $0.499 \cdot S$.

Slightly moving point $C_1$ we turn triangle $ABC_1$ into an acute triangle $ABC$ and the area of the triangle formed by the intersection points of segments remains greater than $0.499 \cdot S_{ABC}$.

**26.19.** It is possible. Let us pave, for instance, infinite angles illustrated on Fig. 102.
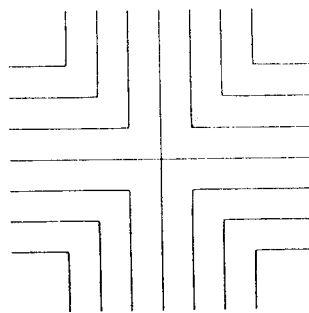


Figure 244 (Sol. 26.19)

**26.20.** Yes, it can. Let us prove the statement by induction replacing 100 with $n$.

For $n = 1$ we can take the endpoints of a segment of length 1. Suppose that the statement is proved for $n$ and $A_1, \ldots, A_k$ is the required set of points. Let $A'_1, \ldots, A'_k$ be the images of points $A_1, \ldots, A_k$ under the parallel transport by unit vector $\mathbf{a}$. To prove the inductive step it suffices to select the unit vector $\mathbf{a}$ so that $\mathbf{a} \neq \overrightarrow{A_i A_j}$ and $A_j A'_i \neq 1$ for $i \neq j$, i.e., $|\overrightarrow{A_j A_i} + \mathbf{a}| \neq 1$ for $i \neq j$. Each of these restrictions excludes from the unit circle not more than 1 point.
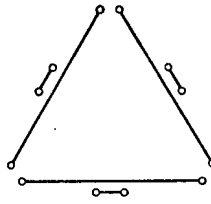
Figure 245 (Sol. 26.21)

**26.21.** Not always. Consider the segments plotted on Fig. 103. The endpoints of each short segment can be connected with the endpoints of the nearest to it long segment only. It is clear that in this way we cannot get a closed nonselfintersecting broken line.

**26.22.** Not necessarily. Let us prove that the center $O$ of the circle inscribed in triangle $ABC$ with sides $AB = 6$, $BC = 4$ and $CA = 8$ is equidistant from the midpoints of sides $AC$ and $BC$. Denote the midpoints of sides $AC$ and $BC$ by $B_1$ and $A_1$ and the bases of the perpendiculars dropped from $O$ to $AC$ and $BC$ by $B_2$ and $A_2$, see Fig. 104. Since $A_1A_2 = 1 = B_1B_2$ (cf. Problem 3.2) and $OA_2 = OB_2$, it follows that $\triangle OA_1A_2 = \triangle OB_1B_2$, i.e., $OA_1 = OB_1$.
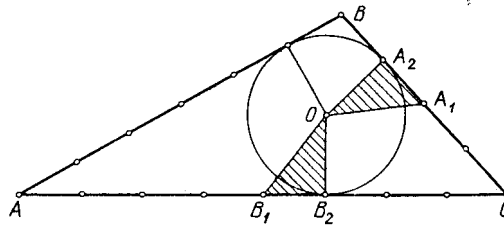


Figure 246 (Sol. 26.22)

**26.23.** This is possible for any $n \geq 2$. Indeed, let us inscribe into the arena a regular $k$-gon, where $k$ is the number of distinct pairs that can be composed of $n$ spotlights, i.e., $k = \frac{n(n-1)}{2}$. Then we can establish a one-to-one correspondence between the segments cut off by the sides of the $k$-gon and the pairs of spotlights. Let each spotlight illuminate the whole $k$-gon and the segments that correspond to pairs of spotlights in which it enters. (Yeah?)It is easy to verify that this illumination possesses the required properties.

# Chapter 27. INDUCTION AND COMBINATORICS

## §1. Induction

**27.1.** Prove that if the plane is divided into parts ("countries") by lines and circles, then the obtained map can be painted two colours so that the parts separated by an arc or a segment are of distinct colours.

**27.2.** Prove that in a convex $n$-gon it is impossible to select more than $n$ diagonals so that any two of them have a common point.

**27.3.** Let $E$ be the intersection point of lateral sides $AD$ and $BC$ of trapezoid $ABCD$, let $B_{n+1}$ be the intersection point of lines $A_nC$ and $BD$ ($A_0 = A$); let $A_{n+1}$ be the intersection point of lines $EB_{n+1}$ and $AB$. Prove that $A_nB = \frac{1}{n+1}AB$.

**27.4.** On a line, there are given points $A_1$, ..., $A_n$ and $B_1$, ..., $B_{n-1}$. Prove that

$$\sum_{i=1}^{n} \left( \frac{\prod_{1 \le k \le n-1} \overline{A_i B_k}}{\prod_{j \ne i} \overline{A_i A_j}} \right) = 1.$$

**27.5.** Prove that if $n$ points do not lie on one line, then among the lines that connect them there are not fewer than $n$ distinct points.

See also Problems 2.12, 5.98, 22.7, 22.9–22.12, 22.20 b, 22.22, 22.23, 22.29, 23.39–23.41, 26.20.

## §2. Combinatorics

**27.6.** Several points are marked on a circle, $A$ is one of them. Which convex polygons with vertices in these points are more numerous: those that contain $A$ or those that do not contain it?

**27.7.** On a circle, nine points are fixed. How many non-closed non-selfintersecting broken lines of nine links with vertices in these points are there?

**27.8.** In a convex $n$-gon ($n \ge 4$) there are drawn all the diagonals and no three of them intersect at one point. Find the number of intersection points of the diagonals.

**27.9.** In a convex $n$-gon ($n \ge 4$) all the diagonals are drawn. Into how many parts do they divide an $n$-gon if no three of them intersect at one point?

**27.10.** Given $n$ points in plane no three of which lie on one line, prove that there exist not fewer than $\frac{\binom{n}{5}}{n-4}$ distinct convex quadrilaterals with vertices in these points.

**27.11.** Prove that the number of nonequal triangles with the vertices in vertices of a regular $n$-gon is equal to the integer nearest to $\frac{n^2}{12}$.

See also Problem 25.6.

### Solutions

**27.1.** Let us carry out the proof by induction on the total number of lines and circles. For one line or circle the statement is obvious. Now, suppose that it is possible to paint any

map given by $n$ lines and circles in the required way and show how to paint a map given by $n + 1$ lines and circles.

Let us delete one of these lines (or circles) and paint the map given by the remaining $n$ lines and circles thanks to the inductive hypothesis. Then retain the colours of all the parts lying on one side of the deleted line (or circle) and replace the colours of all the parts lying on the other side of the deleted line (or circle) with opposite ones.

**27.2.** Let us prove by induction on $n$ that in a convex $n$-gon it is impossible to select more than $n$ sides and diagonals so that any two of them have a common point.

For $n = 3$ this is obvious. Suppose that the statement holds for any convex $n$-gon and prove it for an $(n + 1)$-gon. If from every vertex of the $(n + 1)$-gon there goes not more than two of the selected sides or diagonals, then the total number of selected sides or diagonals does not exceed $n + 1$. Hence, let us assume that from vertex $A$ there goes three of the selected sides or diagonals $AB_1$, $AB_2$ and $AB_3$, where $AB_2$ lies between $AB_1$ and $AB_3$. Since a diagonal or a side coming from point $B_2$ and distinct from $AB_2$ cannot simultaneously intersect $AB_1$ and $AB_3$, it is clear that only one of the chosen diagonals can go from $B_2$. Therefore, it is possible to delete point $B_2$ together with diagonal $AB_2$ and apply the inductive hypothesis.

**27.3.** Clearly, $A_0B = AB$. Let $C_n$ be the intersection point of lines $EA_n$ and $DC$, where $DC : AB = k$, $AB = a$, $A_nB = a_n$ and $A_{n+1}B = x$. Since $CC_{n+1} : A_nA_{n+1} = DC_{n+1} : BA_{n+1}$, it follows that $kx : (a_n - x) = (ka - kx) : x$, i.e., $x = \frac{aa_n}{a+a_n}$. If $a_n = \frac{a}{n+1}$, then $x = \frac{a}{n+2}$.

**27.4.** First, let us prove the desired statement for $n = 2$. Since $\overrightarrow{A_1B_1} + \overrightarrow{B_1A_2} + \overrightarrow{A_2A_1} = \overrightarrow{0}$, it follows that $\frac{\overline{A_1B_1}}{\overline{A_1A_2}} + \frac{\overline{A_2B_1}}{\overline{A_2A_1}} = 1$.

To prove the inductive step let us do as follows. Fix points $A_1, \ldots, A_n$ and $B_1, \ldots, B_{n-2}$ and consider point $B_{n-1}$ variable. Consider the function

$$f(B_{n-1}) = \sum_{i=1}^{n} \left( \frac{\prod_{1 \le k \le n-1} \overline{A_iB_k}}{\prod_{j \ne i} \overline{A_iA_j}} \right) = 1.$$

This function is a linear one and by the inductive hypothesis $f(B_{n-1}) = 1$ if $B_{n-1}$ coincides with one of the points $A_1, \ldots, A_n$. Therefore, this function is identically equal to 1.

**27.5.** Induction on $n$. For $n = 3$ the statement is obvious. Suppose we have proved it for $n - 1$ and let us prove it then for $n$ points. If on every line passing through two of the given points lies one more given point, then all the given points belong to one line (cf. Problem 20.13). Therefore, there exists a line on which there are exactly two given points $A$ and $B$. Let us delete point $A$. The two cases are possible:

1) All the remaining points lie on one line $l$. Then there will be precisely $n$ distinct lines: $l$ and $n - 1$ line passing through $A$.

2) The remaining points do not belong to one line. Then among the lines that connect them there are not fewer than $n - 1$ distinct ones that connect them and all of them differ from $l$. Together with $AB$ they constitute not fewer than $n$ lines.

**27.6.** To any polygon, $P$, that does not contain point $A$ we can assign a polygon that contains $A$ by adding $A$ to the vertices of $P$. The inverse operation, however, that is deleting of the point $A$, can be only performed for $n$-gons with $n \ge 4$. Therefore, there are more polygons that contain $A$ than polygons without $A$ and the difference is equal to the number of triangles with $A$ as a vertex, i.e., $\frac{(n-1)(n-2)}{2}$.

**27.7.** The first point can be selected in 10 ways. Each of the following 8 points can be selected in two ways because it must be neighbouring to one of the points selected earlier (otherwise we get a self-intersecting broken line). Since the beginning and the end do not

differ in this method of calculation, the result should be divided by 2. Hence, the total number of the broken lines is equal to $\frac{10 \cdot 2^8}{2} = 1280$.

**27.8.** Any intersection point of diagonals determines two diagonals whose intersection point it serves and the endpoints of these diagonals fix a convex quadrilateral. Conversely, any four vertices of a polygon determine one intersection point of diagonals. Therefore, the total number of intersection points of diagonals is equal to the number of ways to choose 4 points of $n$, i.e., is equal to $\frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4}$.

**27.9.** Let us consecutively draw diagonals. When we draw a diagonal, the number of parts into which the earlier drawn diagonals divide the polygon increases by $m + 1$, where $m$ is the number of intersection points of the new diagonals with the previously drawn ones, i.e., each new diagonal and each new intersection point of diagonals increase the number of parts by 1. Therefore, the total number of parts into which the diagonals divide an $n$-gon is equal to $D + P + 1$, where $D$ is the number of diagonals, $P$ is the number of intersection points of the diagonals. It is clear that $D = \frac{n(n-3)}{2}$. By the above problem $P = \frac{n(n-1)(n-2)(n-3)}{24}$.

**27.10.** If we choose any five points, then there exists a convex quadrilateral with vertices in these points (Problem 22.2). It remains to notice that a quadruple of points can be complemented to a 5-tuple in $n - 4$ distinct ways.

**27.11.** Let there be $N$ nonequal triangles with vertices in vertices of a regular $n$-gon so that among them there are $N_1$ equailateral, $N_2$ non-equailateral isosceles, and $N_3$ scalane ones. Each equilateral triangle is equal to a triangle with fixed vertex $A$, a non-equilateral isosceles is equal to three triangles with vertex $A$ and a scalane one is equal to 6 triangles. Since the total number of triangles with vertex $A$ is equal to $\frac{(n-1)(n-2)}{2}$, it follows that $\frac{(n-1)(n-2)}{2} = N_1 + 3N_2 + 6N_3$.

Clearly, the number of nonequal equailateral triangles is equal to either 0 or 1 and the number of nonequal isosceles triangles is equal to either $\frac{n-1}{2}$ or $\left(\frac{n}{2}\right) - 1$, i.e., $N_1 = 1 - c$ and $N_1 + N_2 = \frac{n-2+d}{2}$, where $c$ and $d$ are equal to either 0 or 1. Therefore,

$$12N = 12(N_1 + N_2 + N_3) = 2(N_1 + 3N_2 + 6N_3) + 6(N_1 + N_2) + 4N_1 =$$
$$(n-1)(n-2) + 3(n-2+d) + 4(1-c) = n^2 + 3d - 4c.$$

Since $|3d - 4c| < 6$, it follows that $N$ coincides with the nearest integer to $\frac{n^2}{12}$.

# Chapter 28. INVERSION

## Background

1. All the geometric transformations that we have encountered in this book so far turned lines into lines and circles into circles. The *inversion* is a transformation of another type which also preserves the class of lines and circles but can transform a line into a circle and a circle into a line. This and other remarkable properties of inversion serve as a foundation for its astounding effectiveness in solving various geometric problems.

2. In plane, consider circle $S$ centered at $O$ with radius $R$. We call the transformation that sends an arbitrary point $A$ distinct from $O$ into point $A^*$ lying on ray $OA$ at distance $OA^* = \frac{R^2}{OA}$ from $O$ the *inversion relative $S$*. The inversion relative $S$ will be also called the *inversion with center $O$ and degree $R^2$* and $S$ will be called the *circle of inversion*.

3. It follows directly from the definition of inversion that it fixes points of $S$, moves points from inside $S$ outside it and points from outside $S$ inside it. If point $A$ turns into $A^*$ under the inversion, then the inversion sends $A^*$ into $A$, i.e., $(A^*)^* = A$. The image of a line passing through the center of the inversion is this line itself.

Here we should make a reservation connected with the fact that, strictly speaking, the inversion is not a transformation of the plane because $O$ has no image. Therefore, formally speaking, we cannot speak about the "image of the line through $O$" and should consider instead the union of two rays obtained from the line by deleting point $O$. Similar is the case with the circles containing point $O$. Nevertheless, we will use these loose but more graphic formulations and hope that the reader will easily rectify them when necessary.

4. Everywhere in this chapter the image of point $A$ under an inversion is denoted by $A^*$.

5. Let us formulate the most important properties of inversion that are constantly used in the solution of problems.

Under an inversion with center $O$:

a) a line $l$ not containing $O$ turns into a circle passing through $O$ (Problem 28.2);

b) a circle centered at $C$ and passing through $O$ turns into a line perpendicular to $OC$ (Problem 28.3);

c) a circle not passing through $O$ turns into a circle not passing through $O$ (Problem 28.3);

d) the tangency of circles with lines is preserved only if the tangent point does not coincide with the center of inversion; otherwise, the circle and a line turn into a pair of parallel lines (Problem 28.4);

e) the value of the angle between two circles (or between a circle and a line, or between two lines) is preserved (Problem 28.5).

## §1. Properties of inversions

**28.1.** Let an inversion with center $O$ send point $A$ to $A^*$ and $B$ to $B^*$. Prove that triangles $OAB$ and $OB^*A^*$ are similar.

**28.2.** Prove that under any inversion with center $O$ any line $l$ not passing through $O$ turns into a circle passing through $O$.

**28.3.** Prove that under any inversion with center $O$ any circle passing through $O$ turns into a line and any circle not passing through $O$ into a circle.

**28.4.** Prove that tangent circles (any circle tangent to a line) turn under any inversion into tangent circles or in a circle and a line or in a pair of parallel lines.

Let two circles intersect at point $A$. The *angle between the circles* is the angle between the tangents to the circles at point $A$. (Clearly, if the circles intersect at points $A$ and $B$, then the angle between the tangents at point $A$ is equal to the angle between the tangents at point $B$). The *angle between a line and a circle* is similarly defined (as the angle between the line and the tangent to the circle at one of the intersection points).

**28.5.** Prove that inversion preserves the angle between circles (and also between a circle and a line, and between lines).

**28.6.** Prove that two nonintersecting circles $S_1$ and $S_2$ (or a circle and a line) can be transported under an inversion into a pair of concentric circles.

**28.7.** Let $S$ be centered in $O$. Through point $A$ a line $l$ intersecting $S$ at points $M$ and $N$ and not passing through $O$ is drawn. Let $M'$ and $N'$ be points symmetric to $M$ and $N$, respectively, through $OA$ and let $A'$ be the intersection point of lines $MN'$ and $M'N$. Prove that $A'$ coincides with the image of $A$ under the inversion with respect to $S$ (and, therefore, does not depend on the choice of line $l$).

## §2. Construction of circles

While solving problems of this section we will often say "let us perform an inversion ..." . Being translated into a more formal language this should sound as: "Let us construct with the help of a ruler and a compass the images of all the given points, lines and circles under the inversion relative to the given circle". The possibility to perform such constructions follows from properties of inversion and Problem 28.8.

In problems on construction we often make use of the existence of inversion that sends two nonintersecting circles into concentric circles. The solution of Problem 28.6 implies that the center and radius of such an inversion (hence, the images of the circles) can be constructed by a ruler and a compass.

**28.8.** Construct the image of point $A$ under the inversion relative circle $S$ centered in $O$.

**28.9.** Construct the circle passing through two given points and tangent to the given circle (or line).

**28.10.** Through a given point draw the circle tangent to two given circles (or a circle and a line).

**28.11.** (*Apollonius' problem.*) Construct a circle tangent to the three given circles.

**28.12.** Through a given point draw a circle perpendicular to two given circles.

**28.13.** Construct a circle tangent to a given circle $S$ and perpendicular to the two given circles ($S_1$ and $S_2$).

**28.14.** Through given points $A$ and $B$ draw a circle intersecting a given circle $S$ under the angle of $\alpha$.

## §3. Constructions with the help of a compass only

According to the tradition that stems from ancient Greece, in geometry they usually consider constructions with the help of ruler and compass. But we can also make constructions with the help of other instruments, or we can, for instance, consider constructions with the help of one compass only, without a ruler. Clearly, with the help of a compass only one

cannot simultaneously construct all the points of a line. Therefore, let us make a convention: we will consider a line *constructed* if two of its points are constructed.

It turns out that under such convention we can perform with the help of a compass all the constructions which can be performed with the help of a compass and a ruler. This follows from the possibility to construct using only a compass the intersection points of any line given by two points with a given circle (Problem 28.21 a)) and the intersection point of two lines (Problem 28.21 b)). Indeed, any construction with the help of ruler and compass is a sequence of determinations of the intersection points of circles and lines.

In this section we will only consider constructions with a compass only, without a ruler, i.e., the word "construct" means "construct with the help of a compass only". We will consider a segment *constructed* if its endpoints are constructed.

**28.15.** a) Construct a segment twice longer than a given segment.

b) Construct a segment $n$ times longer than a given segment.

**28.16.** Construct the point symmetric to point $A$ through the line passing through given points $B$ and $C$.

**28.17.** Construct the image of point $A$ under the inversion relative a given circle $S$ centered in a given point $O$.

**28.18.** Construct the midpoint of the segment with given endpoints.

**28.19.** Construct the circle into which the given line $AB$ turns into under the inversion relative a given circle with given center $O$.

**28.20.** Construct the circle passing through three given points.

**28.21.** a) Construct the intersection points of the given circle $S$ and the line passing through given points $A$ and $B$.

b) Construct the intersection point of lines $A_1B_1$ and $A_2B_2$, where $A_1$, $B_1$, $A_2$ and $B_2$ are given points.

## §4. Let us perform an inversion

**28.22.** In a disk segment, all possible pairs of tangent circles (Fig. 105) are inscribed. Find the locus of their tangent points.
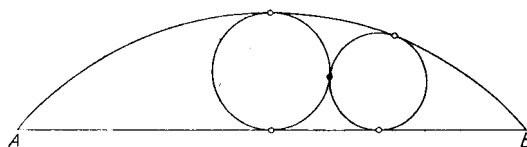


Figure 247 (28.22)

**28.23.** Find the set of tangent points of pairs of circles that are tangent to the legs of the given angle at given points $A$ and $B$.

**28.24.** Prove that the inversion with the center at vertex $A$ of an isosceles triangle $ABC$, where $AB = AC$, of degree $AB^2$ sends the base $BC$ of the triangle into the arc $\smile BC$ of the circumscribed circle.

**28.25.** In a circle segment, all the possible pairs of intersecting circles are inscribed and for each pair a line is drawn through their intersection point. Prove that all these lines pass through one point, cf. Problem 3.44.

**28.26.** No three of the four points $A$, $B$, $C$, $D$ lie on one line. Prove that the angle between the circumscribed circles of triangles $ABC$ and $ABD$ is equal to the angle between the circumscribed circles of triangles $ACD$ and $BCD$.

**28.27.** Through points $A$ and $B$ there are drawn circles $S_1$ and $S_2$ tangent to circle $S$ and circle $S_3$ perpendicular to $S$. Prove that $S_3$ forms equal angles with circles $S_1$ and $S_2$.

**28.28.** Two circles intersecting at point $A$ are tangent to the circle (or line) $S_1$ at points $B_1$ and $C_1$ and to the circle (or line) $S_2$ at points $B_2$ and $C_2$ (and the tangency at $B_2$ and $C_2$ is the same as at respective points $B_1$ and $C_1$, i.e., either inner or outer). Prove that circles circumscribed about triangles $AB_1C_1$ and $AB_2C_2$ are tangent to each other.

**28.29.** Prove that the circle passing through the midpoints of triangle's sides is tangent to its inscribed and three escribed circles. (*Feuerbach's theorem.*)

### §5. Points that lie on one circle and circles passing through one point

**28.30.** Given four circles, $S_1$, $S_2$, $S_3$, $S_4$, where circles $S_1$ and $S_3$ intersect with both circles $S_2$ and $S_4$. Prove that if the intersection points of $S_1$ with $S_2$ and $S_3$ with $S_4$ lie on one circle or line, then the intersection points of $S_1$ with $S_4$ and $S_2$ with $S_3$ lie on one circle or line (Fig. 106).
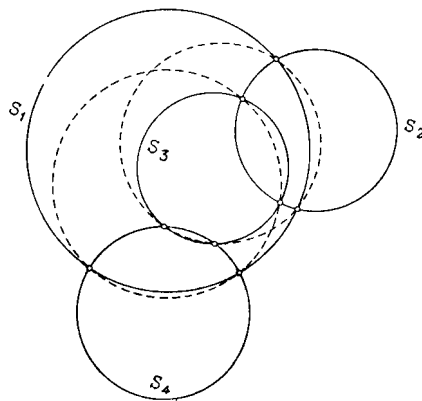


Figure 248 (28.30)

**28.31.** Given four circles $S_1$, $S_2$, $S_3$, $S_4$ such that $S_1$ and $S_2$ intersect at points $A_1$ and $A_2$, $S_2$ and $S_3$ at points $B_1$ and $B_2$, $S_3$ and $S_4$ at points $C_1$ and $C_2$, $S_4$ and $S_1$ at points $D_1$ and $D_2$ (Fig. 107).
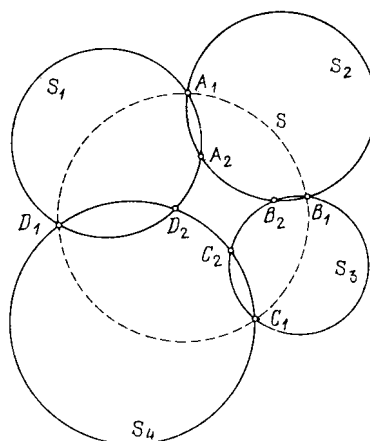


Figure 249 (28.31)

Prove that if points $A_1$, $B_1$, $C_1$, $D_1$ lie on one circle (or line) $S$, then points $A_2$, $B_2$, $C_2$, $D_2$ lie on one circle (or line).

**28.32.** The sides of convex pentagon $ABCDE$ are extended so that five-angled star $AHBKCLDMEN$ (Fig. 108) is formed. The circles are circumscribed about triangles —

the rays of the star. Prove that the five intersection points of these circles distinct from $A$, $B$, $C$, $D$, $E$ lie on one circle.
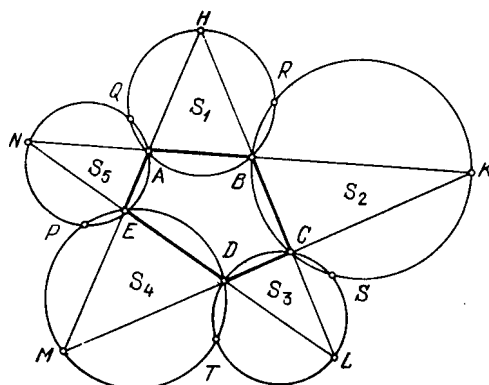


Figure 250 (28.32)

**28.33.** In plane, six points $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, $B_3$ are fixed. Prove that if the circles circumscribed about triangles $A_1A_2B_3$, $A_1B_2A_3$ and $B_1A_2A_3$ pass through one point, then the circles circumscribed about triangles $B_1B_2A_3$, $B_1A_2B_3$ and $A_1B_2B_3$ intersect at one point.

**28.34.** In plane, six points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ are fixed. Prove that if the circles circumscribed about triangles $A_1B_1C_1$, $A_1B_2C_2$, $A_2B_1C_2$, $A_2B_2C_1$ pass through one point, then the circles circumscribed about triangles $A_2B_2C_2$, $A_2B_1C_1$, $A_1B_2C_1$, $A_1B_1C_2$ pass through one point.

**28.35.** In this problem we will consider tuples of $n$ generic lines, i.e., sets of lines no two of which are parallel and no three pass through one point.

To a tuple of two generic lines assign their intersection point and to a tuple of two generic lines assign the circle passing through the three points of their pairwise intersections. If $l_1$, $l_2$, $l_3$, $l_4$ are four generic lines, then the four circles $S_i$ corresponding to four triples of lines obtained by discarding $l_i$ pass through one point (cf. Problem 2.83 a)) that we will assign to the foursome of lines.

This construction can be extended:

a) Let $l_i$, $i = 1$, ..., 5 be five generic points. Prove that five points $A_i$ corresponding to the foursome of lines obtained by discarding $l_i$ lie on one circle.

b) Prove that this construction can be continued in the following way: to every tuple of $n$ generic points assign a point if $n$ is even or a circle if $n$ is odd so that $n$ circles (points) corresponding to tuples of $n - 1$ lines pass through this point (belong to this circle).

**28.36.** On two intersecting lines $l_1$ and $l_2$, select points $M_1$ and $M_2$ not coinciding with the intersection point $M$ of these lines. Assign to this set of lines and points the circle passing through $M_1$, $M_2$ and $M$.

If $(l_1, M_1)$, $(l_2, M_2)$, $(l_3, M_3)$ are three generic lines with fixed points, then by Problem 2.80 a) the three circles corresponding to pairs $(l_1, M_1)$ and $(l_2, M_2)$, $(l_2, M_2)$, $(l_3, M_3)$, $(l_3, M_3)$ and $(l_1, M_1)$ intersect at one point that we will assign to the triple of lines with a fixed point.

a) Let $l_1$, $l_2$, $l_3$, $l_4$ be four generic lines on each of which a point is fixed so that these points lie on one circle. Prove that four points corresponding to the triples obtained by deleting one of the lines lie on one circle.

b) Prove that to every tuple of $n$ generic lines with a point fixed on each of them so that the fixed points lie on one circle one can assign a point (if $n$ is odd) or a circle (if $n$ is even)

so that $n$ circles (if $n$ is odd) or points (if $n$ is even) corresponding to the tuples of $n-1$ lines pass through this point (resp. lie on this circle).

## §6. Chains of circles

**28.37.** Circles $S_1$, $S_2$, ..., $S_n$ are tangent to circles $R_1$ and $R_2$ and, moreover, $S_1$ is tangent to $S_2$ at point $A_1$, $S_2$ is tangent to $S_3$ at point $A_2$, ..., $S_{n-1}$ is tangent to $S_n$ at point $A_{n-1}$. Prove that points $A_1$, $A_2$, ..., $A_{n-1}$ lie on one circle.

**28.38.** Prove that if there exists a chain of circles $S_1$, $S_2$, ..., $S_n$ each of which is tangent to two neighbouring ones ($S_n$ is tangent to $S_{n-1}$ and $S_1$) and two given nonintersecting circles $R_1$ and $R_2$, then there are infinitely many such chains.

(?)Namely, for any circle $T_1$ tangent to $R_1$ and $R_2$ (in the same fashion if $R_1$ and $R_2$ do not lie inside each other, by an inner or an outer way, otherwise) there exists a similar chain of $n$ tangent circles $T_1$, $T_2$, ..., $T_n$. (*Steiner's porism.*)

**28.39.** Prove that for two nonintersecting circles $R_1$ and $R_2$ a chain of $n$ tangent circles (cf. the preceding problem) exists if and only if the angle between the circles $T_1$ and $T_2$ tangent to $R_1$ and $R_2$ at their intersection points with the line that connects the centers of $R_1$ and $R_2$ is equal to an integer multiple of $\frac{360°}{n}$ (Fig. 109).
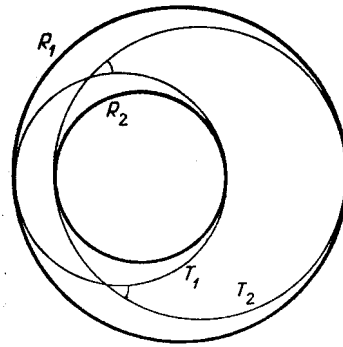


Figure 251 (28.39)

**28.40.** Each of six circles is tangent to four of the remaining five circles, see Fig. 110.
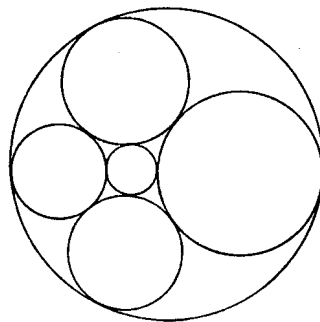


Figure 252 (28.40)

Prove that for any pair of nonintersecting circles (of these six circles) the radii and the distance between their centers are related by the formula

$$d^2 = r_1^2 + r_2^2 \pm 6r_1r_2,$$

where "plus" is taken if the circles are not inside each other and "minus" otherwise.

## Solutions

**28.1.** Let $R^2$ be the degree of the inversion. Then

$$OA \cdot OA^* = OB \cdot OB^* = R^2$$

whence, $OA : OB = OB^* : OA^*$ and $\triangle OAB \sim \triangle OB^*A^*$ because $\angle AOB = \angle B^*OA^*$.

**28.2.** Let us drop perpendicular $OC$ from point $O$ to line $l$ and take an arbitrary point $M$ on $l$. Since triangles $OCM$ and $OM^*C^*$ are similar (Problem 28.1), $\angle OM^*C^* = \angle OCM = 90°$, i.e., point $M^*$ lies on circle $S$ with diameter $OC^*$. If $X$ is a point of $S$ distinct from $O$, then it is the image under the inversion of the intersection point $Y$ of $l$ and $OX$ (since the image of $Y$ lies, on the one hand, on ray $OX$ and, on the other hand, on circle $S$, as is already proved). Thus, the inversion sends line $l$ into circle $S$ (without point $O$).

**28.3.** The case when circle $S$ passes through $O$ is actually considered in the preceding problem (and formally follows from it since $(M^*)^* = M$).

Now, suppose that $O$ does not belong to $S$. Let $A$ and $B$ be the intersection points of circle $S$ with the line passing through $O$ and the center of $S$, let $M$ be an arbitrary point of $S$. Let us prove that the circle with diameter $A^*B^*$ is the image of $S$. To this end it suffices to show that $\angle A^*M^*B^* = 90°$. But by Problem 28.1 $\triangle OAM \sim \triangle OM^*A^*$ and $\triangle OBM \sim \triangle OM^*B^*$; hence, $\angle OMA = \angle OA^*M^*$ and $\angle OMB = \angle OB^*M^*$; more exactly, $\angle(OM, MA) = -\angle(OA^*, M^*A^*)$ and $\angle(OM, MB) = -\angle(OB^*, M^*B^*)$. (In order not to consider various cases of points' disposition we will make use of the properties of oriented angles between lines discussed in Chapter 2.) Therefore,

$$\angle(A^*M^*, M^*B^*) = \angle(A^*M^*, OA^*) + \angle(OB^*, M^*B^*) =$$
$$\angle(OM, MA) + \angle(MB, OM) = \angle(MB, MA) = 90°.$$

**28.4.** If the tangent point does not coincide with the center of inversion, then after the inversion these circles (the circle and the line) will still have one common point, i.e., the tangency is preserved.

If the circles with centers $A$ and $B$ are tangent at point $O$, then under the inversion with center $O$ they turn into a pair of lines perpendicular to $AB$. Finally, if line $l$ is tangent to the circle centered at $A$ at point $O$, then under the inversion with center $O$ the line $l$ turns into itself and the circle into a line perpendicular to $OA$. In each of these two cases we get a pair of parallel lines.

**28.5.** Let us draw tangents $l_1$ and $l_2$ through the intersection point of the circles. Since under the inversion the tangent circles or a circle and a line pass into tangent ones (cf. Problem 28.4), the angle between the images of circles is equal to the angle between the images of the tangents to them. Under the inversion centered at $O$ line $l_i$ turns into itself or into a circle the tangent to which at $O$ is parallel to $l_i$. Therefore, the angle between the images of $l_1$ and $l_2$ under the inversion with center $O$ is equal to the angle between these lines.

**28.6. First solution.** Let us draw the coordinate axis through the centers of the circles. Let $a_1$ and $a_2$ be the coordinates of the intersection points of the axes with $S_1$, let $b_1$ and $b_2$ be the coordinates of the intersection points of the axes with $S_2$. Let $O$ be the point on the axis whose coordinate is $x$. Then under the inversion with center $O$ and degree $k$ our circles turn into the circles whose diameters lie on the axis and whose endpoints have coordinates $a_1'$, $a_2'$ and $b_1'$, $b_2'$, respectively, where

$$a_1' = x + \frac{k}{a_1 - x}, \quad a_2' = x + \frac{k}{a_2 - x}, \quad b_1' = x + \frac{k}{b_1 - x}, \quad b_2' = x + \frac{k}{b_2 - x}.$$

The obtained circles are concentric if $\frac{a_1' + a_2'}{2} = \frac{b_1' + b_2'}{2}$, i.e.,

$$\frac{1}{a_1 - x} + \frac{1}{a_2 - x} = \frac{1}{b_1 - x} + \frac{1}{b_2 - x},$$

wherefrom we have

$$(b_1 + b_2 - a_1 - a_2)x^2 + 2(a_1 a_2 - b_1 b_2)x + b_1 b_2(a_1 + a_2) - a_1 a_2(b + b_2) = 0.$$

The discriminant of this quadratic in $x$ is equal to $4(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)(b_2 - a_2)$. It is positive precisely when the circles do not intersect; this proves the existence of the required inversion.

The existence of such an inversion for the case of a circle and a line is similarly proved.

**Another solution.** On the line that connects centers $O_1$ and $O_2$ of the circles take point $C$ such that the tangents drawn to the circles from $C$ are equal. This point $C$ can be constructed by drawing the radical axis of the circles (cf. Problem 3.53). Let $l$ be the length of these tangents. The circle $S$ of radius $l$ centered in $C$ is perpendicular to $S_1$ and $S_2$. Therefore, under the inversion with center $O$, where $O$ is any of the intersection points of $S$ with line $O_1 O_2$, circle $S$ turns into a line perpendicular to circles $S_1^*$ and $S_2^*$ and, therefore, passing through their centers. But line $O_1 O_2$ also passes through centers of $S_1^*$ and $S_2^*$; hence, circles $S_1^*$ and $S_2^*$ are concentric, i.e., $O$ is the center of the desired inversion.

If $S_2$ is not a circle but a line, the role of line $O_1 O_2$ is played by the perpendicular dropped from $O_1$ to $S_2$, point $C$ is its intersection point with $S_2$, and $l$ is the length of the tangent dropped from $C$ to $S_1$.

**28.7.** Let point $A$ lie outside $S$. Then $A'$ lies inside $S$ and we see that $\angle MA'N = \frac{1}{2}(\smile MN + \smile M'N') = \smile MN = \angle MON$, i.e., quadrilateral $MNOA'$ is an inscribed one. But under the inversion with respect to $S$ line $MN$ turns into the circle passing through points $M$, $N$, $O$ (Problem 28.2). Therefore, point $A^*$ (the image of $A$ under the inversion) lies on the circle circumscribed about quadrilateral $MNOA'$. By the same reason points $A'$ and $A^*$ belong to the circle passing through $M'$, $N'$ and $O$. But these two circles cannot have other common points except $O$ and $A'$. Hence, $A^* = A'$.

If $A$ lies inside $S$, we can apply the already proved to line $MN'$ and point $A'$ (which is outside $S$). We get $A = (A')^*$. But then $A' = A^*$.

**28.8.** Let point $A$ lie outside $S$. Through $A$, draw a line tangent to $S$ at point $M$. Let $MA'$ be a height of triangle $OMA$. Right triangles $OMA$ and $OA'M$ are similar, hence, $A'O : OM = OM : OA$ and $OA' = \frac{R^2}{OA}$, i.e., point $A'$ is the one to be found.

If $A$ lies inside $S$, then we can perform the construction in the reverse order: we drop perpendicular $AM$ to $OA$ (point $M$ lies on the circle). Then the tangent to $S$ at point $M$ intersects with ray $OA$ at the desired point, $A^*$.

Proof is repeated literally.

**28.9.** If both given points $A$ and $B$ lie on the given circle (or line) $S$, then the problem has no solutions. Let now $A$ not lie to $S$. Under the inversion with center $A$ the circle to be found turns into the line passing through $B^*$ and tangent to $S^*$. This implies the following construction. Let us perform the inversion with respect to an arbitrary circle with center $A$. Through $B^*$ draw the tangent $l$ to $S^*$. Perform an inversion once again. Then $l$ turns into the circle to be constructed.

If point $B^*$ lies on $S^*$, then the problem has a unique solution; if $B^*$ lies outside $S^*$, then there are two solutions, and if $B^*$ lies inside $S^*$, then there are no solutions.

**28.10.** The inversion with center at the given point sends circles $S_1$ and $S_2$ into a pair of circles $S_1^*$ and $S_2^*$ (or into circle $S^*$ and line $l$; or into a pair of lines $l_1$ and $l_2$), respectively; the circle tangent to them turns into the common tangent to $S_1^*$ and $S_2^*$ (resp. into the

tangent to $S^*$ parallel to $l$; or into a line parallel to $l_1$ and $l_2$). Therefore, to construct the desired circle we have to construct a line tangent to $S_1^*$ and $S_2^*$ (resp. tangent to $S^*$ and parallel to $l$; or parallel to $l_1$ and $l_2$) and perform an inversion once again.

**28.11.** Let us reduce this problem to Problem 28.10. Let circle $S$ of radius $r$ be tangent to circles $S_1$, $S_2$, $S_3$ of radii $r_1$, $r_2$, $r_3$, respectively. Since the tangency of $S$ with each of $S_i$ ($i = 1, 2, 3$) can be either outer or inner, there are eight possible distinct cases to consider. Let, for instance, $S$ be tangent to $S_1$ and $S_3$ from the outside and to $S_2$ from the inside (Fig. 111).
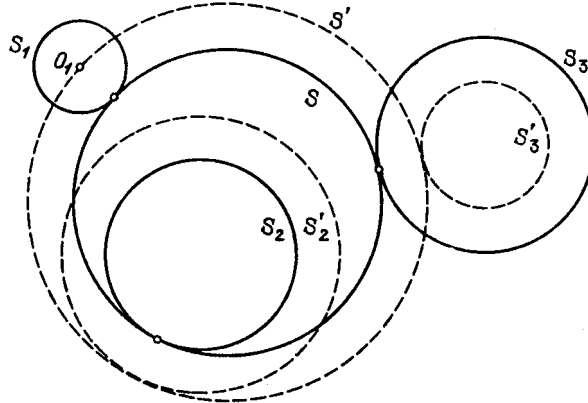


Figure 253 (Sol. 28.11)

Let us replace the circles $S$, $S_2$, $S_3$ with the concentric to them circles $S'$, $S_2'$ and $S_3'$, respectively, so that $S'$ is tangent to $S_2'$ and $S_3'$ and passes through the center $O_1$ of $S_1$. To this end it suffices that the radii of $S'$, $S_2'$, $S_3'$ were equal to $r + r_1$, $r_2 + r_1$, $|r_3 - r_1|$, respectively.

Conversely, from circle $S'$ passing through $O_1$ and tangent to $S_2'$ and $S_3'$ (from the outside if $r_3 - r_1 \geq 0$ and from the inside if $r_3 - r_1 < 0$) we can construct circle $S$ — a solution of the problem — by diminishing the radius of $S'$ by $r_1$. The construction of such a circle $S'$ is described in the solution of Problem 28.10 (if the type of tangency is given, then the circle is uniquely constructed).

One can similarly perform the construction for the other possible types of tangency.

**28.12.** Under the inversion with center at the given point $A$ the circle to be constructed turns into the line perpendicular to the images of both circles $S_1$ and $S_2$, i.e., into the line connecting the centers of $S_1^*$ and $S_2^*$. Therefore, the circle circle to be constructed is the image under this inversion of an arbitrary line passing through the centers of $S_1^*$ and $S_2^*$.

**28.13.** Let us perform an inversion that sends circles $S_1$ and $S_2$ into a pair of lines (if they have a common point) or in a pair of concentric circles (cf. Problem 28.6) with a common center $A$. In the latter case the circle perpendicular to both circles $S_1$ and $S_2$ turns into a line passing through $A$ (since there are no circles perpendicular to two concentric circles); the tangent drawn from $A$ to $S^*$ is the image of the circle circle to be constructed under this inversion.

If $S_1^*$ and $S_2^*$ are parallel lines, then the image of the circle circle to be constructed is any of the two lines perpendicular to $S_1^*$ and $S_2^*$ and tangent to $S^*$. Finally, if $S_1^*$ and $S_2^*$ are lines intersecting at a point $B$, then the circle circle to be constructed is the image under the inversion of any of the two circles with center $B$ and tangent to $S^*$.

**28.14.** Under the inversion with center at point $A$ the problem reduces to the construction of a line $l$ passing through $B^*$ and intersecting circle $S^*$ at an angle of $\alpha$, i.e., to

construction of a point $X$ on $S^*$ such that $\angle B^*XO = 90° - \alpha$, where $O$ is the center of $S^*$. This point $X$ lies on the intersection of $S^*$ with the arc whose points serve as the vertices of angles of $90° - \alpha$ subtending segment $B^*O$.

**28.15.** a) Let $AB$ be the given segment. Let us draw the circle with center $B$ and radius $AB$. On this circle, mark chords $AX, XY$ and $YZ$ of the same length as $AB$; we get equilateral triangles $ABX$, $XBY$ and $YBZ$. Hence, $\angle ABZ = 180°$ and $AZ = 2AB$.

b) In the solution of heading a) we have described how to construct a segment $BZ$ equal to $AB$ on line $AB$. Repeating this procedure $n - 1$ times we get segment $AC$ such that $AC = nAB$.

**28.16.** Let us draw circles with centers $B$ and $C$ passing through $A$. Then the distinct from $A$ intersection point of these circles is the desired one.

**28.17.** First, suppose that point $A$ lies outside circle $S$. Let $B$ and $C$ be the intersection points of $S$ and the circle of radius $AO$ and with center $A$. Let us draw circles with centers $B$ and $C$ of radius $BO = CO$; let $O$ and $A'$ be their intersection points. Let us prove that $A'$ is the desired point.

Indeed, under the symmetry through line $OA$ the circles with centers $B$ and $C$ turn into each other and, therefore, point $A'$ is fixed. Hence, $A'$ lies on line $OA$. Isosceles triangles $OAB$ and $OBA'$ are similar because they have equal angles at the base. Therefore, $OA' : OB = OB : OA$ or $OA' = \frac{OB^2}{OA}$, as required.

Now, let point $A$ lie inside $S$. With the help of the construction from Problem 28.15 a) let us construct on ray $OA$ segments $AA_2$, $A_2A_3$, ..., $A_{n-1}A_n$, ... of length $OA$ until one of the points $A_n$ becomes outside $S$. Applying to $A_n$ the above-described construction we get a point $A_n^*$ on $OA$ such that $OA_n^* = \frac{R^2}{n \cdot OA} = \frac{1}{n}OA^*$. In order to construct point $A^*$ it only remains to enlarge segment $OA_n^*$ $n$ times, cf. Problem 28.15 b).

**28.18.** Let $A$ and $B$ be two given points. If point $C$ lies on ray $AB$ and $AC = 2AB$, then under the inversion with respect to the circle of radius $AB$ centered at $A$ point $C$ turns into the midpoint of segment $AB$. The construction is reduced to Problems 28.15 a) and 28.17.

**28.19.** The center of this circle is the image under an inversion of point $O'$ symmetric to $O$ through $AB$. It remains to apply Problems 28.16 and 28.17.

**28.20.** Let $A$, $B$, $C$ be given points. Let us construct (Problem 28.17) the images of $B$ and $C$ under the inversion with center $A$ and of arbitrary degree. Then the circle passing through $A$, $B$ and $C$ is the image of line $B^*C^*$ under this inversion and its center can be constructed thanks to the preceding problem.

**28.21.** a) Making use of the preceding problem construct the center $O$ of circle $S$. Next, construct points $A^*$ and $B^*$ — the images of $A$ and $B$ under the inversion with respect to $S$. The image of $AB$ is circle $S_1$ passing through points $A^*$, $B^*$ and $O$. Making use of Problem 28.19 we construct $S_1$. The desired points are the images of the intersection points of circles $S$ and $S_1$, i.e., just intersection points of $S$ and $S_1$.

b) Let us consider an inversion with center $A_1$. Line $A_2B_2$ turns under this inversion into the circle $S$ passing through points $A_1$, $A_2^*$ and $B_2^*$. We can construct $S$ making use of Problem 28.19. Further, let us construct the intersection points of $S$ and line $A_1B_1$ making use of the solution of heading a). The desired point is the image of the intersection point distinct from $A_1$ under the inversion considered.

**28.22.** Under the inversion centered in the endpoint $A$ of the segment the configuration plotted on Fig. 105 turns into the pair of tangent circles inscribed into the angle at vertex $B^*$. Clearly, the set of the tangent points of such circles is the bisector of the angle and the desired locus is the image of the bisector under the inversion — the arc of the circle with

endpoints $A$ and $B$ that divides in halves the angle between the arc of the segment and chord $AB$.

**28.23.** Let $C$ be the vertex of the given angle. Under the inversion with center in $A$ line $CB$ turns into circle $S$; circles $S_1$ and $S_2$ turn into circle $S_1^*$ centered in $O_1$ tangent to $S$ at point $B^*$ and line $l$ parallel to $C^*A$ and tangent to $S_1^*$ at $X$, respectively (Fig. 112).
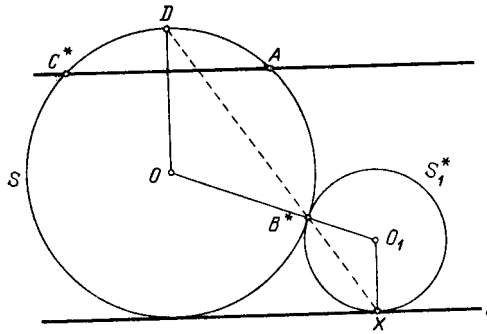


Figure 254 (Sol. 28.23)

In $S$, draw radius $OD$ perpendicular to $C^*A$. Points $O$, $B^*$ and $O_1$ lie on one line and $OD \parallel O_1X$. Hence,

$$\angle OB^*D = 90° - \frac{\angle DOB^*}{2} = 90° - \frac{\angle XO_1B^*}{2} = \angle O_1B^*X,$$

therefore, point $X$ lies on line $DB^*$. Applying inversion once again we see that the desired locus of tangent points is arc $\smile AB$ of the circle passing through points $A$, $B$ and $D^*$.

**28.24.** The given inversion sends line $BC$ into the circle passing through points $A$, $B$ and $C$ so that the image of segment $BC$ should remain inside angle $\angle BAC$.

**28.25.** Let $S_1$ and $S_2$ be circles inscribed into the segment; $M$, $N$ their intersection points (Fig. 113). Let us show that line $MN$ passes through point $P$ of the circle of the segment equidistant from its endpoints $A$ and $B$.
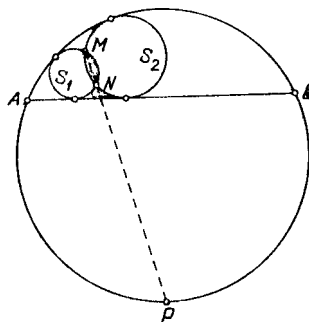


Figure 255 (Sol. 28.25)

Indeed, thanks to the preceding problem the inversion with center $P$ and of degree $PA^2$ sends segment $AB$ to arc $\smile AB$ and circles $S_1$ and $S_2$ to circles $S_1^*$ and $S_2^*$, still inscribed into a segment, respectively. But the tangents to $S_1$ drawn from $P$ are tangent also to $S_1^*$; hence, $S_1^* = S_1$ (since both these circles are similarly tangent to the three fixed points). Analogously, $S_2^* = S_2$; hence, points $M$ and $N$ change places under the inversion, i.e., $M^* = N$ and $MN$ passes through the center of inversion.

**28.26.** Let us perform an inversion with center $A$. The angles of interest to us are then equal (by Problem 28.5) to the respective angles between lines $B^*C^*$ and $B^*D^*$ or between

line $C^*D^*$ and the circle circumscribed about triangle $B^*C^*D^*$. Both these angles are equal to a half arc $\smile C^*D^*$.

**28.27.** Performing an inversion with center $A$ we get three lines passing through $B$: lines $S_1^*$ and $S_2^*$ are tangent to $S^*$ and $S_3^*$ is perpendicular to it. Thus, line $S_3^*$ passes through the center of $S^*$ and is the bisector of the angle formed by $S_1^*$ and $S_2^*$. Therefore, circle $S_3$ divides the angle between $S_1$ and $S_2$ in halves.

**28.28.** The condition of the types of tangency implies that after an inversion with center $A$ we get either two circles inscribed into the same angle or a pair of vertical angles. In either case a homothety with center $A$ turns circles $S_1^*$ and $S_2^*$ into each other. This homothety sends one segment that connects tangent points into another one. Hence, lines $B_1^*C_1^*$ and $B_2^*C_2^*$ are parallel and their images under the inversion are tangent at point $A$.

**28.29.** Let $A_1$, $B_1$ and $C_1$ be the midpoints of sides $BC$, $CA$ and $AB$, respectively. Let us prove that, for instance, the circle circumscribed about triangle $A_1B_1C_1$ is tangent to the inscribed circle $S$ and escribed circle $S_a$ tangent to $BC$. Let points $B'$ and $C'$ be symmetric to $B$ and $C$, respectively, through the bisector of angle $\angle A$ (i.e., $B'C'$ is the second common inner tangent to $S$ and $S_a$), let $P$ and $Q$ be the tangent points of circles $S$ and $S_a$, respectively, with side $BC$ and let $D$ and $E$ be the intersection points of lines $A_1B_1$ and $A_1C_1$, respectively, with line $B'C'$.

By Problem 3.2 $BQ = CP = p - c$ and, therefore, $A_1P = A_1Q = \frac{1}{2}|b - c|$. It suffices to prove that the inversion with center $A_1$ and degree $A_1P^2$ sends points $B_1$ and $C_1$ into $D$ and $E$, respectively, (this inversion sends circles $S$ and $S_a$ into themselves, and the circle circumscribed about triangle $A_1B_1C_1$ into line $B'C'$).

Let $K$ be the midpoint of segment $CC'$. Point $K$ lies on line $A_1B_1$ and

$$A_1K = \frac{BC'}{2} = \frac{|b - c|}{2} = A_1P.$$

Moreover,

$$A_1D : A_1K = BC' : BA = A_1K : A_1B_1,$$

i.e., $A_1D \cdot A_1B_1 = A_1K^2 = A_1P^2$. Similarly, $A_1E \cdot A_1C_1 = A_1P^2$.

**28.30.** After an inversion with center at the intersection point of $S_1$ and $S_2$ we get lines $l_1$, $l_2$ and $l$ intersecting at one point. Line $l_1$ intersects circle $S_4^*$ at points $A$ and $B$, line $l_2$ intersects $S_3^*$ at points $C$ and $D$ and line $l$ passes through the intersection points of these circles. Hence, points $A$, $B$, $C$, $D$ lie on one circle (Problem 3.9).

**28.31.** Let us make an inversion with center at point $A_1$. Then circles $S_1$, $S_2$ and $S_4$ turn into lines $A_2^*D_1^*$, $B_1^*A_2^*$ and $D_1^*B_1^*$; circles $S_3$ and $S_4$ into circles $S_3^*$ and $S_4^*$ circumscribed about triangles $B_2^*C_1^*B_1^*$ and $C_1^*D_1^*D_2^*$, respectively (Fig. 114).
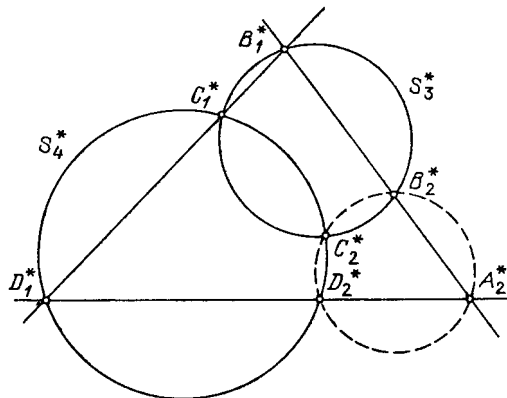


Figure 256 (Sol. 28.31)

Let us draw the circle through points $B_2^*$, $D_2^*$ and $A_2^*$. By Problem 2.80 a) it passes through the intersection point $C_2^*$ of circles $S_3^*$ and $S_4^*$. Thus, points $A_2^*$, $B_2^*$, $C_2^*$, $D_2^*$ lie on one circle. It follows, that points $A_2$, $B_2$, $C_2$, $D_2$ lie on one circle or line.

**28.32.** Let $P$, $Q$, $R$, $S$, $T$ be the intersection points of circles $S_1$, $S_2$, $S_3$, $S_4$, $S_5$ spoken about in the formulation of the problem (cf. Fig. 108).

Let us prove, for instance, that points $P$, $Q$, $R$, $S$ lie on one circle. Let us draw circle $\Sigma$ circumscribed about triangle $NKD$. Applying the result of Problem 2.83 a) (which coincides with that of Problem 19.45) to quadrilaterals $AKDE$ and $BNDC$ we see that circles $S_4$, $S_5$ and $\Sigma$ intersect at one point (namely, $P$) and circles $S_2$, $S_3$, $\Sigma$ also intersect at one point (namely, $S$).

Therefore, circle $\Sigma$ passes through points $P$ and $S$. Now, observe that of eight intersection points of circles $\Sigma$, $S_1$, $S_2$, $S_5$ four, namely, $N$, $A$, $B$, $K$, lie on one line. It follows that by Problem 28.31 the remaining four points $P$, $Q$, $R$, $S$ lie on one circle.

**28.33.** An inversion with center at the intersection point of circumscribed circles of triangles $A_1A_2B_3$, $A_1B_2A_3$ and $B_1A_2A_3$ sends these circles into lines and the statement of the problem reduces to the statement that the circles circumscribed about triangles $B_1^*B_2^*A_3^*$, $B_1^*A_2^*B_3^*$ and $A_1^*B_2^*B_3^*$ pass through one point, i.e., the statement of Problem 2.80 a).

**28.34.** Under an inversion with center at the intersection point of circles circumscribed about triangles $A_1B_1C_1$, $A_1B_2C_2$, $A_2B_1C_2$ and $A_2B_2C_1$ we get four lines and four circles circumscribed about triangles formed by these lines. By Problem 2.83 a) these circles pass through one point.

**28.35.** a) Denote by $M_{ij}$ the intersection point of lines $l_i$ and $l_j$ and by $S_{ij}$ the circle corresponding to the three remaining lines. Then point $A_1$ is distinct from the intersection point $M_{34}$ of circles $S_{15}$ and $S_{12}$.

Repeating this argument for each point $A_i$, we see that thanks to Problem 28.32 they lie on one circle.

b) Let us prove the statement of the problem by induction and considere separately the cases of even and odd $n$.

Let $n$ be odd. Denote by $A_i$ the point corresponding to the tuple of $n-1$ lines obtained by deleting line $l_i$ and by $A_{ijk}$ the point corresponding to the tuple of $n$ given lines without $l_i$, $l_j$ and $l_k$. Similarly, denote by $S_{ij}$ and $S_{ijkm}$ the circles corresponding to tuples of $n-2$ and $n-4$ lines obtained by deleting $l_i$ and $l_j$ or $l_i$, $l_j$, $l_k$ and $l_m$, respectively.

In order to prove that $n$ points $A_1$, $A_2$, ..., $A_n$ lie on the same circle, it suffices to prove that any four of them lie on one circle. Let us prove this, for instance, for points $A_1$, $A_2$, $A_3$ and $A_4$. Since points $A_i$ and $A_{ijk}$ lie on $S_{ij}$, it follows that circles $S_{12}$ and $S_{23}$ intersect at points $A_2$ and $A_{123}$; circles $S_{23}$ and $S_{34}$ intersect at points $A_3$ and $A_{234}$; circles $S_{34}$ and $S_{41}$ at points $A_4$ and $A_{134}$; circles $S_{41}$ and $S_{12}$ at points $A_1$ and $A_{124}$. But points $A_{123}$, $A_{234}$, $A_{134}$ and $A_{124}$ lie on one circle — circle $S_{1234}$ — hence, by Problem 28.31 points $A_1$, $A_2$, $A_3$ and $A_4$ lie on one circle.

Let $n$ be even. Let $S_i$, $A_{ij}$, $S_{ijk}$, $A_{ijkm}$ be circles and points corresponding to tuples of $n-1$, $n-2$, $n-3$ and $n-4$ lines, respectively. In order to prove that circles $S_1$, $S_2$, ..., $S_n$ intersect at one point, let us prove that this holds for any three of them. (This suffices for $n \geq 5$, cf. Problem 26.12.) Let us prove, for instance, that $S_1$, $S_2$ and $S_3$ intersect at one point. By definition of points $A_{ij}$ and circles $S_i$ and $S_{ijk}$, points $A_{12}$, $A_{13}$ and $A_{14}$ lie on circle $S_1$; points $A_{12}$, $A_{23}$ and $A_{24}$ on $S_2$; points $A_{13}$, $A_{14}$ and $A_{34}$ on $S_3$; points $A_{12}$, $A_{14}$ and $A_{24}$ on $S_{124}$; points $A_{13}$, $A_{14}$, $A_{34}$ on $S_{134}$; points $A_{23}$, $A_{24}$, $A_{34}$ on $S_{234}$.

But the three circles $S_{124}$, $S_{134}$ and $S_{234}$ pass through point $A_{1234}$; hence, by Problem 28.33 circles $S_1$, $S_2$ and $S_3$ also intersect at one point.

**28.36.** a) Denote by $M_{ij}$ the intersection point of lines $l_i$ and $l_j$. Then point $A_1$ corresponding to the triple $(l_2, l_3, l_4)$ is the intersection point of the circles circumscribed about triangles $M_2 M_3 M_{23}$ and $M_3 M_4 M_{34}$. By the similar arguments applied to $A_2$, $A_3$ and $A_4$ we see that points $A_1$, $A_2$, $A_3$ and $A_4$ lie on one circle thanks to Problem 28.31 because points $M_1$, $M_2$, $M_3$, $M_4$ lie on one circle.

b) As in Problem 28.35 b), let us prove our statement by induction; consider the cases of even and odd $n$ separately.

Let $n$ be even; let $A_i$, $S_{ij}$, $A_{ijk}$ and $S_{ijkm}$ denote points and circles corresponding to tuples of $n-1$, $n-2$, $n-3$ and $n-4$ lines, respectively. Let us prove that points $A_1$, $A_2$, $A_3$, $A_4$ lie on one circle. By definition of points $A_i$ and $A_{ijk}$, circles $S_{12}$ and $S_{23}$ intersect at points $A_2$ and $A_{123}$; circles $S_{23}$ and $S_{34}$ at points $A_3$ and $A_{234}$; circles $S_{34}$ and $S_{41}$ at points $A_4$ and $A_{134}$; circles $S_{41}$ and $S_{12}$ at points $A_1$ and $A_{124}$.

Points $A_{123}$, $A_{234}$, $A_{134}$ and $A_{124}$ lie on circle $S_{1234}$; hence, by Problem 28.31 points $A_1$, $A_2$, $A_3$, $A_4$ lie on one circle. We similarly prove that any four of points $A_i$ (hence, all of them) lie on one circle.

Proof for $n$ odd, $n \geq 5$, literally repeats the proof of heading b) of Problem 28.35 for the case of $n$ even.

**28.37.** If circles $R_1$ and $R_2$ intersect or are tangent to each other, then an inversion with the center at their intersection point sends circles $S_1$, $S_2$, ..., $S_n$ into the circles that are tangent to a pair of straight lines and to each other at points $A_1^*$, $A_2^*$, ..., $A_{n-1}^*$ lying on the bisector of the angle formed by lines $R_1^*$ and $R_2^*$ if $R_1^*$ and $R_2^*$ intersect, or on the line parallel to $R_1^*$ and $R_2^*$ if these lines do not intersect. Applying the inversion once again we see that points $A_1^*$, $A_2^*$, ..., $A_{n-1}^*$ lie on one circle.

If circles $R_1$ and $R_2$ do not intersect, then by Problem 28.6 there is an inversion sending them into a pair of concentric circles. In this case points $A_1^*$, $A_2^*$, ..., $A_{n-1}^*$ lie on a circle concentric with $R_1^*$ and $R_2^*$; hence, points $A_1$, $A_2$, ..., $A_{n-1}$ lie on one circle.

**28.37.** Let us make an inversion sending $R_1$ and $R_2$ into a pair of concentric circles. Then circles $S_1^*$, $S_2^*$, ..., $S_n^*$ and $T_1^*$ are equal (Fig. 115).
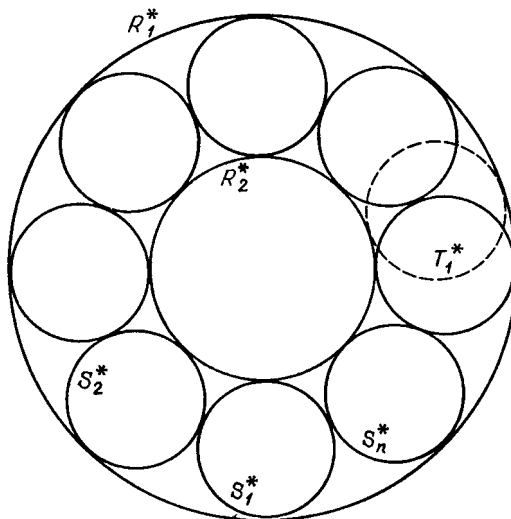


Figure 257 (Sol. 28.37)

Turning the chain of circles $S_1^*$, ..., $S_n^*$ about the center of the circle $R_1^*$ so that $S_1^*$ becomes $T_1^*$ and making an inversion once again we get the desired chain $T_1$, $T_2$, ..., $T_n$.

**28.39.** The center of inversion that sends circles $R_1$ and $R_2$ into concentric circles lies (see the solution of Problem 28.6) on the line that connects their centers. Therefore, making

this inversion and taking into account that the angle between circles, as well as the type of tangency, are preserved under an inversion, we reduce the proof to the case of concentric circles $R_1$ and $R_2$ with center $O$ and radii $r_1$ and $r_2$, respectively.

Let us draw circle $S$ with center $P$ and of radius $\frac{1}{2}(r_1 - r_2)$ tangent to $R_1$ from the inside and to $R_2$ from the outside and let us draw circles $S'$ and $S''$ each of radius $\frac{1}{2}(r_1 + r_2)$ with centers $A$ and $B$, respectively, tangent to $R_1$ and $R_2$ at their intersection points with line $OP$ (Fig. 116).
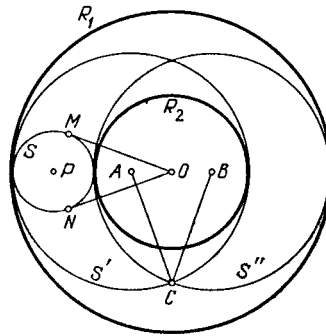


Figure 258 (Sol. 28.39)

Let $OM$ and $ON$ be tangent to $S$ drawn at $O$. Clearly, the chain of $n$ circles tangent to $R_1$ and $R_2$ exists if and only if $\angle MON = m\frac{360°}{n}$. (In this case the circles of the chain run $m$ times about the circle $R_2$.)

Therefore, it remains to prove that the angle between circles $S'$ and $S''$ is equal to $\angle MON$. But the angle between $S'$ and $S''$ is equal to the angle between their radii drawn to the intersection point $C$. Moreover, since

$$PO = r_1 - \frac{r_1 - r_2}{2} = \frac{r_1 + r_2}{2} = AC,$$
$$PN = \frac{r_1 - r_2}{2} = r_1 - \frac{r_1 + r_2}{2} = OA,$$
$$\angle PNO = \angle AOC = 90°,$$

we have $\triangle ACO = \triangle PON$. Therefore,
$$\angle ACB = 2\angle ACO = 2\angle PON = \angle NOM.$$

**28.40.** Let $R_1$ and $R_2$ be a pair of circles without common points. The remaining four circles constitute a chain and, therefore, by the preceding problem circles $S'$ and $S''$ tangent to $R_1$ and $R_2$ at the intersection points of the latter with the line connecting their centers intersect at right angle (Fig. 117). If $R_2$ lies inside $R_1$, then the radii $r'$ and $r''$ of circles $S'$ and $S''$ are equal to $\frac{1}{2}(r_1 + r_2 + d)$ and $\frac{1}{2}(r_1 + r_2 - d)$, respectively, and the distance between their centers is equal to $d' = 2r_1 - r_1 - r_2 = r_1 - r_2$. The angle between $S'$ and $S''$ is equal to the angle between the radii drawn to the intersection point, hence, $(d')^2 = (r')^2 + (r'')^2$ or, after simplification, $d^2 = r_1^2 + r_2^2 - 6r_1r_2$.

If $R_1$ and $R_2$ are not inside one another, then the radii of $S'$ and $S''$ are equal to $\frac{1}{2}(d + (r_1 - r_2))$ and $\frac{1}{2}(d - (r_1 - r_2))$, respectively, and the distance between their centers is $d' = r_1 + r_2 + d - (r_1' + r_2') = r_1 + r_2$. As a result we get $d^2 = r_1^2 + r_2^2 + 6r_1r_2$.
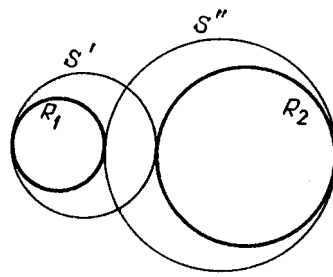
Figure 259 (Sol. 28.40)

# Chapter 29. AFFINE TRANSFORMATIONS

## §1. Affine transformations

A transformation of the plane is called an *affine* one if it is continuous, one-to-one, and the image of every line is a line.

*Shifts* and *similarity transformations* are particular cases of affine transformations.

A *dilation* of the plane relative axis $l$ with coefficient $k$ is a transformation of the plane under which point $M$ turns into point $M'$ such that $\overrightarrow{OM'} = k\overrightarrow{OM}$, where $O$ is the projection of $M$ to $l$. (A dilation with coefficient smaller than 1 is called a *contraction*.)

**29.1.** Prove that a dilation of the plane is an affine transformation.

**29.2.** Prove that under an affine transformation parallel lines turn into parallel ones.

**29.3.** Let $A_1$, $B_1$, $C_1$, $D_1$ be images of points $A$, $B$, $C$, $D$, respectively, under an affine transformation. Prove that if $\overrightarrow{AB} = \overrightarrow{CD}$, then $\overrightarrow{A_1B_1} = \overrightarrow{C_1D_1}$.

Problem 29.3 implies that we can define the image of vector $\overrightarrow{AB}$ under an affine transformation $L$ as $\overrightarrow{L(A)L(B)}$ and this definition does not depend on the choice of points $A$ and $B$ that determine equal vectors.

**29.4.** Prove that if $L$ is an affine transformation, then

a) $L(\overrightarrow{0}) = \overrightarrow{0}$;

b) $L(\mathbf{a} + \mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b})$;

c) $L(k\mathbf{a}) = kL(\mathbf{a})$.

**29.5.** Let $A'$, $B'$, $C'$ be images of points $A$, $B$, $C$ under an affine transformation $L$. Prove that if $C$ divides segment $AB$ in the ratio $AC : CB = p : q$, then $C'$ divides segment $A'B'$ in the same ratio.

**29.6.** Given two points $O$ and $O'$ in plane and two bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_1', \mathbf{e}_2'\}$.

a) Prove that there exists a unique affine transformation that sends $O$ into $O'$ and a the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ into the basis $\{\mathbf{e}_1', \mathbf{e}_2'\}$.

b) Given two triangles $ABC$ and $A_1B_1C_1$ prove that there exists a unique affine transformation that sends $A$ into $A_1$, $B$ into $B_1$ and $C$ into $C_1$.

c) Given two parallelograms, prove that there exists a unique affine transformation that sends one of them into another one.

**29.7.** Prove that if a non-identity affine transformation $L$ sends each point of line $l$ into itself, then all the lines of the form $ML(M)$, where $M$ is an arbitrary point not on $l$, are parallel to each other.

**29.8.** Prove that any affine transformation can be represented as a composition of two dilations and an affine transformation that sends any triangle into a similar triangle.

**29.9.** Prove that any affine transformation can be represented as a composition of a dilation (contraction) and an affine transformation that sends any triangle into a similar triangle.

**29.10.** Prove that if an affine transformation sends a circle into itself, then it is either a rotation or a symmetry.

**29.11.** Prove that if $M'$ and $N'$ are the images of polygons $M$ and $N$, respectively, under an affine transformation, then the ratio of areas of $M$ and $N$ is equal to the ratio of areas of $M'$ and $N'$.

## §2. How to solve problems with the help of affine transformations

**29.12.** Through every vertex of a triangle two lines are drawn. The lines divide the opposite side of the triangle into three equal parts. Prove that the diagonals connecting opposite vertices of the hexagon formed by these lines intersect at one point.

**29.13.** On sides $AB$, $BC$ and $CD$ of parallelogram $ABCD$ points $K$, $L$ and $M$, respectively, are taken. The points divide the sides in the same ratio. Let $b$, $c$, $d$ be lines passing through points $B$, $C$, $D$ parallel to lines $KL$, $KM$, $ML$, respectively. Prove that lines $b$, $c$, $d$ pass through one point.

**29.14.** Given triangle $ABC$, let $O$ be the intersection point of its medians and $M$, $N$ and $P$ be points on sides $AB$, $BC$ and $CA$, respectively, that divide these sides in the same ratio (i.e., $AM : MB = BN : NC = CP : PA = p : q$). Prove that:

a) $O$ is the intersection point of the medians of triangle $MNP$;

b) $O$ is the intersection point of the medians of the triangle formed by lines $AN$, $BP$ and $CM$.

**29.15.** In trapezoid $ABCD$ with bases $AD$ and $BC$, a line is drawn through point $B$ parallel to side $CD$ and intersecting diagonal $AC$ at point $P$; through point $C$ a line is drawn parallel to $AB$ and intersecting diagonal $BD$ at $Q$. Prove that $PQ$ is parallel to the bases of the trapezoid.

**29.16.** In parallelogram $ABCD$, points $A_1$, $B_1$, $C_1$, $D_1$ lie on sides $AB$, $BC$, $CD$, $DA$, respectively. On sides $A_1B_1$, $B_1C_1$, $C_1D_1$, $D_1A_1$ of quadrilateral $A_1B_1C_1D_1$ points $A_2$, $B_2$, $C_2$, $D_2$, respectively, are taken. It is known that

$$\frac{AA_1}{BA_1} = \frac{BB_1}{CB_1} = \frac{CC_1}{DC_1} = \frac{DD_1}{AD} = \frac{AD_2}{D_1D_2} = \frac{D_1C_2}{C_1C_2} = \frac{C_1B_2}{B_1B_2} = \frac{B_1A_2}{A_1A_2}.$$

Prove that $A_2B_2C_2D_2$ is a parallelogram with sides parallel to the sides of $ABCD$.

**29.17.** On sides $AB$, $BC$ and $AC$ of triangle $ABC$, points $M$, $N$ and $P$, respectively, are taken. Prove that:

a) if points $M_1$, $N_1$ and $P_1$ are symmetric to points $M$, $N$ and $P$ through the midpoints of the corresponding sides, then $S_{MNP} = S_{M_1N_1P_1}$.

b) if $M_1$, $N_1$ and $P_1$ are points on sides $AC$, $BA$ and $CB$, respectively, such that $MM_1 \parallel BC$, $NN_1 \parallel CA$ and $PP_1 \parallel AB$, then $S_{MNP} = S_{M_1N_1P_1}$.

## Solutions

**29.1.** We have to prove that if $A'$, $B'$, $C'$ are images of points $A$, $B$, $C$ under the dilation with respect to line $l$ with coefficient $k$ and point $C$ lies on line $AB$, then point $C'$ lies on line $A'B'$. Let $\overrightarrow{AC} = t\overrightarrow{AB}$. Denote by $A_1$, $B_1$, $C_1$ the projections of points $A$, $B$, $C$, respectively, on line $l$ and let

$$\mathbf{a} = \overrightarrow{A_1A}, \ \ \mathbf{b} = \overrightarrow{B_1B}, \ \ \mathbf{c} = \overrightarrow{C_1C},$$

$$\mathbf{a}' = \overrightarrow{A_1A'}, \ \ \mathbf{b}' = \overrightarrow{B_1B'}, \ \ \mathbf{c}' = \overrightarrow{C_1C'},$$

$$\mathbf{x} = \overrightarrow{A_1B_1}, \ \ \mathbf{y} = \overrightarrow{A_1C_1}.$$

Since the ratio of lengths of proportional vectors under the projection on line $l$ is preserved, then $\mathbf{y} = t\mathbf{x}$ and $\mathbf{y} + (\mathbf{c} - \mathbf{a}) = t(\mathbf{y} + (\mathbf{b} - \mathbf{a}))$. By subtracting the first equality from the second one we get $(\mathbf{c} - \mathbf{a}) = t(\mathbf{b} - \mathbf{a})$. By definition of a dilation $\mathbf{a}' = k\mathbf{a}$, $\mathbf{b}' = k\mathbf{b}$, $\mathbf{c}' = k\mathbf{c}$; hence,

$$\overrightarrow{A'C'} = \mathbf{y} + k(\mathbf{c} - \mathbf{a}) = t\mathbf{x} + k(t(\mathbf{b} - \mathbf{a})) = t(\mathbf{x} + k(\mathbf{b} - \mathbf{a})) = t\overrightarrow{A'B'}.$$

**29.2.** By definition, the images of lines are lines and from the property of an affine transformation to be one-to-one it follows that the images of nonintersecting lines do not intersect.

**29.3.** Let $\overrightarrow{AB} = \overrightarrow{CD}$. First, consider the case when points $A$, $B$, $C$, $D$ do not lie on one line. Then $ABCD$ is a parallelogram. The preceding problem implies that $A_1B_1C_1D_1$ is also a parallelogram; hence, $\overrightarrow{A_1B_1} = \overrightarrow{C_1D_1}$.

Now, let points $A$, $B$, $C$, $D$ lie on one line. Take points $E$ and $F$ that do not lie on this line and such that $\overrightarrow{EF} = \overrightarrow{AB}$. Let $E_1$ and $F_1$ be their images. Then $\overrightarrow{A_1B_1} = \overrightarrow{E_1F_1} = \overrightarrow{C_1D_1}$.

**29.4.** a) $L(\overrightarrow{0}) = L(\overrightarrow{AA}) = \overrightarrow{L(A)L(A)} = \overrightarrow{0}$.

b)
$$L(\overrightarrow{AB} + \overrightarrow{BC}) = L(\overrightarrow{AC}) = \overrightarrow{L(A)L(C)} = \overrightarrow{L(A)L(B)} + \overrightarrow{L(B)L(C)} =$$
$$L(\overrightarrow{AB}) + L(\overrightarrow{BC}).$$

c) First, suppose $k$ is an integer. Then

$$L(k\mathbf{a}) = L(\mathbf{a} + \cdots + \mathbf{a}) = L(\mathbf{a}) + \cdots + L(\mathbf{a}) = kL(\mathbf{a}).$$

Now, let $k = \frac{m}{n}$ be a rational number. Then

$$nL(k\mathbf{a}) = L(nk\mathbf{a}) = L(m\mathbf{a}) = mL(\mathbf{a});$$

hence,

$$L(k\mathbf{a}) = \frac{mL(\mathbf{a})}{n} = kL(\mathbf{a}).$$

Finally, if $k$ is an irrational number, then there always exists a sequence $k_n$ $(n \in \mathbb{N})$ of rational numbers tending to $k$ (for instance, the sequence of decimal approximations of $k$). Since $L$ is continuous,

$$L(k\mathbf{a}) = L(\lim_{n \to \infty} k_n\mathbf{a}) = \lim_{n \to \infty} k_nL(\mathbf{a}) = kL(\mathbf{a}).$$

**29.5.** By Problem 29.4 c) the condition $q\overrightarrow{AC} = p\overrightarrow{CB}$ implies that

$$q\overrightarrow{A'C'} = qL(\overrightarrow{AC}) = L(q\overrightarrow{AC}) = L(p\overrightarrow{CB}) = pL(\overrightarrow{CB}) = p\overrightarrow{C'B'}.$$

**29.6.** a) Define the map $L$ as follows. Let $X$ be an arbitrary point. Since $\mathbf{e}_1$, $\mathbf{e}_2$ is a basis, it follows that there exist the uniquely determined numbers $x_1$ and $x_2$ such that $\overrightarrow{OX} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Assign to $X$ point $X' = L(X)$ such that $\overrightarrow{O'X'} = x_1\mathbf{e}_1' + x_2\mathbf{e}_2'$. Since $\mathbf{e}_1'$, $\mathbf{e}_2'$ is also a basis, the obtained map is one-to-one. (The inverse map is similarly constructed.)

Let us prove that the image of any line $AB$ under $L$ is a line. Let $A' = L(A)$, $B' = L(B)$; let $a_1$, $a_2$, and $b_1$, $b_2$ be the coordinates of points $A$ and $B$, respectively, in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, i.e., $\overrightarrow{OA} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$, $\overrightarrow{OB} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$. Let us consider an arbitrary point $C$ on line $AB$.

Then $\overrightarrow{AC} = k\overrightarrow{AB}$ for some $k$, i.e.,

$$\overrightarrow{OC} = \overrightarrow{OA} + k(\overrightarrow{OB} - \overrightarrow{OA}) =$$

$$((1-k)a_1 + kb_1)\mathbf{e}_1 + ((1-k)a_2 + kb_2)\mathbf{e}_2.$$

Hence, if $C' = L(C)$, then

$$\overrightarrow{O'C'} = ((1-k)a_1 + kb_1)\mathbf{e}_1' + ((1-k)a_2 + kb_2)\mathbf{e}_2' =$$

$$\overrightarrow{O'A'} + k(\overrightarrow{O'B'} - \overrightarrow{O'A'}),$$

i.e., point $C'$ lies on line $A'B'$.

The uniqueness of $L$ follows from the result of Problem 29.4. Indeed, $L(\overrightarrow{OX}) = x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2)$, i.e., the image of $X$ is uniquely determined by the images of vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and point $O$.

b) To prove it, it suffices to make use of the previous heading setting $O = A$, $\mathbf{e}_1 = \overrightarrow{AB}$, $\mathbf{e}_2 = \overrightarrow{AC}$, $O' = A_1$, $\mathbf{e}_1' = \overrightarrow{A_1B_1}$, $\mathbf{e}_2' = \overrightarrow{A_1C_1}$.

c) Follows from heading b) and the fact that parallel lines turn into parallel lines.

**29.7.** Let $M$ and $N$ be arbitrary points not on line $l$. Denote by $M_0$ and $N_0$ their projections to $l$ and by $M'$ and $N'$ the images of $M$ and $N$ under $L$. Lines $M_0M$ and $N_0N$ are parallel because both of them are perpendicular to $l$, i.e., there exists a number $k$ such that $\overrightarrow{M_0M} = k\overrightarrow{N_0N}$. Then by Problem 29.4 c) $\overrightarrow{M_0M'} = k\overrightarrow{N_0N'}$. Hence, the image of triangle $M_0MM'$ under the parallel translation by vector $\overrightarrow{M_0N_0}$ is homothetic with coefficient $k$ to triangle $N_0NN'$ and, therefore, lines $MM'$ and $NN'$ are parallel.

**29.8.** Since an affine map is uniquely determined by the images of vertices of any fixed triangle (see Problem 29.6 b)), it suffices to prove that with the help of two dilations one can get from any triangle an arbitrary triangle similar to any before given one, for instance, to an isosceles right triangle. Let us prove this.

Let $ABC$ be an arbitrary triangle, $BN$ the bisector of the outer angle $\angle B$ adjacent to side $BC$. Then under the dilation with respect to $BN$ with coefficient $\frac{\tan 45°}{\tan \angle CBN}$ we get from triangle $ABC$ triangle $A'B'C'$ with right angle $\angle B'$. With the help of a dilation with respect to one of the legs of a right triangle one can always get from this triangle an isosceles right triangle.

**29.9.** Let $L$ be a given affine transformation, $O$ an arbitrary point, $T$ the shift by vector $\overrightarrow{L(O)O}$ and $L_1 = T \circ L$. Then $O$ is a fixed point of $L_1$. Among the points of the unit circle with center $O$, select a point $A$ for which the vector $L(\overrightarrow{OA})$ is the longest. Let $H$ be a rotational homothety with center $O$ that sends point $L_1(A)$ into $A$ and let $L_2 = H \circ L_1 = H \circ T \circ L$. Then $L_2$ is an affine transformation that preserves points $O$ and $A$; hence, by Problem 29.4 c) it preserves all the other points of line $OA$ and thanks to the choice of point $A$ for all points $M$ we have $|\overrightarrow{OM}| \geq |L(\overrightarrow{OM})|$.

Let us prove (which will imply the statement of the problem) that $L_2$ is a contraction with respect to line $OA$. If $L_2$ is the identity transformation, then it is a contraction with coefficient 1, so let us assume that $L_2$ is not the identity.

By Problem 29.9 all the lines of the form $\overrightarrow{ML_2(M)}$, where $M$ is an arbitrary point not on $OA$, are parallel to each other. Let $\overrightarrow{OB}$ be the unit vector perpendicular to all these lines.

Then $B$ is a fixed point of $L_2$ because otherwise we would have had

$$|\overrightarrow{OL_2(B)}| = \sqrt{OB^2 + BL_2(B)^2} > |OB|.$$

If $B$ does not lie on line $OA$, then by Problem 29.6 b) transformation $L_2$ is the identity. If $B$ lies on $OA$, then all the lines of the form $ML_2(M)$ are perpendicular to the fixed line of transformation $L_2$. With the help of Problem 29.4 c) it is not difficult to show that the map with such a property is either a dilation or a contraction.

**29.10.** First, let us prove that an affine transformation $L$ that sends a given circle into itself sends diametrically opposite points into diametrically opposite ones. To this end let us notice that the tangent to the circle at point $A$ turns into the line that, thanks to the property of $L$ to be one-to-one, intersects with the circle at a (uniquely determined) point $L(A)$, i.e., is the tangent at point $L(A)$. Therefore, if the tangents at points $A$ and $B$ are parallel to each other (i.e., $AB$ is a diameter), then the tangents at points $L(A)$ and $L(B)$ are also parallel, i.e., $L(A)L(B)$ is also a diameter.

Fix a diameter $AB$ of the given circle. Since $L(A)L(B)$ is also a diameter, there exists a movement $P$ of the plane which is either a rotation or a symmetry that sends $A$ and $B$ into $L(A)$ and $L(B)$, respectively, and each of the arcs $\alpha$ and $\beta$ into which points $A$ and $B$ divide the given circle into the image of these arcs under $L$.

Let us prove that the map $F = P^{-1} \circ L$ is the identity. Indeed, $F(A) = A$ and $F(B) = B$; hence, all points of line $AB$ are fixed. Hence, if $X$ is an arbitrary point of the circle, then the tangent at $X$ intersects line $AB$ at the same place where the tangent at point $X' = F(X)$ does because the intersection point is fixed. Since $X$ and $X'$ lie on one and the same of the two arcs $\alpha$ or $\beta$, it follows that $X$ coincides with $X'$. Thus, $P^{-1} \circ L = E$, i.e., $L = P$.

**29.11.** Let $a_1$ and $a_2$ be two perpendicular lines. Since an affine transformation preserves the ratio of the lengths of (the segments of the) parallel lines, the lengths of all the segments parallel to one line are multiplied by the same coefficient. Denote by $k_1$ and $k_2$ these coefficients for lines $a_1$ and $a_2$. Let $\varphi$ be the angle between the images of these lines. Let us prove that the given affine transformation multiplies the areas of all polygons by $k$, where $k = k_1 k_2 \sin \varphi$.

For rectangles with sides parallel to $a_1$ and $a_2$ and also for a right triangle with legs parallel to $a_1$ and $a_2$ the statement is obvious. Any other triangle can be obtained by cuttting off the rectangle with sides parallel to $a_1$ and $a_2$ several right triangles with legs parallel to $a_1$ and $a_2$ as shown on Fig. 118 and, finally, by Problem 22.22 any polygon can be cut into triangles.
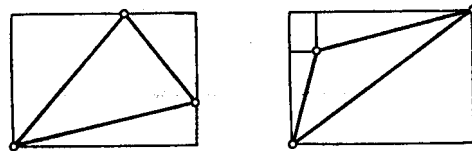


Figure 260 (Sol. 29.11)

**29.12.** Since an affine transformation sends an arbitrary triangle into an equilateral one (Problem 29.6 b)), the ratio of lengths of parallel segments are preserved (Problem 29.5). It suffices to prove the statement of the problem for an equilateral triangle $ABC$. Let points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ divide the sides of the triangle into equal parts and $A'$, $B'$, $C'$ be the midpoints of the sides (Fig. 119). Under the symmetry through $AA'$ line $BB_1$ turns into $CC_2$ and $BB_2$ into $CC_1$. Since symmetric lines intersect on the axis of symmetry, $AA'$ contains a diagonal of the considered hexagon. Similarly, the remaining diagonals lie on $BB'$ and $CC'$. It is clear that the medians $AA'$, $BB'$, $CC'$ intersect at one point.
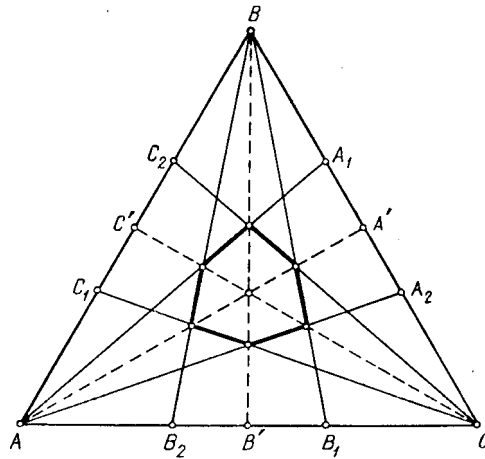
Figure 161 (Sol. 29.12)

**29.13.** Problem 29.6 b) implies that an affine transformation sends an arbitrary parallelogram into a square. Since this preserves the ratio of lengths of parallel segments (Problem 29.5), it suffices to prove the statement of the problem for the case when $ABCD$ is a square. Denote by $P$ the intersection point of lines $b$ and $d$. It suffices to prove that $PC \parallel MK$. Segment $KL$ turns under the rotation through the angle of $90°$ about the center of square $ABCD$ into $LM$, hence, lines $b$ and $d$ which are parallel to these respective segments are perpendicular; hence, $P$ lies on the circle circumscribed about $ABCD$. Then $\angle CPD = \angle CBD = 45°$. Therefore, the angle between lines $CP$ and $b$ is equal to $45°$ but the angle between lines $MK$ and $KL$ is also equal to $45°$ and $b \parallel KL$ implying $CP \parallel MK$.

**29.14.** a) Let us consider an affine transformation that sends triangle $ABC$ into a equilateral triangle $A'B'C'$. Let $O'$, $M'$, $N'$, $P'$ be the images of points $O$, $M$, $N$, $P$. Under the rotation through the angle of $120°$ about point $O'$ triangle $M'N'P'$ turns into itself and, therefore, this triangle is a equilateral one and $O'$ is the intersection point of its medians. Since under an affine transformation any median turns into a median, $O$ is the intersection point of the medians of triangle $MNP$.

b) Solution is similar to the solution of heading a).

**29.15.** Let us consider an affine transformation that sends $ABCD$ into an isosceles trapezoid $A'B'C'D'$. For such a transformation one can take the affine transformation that sends triangle $ADE$, where $E$ is the intersection point of $AB$ and $CD$, into an isosceles triangle. Then the symmetry through the midperpendicular to $A'D'$ sends point $P'$ into point $Q'$, i.e., lines $P'Q'$ and $A'D'$ are parallel.

**29.16.** Any parallelogram $ABCD$ can be translated by an affine transformation into a square (for this we only have to transform triangle $ABC$ into an isosceles right triangle). Since the problem only deals with parallel lines and ratios of segments that lie on one line, we may assume that $ABCD$ is a square. Let us consider a rotation through an angle of $90°$ sending $ABCD$ into itself. This rotation sends quadrilaterals $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ into themselves; hence, the quadrilaterals are also squares. We also have

$$\tan \angle BA_1B_1 = BB_1 : BA_1 = A_1D_2 : A_1A_2 = \tan \angle A_1A_2D_2,$$

i.e., $AB \parallel A_2D_2$ (Fig. 120).

**29.17.** a) Since an affine transformation sends any triangle into a equilateral one, the midpoints of the sides into the midpoints, the centrally symmetric points into centrally symmetric and triangles of the same area into triangles of the same area (Problem 29.11), it follows that we can assume that triangle $ABC$ is an equilateral one with side $a$. Denote
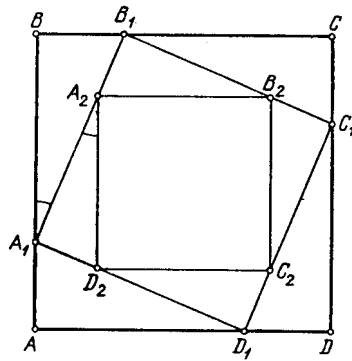
Figure 262 (Sol. 29.16)

the lengths of segments $AM$, $BN$, $CP$ by $p$, $q$, $r$, respectively. Then

$$S_{ABC} - S_{MNP} = S_{AMP} + S_{BMN} + S_{CNP} =$$

$$\frac{1}{2}\sin 60° \cdot (p(a-r) + q(a-p) + r(a-q)) = \frac{1}{2}\sin 60° \cdot (a(p+q+r) - (pq+qr+rp)).$$

Similarly,

$$S_{ABC} - S_{M_1 N_1 P_1} = \frac{1}{2}\sin 60° \cdot (r(a-p) + p(a-q) + q(a-r)) =$$

$$\frac{1}{2}\sin 60° \cdot (a(p+q+r) - (pq+qr+rp)).$$

b) By the same reasons as in heading a) let us assume that $ABC$ is an equilateral triangle. Let $M_2 N_2 P_2$ be the image of triangle $M_1 N_1 P_1$ under the rotation about the center of triangle $ABC$ through the angle of $120°$ in the direction from $A$ to $B$ (Fig. 121).
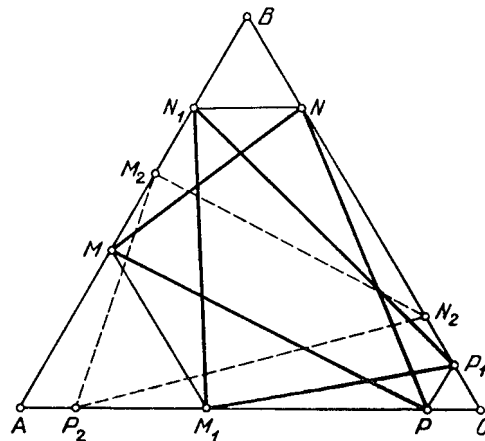


Figure 263 (Sol. 29.17)

Then $AM_2 = CM_1 = BM$. Similarly, $BN_2 = CN$ and $CP_2 = AP$, i.e., points $M_2$, $N_2$, $P_2$ are symmetric to points $M$, $N$, $P$ through the midpoints of the corresponding sides. Therefore, this heading is reduced to heading a).

# Chapter 30. PROJECTIVE TRANSFORMATIONS

## §1. Projective transformations of the line

1. Let $l_1$ and $l_2$ be two lines on the plane, $O$ a point that does not lie on any of these lines. The *central projection* of line $l_1$ to line $l_2$ with center $O$ is the map that to point $A_1$ on line $l_1$ assigns the intersection point of lines $OA_1$ and $l_2$.

2. Let $l_1$ and $l_2$ be two lines on the plane, $l$ a line not parallel to either of the lines. The *parallel projection* of $l_1$ to $l_2$ along $l$ is the map that to point $A_1$ of line $l_1$ assigns the intersection point of $l_2$ with the line passing through $A_1$ parallel to $l$.

3. A map $P$ of line $a$ to line $b$ is called a *projective* one if it is the composition of central or parallel projections, i.e., if there exist lines $a_0 = a$, $a_1$, ..., $a_n = b$ and maps $P_i$ of the line $a_i$ to $a_{i+1}$ each of which is either a central or a parallel projection and $P$ is the composition of the maps $P_i$ in some order. If $b$ coincides with $a$, then $P$ is called a *projective transformation* of line $a$.

**30.1.** Prove that there exists a projective transformation that sends three given points on one line into three given points on another line.

The *cross ratio* of a quadruple of points $A$, $B$, $C$, $D$ lying on one line is the number

$$(ABCD) = \frac{c-a}{c-b} : \frac{d-a}{d-b},$$

where $a$, $b$, $c$, $d$ are the coordinates of points $A$, $B$, $C$, $D$, respectively. It is easy to verify that the cross ratio does not depend on the choice of the coordinate system on the line. We will also write

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD}$$

in the sence that $\frac{AC}{BC}$ (resp. $\frac{AD}{BD}$) denotes the ratio of the lengths of these segments, if vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ (resp. $\overrightarrow{AD}$ and $\overrightarrow{BD}$) are similarly directed or the ratio of the lengths of these segments taken with minus sign, if these vectors are pointed in the opposite directions.

The *double ratio* of the quadruple of lines $a$, $b$, $c$, $d$ passing through one point is the number

$$(abcd) = \pm \frac{\sin(a,c)}{\sin(b,c)} : \frac{\sin(a,d)}{\sin(b,d)}$$

whose sign is determined as follows: if one of the angles formed by lines $a$ and $b$ does not intersect with one of the lines $c$ or $d$ (in this case we say that *the pair of lines $a$ and $b$ does not divide the pair of lines $c$ and $d$*) then $(abcd) > 0$; otherwise $(abcd) < 0$.

**30.2.** a) Given lines $a$, $b$, $c$, $d$ passing through one point and line $l$ that does not pass through this point. Let $A$, $B$, $C$, $D$ be intersection points of $l$ with lines $a$, $b$, $c$, $d$, respectively. Prove that $(abcd) = (ABCD)$.

b) Prove that the double ratio of the quadruple of points is preserved under projective transformations.

**30.3.** Prove that if $(ABCX) = (ABCY)$, then $X = Y$ (all points are assumed to be pairwise distinct except, perhaps, points $X$ and $Y$, and lie on one line).

**30.4.** Prove that any projective transformation of the line is uniquely determined by the image of three arbitrary points.

**30.5.** Prove that any non-identity projective transformation of the line has not more than two fixed points.

**30.6.** A map sends line $a$ into line $b$ and preserves the double ratio of any quadruple of points. Prove that this map is a projective one.

**30.7.** Prove that transformation $P$ of the real line is projective if and only if it can be represented in the form

$$P(x) = \frac{ax + b}{cx + d},$$

where $a$, $b$, $c$, $d$ are numbers such that $ad - bc \neq 0$. (Such maps are called *fractionally-linear* ones.)

**30.8.** Points $A$, $B$, $C$, $D$ lie on one line. Prove that if $(ABCD) = 1$, then either $A = B$ or $C = D$.

**30.9.** Given line $l$, a circle and points $M$, $N$ that lie on the circle and do not lie on $l$. Consider map $P$ of line $l$ to itself,; let $P$ be the composition of the projection of $l$ to the given circle from point $M$ and the projection of the circle to $l$ from point $N$. (If point $X$ lies on line $l$, then $P(X)$ is the intersection of line $NY$ with line $l$, where $Y$ is the distinct from $M$ intersection point of line $MX$ with the given circle.) Prove that $P$ is a projective transformation.

**30.10.** Given line $l$, a circle and point $M$ that lies on the circle and does not lie on $l$, let $P_M$ be the projection map of $l$ to the given circle from point $M$ (point $X$ of line $l$ is mapped into the distinct from $M$ intersection point of line $XM$ with the circle), $R$ the movement of the plane that preserves the given circle (i.e., a rotation of the plane about the center of the circle or the symmetry through a diameter). Prove that the composition $P_M^{-1} \circ R \circ P_M$ is a projective transformation.

REMARK. If we assume that the given circle is identified with line $l$ via a projection map from point $M$, then the statement of the problem can be reformulated as follows: the map of a circle to itself with the help of a movement of the plane is a projective transformation of the line.

## §2. Projective transformations of the plane

Let $\alpha_1$ and $\alpha_2$ be two planes in space, $O$ a point that does not belong to any of these planes. The *central projection map* of $\alpha_1$ to $\alpha_2$ with center $O$ is the map that to point $A_1$ of plane $\alpha_1$ assigns the intersection point of $OA_1$ with plane $\alpha_2$.

**30.11.** Prove that if planes $\alpha_1$ and $\alpha_2$ intersect, then the central projection map of $\alpha_1$ to $\alpha_2$ with center $O$ determines a one-to-one correspondence of plane $\alpha_1$ with deleted line $l_1$ onto plane $\alpha_2$ with deleted line $l_2$, where $l_1$ and $l_2$ are the intersection lines of planes $\alpha_1$ and $\alpha_2$, respectively, with planes passing through $O$ and parallel to $\alpha_1$ and $\alpha_2$. On $l_1$, the map is not defined.

A line on which the central projection map is not defined is called the *singular line of the given projection map*.

**30.12.** Prove that under a central projection a nonsingular line is projected to a line.

In order to define a central projection everywhere it is convenient to assume that in addition to ordinary points every line has one more so-called *infinite point* sometimes denoted by $\infty$. If two points are parallel, then we assume that their infinite points coincide; in other words, parallel lines intersect at their infinite point.

*We will also assume that on every plane in addition to ordinary lines there is one more, infinite line, which hosts all the infinite points of the lines of the plane. The infinite line intersects with every ordinary line $l$ lying in the same plane in the infinite point of $l$.*

*If we introduce infinite points and lines, then the central projection map of plane $\alpha_1$ to plane $\alpha_2$ with center at point $O$ is defined through(?) points of $\alpha_1$ and the singular line is mapped into the infinite line of $\alpha_2$, namely, the image of point $M$ of the singular line is the infinite point of line $OM$; this is the point at which the lines of plane $\alpha_2$ parallel to $OM$ intersect.*

**30.13.** Prove that if together with the usual (finite) points and lines we consider infinite ones, then

a) through any two points only one line passes;

b) any two lines lying in one plane intersect at one point;

c) a central projection map of one plane to another one is a one-to-one correspondence.

*A map $P$ of plane $\alpha$ to plane $\beta$ is called a projective one if it is the composition of central projections and affine transformations, i.e., if there exist planes $\alpha_0 = \alpha$, $\alpha_1$, ..., $\alpha_n = \beta$ and maps $P_i$ of plane $\alpha_i$ to $\alpha_{i+1}$ each of which is either a central projection or an affine transformation and $P$ is the composition of the $P_i$. If plane $\alpha$ coincides with $\beta$, map $P$ is called a projective transformation of $\alpha$. The preimage of the infinite line will be called the singular line of the given projective transformation.*

**30.14.** a) Prove that a projective transformation $P$ of the plane sending the infinite line into the infinite line is an affine transformation.

b) Prove that if points $A$, $B$, $C$, $D$ lie on a line parallel to the singular line of a projective transformation $P$ of plane $\alpha$, then $P(A)P(B) : P(C)P(D) = AB : CD$.

c) Prove that if a projective transformation $P$ sends parallel lines $l_1$ and $l_2$ into parallel lines, then either $P$ is affine or its singular line is parallel to $l_1$ and $l_2$.

d) Let $P$ be a one-to-one transformation of the set of all finite and infinite points of the plane, let $P$ send every line into a line. Prove that $P$ is a projective map.

**30.15.** Given points $A$, $B$, $C$, $D$ no three of which lie on one line and points $A_1$, $B_1$, $C_1$, $D_1$ with the same property.

a) Prove that there exists a projective transformation sending points $A$, $B$, $C$, $D$ to points $A_1$, $B_1$, $C_1$, $D_1$, respectively.

b) Prove that the transformation from heading a) is unique, i.e., any projective transformation of the plane is determined by the images of four generic points (cf. Problem 30.4).

c) Prove statement of heading a) if points $A$, $B$, $C$ lie on one line $l$ and points $A_1$, $B_1$, $C_1$, $D_1$ on one line $l_1$.

d) Is transformation from heading c) unique?

*In space, consider the unit sphere with center in the origin. Let $N(0, 0, 1)$ be the sphere's north pole. The stereographic projection of the sphere to the plane is the map that to every point $M$ of the sphere assigns distinct from $N$ intersection point of line $MN$ with plane $Oxy$. It is known (see, for example, Solid Problem 16.19 b)) that the stereographic projection sends a circle on the sphere into a circle in plane. Make use of this fact while solving the following two problems:*

**30.16.** Givin a circle and a point inside it.

a) Prove that there exists a projective transformation that sends the given circle into a circle and the given point into the center of the given circle's image.

b) Prove that if a projective transformation sends the given circle into a circle and point $M$ into the center of the given circle's image, then the singular line of this transformation is perpendicular to a diameter through $M$.

**30.17.** In plane, there are given a circle and a line that does not intersect the circle. Prove that there exists a projective transformation sending the given circle into a circle and the given line into the infinite line.

**30.18.** Given a circle and a chord in it. Prove that there exists a projective transformation that sends the given circle into a circle and the given chord into the diameter of the given circle's image.

**30.19.** Given circle $S$ and point $O$ inside it, consider all the projective maps that send $S$ into a circle and $O$ into the center of the image of $S$. Prove that all such transformations map one and the same line into the infinite line.

The preimage of the infinite line under the above transformations is called *the polar line* of point $O$ relative circle $S$.

**30.20.** A projective transformation sends a circle into itself so that its center is fixed. Prove that this transformation is either a rotation or a symmetry.

**30.21.** Given point $O$ and two parallel lines $a$ and $b$. For every point $M$ we perform the following construction. Through $M$ draw a line $l$ not passing through $O$ and intersecting lines $a$ and $b$. Denote the intersection points pf $l$ with $a$ and $b$ by $A$ and $B$, respectively, and let $M'$ be the intersection point of $OM$ with the line parallel to $OB$ and passing through $A$.

a) Prove that point $M'$ does not depend on the choice of line $l$.

b) Prove that the transformation of the plane sending $M$ into $M'$ is a projective one.

**30.22.** Prove that the transformation of the coordinate plane that every point with coordinates $(x, y)$ sends into the point with coordinates $(\frac{1}{x}, \frac{y}{x})$ is a projective one.

(?)**30.23.** Let $O$ be the center of a lens, $\pi$ a plane passing through the optic axis $a$ of the lens, $a$ and $f$ the intersection lines of $\pi$ with the plane of the lens and the focal plane, respectively, $(a \parallel f)$. In the school course of physics it is shown that if we neglect the lens, then the image $M'$ of point $M$ that lies in plane $\pi$ is constructed as follows, see Fig. 122.
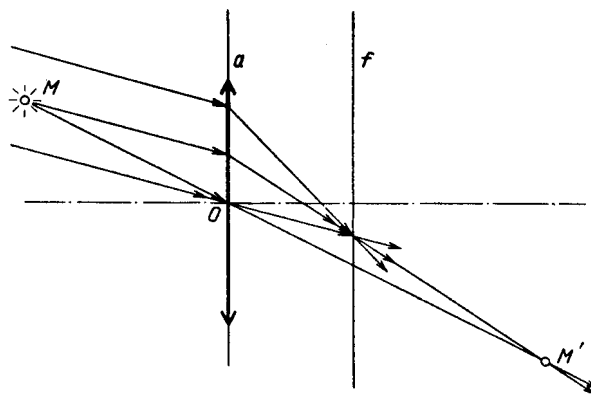


Figure 264 (30.23)

Through point $M$ draw an arbitrary line $l$; let $A$ be the intersection point of lines $a$ and $l$, let $B$ be the intersection point of $f$ with the line passing through $O$ parallel to $l$. Then $M'$ is defined as the intersection point of lines $AB$ and $OM$.

Prove that the transformation of plane $\pi$ assigning to every of its points its image is a projective one.

Thus, through a magnifying glass we can see the image of our world thanks to projective transformations.

## §3. Let us transform the given line into the infinite one

**30.24.** Prove that the locus of the intersection points of quadrilaterals $ABCD$ whose sides $AB$ and $CD$ belong to two given lines $l_1$ and $l_2$ and sides $BC$ and $AD$ intersect at a given point $P$ is a line passing through the intersection point $Q$ of lines $l_1$ and $l_2$.

**30.25.** Let $O$ be the intersection point of the diagonals of quadrilateral $ABCD$; let $E$ (resp. $F$) be the intersection point of the continuations of sides $AB$ and $CD$ (resp. $BC$ and $AD$). Line $EO$ intersects sides $AD$ and $BC$ at points $K$ and $L$, respectively, and line $FO$ intersects sides $AB$ and $CD$ at points $M$ and $N$, respectively. Prove that the intersection point $X$ of lines $KN$ and $LM$ lies on line $EF$.

**30.26.** Lines $a$, $b$, $c$ intersect at one point $O$. In triangles $A_1B_1C_1$ and $A_2B_2C_2$, vertices $A_1$ and $A_2$ lie on line $a$; $B_1$ and $B_2$ lie on line $b$; $C_1$ and $C_2$ lie on line $c$. Let $A$, $B$, $C$ be the intersection points of lines $B_1C_1$ and $B_2C_2$, $C_1A_1$ and $C_2A_2$, $A_1B_1$ and $A_2B_2$, respectively. Prove that points $A$, $B$, $C$ lie on one line (*Desargue's theorem.*)

**30.27.** Points $A$, $B$, $C$ lie on line $l$ and points $A_1$, $B_1$, $C_1$ on line $l_1$. Prove that the intersection points of lines $AB_1$ and $BA_1$, $BC_1$ and $CB_1$, $CA_1$ and $AC_1$ lie on one line (*Pappus's theorem.*)

**30.28.** Given convex quadrilateral $ABCD$. Let $P$, $Q$ be the intersection points of the continuations of the opposite sides $AB$ and $CD$, $AD$ and $BC$, respectively, $R$ an arbitrary point inside the quadrilateral. Let $K$, $L$, $M$ be the intersection point of lines $BC$ and $PR$, $AB$ and $QR$, $AK$ and $DR$, respectively. Prove that points $L$, $M$ and $C$ lie on one line.

**30.29.** Given two triangles $ABC$ and $A_1B_1C_1$ so that lines $AA_1$, $BB_1$ and $CC_1$ intersect at one point $O$ and lines $AB_1$, $BC_1$ and $CA_1$ intersect at one point $O_1$. Prove that lines $AC_1$, $BA_1$ and $CB_1$ also intersect at one point $O_2$. (*Theorem on doubly perspective triangles.*)

**30.30.** Given two triangles $ABC$ and $A_1B_1C_1$ so that lines $AA_1$, $BB_1$ and $CC_1$ intersect at one point $O$, lines $AA_1$, $BC_1$ and $CB_1$ intersect at one point $O_1$ and lines $AC_1$, $BB_1$ and $CA_1$ intersect at one point $O_2$, prove that lines $AB_1$, $BA_1$ and $CC_1$ also intersect at one point $O_3$. (*Theorem on triply perspective triangles.*)

**30.31.** Prove that the orthocenters of four triangles formed by four lines lie on one line.

**30.32.** Given quadrilateral $ABCD$ and line $l$. Denote by $P$, $Q$, $R$ the intersection points of lines $AB$ and $CD$, $AC$ and $BD$, $BC$ and $AD$, respectively. Denote by $P_1$, $Q_1$, $R_1$ the midpoints of the segments which these pairs of lines cut off line $l$. Prove that lines $PP_1$, $QQ_1$ and $RR_1$ intersect at one point.

**30.33.** Given triangle $ABC$ and line $l$. Denote by $A_1$, $B_1$, $C_1$ the midpoints of the segments cut off line $l$ by angles $\angle A$, $\angle B$, $\angle C$ and by $A_2$, $B_2$, $C_2$ the intersection points of lines $AA_1$ and $BC$, $BB_1$ and $AC$, $CC_1$ and $AB$, respectively. Prove that points $A_2$, $B_2$, $C_2$ lie on one line.

**30.34.** (*Theorem on a complete quadrilateral.*) Given four points $A$, $B$, $C$, $D$ and the intersection points $P$, $Q$, $R$ of lines $AB$ and $CD$, $AD$ and $BC$, $AC$ and $BD$, respectively; the intersection points $K$ and $L$ of line $QR$ with lines $AB$ and $CD$, respectively. Prove that $(QRKL) = -1$.

**30.35.** Is it possible to paint 1991 points of the plane red and 1991 points blue so that any line passing through two points of distinct colour contains one more of coloured points? (We assume that coloured points are distinct and do not belong to one line.)

## §4. Application of projective maps that preserve a circle

The main tools in the solution of problems of this section are the results of Problems 30.16 and 30.17.

**30.36.** Prove that the lines that connect the opposite tangent points of a circumscribed quadrilateral pass through the intersection point of the diagonals of this quadrilateral.

**30.37.** Consider a triangle and the inscribed circle. Prove that the lines that connect the triangle's vertices with the tangent points of the opposite sides intersect at one point.

**30.38.** a) Through point $P$ all secants of circle $S$ are drawn. Find the locus of the intersection points of the tangents to $S$ drawn through the two intersection points of $S$ with every secant.

b) Through point $P$ the secants $AB$ and $CD$ of circle $S$ are drawn, where $A$, $B$, $C$, $D$ are the intersection points of the secants with the circle. Find the locus of the intersection points of $AC$ and $BD$.

**30.39.** Given circle $S$, line $l$, point $M$ on $S$ and not on $l$ and point $O$ not on $S$. Consider a map $P$ of line $l$ which is the composition of the projection map of $l$ to $S$ from $M$, of $S$ to itself from $O$ and $S$ to $l$ from $M$, i.e., for any point $A$ point $P(A)$ is the intersection point of lines $l$ and $MC$, where $C$ is the distinct from $B$ intersection point of $S$ with line $OB$ and $B$ is the distinct from $A$ intersection point of $S$ with line $MA$. Prove that $P$ is a projective map.

REMARK. If we assume that a projection map from point $M$ identifies circle $S$ with line $l$, then the statement of the problem can be reformulated as follows: every central projection of a circle to itself is a projective transformation.

**30.40.** Consider disk $S$, point $P$ outside $S$ and line $l$ passing through $P$ and intersecting the circle at points $A$ and $B$. Denote the intersection point of the tangents to the disk at points $A$ and $B$ by $K$.

a) Consider all the lines passing through $P$ and intersecting $AK$ and $BK$ at points $M$ and $N$, respectively. Prove that the locus of the tangents to $S$ drawn through $M$ and $N$ and distinct from $AK$ and $BK$ is a line passing through $K$ and having the empty intersection with the interior of $S$.

b) Let us select various points $R$ on the circle and draw the line that connects the distinct from $R$ intersection points of lines $RK$ and $RP$ with $S$. Prove that all the obtained lines pass through one point and this point belongs to $l$.

**30.41.** An escribed circle of triangle $ABC$ is tangent to side $BC$ at point $D$ and to the extensions of sides $AB$ and $AC$ at points $E$ and $F$, respectively. Let $T$ be the intersection point of lines $BF$ and $CE$. Prove that points $A$, $D$ and $T$ lie on one line.

**30.42.** Let $ABCDEF$ be a circumscribed hexagon. Prove that its diagonals $AD$, $BE$ and $CF$ intersect at one point. (*Brianchon's theorem.*)

**30.43.** Hexagon $ABCDEF$ is inscribed in circle $S$. Prove that the intersection points of lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$ lie on one line. (*Pascal's theorem.*)

**30.44.** Let $O$ be the midpoint of chord $AB$ of circle $S$, let $MN$ and $PQ$ be arbitrary chords through $O$ such that points $P$ and $N$ lie on one side of $AB$; let $E$ and $F$ be the intersection points of chord $AB$ with chords $MP$ and $NQ$, respectively. Prove that $O$ is the midpoint of segment $EF$. (*The butterfly problem.*)

**30.45.** Points $A$, $B$, $C$ and $D$ lie on a circle, $SA$ and $SD$ are tangents to this circle, $P$ and $Q$ are the intersection points of lines $AB$ and $CD$, $AC$ and $BD$, respectively. Prove that points $P$, $Q$ and $S$ lie on line line.

## §5. Application of projective transformations of the line

**30.46.** On side $AB$ of quadrilateral $ABCD$ point $M_1$ is taken. Let $M_2$ be the projection of $M_1$ to line $BC$ from $D$, let $M_3$ be the projection of $M_2$ to $CD$ from $A$, $M_4$ the projection of $M_3$ on $DA$ from $B$, $M_5$ the projection of $M_4$ to $AB$ from $C$, etc. Prove that $M_{13} = M_1$ (hence, $M_{14} = M_2$, $M_{15} = M_3$, etc.).

**30.47.** Making use of projective transformations of the line prove the theorem on a complete quadrilateral (Problem 30.34).

**30.48.** Making use of projective transformations of the line prove Pappus's theorem (Problem 30.27).

**30.49.** Making use of projective transformations of the line prove the butterfly problem (Problem 30.44).

**30.50.** Points $A$, $B$, $C$, $D$, $E$, $F$ lie on one circle. Prove that the intersection points of lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$ lie on one line. (*Pascal's theorem.*)

**30.51.** Given triangle $ABC$ and point $T$, let $P$ and $Q$ be the bases of perpendiculars dropped from point $T$ to lines $AB$ and $AC$, respectively; let $R$ and $S$ be the bases of perpendiculars dropped from point $A$ to lines $TC$ and $TB$, respectively. Prove that the intersection point $X$ of lines $PR$ and $QS$ lies on line $BC$.

## §6. Application of projective transformations of the line in problems on construction

**30.52.** Given a circle, a line, and points $A$, $A'$, $B$, $B'$, $C$, $C'$, $M$ on this line. By Problems 30.1 and 30.3 there exists a unique projective transformation of the given line to itself that maps points $A$, $B$, $C$ into $A'$, $B'$, $C'$, respectively. Denote this transformation by $P$. Construct with the help of a ruler only a) point $P(M)$; b) fixed points of map $P$. (*J. Steiner's problem.*)

The problem of constructing fixed points of a projective transformation is the key one for this section in the sense that all the other problems can be reduced to it, cf. also remarks after Problems 30.10 and 30.39.

**30.53.** Given two lines $l_1$ and $l_2$, two points $A$ and $B$ not on these lines, and point $E$ of line $l_2$. Construct with a ruler and compass point $X$ on $l_1$ such that lines $AX$ and $BX$ intercept on line $l_2$ a segment a) of given length $a$; b) divisible in halves by $E$.

**30.54.** Points $A$ and $B$ lie on lines $a$ and $b$, respectively, and point $P$ does not lie on any of these lines. With the help of a ruler and compass draw through $P$ a line that intersects lines $a$ and $b$ at points $X$ and $Y$, respectively, so that the lengths of segments $AX$ and $BY$ a) are of given ratio; b) have a given product.

**30.55.** With the help of a ruler and compass draw through a given point a line on which three given lines intercept equal segments.

**30.56.** Consider a circle $S$, two chords $AB$ and $CD$ on it, and point $E$ of chord $CD$. Construct with a ruler and compass point $X$ on $S$ so that lines $AX$ and $BX$ intercept on $CD$ a segment a) of given length $a$; b) divided in halves by $E$.

**30.57.** a) Given line $l$, point $P$ outside it, a given length, and a given angle $\alpha$. Construct with a ruler and compass segment $XY$ on $l$ of the given length and subtending an angle of value $\alpha$ and with vertex in $P$.

b) Given two lines $l_1$ and $l_2$, points $P$ and $Q$ outside them, and given angles $\alpha$ and $\beta$. Construct with the help of a ruler and compass point $X$ on $l_1$ and point $Y$ on $l_2$ such that segment $XY$ subtends an angle of value $\alpha$ with vertex in $P$ and another angle equal to $\beta$ with vertex in $Q$.

**30.58.** a) Given a circle, $n$ points and $n$ lines. Construct with the help of a ruler only an $n$-gon whose sides pass through the given points and whose vertices lie on the given lines.

b) With the help of ruler only inscribe in the given circle an $n$-gon whose sides pass through $n$ given points.

c) With the help of a ruler and compass inscribe in a given circle a polygon certain sides of which pass through the given points, certain other sides are parallel to the given lines and the remaining sides are of prescribed lengths (about each side we have an information of one of the above three types).

## §7. Impossibility of construction with the help of a ruler only

**30.59.** Prove that with the help of a ruler only it is impossible to divide a given segment in halves.

**30.60.** Given a circle on the plane, prove that its center is impossible to construct with the help of a ruler only.

## Solutions

**30.1.** Denote the given lines by $l_0$ and $l$, the given points on $l_0$ by $A_0$, $B_0$, $C_0$ and the given points on $l$ by $A$, $B$, $C$. Let $l_1$ be an arbitrary line not passing through $A$. Take an arbitrary point $O_0$ not on lines $l_0$ and $l_1$. Denote by $P_0$ the central projection map of $l_0$ to $l_1$ with center at $O_0$ and by $A_1$, $B_1$, $C_1$ the projections of points $A_0$, $B_0$, $C_0$, respectively, under $P_0$.

Let $l_2$ be an arbitrary line through point $A$ not coinciding with $l$ and not passing through $A_1$. Take point $O_1$ on line $AA_1$ and consider the central projection map $P_1$ of $l_1$ to $l_2$ with center at $O_1$. Denote by $A_2$, $B_2$, $C_2$ the projections of points $A_1, B_1, C_1$, respectively, under $P_1$. Clearly, $A_2$ coincides with $A$.

Finally, let $P_2$ be the projection map of $l_2$ to $l$ which in the case when lines $BB_2$ and $CC_2$ are not parallel is the central projection with center at the intersection point of these lines; if lines $BB_2$ and $CC_2$ are parallel this is the parallel projection along either of these lines.

The composition $P_2 \circ P_1 \circ P_0$ is the required projective transformation.

**30.2.** a) Denote the intersection point of the four given lines by $O$; let $H$ be the projection of $H$ on $l$ and $h = OH$. Then

$$
\begin{aligned}
2S_{OAC} &= OA \cdot OC \sin(a,c) = h \cdot AC, \\
2S_{OBC} &= OB \cdot OC \sin(b,c) = h \cdot BC, \\
2S_{OAD} &= OA \cdot OD \sin(a,d) = h \cdot AD, \\
2S_{OBD} &= OB \cdot OD \sin(b,d) = h \cdot BD.
\end{aligned}
$$

Dividing the first equality by the second one and the third one by the fourth one we get

$$
\frac{OA \sin(a,c)}{OB \sin(b,c)} = \frac{AC}{BC}, \qquad \frac{OA \sin(a,d)}{OB \sin(b,d)} = \frac{AD}{BD}.
$$

Dividing the first of the obtained equalities by the second one we get $|(ABCD)| = |(abcd)|$. To prove that the numbers $(ABCD)$ and $(abcd)$ are of the same sign, we can, for example, write down all the possible ways to arrange points on the line (24 ways altogether) and verify case by case that $(ABCD)$ is positive if and only if the pair of lines $a$, $b$ does not separate the pair of lines $c$, $d$.

b) follows immediately from heading a).

**30.3.** Let $a$, $b$, $c$, $x$, $y$ be the coordinates of points $A$, $B$, $C$, $X$, $Y$, respectively. Then

$$\frac{x-a}{x-b} : \frac{c-a}{c-b} = \frac{y-a}{y-b} : \frac{c-a}{c-b}.$$

Therefore, since all the points are distinct, $(x-a)(y-b) = (x-b)(y-a)$. By simplifying we get $ax - bx = ay - by$. Dividing this equality by $a - b$ we get $x = y$.

**30.4.** Let the image of each of the three given points under one projective transformation coincide with the image of this point under another projective transformation. Let us prove then that the images of any other point under these transformations coincide. Let us denote the images of the given points by $A$, $B$, $C$. Take an arbitrary point and denote by $X$ and $Y$ its images under the given projective transformations. Then by Problem 30.2 $(ABCX) = (ABCY)$ and, therefore, $X = Y$ by Problem 30.3.

**30.5.** This problem is a corollary of the preceding one.

**30.6.** On line $a$, fix three distinct points. By Problem 30.1 there exists a projective map $P$ which maps these points in the same way as the given map. But in the solution of Problem 30.4 we actually proved that any map that preserves the cross ratio is uniquely determined by the images of three points. Therefore, the given map coincides with $P$.

**30.7.** First, let us show that the fractionally linear transformation

$$P(x) = \frac{ax+b}{cx+d}, \quad ad - bc \neq 0$$

preserves the cross ratio. Indeed, let $x_1$, $x_2$, $x_3$, $x_4$ be arbitrary numbers and $y_i = P(x_i)$. Then

$$y_i - y_j = \frac{ax_i + b}{cx_i + d} - \frac{ax_j + b}{cx_j + d} = \frac{(ad - bc)(x_i - x_j)}{(cx_i + d)(cx_j + d)};$$

hence, $(y_1 y_2 y_3 y_4) = (x_1 x_2 x_3 x_4)$.

In the solution of Problem 30.4 we have actually proved that if a transformation of the line preserves the cross ratio, then it is uniquely determined by the images of three arbitrary distinct points. By Problem 30.2 b) projective transformations preserve the cross ratio. It remains to prove that for any two triples of pairwise distinct points $x_1$, $x_2$, $x_3$ and $y_1$, $y_2$, $y_3$ there exists a fractionally linear transformation $P$ such that $P(x_i) = y_i$.

For this, in turn, it suffices to prove that for any three pairwise distinct points there exists a fractionally linear transformation that sends them into points $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$.

Indeed, if $P_1$ and $P_2$ be fractionally linear transformations such that $P_1(x_i) = z_i$ and $P_2(y_i) = z_i$, then $P_2^{-1}(P_1(x_i)) = y_i$. The inverse to a fractionally linear transformation is a fractionally linear transformation itself because if $y = \frac{ax+c}{cx+d}$, then $x = \frac{dy-b}{-cy+a}$; the verification of the fact that the composition of fractionally linear transformations is a fractionally linear transformation is left for the reader.

Thus, we have to prove that if $x_1$, $x_2$, $x_3$ are arbitrary distinct numbers, then there exist numbers $a$, $b$, $c$, $d$ such that $ad - bc \neq 0$ and

$$ax_1 + b = 0, \quad ax_2 + b = cx_2 + d, \quad cx_3 + d = 0.$$

Find $b$ and $d$ from the first and third equations and substitute the result into the third one; we get

$$a(x_2 - x_1) = c(x_2 - x_3)$$

wherefrom we find the solution: $a = (x_2 - x_3)$, $b = x_1(x_3 - x_2)$, $c = (x_2 - x_1)$, $d = x_3(x_1 - x_2)$. We, clearly, have $ad - bc = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \neq 0$.

**30.8. First solution.** Let $a$, $b$, $c$, $d$ be the coordinates of the given points. Then by the hypothesis $(c-a)(d-b) = (c-b)(d-a)$. After simplification we get $cb + ad = ca + bd$.

Transfer everything to the left-hand side and factorize; we get $(d-c)(b-a) = 0$, i.e., either $a = b$ or $c = d$.

**Second solution.** Suppose that $C \neq D$, let us prove that in this case $A = B$. Consider the central projection map of the given line to another line, let the projection send point $D$ into $\infty$. Let $A'$, $B'$, $C'$ be the projections of points $A$, $B$, $C$, respectively. By Problem 30.2 $(ABCD) = (A'B'C'\infty) = 1$, i.e., $\overrightarrow{AC} = \overrightarrow{BC}$. But this means that $A = B$.

**30.9.** By Problem 30.6 it suffices to prove that the map $P$ preserves the cross ratio. Let $A$, $B$, $C$, $D$ be arbitrary points on line $l$. Denote by $A'$, $B'$, $C'$, $D'$ their respective images under $P$ and by $a$, $b$, $c$, $d$ and $a'$, $b'$, $c'$, $d'$ the lines $MA$, $MB$, $MC$, $MD$ and $NA'$, $NB'$, $NC'$, $ND'$, respectively. Then by Problem 30.2 a) we have $(ABCD) = (abcd)$ and $(A'B'C'D') = (a'b'c'd')$ and by the theorem on an inscribed angle $\angle(a, c) = \angle(a', c')$, $\angle(b, c) = \angle(b', c')$, etc.; hence, $(abcd) = (a'b'c'd')$.

**30.10.** Let $N = R^{-1}(M)$, $m = R(l)$, $P_N$ be the projection map of $l$ to the circle from point $N$, $Q$ the projection map of line $m$ to $l$ from point $M$. Then $P_M^{-1} \circ R \circ P_M = Q \circ R \circ P_N^{-1} \circ P_M$. But by the preceding problem the map $P_N^{-1} \circ P_M$ is a projective one.

**30.11.** Lines passing through $O$ and parallel to plane $\alpha_1$ (resp. $\alpha_2$) intersect plane $\alpha_2$ (resp. $\alpha_1$) at points of line $l_2$ (resp. $l_1$). Therefore, if a point lies on one of the planes $\alpha_1$, $\alpha_2$ and does not lie on lines $l_1$, $l_2$, then its projection to another plane is well-defined. Clearly, the distinct points have distinct images.

**30.12.** The central projection to plane $\alpha_2$ with center $O$ sends line $l$ into the intersection of the plane passing through $O$ and $l$ with $\alpha_2$.

**30.13.** This problem is a direct corollary of the axioms of geometry and the definition of infinite lines and points.

**30.14.** a) Problem 30.13 c) implies that if together with the ordinary (finite) points we consider infinite ones, then $P$ is a one-to-one correspondence. Under such an assumption the infinite line is mapped to the infinite line. Therefore, the set of finite points is also mapped one-to-one to the set of finite points. Since $P$ sends lines into lines, $P$ is an affine map.

b) Denote by $l$ the line on which points $A$, $B$, $C$, $D$ lie and by $l_0$ the singular line of map $P$. Take an arbitrary point $O$ outside plane $\alpha$ and consider plane $\beta$ that passes through line $l$ and is parallel to the plane passing through line $l_0$ and point $O$. Let $Q$ be the composition of the central projection of $\alpha$ on $\beta$ with center $O$ with the subsequent rotation of the space about axis $l$ that sends $\beta$ into $\alpha$. The singular line of map $Q$ is $l_0$.

Therefore, the projective transformation $R = P \circ Q^{-1}$ of $\alpha$ sends the infinite line into the infinite line and by heading a) is an affine transformation, in particular, it preserves the ratio of segments that lie on line $l$. It only remains to notice that transformation $Q$ preserves the points of line $l$.

c) The fact that the images of parallel lines $l_1$ and $l_2$ are parallel lines means that the infinite point $A$ of these lines turns into an infinite point, i.e., $A$ lies on the preimage $l$ of the infinite line. Therefore, either $l$ is the infinite line and then by heading a) $P$ is an affine transformation or $l$ is parallel to lines $l_1$ and $l_2$.

d) Denote by $l_\infty$ the infinite line. If $P(l_\infty) = l_\infty$, then $P$ determines a one-to-one transformation of the plane that sends every line into a line and, therefore, by definition is an affine one.

Otherwise denote $P(l_\infty)$ by $a$ and consider an arbitrary projective transformation $Q$ for which $a$ is the singular line. Denote $Q \circ P$ by $R$. Then $R(l_\infty) = l_\infty$ and, therefore, as was shown above, $R$ is an affine map. Hence, $P = Q^{-1} \circ R$ is a projective map.

**30.15.** a) It suffices to prove that points $A$, $B$, $C$, $D$ can be transformed by a projective transformation into vertices of a square. Let $E$ and $F$ be (perhaps, infinite) intersection

points of line $AB$ with line $CD$ and $BC$ with $AD$, respectively. If line $EF$ is not infinite, then there exists a central projection of plane $ABC$ to a plane $\alpha$ for which $EF$ is the singular line. For the center of projection one may take an arbitrary point $O$ outside plane $ABC$ and for plane $\alpha$ an arbitrary plane parallel to plane $OEF$ and not coinciding with it. This projection maps points $A$, $B$, $C$, $D$ into the vertices of a parallelogram which can be now transformed into a square with the help of an affine transformation.

If line $EF$ is an infinite one, then $ABCD$ is already a parallelogram.

b) Thanks to heading a) it suffices to consider the case when $ABCD$ and $A_1B_1C_1D_1$ is one and the same parallelogram. In this case its vertices are fixed and, therefore, two points on an infinite line in which the extensions of the opposite sides of the parallelogram intersect are also fixed. Hence, by Problem 30.14 a) the map should be an affine one and, therefore, by Problem 20.6 the identity one.

c) Since with the help of a projection we can send lines $l$ and $l_1$ into the infinit line (see the solution of heading a)), it suffices to prove that there exists an affine transformation that maps every point $O$ into a given point $O_1$ and lines parallel to given lines $a$, $b$, $c$ into lines parallel to given lines $a_1$, $b_1$, $c_1$, respectively.

We may assume that lines $a$, $b$, $c$ pass through $O$ and lines $a_1$, $b_1$, $c_1$ pass through $O_1$. On $c$ and $c_1$, select arbitrary points $C$ and $C_1$, respectively, and draw through each of them two lines $a'$, $b'$ and $a_1'$, $b_1'$ parallel to lines $a$, $b$ and $a_1$, $b_1$, respectively. Then the affine transformation that sends the parallelogram bounded by lines $a$, $a'$, $b$, $b'$ into the parallelogram bounded by lines $a_1$, $a_1'$, $b_1$, $b_1'$ (see Problem 29.6 c)) is the desired one.

d) Not necessarily. The transformation from Problem 30.21 (as well as the identity transformation) preserves point $O$ and line $a$.

**30.16.** a) On the coordinate plane $Oxz$ consider points $O(0,0)$, $N(0,1)$, $E(1,0)$. For an arbitrary point $M$ that lies on arc $\smile NE$ of the unit circle (see Fig. 123), denote by $P$ the midpoint of segment $EM$ and by $M^*$ and $P^*$ the intersection points of lines $NM$ and $NP$, respectively, with line $OE$.
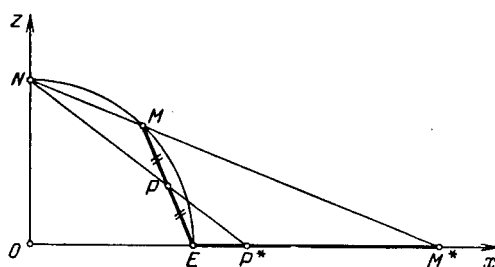


Figure 265 (Sol. 30.16)

Let us prove that for an arbitrary number $k > 2$ we can select point $M$ soy that $M^*E : P^*E = k$. Let $A(a,b)$ be an arbitrary point on the plane, $A^*(t,0)$ the intersection point of lines $NA$ and $OE$, $B(0,b)$ the projection of point $A$ to line $ON$. Then

$$t = \frac{A^*O}{ON} = \frac{AB}{BN} = \frac{a}{1-b}.$$

Therefore, if $(x, z)$ are coordinates of point $M$, then points $P$, $M^*$, $P^*$ have coordinates

$$P\left(\frac{x+1}{2}, \frac{z}{2}\right), \quad M^*\left(\frac{x}{1-z}, 0\right), \quad P^*\left(\frac{(x+1)/2}{1-(z/2)}, 0\right),$$

respectively, and, therefore,

$$M^*E : P^*E = \left(\frac{x}{1-z} - 1\right) : \left(\frac{x+1}{2-z} - 1\right) = \frac{x+z-1}{1-z} : \frac{x+z-1}{2-z} = \frac{2-z}{1-z}.$$

Clearly, the solution of the equation $\frac{2-z}{1-z} = k$ is $z = \frac{k-2}{k-1}$ and, if $k > 2$, then $0 < z < 1$ and, therefore, point $M(\sqrt{1-z^2}, z)$ is the desired one.

Now, let us prove the main statement of the problem. Denote the given circle and point inside it, respectively, by $S$ and $C$. If point $C$ is the center of $S$, then the identity transformation is the desired projective transformation.

Therefore, let us assume that $C$ is not the center. Denote by $AB$ the diameter that contains point $C$. Let, for definiteness, $BC > CA$. Set $k = BA : AC$. Then $k > 2$ and, therefore, as was proved, we can place point $M$ on the unit circle in plane $Oxz$ so that $M^*E : P^*E = k = BA : CA$. Therefore, by a similarity transformation we can translate circle $S$ into a circle $S_1$ constructed in plane $Oxy$ with segment $EM^*$ as a diameter so that the images of points $A$, $B$, $C$ are $E$, $M^*$, $P^*$, respectively.

The stereographic projection maps $S_1$ into circle $S_2$ on the unit sphere symmetric through plane $Oxz$; hence, through line $EM$ as well. Thus, $EM$ is a diameter of $S_2$ and the midpoint $P$ of $EM$ is the center of $S_2$.

Let $\alpha$ be the plane containing circle $S_2$. Clearly, the central projection of plane $Oxy$ to plane $\alpha$ from the north pole of the unit sphere sends $S_1$ into $S_2$ and point $P^*$ into the center $P$ of $S_2$.

b) The diameter $AB$ passing through $M$ turns into a diameter. Therefore, the tangents at points $A$ and $B$ turn into tangents. But if the parallel lines pass into parallel lines, then the singular line is parallel to them (see Problem 30.14 c)).

**30.17.** On the coordinate plane $Oxz$ consider points $O(0,0)$, $N(0,1)$, $E(1,0)$. For an arbitrary point $M$ on arc $\smile NE$ of the unit circle denote by $P$ the intersection of segment $EM$ with line $z = 1$. Clearly, by moving point $M$ along arc $NE$ we can make the ratio $EM : MP$ equal to an arbitrary number. Therefore, a similarity transformation can send the given circle $S$ into circle $S_1$ constructed on segment $EM$ as on diameter in plane $\alpha$ perpendicular to $Oxz$ so that the given line $l$ turns into the line passing through $P$ perpendicularly to $Oxz$. Circle $S_1$ lies on the unit sphere with the center at the origin and, therefore, the stereographic projection sends $S_1$ to circle $S_2$ in plane $Oxy$. Thus, the central projection of plane $\alpha$ to plane $Oxy$ from $N$ sends $S_1$ to $S_2$ and line $l$ into the infinite line.

**30.18.** Let $M$ be an arbitrary point on the given chord. By Problem 30.16 there exists a projective transformation that sends the given circle into a circle $S$ and point $M$ into the center of $S$. Since under a projective transformation a line turns into a line, the given chord will turn into a diameter.

**30.19.** Let us pass through point $O$ two arbitrary chords $AC$ and $BD$. Let $P$ and $Q$ be the intersection points of the extensions of opposite sides of quadrilateral $ABCD$. Consider an arbitrary projective transformation that maps $S$ into a circle, $S_1$, and $O$ into the center of $S_1$. It is clear that this transformation sends quadrilateral $ABCD$ into a rectangle and, therefore, it sends line $PQ$ into the infinite line.

**30.20.** A projective transformation sends any line into a line and since the center is fixed, every diameter turns into a diameter. Therefore, every infinite point — the intersection point of the lines tangent to the circle in diametrically opposite points — turns into an infinite point. Therefore, by Problem 30.14 a) the given transformation is an affine one and by Problem 29.12 it is either a rotation or a symmetry.

**30.21.** a) Point $M'$ lies on line $OM$ and, therefore, its position is uniquely determined by the ratio $MO : OM'$. But since triangles $MBO$ and $MAM'$ are similar, $MO : OM' =$

$MB : BA$ and the latter relation does not depend on the choice of line $l$ due to Thales' theorem.

b) **First solution.** If we extend the given transformation (let us denote it by $P$) by defining it at point $O$ setting $P(O) = O$, then, as is easy to verify, $P$ determines a one-to-one transformation of the set of all finite and infinite points of the plane into itself. (In order to construct point $M$ from point $M'$ we have to take an arbitrary point $A$ on line $a$ and draw lines $AM'$, $OB$ so that it is parallel to $AM'$, and $AB$.) It is clear that every line passing through $O$ turns into itself. Every line $l$ not passing through $O$ turns into the line parallel to $OB$ and passing through $M$. Now, it only remains to make use of Problem 30.14 c).

**Second solution** (sketch). Denote the given plane by $\pi$ and let $\pi' = R(\pi)$, where $R$ is a rotation of the space about axis $a$. Denote $R(O)$ by $O'$ and let $P$ be the projection map of plane $\pi$ to plane $\pi'$ from the intersection point of line $OO'$ with the plane passing through $b$ parallel to $\pi'$. Then $R^{-1} \circ P$ coincides (prove it on your own) with the transformation mentioned in the formulation of the problem.

**30.22. First solution.** Denote the given transformation by $P$. Let us extend it to points of the line $x = 0$ and infinite points by setting $P(0, k) = M_k$, $P(M_k) = (0, k)$, where $M_k$ is an infinite point on the line $y = kx$. It is easy to see that the map $P$ extended in this way is a one-to-one correspondence.

Let us prove that under $P$ every line turns into a line. Indeed, the line $x = 0$ and the infinite line turn into each other. Let $ax + by + c = 0$ be an arbitrary other line (i.e., either $b$ or $c$ is nonzero). Since $P \circ P = E$, the image of any line coincides with its preimage. Clearly, point $P(x, y)$ lies on the considered line if and only if $\frac{a}{x} + \frac{by}{x} + c = 0$, i.e., $cx + by + a = 0$. It remains to make use of Problem 30.14 d).

**Second solution** (sketch). Denote lines $x = 1$ and $x = 0$ by $a$ and $b$, respectively, and point $(-1, 0)$ by $O$. Then the given transformation coincides with the transformation from the preceding problem.

**30.23.** If we denote line $f$ by $b$, then the transformation mentioned in this problem is the inverse to the transformation of Problem 30.21.

**30.24.** Consider a projective transformation for which line $PQ$ is the singular one. The images $l_1'$ and $l_2'$ of lines $l_1$ and $l_2$ under this transformation are parallel and the images of the considered quadrilaterals are parallelograms two sides of which lie on lines $l_1'$ and $l_2'$ and the other two sides are parallel to a fixed line (the infinite point of this line is the image of point $P$). It is clear that the locus of the intersection points of the diagonals of such parallelograms is the line equidistant from $l_1'$ and $l_2'$.

**30.25.** Let us make a projective transformation whose singular line is $EF$. Then quadrilateral $ABCD$ turns into a parallelogram and lines $KL$ and $MN$ into lines parallel to the sides of the parallelogram and passing through the intersection point of its diagonals, i.e., into the midlines. Therefore, the images of points $K$, $L$, $M$, $N$ are the midpoints of the parallelogram and, therefore, the images of lines $KN$ and $LM$ are parallel, i.e., point $X$ turns into an infinite point and, therefore, $X$ lies on the singular line $EF$.

**30.26.** Let us make the projective transformation with singular line $AB$. The images of points under this transformation will be denoted by primed letters. Let us consider a homothety with center at point $O'$ (or a parallel translation if $O'$ is an infinite point) that sends $C_1'$ to $C_2'$. Under this homothety segment $B_1'C_1'$ turns into segment $B_2'C_2'$ because $B_1'C_1' \parallel B_2'C_2'$. Similarly, $C_1'A_1'$ turns to $C_2'A_2'$. Therefore, the corresponding sides of triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ are parallel, i.e., all three points $A'$, $B'$, $C'$ lie on the infinite line.

**30.27.** Let us consider the projective transformation whose singular line passes through the intersection points of lines $AB_1$ and $BA_1$, $BC_1$ and $CB_1$ and denote by $A'$, $B'$, ...the images of points $A$, $B$, .... Then $A'B_1' \parallel B'A_1'$, $B'C_1' \parallel C'B_1'$ and we have to prove that $C'A_1' \parallel A'C_1'$ (see Problem 1.12 a)).

**30.28.** As a result of the projective transformation with singular line $PQ$ the problem is reduced to Problem 4.54.

**30.29.** This problem is a reformulation of the preceding one. Indeed, suppose that the pair of lines $OO_1$ and $OB$ separates the pair of lines $OA$ and $OC$ and the pair of lines $OO_1$ and $O_1B$ separates the pair of lines $O_1A$ and $O_1C$ (consider on your own in a similar way the remaining ways of disposition of these lines). Therefore, if we redenote points $A_1$, $B$, $B_1$, $C_1$, $O$, $O_1$ and the intersection point of lines $AB_1$ and $CC_1$ by $D$, $R$, $L$, $K$, $Q$, $P$ and $B$, respectively, then the preceding problem implies that the needed lines pass through point $M$.

**30.30.** Let us consider the projective transformation with singular line $O_1O_2$ and denote by $A'$, $B'$, ...the images of points $A$, $B$, .... Then $A'C_1' \parallel C_1'A_1' \parallel B'B_1'$, $B'C_1' \parallel C'B_1' \parallel A'A_1'$. Let us, for definiteness sake, assume that point $C$ lies inside angle $\angle A'O'B'$ (the remaining cases can be reduced to this one after a renotation). Making, if necessary, an affine transformation we can assume that the parallelogram $O'A'C_1'B'$ is a square and, therefore, $O'A_1'C'B_1'$ is also a square and the diagonals $O'C_1'$ and $O'C'$ of these squares lie on one line. It remains to make use of the symmetry through this line.

**30.31.** It suffices to prove that the orthocenters of each triple of triangles formed by the given lines lie on one line. Select some three triangles. It is easy to see that one of the given lines (denoted by $l$) is such that one of the sides of each of the chosen triangles lies on $l$. Denote the remaining lines by $a$, $b$, $c$ and let $A$, $B$, $C$, respectively, be their intersection points with $l$.

Denote by $l_1$ the infinite line and by $A_1$ (resp. $B_1$, $C_1$) the infinite points of the lines perpendicular to $a$ (resp. $b$, $c$). Then the fact that the orthocenters of the three selected triangles lie on one line is a direct corollary of Pappus's theorem (Problem 30.27).

**30.32.** Perform a projective transformation with singular line parallel to $l$ and passing through the intersection point of lines $PP_1$ and $QQ_1$; next, perform an affine transformation that makes the images of lines $l$ and $PP_1$ perpendicular to each other. We may assume that lines $PP_1$ and $QQ_1$ are perpendicular to line $l$ and our problem is to prove that line $RR_1$ is also perpendicular to $l$ (points $P_1$, $Q_1$, $R_1$ are the midpoints of the corresponding segments because these segments are parallel to the singular line; see Problem 30.14 b)). Segment $PP_1$ is both a median and a hight, hence, a bisector in the triangle formed by lines $l$, $AB$ and $CD$.

Similarly, $QQ_1$ is a bisector in the triangle formed by lines $l$, $AC$ and $BD$. This and the fact that $PP_1 \parallel QQ_1$ imply that $\angle BAC = \angle BDC$. It follows that quadrilateral $ABCD$ is an inscribed one and $\angle ADB = \angle ACB$. Denote the points at which $l$ intersects lines $AC$ and $BD$ by $M$ and $N$, respectively (Fig. 124). Then the angle between $l$ and $AD$ is equal to $\angle ADB - \angle QNM = \angle ACB - \angle QMN$, i.e., it is equal to the angle between $l$ and $BC$. It follows that the triangle bounded by lines $l$, $AD$ and $BC$ is an isosceles one and segment $RR_1$ which is its median is also its hight, i.e., it is perpendicular to line $l$, as required.

**30.33.** Perform a projective transformation with singular line parallel to $l$ and passing through point $A$. We may assume that point $A$ is infinite, i.e., lines $AB$ and $AC$ are parallel. Then by Problem 30.14 b) points $A_1$, $B_1$, $C_1$ are, as earlier, the midpoints of the corresponding segments because these segments lie on the line parallel to the singular one. Two triangles formed by lines $l$, $AB$, $BC$ and $l$, $AC$, $BC$ are homothetic and, therefore, lines $BB_1$ and $CC_1$, which are medians of these triangles, are parallel. Therefore, quadrilateral
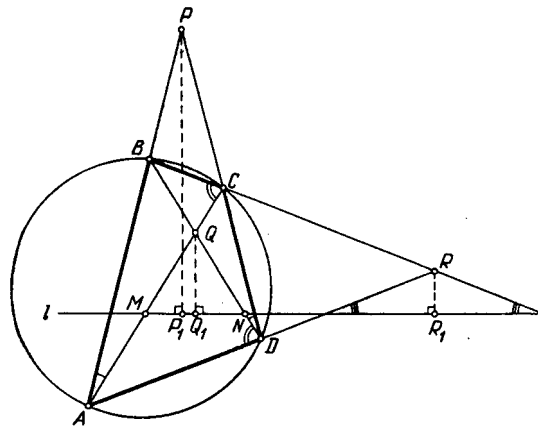
Figure 266 (Sol. 30.32)

$BB_2CC_2$ is a parallelogram because its opposite sides are parallel. It remains to notice that point $A_2$ is the midpoint of diagonal $BC$ of this parallelogram and, therefore, it is also the midpoint of diagonal $B_2C_2$.

**30.34.** Let us make the projective transformation whose singular line is line $PQ$. Denote by $A'$, $B'$, ... the images of points $A$, $B$, .... Then $A'B'C'D'$ is a parallelogram, $R'$ the intersection point of its diagonals, $Q'$ is the infinite point of line $Q'R'$, $K'$ and $L'$ the intersection points of the sides of the parallelogram on line $Q'R'$. Clearly, points $K'$ and $L'$ are symmetric through point $R'$. Hence,

$$(Q'R'K'L') = \frac{Q'K'}{Q'L'} : \frac{R'K'}{R'L'} = 1 : \frac{R'K'}{R'L'} = -1.$$

It remains to notice that $(QRKL) = (Q'R'K'L')$ by Problem 30.2 b).

**30.35.** Answer: It is possible. Indeed, consider the vertices of a regular 1991-gon (red points) and points at which the extensions of the sides of this polygon intersect the infinite line (blue points). This set of points has the required properties. Indeed, for any regular $n$-gon, where $n$ is odd, the line passing through its vertex and parallel to one of the sides passes through one more vertex. Any given finite set of points can be transformed by a projective transformation into a set of finite (i.e., not infinite) points.

**30.36.** Let us make a projective transformation that sends the circle inscribed into the quadrilateral into a circle $S$ and the intersection point of the lines connecting the opposite tangent points into the center of $S$, cf. Problem 30.16 a). The statement of the problem now follows from the fact that the obtained quadrilateral is symmetric with respect to the center of $S$.

**30.37.** Let us make a projective transformation that sends the inscribed circle into a circle $S$ and the intersection point of two of the three lines under consideration into the center of $S$, cf. Problem 30.16 a). Then the images of these two lines are simultaneously bisectors and hights of the image of the given triangle and, therefore, this triangle is an equilateral one. For an equilateral triangle the statement of the problem is obvious.

**30.38.** Let us consider, separately, the following two cases.

1) Point $P$ lies outside $S$. Let us make the projective transformation that sends circle $S$ into circle $S_1$ and point $P$ into $\infty$ (see Problem 30.17), i.e., the images of all lines passing through $P$ are parallel to each other. Then in heading b) the image of the locus to be found is line $l$, their common perpendicular passing through the center of $S_1$, and in heading a) the line $l$ with the diameter of $S_1$ deleted.

To prove this, we have to make use of the symmetry through line $l$. Therefore, the locus itself is: in heading b), the line passing through the tangent points of $S$ with the lines drawn through point $P$ and in heading a), the part of this line lying outside $S$.

2) Point $P$ lies inside $S$. Let us make a projective transformation that sends circle $S$ into circle $S_1$ and point $P$ into its center, cf. Problem 30.16 a). Then the image of the locus to be found in both headings is the infinite line. Therefore, the locus itself is a line.

The obtained line coincides for both headings with the polar line of point $P$ relative to $S$, cf. Problem 30.19.

**30.39.** Denote by $m$ the line which is the locus to be found in Problem 30.38 b) and by $N$ the distinct from $M$ intersection point of $S$ with line $OM$. Denote by $Q$ the composition of the projection of $l$ to $S$ from $M$ and $S$ to $M$ from $N$. By Problem 30.9 this composition is a projective map.

Let us prove that $P$ is the composition of $Q$ with the projection of $m$ to $l$ from $M$. Let $A$ be an arbitrary point on $l$, $B$ its projection to $S$ from $M$, $C$ the projection of $B$ to $S$ from $O$, $D$ the intersection point of lines $BN$ and $CM$. By Problem 30.38 b) point $D$ lies on line $m$, i.e., $D = Q(A)$. Clearly, $P(A)$ is the projection of $D$ to $l$ from $M$.

**30.40.** Both headings of the problem become obvious after a projective transformation that sends circle $S$ into a circle and line $KP$ into the infinite line, cf. Problem 30.17. The answer is as follows:

a) The locus to be found lies on the line equidistant from the images of lines $AK$ and $BK$.

b) The point to be found is the center of the image of $S$.

**30.41.** Let $A'$, $B'$, ... be the images of points $A$, $B$, ... under the projective transformation that sends an escribed circle of triangle $ABC$ into circle $S$, and chord $EF$ into a diameter of $S$ (see Problem 30.18). Then $A'$ is the infinite point of lines perpendicular to diameter $E'F'$ and we have to prove that line $D'T'$ contains this point, i.e., is also perpendicular to $E'F'$.

Since $\triangle T'B'E' \sim \triangle T'F'C'$, it follows that $C'T' : T'E' = C'F' : B'E'$. But $C'D' = C'F'$ and $B'D' = B'E'$ as tangents drawn from one point; hence, $C'T' : T'E' = C'D' : D'B'$, i.e., $D'T' \parallel B'E'$.

**30.42.** By Problem 30.16 a) it suffices to consider the case when diagonals $AD$ and $BE$ pass through the center of the circle. It remains to make use of the result of Problem 6.83 for $n = 3$.

**30.43.** Consider the projective transformation that sends circle $S$ into a circle and the intersection points of lines $AB$ and $DE$, $BC$ and $EF$ into infinite points (see Problem 29.17). Our problem is reduced to Problem 2.11.

**30.44.** Consider a projective transformation that sends circle $S$ into circle $S_1$ and point $O$ into the center $O'$ of $S_1$, cf. Problem 30.16 a). Let $A'$, $B'$, ... be the images of points $A$, $B$, .... Then $A'B'$, $M'N'$ and $P'Q'$ are diameters. Therefore, the central symmetry through $O'$ sends point $E'$ into $F'$, i.e., $O'$ is the midpoint of segment $E'F'$. Since chord $AB$ is perpendicular to the diameter passing through $O$, Problem 30.16 b) implies that $AB$ is parallel to the singular line. Therefore, by Problem 30.14 b) the ratio of the lengths of the segments that lie on line $AB$ is preserved and, therefore, $O$ is the midpoint of segment $EF$.

**30.45.** Let us consider the projective transformation that maps the given circle into circle $S'$ and segment $AD$ into a diameter of $S'$ (see Problem 30.18). Let $A'$, $B'$, ... be the images of $A$, $B$, .... Then $S$ turns into the infinite point $S'$ of lines perpendicular to line $A'D'$. But $A'C'$ and $B'D'$ are hights in $\triangle A'D'P'$ and, therefore, $Q'$ is the orthocenter of this triangle. Therefore, line $P'Q'$ is also a hight; hence, it passes through point $S'$.

**30.46.** By Problem 30.15 it suffices to consider only the case when $ABCD$ is a square. We have to prove that the composition of projections described in the formulation of the problem is the identity transformation. By Problem 30.4 a projective transformation is the identity if it has three distinct fixed points. It is not difficult to verify that points $A$, $B$ and the infinite point of line $AB$ are fixed for this composition.

**30.47.** Under the projection of line $QR$ from point $A$ to line $CD$ points $Q$, $R$, $K$, $L$ are mapped into points $D$, $C$, $P$, $L$, respectively. Therefore, by Problem 30.2 b) $(QRKL) = (DCPL)$. Similarly, by projecting line $CD$ to line $QR$ from point $B$ we get $(DCPL) = (RQKL)$; hence, $(QRKL) = (RQKL)$. On the other hand,

$$(RQKL) = \frac{RK}{RL} : \frac{QK}{QL} = \left(\frac{QK}{QL} : \frac{RK}{RL}\right)^{-1} = (QRKL)^{-1}.$$

These two equalities imply that $(QRKL)^2 = 1$, i.e., either $(QRKL) = 1$ or $(QRKL) = -1$. But by Problem 30.8 the cross ratio of distinct points cannot be equal to one.

**30.48.** Denote the intersection points of lines $AB_1$ and $BA_1$, $BC_1$ and $CB_1$, $CA_1$ and $AC_1$ by $P$, $Q$, $R$, respectively, and the intersection point of lines $PQ$ and $CA_1$ by $R_1$. We have to prove that points $R$ and $R_1$ coincide. Let $D$ be the intersection point of $AB_1$ and $CA_1$. Let us consider the composition of projections: of line $CA_1$ to line $l_1$ from point $A$, of $l_1$ to $CB_1$ from $B$, and of $CB_1$ to $CA_1$ from $P$. It is easy to see that the obtained projective transformation of line $CA_1$ fixes points $C$, $D$ and $A_1$ and sends $R$ into $R_1$. But by Problem 30.5 a projective transformation with three distinct fixed points is the identity one. Hence, $R_1 = R$.

**30.49.** Let $F'$ be the point symmetric to $F$ through $O$. We have to prove that $F' = F$. By Problem 30.9 the composition of the projection of line $AB$ to circle $S$ from point $M$ followed by the projection of $S$ back to $AB$ from $Q$ is a projective transformation of line $AB$. Consider the composition of this transformation with the symmetry through point $O$. This composition sends points $A$, $B$, $O$, $E$ to $B$, $A$, $F'$, $O$, respectively. Therefore, by Problem 30.2 b)

$$(ABOE) = (BAF'O).$$

On the other hand, it is clear that

$$(BAF'O) = \frac{BF'}{AF'} : \frac{BO}{AO} = \frac{AO}{BO} : \frac{AF'}{BF'} = (ABOF')$$

i.e., $(ABOE) = (ABOF')$; hence, by Problem 30.3, $E = F'$.

(?)**30.50.** Denote the intersection points of lines $AB$ and $DE$, $BC$ and $EF$, $CD$ and $FA$ by $P$, $Q$, $R$, respectively, and the intersection point of lines $PQ$ and $CD$ by $R'$. We have to prove that points $R$ and $R'$ coincide. Let $G$ be the intersection point of $AB$ and $CD$. Denote the composition of the projection of line $CD$ on the given circle from point $A$ with the projection of circle \$ to line $BC$ from point $E$.

By Problem 30.9 this composition is a projective map. It is easy to see that its composition with the projection of $BC$ to $CD$ from point $P$ fixes points $C$, $D$ and $G$ and sends point $R$ to $R'$. But by Problem 30.5 a projective transformation with three fixed points is the identity one. Hence, $R' = R$.

**30.51.** Since angles $\angle APT$, $\angle ART$, $\angle AST$ and $\angle AQT$ are right ones, points $A$, $P$, $R$, $T$, $S$, $Q$ lie on the circle constructed on segment $AT$ as on diameter. Hence, by Pascal's theorem (Problem 30.50) points $B$, $C$ and $X$ lie on one line.

**30.52.** Denote the given line and circle by $l$ and $S$, respectively. Let $O$ be an arbitrary point of the given circle and let $A_1$, $A_1'$, $B_1$, $B_1'$, $C_1$, $C_1'$ be the images of points $A$, $A'$, $B$, $B'$,

$C$, $C'$ under the projection map of $l$ to $S$ from point $O$, i.e., $A_1$ (resp. $A_1'$, $B_1$, ...) is the distinct from $O$ intersection point of line $AO$ (resp. $A'O$, $BO$, ...) with circle $S$.

Denote by $B_2$ the intersection point of lines $A_1'B_1$ and $A_1B_1'$ and by $C_2$ the intersection point of lines $A_1'C_1$ and $A_1C_1'$. Let $P_1$ be the composition of the projection of line $l$ to circle $S$ from point $O$ with the projection of $S$ to line $B_2C_2$ from point $A_1'$; let $P_2$ be the composition of the projection of $B_2C_2$ to $S$ from point $A_1$ with the projection of $S$ to $l$ from point $O$. Then by Problem 30.9 transformations $P_1$ and $P_2$ are projective ones and their composition sends points $A$, $B$, $C$ to $A'$, $B'$, $C'$, respectively.

It is clear that all the considered points can be constructed with the help of a ruler (in the same order as they were introduced).

a) Let $M_1$ be the distinct from $O$ intersection point of line $MO$ with circle $S$; $M_2 = P_1(M)$ the intersection point of lines $A_1'M_1$ and $B_2C_2$; $M_3$ the distinct from $A_1$ intersection point of line $M_2A_1$ with circle $S$; $P(M) = P_2(P_1(M))$ the intersection point of lines $l$ and $OM_3$.

b) Let $M_1$ and $N_1$ be the intersection points of circle $S$ with line $B_2C_2$. Then the fixed points of transformation $P$ are the intersection points of lines $OM_1$ and $ON_1$ with line $l$.

**30.53.** a) The point $X$ to be found is the fixed point of the composition of the projection of $l_1$ to $l_2$ from point $A$, the translation along line $l_2$ at distance $a$ and the projection of $l_2$ to $l_1$ from point $B$. The fixed point of this projective map is constructed in Problem 30.52.

b) Replace the shift from the solution of heading a) with the central symmetry with respect to $E$.

**30.54.** a) Denote by $k$ the number to which the ratio $\frac{AX}{BY}$ should be equal to. Consider the projective transformation of line $a$ which is the composition of the projection of $a$ to line $b$ from point $P$, the movement of the plane that sends $b$ to $a$ and $B$ to $A$ and, finally, the homothety with center $A$ and coefficient $k$. The required point $X$ is the fixed point of this transformation. The construction of point $Y$ is obvious.

b) Denote by $k$ the number to which the product $AX \cdot BY$ should equal to and by $Q$ the intersection point of the lines passing through points $A$ and $B$ parallel to lines $b$ and $a$, respectively; let $p = AQ \cdot BQ$. Consider the projective transformation of line $a$ which is the composition of the projection of $a$ to line $b$ from point $P$, projection of $b$ to $a$ from $Q$ and the homothety with center $A$ and coefficient $\frac{k}{p}$.

Let $X$ be the fixed point of this transformation, $Y$ its image under the first projection and $X_1$ the image of $Y$ under the second projection. Let us prove that line $XY$ is the desired one. Indeed, since $\triangle AQX_1 \sim \triangle BYQ$, it follows that

$$AX_1 \cdot BY = AQ \cdot BQ = p$$

and, therefore,

$$AX \cdot BY = \frac{k}{p} \cdot AX_1 \cdot BY = k.$$

**30.55.** Let $P$ be the given point; $A$, $B$, $C$ the points of pairwise intersections of the given lines $a$, $b$, $c$; let $X$, $Y$, $Z$ be the intersection points of the given lines with line $l$ to be found (Fig. 125).

By the hypothesis $XZ = ZY$. Let $T$ be the intersection point of line $c$ with the line passing through $X$ parallel to $b$. Clearly, $XT = AY$. Since $\triangle XTB \sim \triangle CAB$, it follows that $XB : XT = CB : CA$ which implies $BX : YA = CB : CA$, i.e., the ratio $BX : YA$ is known. Thus, our problem is reduced to Problem 30.54 a).

**30.56.** a) By Problem 30.9 the composition of the projection of $CD$ on $S$ from $A$ with the projection of $S$ on $CD$ from $B$ is a projective transformation of line $CD$. Let $M$ be a fixed point of the composition of this transformation with the shift along line $CD$ by
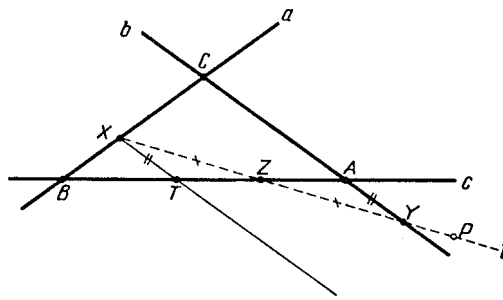
Figure 267 (Sol. 30.55)

distance $a$. Then the projection of $M$ on $S$ from $A$ is the desired point. The fixed point of any projective transformation is constructed in Problem 30.52.

b) In the solution of heading a) replace the shift by the central symmetry through $E$.

**30.57.** a) Let us draw an arbitrary circle $S$ through point $P$. By Problem 30.10 the composition of the projection of $l$ to $S$ from $P$, the rotation about the center of $S$ through an angle of $2\alpha$ and the projection of $S$ to $l$ from $P$ is a projective transformation of line $l$. Then (by the theorem on an escribed angle) the fixed point of the composition of this transformation with the shift along line $CD$ by given distance $XY$ is the desired point. The fixed point of any projective transformation is constructed in Problem 30.52.

b) Let us construct arbitrary circles $S_1$ and $S_2$ passing through points $P$ and $Q$, respectively. Consider the composition of projection of $l_1$ to $S_1$ from $P$, the rotation about the center of $S_1$ through an angle of $2\alpha$ and the projection of $S_1$ to $l_2$ from $P$. By Problem 30.10 this composition is a projective map. Similarly, the composition of the projection of $l_2$ to $S_2$ from $Q$, the rotation about the center of $S_2$ through an angle of $2\beta$ and projection of $S_2$ to $l_1$ from $Q$ is also a projective map. By the theorem on an escribed angle the fixed point of the composition of these maps is the desired point $X$ and in order to construct it we can make use of Problem 30.52.

**30.58.** a) Denote the given points by $M_1$, $\ldots$, $M_n$ and the given lines by $l_1$, $\ldots$, $l_n$. A vertex of the polygon to be found is the fixed point of the projective transformation of line $l_1$ which is the composition of projections of $l_1$ to $l_2$ from $M_1$, $l_2$ to $l_3$ from $M_2$, $\ldots$, $l_n$ to $l_1$ from $M_n$. The fixed point of a projective transformation is constructed in Problem 30.52.

b) Select an arbitrary point on a given circle and with the help of projection from the given point let us identify the given circle with line $l$. By Problem 30.39 the central projecting of the circle to itself is a projective transformation of line $l$ under this identification. Clearly, a vertex of the desired polygon is the fixed point of the composition of consecutive projections of the given circle to itself from given points. The fixed point of a projective transformation is constructed in Problem 30.52.

c) In the solution of heading b) certain central projections should be replaced by either rotations about the center of the circle if the corresponding side is of the given length or by symmetries if the corresponding side has the prescribed direction (the axis of the symmetry should be the diameter perpendicular to the given direction).

**30.59.** Suppose that we managed to find the required construction, i.e., to write an instruction the result of fulfilment of which is always the midpoint of the given segment. Let us perform this construction and consider the projective transformation that fixes the endpoints of the given segment and sends the midpoint to some other point. We can select this transformation so that the singular line would not pass through neither of the points obtained in the course of intermediate constructions.

Let us perform our imaginary procedure once again but now every time that we will encounter in the instruction words "take an arbitrary point (resp. line)" we shall take the image of the point (resp. line) that was taken in the course of the first construction.

Since a projective transformation sends any line into a line and the intersection of lines into the intersection of their images and due to the choice of the projective transformation this intersection is always a finite point, it follows that at each step of the second construction we obtain the image of the result of the first construction and, therefore, we will finally get not the midpoint of the interval but its image instead. Contradiction.

REMARK. We have, actually, proved the following statement: *if there exists a projective transformation that sends each of the objects $A_1$, ..., $A_n$ into themselves and does not send an object $B$ into itself, then it is impossible to construct object $B$ from objects $A_1$, ..., $A_n$ with the help of a ruler only.*

**30.60.** The statement of the problem follows directly from Remark 30.59 above and from Problem 30.16 a).

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