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CONVEX FUNCTIONS

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A single valued function $f(x)$ defined on the interval $\triangle = (a, b)$ is called a convex function, if for every pair $x, y \in \triangle$ the relation

$$2f\left(\frac{x+y}{2}\right) \leq f(x) + f(y) \tag{1}$$

holds.

It is simple to see that if a convex function is bounded on $\triangle$ that it is continuous in $\triangle$, and if it is unbounded on $\triangle$ that it is unbounded on any subinterval on $\triangle$. Therefore, if a convex function is bounded on any subinterval of $\triangle$ it is continuous in $\triangle$ [1].

One simple example of such a function is a function which satisfies the following functional equation:

$$f(x+y) = f(x) + f(y). \tag{2}$$

M. Fréchet [2] has proved that a measurable function which satisfies (2) is continuous. W. Sierpiński ([5], II) has proved that a measurable convex function is also continuous.

Here we want to prove a more general theorem:

Theorem 1. I. *If a convex function* f(x) *is bounded on a set* T *which has property that*

$$m_i\,(T+T) > 0 \tag{3}$$

then f *is continuous*[1].

II. *Condition* (3) *can not be replaced by the condition*

$$m_e\,(T+T) > 0. \tag{4}$$

From now on, we shall denote by $m_i S$, $m_e S$ and mS respectively the inner measure, the outer measure and the measure of the set S in Lebesgue's sense.

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[1]) When this paper was ready for publication we were told that A. Ostrowski has proved the theorem according to which a convex function which is bounded on a set of the positive measure is continuous (see: *A. Ostrowski*, Mathematische Miszellen XIV, Über die Funktionalgleichungen der Exponentialfunktion und verwandte Funktionalgleichungen, Jahresber. Deutsch. Math. Ver. **38**, (1929), pp. 54—62). Therefore the first part of Theorem 1. appears as a slight generalisation of the result obtained by A. Ostrowski.

The following lemma will be used:

Lemma 1. *Let us put* $\delta = (0, 2c) = P \cup S$, $c > 0$, *where* P, S *are two measurable disjoint sets. If*

$$m P > 3 m S , \tag{5}$$

then for every

$$x \in S \cap (m S , c) \tag{6}$$

there is a pair of numbers p, q $\in$ P *such that the relation:*

$$2 x = p + q \tag{7}$$

holds.

Proof. Let us denote:

$$\delta' = (0, 2 x) , \quad P' = P \cap \delta' , \quad S' = S \cap \delta' . \tag{8}$$

We have:

$$S' \cap P' = v \text{ (void)}, \quad S' \cup P' = \delta', \quad (2 x - P') \subseteq \delta'. \tag{9}$$

Since $x > mS$, we get:

$$m P' + m S' = m \delta' = 2 x > 2 m S > 2 m S', \quad \text{i. e.} \quad m P' > m S'. \tag{10}$$

We assert that

$$(2 x - P') \cap P' > v . \tag{11}$$

Otherwise we should have

$$(2 x - P') \cap P' = v , \quad (2 x - P') \cup P' \subseteq \delta' . \tag{12}$$

This implies:

$m (2x - P') + mP' = 2mP' \leq m \delta' = m (P' \cup S') = mP' + mS'$ i. e.

$$m P' \leq m S' . \tag{13}$$

which contradicts (10).

Now we can prove the first part of Theorem 1. The second part will be proved after Theorem 3.

Let $f(t)$ be a convex function which is bounded on the set T which has the property (3). Then, there exists at least one set $Q \subseteq T + T$ with positive measure and there exists a constant A such that $|f(t)| \leq A$ for all $t \in T$. The set $Q_1 = Q/2$ has positive measure. If $x \in Q_1$, then there is a couple of numbers $t, s \in T$ such that $x = (t + s)/2$. We have:

$$2 f (x) = 2 f \left(\frac{t + s}{2} \right) \leq f (t) + f (s), \tag{14}$$

or $|f(x)| \leq A$ for $x \in Q_1$. Since $m Q_1 > 0$, there is an interval $\delta = (a, b)$, $(b > a)$ such that

$$m (Q_1 \cap \delta) > \frac{3}{4} m \delta . \tag{15}$$

Denote: $Q_2 = Q_1 \cap \delta$, $S_2 = \delta \setminus Q_2$. The relation (15) implies $mQ_2 > 3mS_2$. On the set Q_2 the function f is bounded. The function

$g(x') = g(x - a) = f(x)$ is convex on the interval $\triangle = (0, b - a) = \delta - a$, and bounded on the set $P = Q_2 - a$.

Denote: $S = S_2 - a$. We have: $\triangle = P \cup S$, $P \cap S = v$, $mP > 3mS$, $|g(x')| \leq A$ for all $x' \in P$. We assert that $g(x')$ is bounded on the interval $\triangle' = (mS, (b-a)/2)$. Indeed, if $S \cap \triangle' = v$, then $\triangle' \subseteq P$ and g is bounded on $\triangle'$. If $S \cap \triangle' \supset v$, then (according to Lemma 1) for any $x' \in \triangle'$, $x' \in S$ we have: $2x' = p + q$, $p, q \in P$. This implies: $2g(x') = 2g((p+q)/2) \leq g(p) + g(q)$ i. e. $|g(x')| \leq A$ for $x' \in (mS, (b-a)/2)$ or $|f(x)| \leq A$ for $x \in (mS + a, (b + a)/2)$.

Since f is bounded on one interval it is, therefore, a continuous function.

Corrolary 1. If a convex function is measurable, it is continuous ([5], II).

Corrolary 2. If a convex function $f(x)$ is bounded on the set T of measure zero, which has the property that $T + T$ contains a set of positive measure then $f(x)$ is continuous.

For example, if f is bounded on Cantor's set, then f is continuous.

Corrolary 3. Let a real function $f(x)$ satisfy the functional equation $f(x + y) = f(x) \cdot f(y)$ for all real numbers x and y.

If $f(x)$ is bounded on a set T which has the property that $m_i(T + T) > 0$ then $f(x) = \exp(cx)$, where c is an arbitrary real number.

Theorem 2. *Let* f(x) *be a function which satisfies the following conditions:*

$$\text{a)}\ |f(x)| = 1, \qquad \text{b)}\ f(x+y) = f(x) \cdot f(y), \tag{16}$$

for every pair of real numbers x *and* y. *If the function* f *is continuous on a closed bounded set* T *which has the property that* $m_i(T + T) > 0$, *then* f *is a continuous function on the set of real numbers.*

Proof. Let $P \subseteq T + T$ be a perfect set with positive measure. We assert that f is continuous on the set P. Suppose that this is not the case. Then there exists at least one sequence $x_0, x_n \in P$, $x_n \to x_0$, and a real number $a > 0$ sucht that

$$|f(x_n) - f(x_0)| \geq a \tag{17}$$

for all n. Because $x_n \in P$ there exists a pair of numbers $p_n, q_n \in T$ such that $x_n = p_n + q_n$. Let $p_{n'}$ be a convergent subsequence of the sequence p_n, and let it tend to p_0. Then the sequence $q_{n'} = x_{n'} - p_{n'}$ tends to $x_0 - p_0 = q_0$. The assumption in Theorem 2 implies $f(p_{n'}) \to f(p_0)$, $f(q_{n'}) \to f(q_0)$. Therefore: $f(x_{n'}) = f(p_{n'} + q_{n'}) = f(p_{n'}) \cdot f(q_{n'}) \to f(p_0) \cdot f(q_0) = f(p_0 + q_0) = f(x_0)$ i. e.

$$f(x_{n'}) \to f(x_0) \tag{18}$$

However (18) contradicts (17). So, f is continuous on the set P. Since $mP > 0$, there is an interval δ such that $mP_1 > 3mS_1$, where $P_1 = P \cap \delta$ and $S_1 = \delta \setminus P_1$. Without loss of generality we can assume $\delta = [0,2c]$, $c > 0$. According to Lemma 1 we have $\delta' = (2mS_1, 2c) \subseteq P_1 + P_1$. On the other hand it is easy to prove that $f(x)$ is a continuous function on $P_1 + P_1$. Thus $f(x)$ is a continuous function in the interval δ'. This implies that f is continuous on the set of all real numbers.

Theorem 3. *If* H *is a Hamel's base* [3] *and if* $r_1, \ldots, r_n$ *is any system of* $n \geq 1$ *rational numbers, then*

$$m_i (r_1 H + \ldots + r_n H) = 0. \tag{19}$$

Proof. Let $H = \{H_\alpha\}$ be Hamel's base of the set of real numbers. For any real number x the representation $x = \Sigma x_\alpha H_\alpha$ is unique, where x_α are rational numbers and in the sum there are only finite number of terms different from zero. The function $f(x) = \Sigma x_\alpha f(H_\alpha)$ satisfies equation (2) for any $f(H_\alpha)$. This function is continuous if and only if $f(H_\alpha)/H_\alpha$ is a constant independent of α. If we define f in such a way that $f(H_\alpha) = 1$ for all α then: a) $f(x)$ is a convex function, b) $f(x)$ is not continuous and c) $f(x)$ is bounded on the set

$$r_1 H + r_2 H + \ldots + r_n H. \tag{20}$$

This, together with Theorem 1 and the fact that $m_i(A + A) = 0$ implies $m_i A = 0$, gives (19).

Corrolary 4. A measurable Hamel's base has measure zero.

Proof. In Theorem 3 put $n = 1$ and $r_1 = 1$.

The existence of measurable Hamel's base is obvious. It is sufficient to consider any set T such that $mT = 0$, and that $(T + T)$ is the set of all real numbers. Such a set does exist. For example we can take on every interval (a, b) (a, b are integers) Cantor's set. The union of all these sets is the set T with above properties. Now, take any Hamel's base from the set T for the set T. The base which is so obtained is measurable and of course is the Hamel's base for the set of real numbers.

Corrolary 5. There exist two measurable sets A and B such that the set $A + B$ is not measurable ([5], III).

Proof. Suppose that Corrolary 5 does not hold. If H is a measurable Hamel's base then the set (20) has a measure zero for every system $r_1, r_2, \ldots, r_n$ of rational numbers. The set of all real numbers is the union of sets of the form (20) as $r_1, r_2, \ldots, r_n$ runs through the set of rational numbers. Thus, the set of all real numbers is the union of countable many sets each of which has a measure zero. This is an absurdity. Therefore there exists at least one $n > 1$, and a system of rational numbers $r_1, \ldots, r_n$ such that the corresponding set (20) is not measurable. With m we denote the smallest n with this property. Thus there exists a system $r_1, \ldots, r_m$ of rational numbers such that the set:

$$Q = r_1 H + \ldots + r_m H = (r_1 H) + (r_2 H + \ldots + r_m H) \tag{21}$$

is not measurable, and that every set in brackets is measurable. Therefore, the set Q is a non-measurable set and it is the sum of two sets each of which has measure zero.

From here follows the second part of Theorem 1. For, the function $f(x)$ which was defined in the proof of Theorem 3 is a convex, discontinuous function and bounded on the set Q which has the property that $m_e(Q+Q)>0$.

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KONVEKSNE FUNKCIJE

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Sadržaj

Na intervalu $\triangle=(a,b)$ definiranu jednoznačnu funkciju $f(x)$ zovemo konveksnom, ako za svaki par $x, y \in \triangle$ vrijedi (2). Ako je konveksna funkcija ograničena na bar jednom podintervalu od $\triangle$, tada je ona neprekidna u $\triangle$. Sierpiński je dokazao, da izmjerivost konveksne funkcije povlači njenu neprekidnost.

Mi smo dokazali ove teoreme:

Teorem 1. *Ako je konveksna funkcija* f(x) *ograničena na skupu* T, *koji ima svojstvo* (3), *tada je* f *neprekidna.*

Teorem 2. *Neka funkcija* f *zadovoljava uvjete* a) *i* b) *u relaciji* (16), *za svaki par realnih brojeva* x, y. *Ako je funkcija neprekidna na zatvorenom i ograničenom skupu* T, *koji ima svojstvo* (3), *tada je* f *neprekidna funkcija na skupu realnih brojeva.*

Teorem 3. *Ako je* H *Hamelova baza, tada je* $m_i(H+H)=m_iH=0$.

$m_i S$ znači nutarnju mjeru skupa S, $m_e S$ znači vanjsku mjeru skupa S, a mS znači mjeru skupa S. Izmjerivost i sve što se mjere tiče odnosi se na Lebesgue-ovu mjeru.

Prvi dio teorema 1 i teorem 2 dokazani su pomoću ove leme:

Lema 1. Neka je $\delta = (0,2c) = P \cup S, c > 0$, gdje su P i S disjunktni izmjerivi skupovi. Ako vrijedi (5), tada svako x, koje zadovoljava uvjet (6), možemo pisati u obliku (7), gdje su $p, q \in P$.

Dokaz leme teče ovako: Uz oznake (8) vrijedi (9), a jer je $x > mS$, to vrijedi (10). Tvrdimo da vrijedi (11). U protivnom bi vrijedilo (12) dakle i (13). No (13) i (10) su protivrječne relacije. Dakle vrijedi (10), a ovo odmah daje ispravnost leme 1.

Prvi dio teorema 1 se dokazuje ovako: Iz (3) i pretpostavke da je konveksna funkcija ograničena na T slijedi, da je ona ograničena na skupu Q_1 pozitivne mjere. Budući je Q_1 pozitivne mjere, to postoji interval δ takav, da vrijedi (15). Odavde koristeći lemu 1 lako se može zaključiti, da je f ograničena na jednom intervalu. Dakle je ona i neprekidna.

Drugi dio teorema 1 slijedi iz teorema 3 i činjenice da postoji neizmjeriva Hamelova baza.

Teorem 2 smo dokazali ovako: Neka je $P \subseteq T + T$ perfektan skup, koji ima pozitivnu mjeru. Tvrdimo, da je f neprekidna na P. U protivnom bi postojao niz $x_0, x_n \in P$ $x_n \rightarrow x_0$ i realni broj $a > 0$ takav, da vrijedi za svako n (17). Lako je vidjeti, da postoji podniz $x_{n'}$ niza x_n takav, da vrijedi (18). Budući da su relacije (17) i (18) protivrječne, to je f neprekidna funkcija na P. Odavde se uz pomoć leme 1 lako dokazuje, da je f neprekidna na bar jednom intervalu. Dakle je f neprekidna na skupu realnih brojeva.

Teorem 3 smo dokazali tako, da smo pokazali kako se za svaku Hamelovu bazu dade konstruirati konveksna funkcija, koja je ograničena na toj Hamelovoj bazi i koja je diskontinuirana. Tada prvi dio teorema 1 daje teorem 3. Odavde specijalno slijedi, da svaka izmjeriva Hamelova baza ima mjeru nula.

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