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CONTINUOUS IMAGES OF ORDERED COMPACTA, THE SUSLIN PROPERTY AND DIADIC COMPACTA

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In this paper[1] we study spaces X, which are obtainable as images of ordered compacta K, under continuous mappings $f: K \to X$ onto X. To these spaces we refer in the following merely as to *continuous images of ordered compacta.*[2]

Our attention is centered on relations between the degree of cellularity[3] $c(X)$ of continuous images of ordered compacta and their local weight $lw(X)$ (§ 4). We prove that $lw(X) \leq c(X)$ (Theorem 2). In particular, if $c(X) \leq \aleph_0$, i. e. if X has the Suslin property[3], then X satisfies the first axiom of countability. This result together with known facts about diadic compacta (see § 9) proves a recent conjecture of P. S. Aleksandrov to the effect that a diadic compactum is the continuous image of an ordered compactum if and only if it is metrizable (Theorem 14).

The question of the equality of $c(X)$ and the degree of separability $s(X)$[3], for continuous images of ordered compacta is reduced in § 8 to the Suslin problem.

We also study the behaviour of c, s, lw, and weight w under mappings $f: K \to X$ onto X (§§ 3, 5, 6, 7), in particular when f is quasi-open and light in the sense of ordering (see § 2). One of our main results in this direction is Theorem 1, which establishes equality of $c(K)$ and $c(X)$ under light quasi-open mappings f.

Results about weight (Theorem 6) enable us to strengthen one of our earlier results of [8]. We also find that c and s are monotone functions on closed subsets of X (Theorem 12).

§ 1. *Preliminaries*

All spaces in this paper are assumed to be Hausdorff topological spaces. By a compactum we mean any Hausdorff compact space (not necessarily metrizable) and by a continuum any connected compactum.

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[1] Part of the results of this paper were announced in the authors' note [9].

[2] The authors have already studied this class of spaces in [8], also cf. [7].

[3] For definition of notions appearing in this introduction cf. § 1.

An *ordered compactum* K is a compactum provided with a total ordering $<$ such that the topology of K is the order topology induced by $<$. In other words, a subbasis for the topology of K is formed by all the sets of the form $(\cdot, t) = \{s \in K \mid s < t\}$, or $(t, \cdot) = \{s \in K \mid t < s\}$. An ordered continuum C is a connected ordered compactum. The only metrizable ordered continuum is the arc, i. e. the homeomorph of the real line segment $I = [0, 1]$. Its closed subsets are the only metrizable ordered compacta.

A useful example of a non-metrizable ordered continuum is obtained by considering the square $I \times I = \{(s, t) \mid s \in I,\ t \in I\}$ in the »lexicographic order« $<$. We set $(s, t) < (s', t')$ if and only if either $s < s'$ or $s = s'$ and $t < t'$. We denote this continuum by Q and refer to it as to the »square in lexicographic order«. Another interesting example is the ordered compactum $Q_1 \subset Q$ defined as $Q_1 = (I \times 0) \cup (I \times 1)$ with the lexicographic order.

Throughout this paper we denote by $k(A)$ the cardinal of the set A. With every space X several cardinal numbers are associated. The *weight* $w(X)$ is the least cardinal k having the property that X admits a basis for its topology with $\leq k$ elements. Clearly, a compactum X is metrizable if and only if $w(X) \leq \aleph_0$. The weight $w(x, X)$ of a space X at a point $x \in X$ is the least cardinal k having the property that there is at x a basis of neighbourhoods of cardinality $\leq k$. The *local weight* $lw(X)$ of X is defined as $\operatorname{Sup}_{x \in X} w(x, X)$. Clearly,

$$lw(X) \leq w(X). \tag{1}$$

The *degree of separability* $s(X)$ is the least cardinal k having the property that X contains a subset $R \subset X$, dense in X, and of cardinality $k(R) \leq k$. Clearly,

$$s(X) \leq w(X). \tag{2}$$

Spaces X satisfying $s(X) \leq \aleph_0$ are usually called *separable*. Finally, the *degree of cellularity* $c(X)$ is defined as Sup $k(\mathfrak{U})$, where $\mathfrak{U}$ runs through all families $\mathfrak{U} = \{U_\alpha\}$ of disjoint non-empty open sets $U_\alpha \subset X$. Clearly, $c(X)$ is well-defined and

$$c(X) \leq s(X) \leq w(X). \tag{3}$$

This notion is due to Đ. K u r e p a ([3], p. 131; also cf. [4]). A space X is said to posses the *Suslin property* provided $c(X) \leq \aleph_0$. In other words, every family of non-empty disjoint open sets in X is at most countable. In the following we refer to compacta (continua) having the Suslin property merely as to *Suslin compacta (continua)*.

The above mentioned inequalities (1), (2) and (3) are the only inequalities relating $w(X)$, $lw(X)$, $s(X)$ and $c(X)$, valid for all spaces X. Any other inequality is violated already in the class of compacta. E. g. for the ordered continuum Q we have

$$w(Q)=2^{\aleph_0},\quad lw(Q)=\aleph_0,\quad s(Q)=2^{\aleph_0},\quad c(Q)=2^{\aleph_0},$$

which shows that we can have

$$lw(X)<w(X),\quad lw(X)<s(X),\quad lw(X)<c(X).$$

For the ordered compactum $Q_1\subset Q$ we have

$$w(Q_1)=2^{\aleph_0},\quad lw(Q_1)=\aleph_0,\quad s(Q_1)=\aleph_0,\quad c(Q_1)=\aleph_0,$$

which shows that we can have

$$lw(X)<w(X),\quad s(X)<w(X),\quad c(X)<w(X).$$

An example, showing that $s(X)<lw(X)$ can occur, is furnished by the direct product $P=\prod_{a\in A} I_a$, where $I_a=I$, and $k(A)=2^{\aleph_0}$. For this space $s(P)=\aleph_0$ (see e. g. [2], N, p. 103). On the other hand, it is well known that $lw(P)=w(P)=2^{\aleph_0}$. Furthermore, $c(P)=\aleph_0$, because of the following theorem due to E. Szpilrajn [12].

Let $\{X_a, a\in A\}$ be any family of topological spaces X_a of weight $w(X_a)\leq\aleph_0$. Then $\prod_{a\in A} X_a$ has the Suslin property.

Thus our example also shows that $c(X)<lw(X)$ can occur. Finally, consider the space $T=\prod_{\beta\in B} I_\beta$, where $I_\beta=I$ and $k(B)>2^{\aleph_0}$. Clearly, $w(T)=lw(T)>2^{\aleph_0}$. By the Szpilrajn theorem $c(T)=\aleph_0$. However, $s(T)>\aleph_0=c(T)$, because of the following proposition:

If X is a regular space with $s(X)\leq\aleph_0$, then $w(X)\leq 2^{\aleph_0}$.

Proof. Let $R\subset X$ be a countable set, dense in X. Assign to each subset $S\subset R$ the open set $U_S=$ Interior $Cl(S)$. Clearly, the family $\mathfrak{U}=\{U_S\}$, where S runs through all subsets of R is of cardinality $\leq 2^{\aleph_0}$. But $\mathfrak{U}$ is readily seen to be a basis for the topology of X. Indeed, if $x\in X$ and $V\subset X$ is open, $x\in V$, then choose an open set W such that $x\in W\subset Cl\,W\subset V$. Put $S=R\cap W$. Clearly, $W\subset Cl\,S\subset Cl\,W\subset V$, and therefore, $x\in W\subset$ Interior $Cl\,S=U_S\subset V$.

Notice, that for ordered compacta K, in addition to (1), (2) and (3) we always have

$$lw(K)\leq s(K)\ \text{and} \tag{4}$$

$$lw(K)\leq c(K) \tag{5}$$

(see Lemma 5.).

As to the relations between c (X) and $s(X)$, *it is not known whether one can have* c (K) $<$ s (K) *for some ordered compactum* K. Actually, the question: *does* c (K) $= \aleph_0$ *imply* s (K) $= \aleph_0$, *for ordered compacta* K? is the famous unsolved *Suslin problem,* raised by M. Ya. Suslin in 1920 (Fund. Math. 1 (1920), p. 223, Problem 3.).

§ 2. *Monotone, light and quasi-open mappings*

In an ordered compactum K an interval (a, b), $a < b$, is the set

$$(a, \cdot) \cap (\cdot, b) = \{t \in K \mid a < t < b\}.$$

A segment $[a, b]$, $a \leq b$, is the set

$$\{t \in K \mid a \leq t \leq b\}.$$

If $M \subset (K, <)$ is any subset, we call a non-empty subset $N \subset M$ an *order component* of M provided
(α) $a, b \in N$ implies $[a, b] \subset N$ (here $[a, b]$ denotes a segment of K) and
(β) whenever a subset $N' \subset M$ has property (α), then $N' \subset N$.

Clearly, the order components of M give a decomposition of M into disjoint subsets. If M is open (closed) its components are intervals (segments) of K.

Definition 1. *A mapping* f : K $\to$ X *of an ordered compactum* (K $<$) *onto* X *is said to be* monotone *in the sense of ordering provided, for each* x $\in$ X, f^{-1} (x) *is a segment of* K.

Definition 2. *A mapping* f : K $\to$ X *of an ordered compactum* (K, $<$) *onto* X *is said to be* light *in the sense of ordering provided, for each* x $\in$ X, *every order component of* f^{-1} (x) *has but one single point.*

Remark. If $K = C$ is an ordered continuum, then these definitions give monotone and light mappings in the usual sense, as used in topology.

For simplicity we shall often leave out the attribute »in the sense of ordering«.

Lemma 1. *Let* K *be an ordered compactum and* f : K $\to$ X *a mapping onto* X. *Then there is a compactum* K', *a mapping* m : K $\to$ K' *and a mapping* g : K' $\to$ X *such that* f = g m. *Moreover,* m *is monotone and* g *is light in the sense of ordering. This factorization is uniquely determined.*

This lemma is the order analogue of the well-known Whyburn monotone-light factorization theorem (see [13] and [10]).

Proof. It suffices to consider the decomposition of K produced by the order components of the sets $f^{-1}(x), x \in X$. K' is defined as the corresponding quotient space and $m : K \to K'$ as the corresponding natural mapping. The definition of g follows from the requirement $f = g\, m$.

In the following we shall also need another class of mappings that we shall call, for brevity, *quasi-open* mappings.

Definition 3. *Let* X *and* Y *be topological spaces and* $f: X \to Y$ *a mapping.* f *is called* quasi-open, *provided for each non-empty open set* $U \subset X$ *the set* $f(U)$ *has a non-empty interior* $\mathrm{Int}\, f(U) \neq 0$.

Lemma 2. *Let* $f: X \to Y$ *be a mapping of the compactum* X *onto* Y. *Then there exists a compactum* $X_1 \subset X$ *such that* $f(X_1) = Y$ *and that the restriction* $f_1 = f \mid X_1$ *is a quasi-open mapping* $f: X_1 \to Y$.

Proof. Denote by $\mathfrak{F}$ the family of all closed subsets $X_\alpha \subset X$ for which

$$f(X_\alpha) = Y. \tag{1}$$

Define in $\mathfrak{F}$ a partial order $\leq$ by setting $X_\alpha \leq X_\beta$ if and only if $X_\alpha \supset X_\beta$.

Let us prove that each totally ordered subset $\mathfrak{H} \subset \mathfrak{F}$ has an upper bound in $(\mathfrak{F}, \leq)$.

Clearly, it suffices to show that the set

$$X' = \bigcap_\beta X_\beta, \; X_\beta \in \mathfrak{H} \tag{2}$$

belongs to $\mathfrak{F}$, i. e. that

$$f(X') = Y. \tag{3}$$

Thus take any $y \in Y$. In any finite subfamily $\{X_{\beta_1}, \ldots, X_{\beta_n}\} \subset \mathfrak{H}$ one of the members, say X_{β_n}, is contained in the intersection of this subfamily ($\mathfrak{H}$ is totally ordered). Therefore, by (1),

$$0 \neq f^{-1}(y) \cap X_{\beta_n} \subset \bigcap_{i=1}^{n} (f^{-1}(y) \cap X_{\beta_i}), \tag{4}$$

which shows that $\{f^{-1}(y) \cap X_\beta\}$, $X_\beta \in \mathfrak{H}$, is a centered system of closed sets. Hence, by compactness,

$$0 \neq \bigcap (f^{-1}(y) \cap X_\beta), \; X_\beta \in \mathfrak{H}, \tag{5}$$

which proves that

$$f^{-1}(y) \cap X' \neq 0. \tag{6}$$

Thus we can apply Zorn's lemma and obtain a maximal element $X_1 \in \mathfrak{F}$ which, we claim, satisfies the assertion of the lemma. Assuming that this were not the case, we could find a set $U \subset X$, open in X and such that $U_1 = U \cap X_1 \neq 0$ and $\mathrm{Int}\, f(U_1) = 0$. Then we could prove that $X_2 = X_1 \setminus U_1 \in \mathfrak{F}$. Indeed $Y \setminus f(U_1)$, and a fortiori $f(X_2) = f(X_1 \setminus U_1) \supset Y \setminus f(U_1)$, would be sets dense in Y (notice that $f(X_1) = Y$), which would yield $f(X_2) = Y$, $f(X_2)$ being a closed set. Moreover X_2 being a proper subset of X_1, we would have $X_1 < X_2$, which is in contradiction with the assumption that X_1 is maximal in $\mathfrak{F}$. This completes the proof.

Remark. We shall need this lemma only in the case when $X = K$ is an ordered compactum.

Applying subsequently Lemma 2 and Lemma 1 we obtain this

Lemma 3. *Let* X *be the continuous image of an ordered compactum. Then there exist an ordered compactum* K *and a mapping* $f : K \to X$ *onto* X, *which is at the same time light in the sense of ordering and quasi-open.*

We conclude this section by a very simple but important lemma concerning arbitrary continuous mappings of ordered compacta.

Lemma 4. *Let* $f : K \to X$ *be a mapping of an ordered compactum* K *into* X. *Let* F *and* F′ *be two disjoint closed subsets of* X *and* $\{V_\lambda\}$, $\lambda \in \Lambda$, *a family of disjoint non-empty intervals* $V_\lambda = (a_\lambda, b_\lambda)$ *of* K. *If, for each* $\lambda \in \Lambda$, $f(V_\lambda) \cap F \neq 0$ *and* $f(V_\lambda) \cap F' \neq 0$, *then* Λ *is a finite set.*

Proof. Assume on the contrary that Λ is infinite and choose an infinite sequence of different indices $\lambda_1, \ldots, \lambda_n, \ldots \in \Lambda$. There is no loss of generality in assuming that the left end-points a_{λ_n} of the intervals V_{λ_n} converge to some point $a_0 \in K$. We can also assume that the sequence a_{λ_n} is monotone. Since the intervals V_{λ_n} are disjoint, each neighbourhood of a_0 contains all but a finite number of sets V_{λ_n}. Now, choose in V_{λ_n} two points t_n and t'_n such that $f(t_n) \in F$ and $f(t'_n) \in F'$. Then, clearly, $a_0 = \lim_n t_n = \lim_n t'_n$. We conclude, by continuity of f, that $f(a_0) = \lim_n f(t_n) \in F$ and at the same time $f(a_0) = \lim_n f(t'_n) \in F'$, which contradicts the assumption $F \cap F' = 0$.

§ 3. *Light quasi-open mappings of ordered compacta and the degree of cellularity.*

In this section we establish the central theorem of the whole paper.

Theorem 1. *Let* K *be an ordered compactum and* $f : K \to X$ *a map onto* X *which is quasi-open and light in the sense of ordering. Then the degree of cellularity* $c(K) = c(X)$, *whenever* $c(X)$ *is infinite; if* $c(X)$ *is finite, then* $c(K)$ *is finite too.*

Corollary 1. *Let* K *be an ordered compactum and* $f : K \to X$ *a map onto* X *which is quasi-open and light in the sense of ordering. Then* X *has the Suslin property if and only if* K *too has the Suslin property.*

Proof of Theorem 1. First observe that for any mapping $f : K \to X$ onto X we have $c(X) \leq c(K)$. As to the reversed inequality, first consider the set $Z \subset X$ of all the isolated points of X. We shall show that

$$k(f^{-1}(Z)) \leq c(X). \tag{1}$$

Indeed, for any $z \in Z$, $\{z\}$ is open and closed. Therefore, $f^{-1}(z)$ decomposes into order components, each one of which is at the same time an interval and a segment. There is only a finite number of these components, because they cover the compactum $f^{-1}(z)$. f being light, each of the components reduces to a point. Thus, for each $z \in Z$, $f^{-1}(z)$ is a finite set. Furthermore, $k(Z) \leq c(X)$, because $\{z\}$, $z \in Z$, is a family of disjoint open sets of X. This establishes (1), if $c(X)$ is infinite.

If $c(X)$ is finite, then clearly X itself is a finite set and thus $X = Z$. Therefore, by the argument used in proving (1), K is finite, which implies that $c(K)$ is finite too. Thus, we can assume from now on that $c(X)$ is infinite.

Given any family $\{U_a\}$, $a \in A$, of disjoint open non-empty sets U_a of K, we have to prove that $k(A) \leq c(X)$.

Because of (1), $f^{-1}(Z)$ can intersect at most $c(X)$ sets U_a, $a \in A$. Therefore, we can assume in the following (with no loss of generality) that $U_a \cap f^{-1}(Z) = 0$; for all $a \in A$.

Now we shall assign to each $a \in A$ a non-empty open F_σ-set $U_a^* \subset X$ having the property that

$$U_a^* \subset f(U_a) . \tag{2}$$

This is readily done by taking a point $x_0 \in \operatorname{Int} f(U_a) \neq 0$ (recall that f is quasi-open) and constructing, by normality, a sequence of open sets V_n such that

$$x_0 \in V_1 \subset Cl(V_1) \subset \ldots \subset V_n \subset Cl(V_n) \subset \ldots \subset \operatorname{Int} f(U_a) \subset f(U_a) . \tag{3}$$

Clearly, the set

$$U_a^* = \bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} Cl(V_n) \tag{4}$$

has all the required properties.

Notice, that U_a^* always contains more than one point, because of $U_a \cap f^{-1}(Z) = 0$.

Now, we define in A a partial ordering $<$ by setting $a < a'$, $a, a' \in A$, if and only if $U_a^* \supset U_{a'}^*$. We shall prove that $(A, <)$ has the following properties

(i) for any fixed $a \in A$, the set of all $a' \in A$ with $a' < a$ is finite,

(ii) for each totally unordered[4] subset $A' \subset A$ we have $k(A') \leq c(X)$.

From (i) and (ii) it readily follows that $k(A) \leq c(X)$. Indeed, denote by $R_0(A)$ the set of all minimal elements of A (cf. [3], p. 72) and define by induction $R_n(A)$ as the set

$$R_n(A) = R_0(A \setminus (R_0(A) \cup \ldots \cup R_{n-1}(A))) . \tag{5}$$

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[4] A subset of a partially ordered set is said to be totally unordered provided no pair of its elements is in the order relation.

Clearly, for any n, $R_n(A)$ is totally unordered and thus, by (ii) we obtain

$$k(R_n(A)) \leq c(X). \tag{6}$$

On the other hand, by (i), we have

$$A = \bigcup_{n<\omega_0} R_n(A), \tag{7}$$

because an element $\alpha \in A$ with n predecessors in A surely belongs to $R_0(A) \cup \ldots \cup R_n(A)$.

Thus, all that remains to be done is to prove (i) and (ii).

Proof of (i). Let $\alpha_0 \in A$ be any fixed element. Choose in $U_{\alpha_0}^*$ two distinct points x and x'. Let $A_1 \subset A$ be the set of all $\alpha \in A$ which precede α_0. In other words, $\alpha \in A_1$ means that $\alpha < \alpha_0$ and therefore,

$$f(U_\alpha) \supset U^*_\alpha \supset U^*_{\alpha_0} \supset \{x, x'\}. \tag{8}$$

Let V be an open set of $X \setminus \{x\}$ containing x' and put $F = X \setminus V$ and $F' = \{x'\}$. Clearly, F and F' are disjoint closed sets and since $x \in F$ and $x' \in F'$, we have $f(U_\alpha) \cap F \neq 0$ and $f(U_\alpha) \cap F' \neq 0$, for each $\alpha \in A_1$. Therefore, by Lemma 4, we conclude that A_1 is a finite set.

Proof of (ii). Let $A' \subset A$ be any infinite totally unordered subset of $(A, <)$. We have to prove that $k(A') \leq c(X)$.

Let $A'(\alpha')$, $\alpha' \in A'$, denote the set of all elements $\alpha \in A'$ such that $U_{\alpha'}^* \cap U_\alpha^* \neq 0$. We shall define a subset $B \subset A'$ such that

$$A' = \bigcup_{\beta \in B} A'(\beta) \tag{9}$$

and that $\{U_\beta^*\}$, $\beta \in B$, is a family of disjoint sets U_β^*.

B is defined by transfinite induction as follows. Let $\alpha_0 < \alpha_1 < \ldots < \alpha_\xi < \ldots, \xi < \omega_\tau$ be a well-ordering of A'. We set $\alpha_0 \in B$. Assume that we have already determined, for each α_η, $\eta < \xi < \omega_\tau$, does α_η belong to B or not. We set $\alpha_\xi \in B$ if and only if $U_{\alpha_\xi}^* \cap U_{\alpha_\eta}^* = 0$, for all $\eta < \xi$. Clearly, B is well-defined and has the two required properties.

$\{U_\beta^*\}$, $\beta \in B$, is a family of disjoint non-empty open sets of X. Therefore, we have

$$k(B) \leq c(X). \tag{10}$$

Taking into account (9) and (10), our proof will be completed, if we show that

$$k(A'(\beta)) \leq \aleph_0, \tag{11}$$

for each $\beta \in B$.

In order to establish (11) recall that U_β^* is an open F_σ-set and thus

$$U_\beta^* = \bigcup_{i=1}^{\infty} F_i, \tag{12}$$

where $F_i \subset U_\beta^*$ are closed sets. Therefore, it suffices to show, that, for each $i \in \{1, 2, \ldots\}$, the set $A_i' \subset A'(\beta)$, consisting of all $\alpha \in A'(\beta)$ with $U_\alpha^* \cap F_i \neq 0$, is a finite set.

For this purpose put $F = F_i$ and $F' = X \setminus U_\beta^*$. For $\alpha \in A_i' \subset A'$, $\alpha \neq \beta$, we have, by definition, $U_\alpha^* \cap F \neq 0$. Moreover, we have $U_\alpha^* \cap F' \neq 0$, for otherwise we would have $U_\alpha^* \subset U_\beta^*$ and thus $\beta < \alpha$, contrary to the assumption that A' is totally unordered and $\alpha, \beta \in A'$. Since $U_\alpha^* \subset f(U_\alpha)$, and $\{U_\alpha\}$, $\alpha \in A$, is a disjoint family of open sets, Lemma 4 yields the conclusion that A_i' is indeed a finite set. This ends the proof of Theorem 1.

Remark. The constant mapping of a non-Suslin ordered compactum shows that lightness is not a redundant condition in Theorem 1.

Problem 1. *Does Theorem 1 remain true if one only assumes that* f *is light in the sense of ordering and do not require that* f *be quasi-open?*

§ 4. *The degree of cellularity and local weight of continuous images of ordered compacta*

We open this section by a simple lemma.

Lemma 5. *For ordered compacta* K *the weight* w (K), *the degree of separability* s (K), *the degree of cellularity* c (K) *and the local weight* lw (K) *always satisfy the inequality*

$$lw\,(K) \leq c\,(K) \leq s(K) \leq w\,(K) \tag{1}$$

Proof. It suffices to prove that $lw\,(K) \leq c\,(K)$, because $c\,(X) \leq s\,(X) \leq w\,(X)$ holds for all spaces X. Thus, we have to show that each point $t \in K$ admits a basis containing at most $c\,(K)$ neighbourhoods. This is trivial if t is an isolated point. Therefore, assume that t is an accumulation point of the set $(\cdot, t) = \{s \in K \mid s < t\}$. By transfinite induction we can easily define such a transfinite sequence

$$s_0 < s_1 < \ldots < s_\xi < \ldots, \ \xi < \eta, \tag{2}$$

that each interval $(s_\xi, s_{\xi+1})$, $\xi < \eta$, is non-empty and

$$t = \operatorname*{Sup}_{\xi < \eta} \{s_\xi\}. \tag{3}$$

Since $\{(s_\xi, s_{\xi+1})\}$ is a family of disjoint non-empty open sets of K containing $k\,(\eta)$ members, it follows that $k\,(\eta) \leq c\,(K)$. Hence t is the least upper bound of a sequence of $\leq c\,(K)$ points $s < t$.

If t is also a point of accumulation of $(t, \cdot) = \{u \in K \mid t < u\}$, we obtain a decreasing sequence of $\leq c\,(K)$ points u_ζ with $t = \inf u_\zeta$. Clearly, (s_ξ, u_ζ) give a basis of intervals at the point t, containing at most $c\,(K) \cdot c\,(K) = c\,(K)$ members. We proceed similarly in the case when t is isolated from one side.

Now, we shall establish one of the main results of this paper, asserting that (1) remains true also for continuous images of ordered compacta.

T h e o r e m 2. *If* X *is the continuous image of an ordered compactum, then its local weight* lw(X) *does not surpass its degree of cellularity* c(X), *so that we have*

$$l\,w(X) \leq c(X) \leq s(X) \leq w(X). \tag{4}$$

C o r o l l a r y 2. *If the Suslin compactum* X *is the continuous image of an ordered compactum, then* X *satisfies the first axiom of countability, i. e. its local weight* $\mathrm{lw}(X) \leq \aleph_0$.

Theorem 2 will be derived as a consequence of this

T h e o r e m 3. *Let* X *be the continuous image of an ordered compactum. Then the degree of cellularity* $c(X) \leq \aleph_\alpha$, $\alpha \geq 0$, *if and only if each open subset* $V \subset X$ *is the union of* $\leq \aleph_\alpha$ *closed sets of* X.

Proof of sufficiency. By Lemma 3 we can assume (with no loss of generality) that $X = f(K)$, where f is quasi-open and light in the sense of ordering. Then, by Theorem 1, $c(K) = c(X)$, provided $c(X)$ is infinite. Therefore, by Lemma 5, $l\,w(K) \leq c(X) \leq \leq \aleph_\alpha$. However, this implies that each interval (a, b) in K is the union of $\leq \aleph_\alpha$ segments. Indeed, let for instance a have an immediate successor a', $a < a'$, $(a, a') = 0$, and let b be a point of accumulation of $(\cdot, b)$. Then

$$(a, b) = \bigcup_{\xi} [a', b_\xi], \tag{5}$$

where $\{b_\xi\}$, $\xi < \eta$, is a monotone increasing sequence of points from (a, b) with $\mathrm{Sup}_\xi\, b_\xi = b$ and $k(\eta) \leq lw(K) \leq \aleph_\alpha$.

Now, if $V \subset X$ is any open set, then $f^{-1}(V)$ decomposes into at most $c(K) = c(X) \leq \aleph_\alpha$ disjoint intervals, and since each of these intervals is the union of at most $\aleph_\alpha$ segments, we conclude that $f^{-1}(V)$ itself is the union of at most $\aleph_\alpha$ segments. f being closed, we obtain that $V = f\,f^{-1}(V)$ is indeed the union of at most $\aleph_\alpha$ closed sets.

Necessity follows from this

L e m m a 6. *Let* X *be a compactum such that each open set* $V \subset X$ *is the union of at most* $\aleph_\alpha$ *closed sets. Then* $c(X) \leq \aleph_\alpha$.

Proof. Let $\{V_\alpha\}$, $\alpha \in A$, be a family of non-empty disjoint open sets. Then

$$V = \bigcup_{\alpha \in A} V_\alpha \tag{6}$$

is an open set of X and, by assumption,

$$V = \bigcup_{\beta \in B} F_\beta, \tag{7}$$

where $F_\beta \subset X$ is closed and $k(B) \leq \aleph_\alpha$.

$\{V_\alpha\}$ is an open covering for each F_β, so that F_β must be contained already in finitely many sets V_α.

Therefore, $V = \bigcup_{\beta \in B} F_\beta$ must be contained in $k(B) \leq \aleph_\alpha$ sets V_α, which proves that $k(A) \leq \aleph_\alpha$ and therefore $c(X) \leq \aleph_\alpha$.

Corollary 3. *Let* X *be the continuous image of an ordered compactum. Then* X *has the Suslin property if and only if each open set* $V \subset X$ *is an* F_σ*-set.*

Proof of Theorem 2. If $c(X)$ is finite, then X is finite, and (4) is fulfilled. Therefore, assume that $c(X) = \aleph_\alpha$. Then, by Theorem 3, each open set $V \subset X$ is the union of $\leq \aleph_\alpha$ closed sets of X and dually each closed set $F \subset X$ is the intersection of $\leq \aleph_\alpha$ open sets of X. In particular, for each $x_0 \in X$, there is a family $\{V_\lambda\}$, $\lambda \in \Lambda$, of open sets $V_\lambda \subset X$ such that

$$\bigcap_{\lambda \in \Lambda} V_\lambda = \{x_0\} \tag{8}$$

and $k(\Lambda) \leq \aleph_\alpha = c(X)$.

Choose, for each $\lambda \in \Lambda$, an open set U_λ, $x_0 \in U_\lambda$, such that

$$Cl(U_\lambda) \subset V_\lambda . \tag{9}$$

We shall prove that the family $\mathfrak{U}$ of all finite intersections $U = U_{\lambda_1} \cap \ldots \cap U_{\lambda_n}$, $\lambda_1, \ldots, \lambda_n \in \Lambda$, is a basis of neighbourhoods of x_0. Observe that

$$k(\mathfrak{U}) \leq \aleph_\alpha = c(X), \tag{10}$$

so that (10) implies (4). Thus, our proof will be completed if we show that, for any open $V \subset X$, $x_0 \in V$, there is a finite subset $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda$, such that

$$U_{\lambda_1} \cap \ldots \cap U_{\lambda_n} \subset Cl(U_{\lambda_1}) \cap \ldots \cap Cl(U_{\lambda_n}) \subset V . \tag{11}$$

Assuming that this is not the case, we would have

$$[Cl(U_{\lambda_1}) \cap (X \setminus V)] \cap \ldots \cap [Cl(U_{\lambda_n}) \cap (X \setminus V)] \neq 0, \tag{12}$$

for all finite subsets $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda$, which would mean that $\{Cl(U_\lambda) \cap (X \setminus V)\}$, $\lambda \in \Lambda$, is a centered system of closed sets. By compactness of X it would follow that

$$[\bigcap_{\lambda \in \Lambda} Cl(U_\lambda)] \cap (X \setminus V) \neq 0, \tag{13}$$

and a fortiori

$$\bigcap_{\lambda \in \Lambda} V_\lambda \cap (X \setminus V) \neq 0, \tag{14}$$

which, however, contradicts (8). This completes the proof of Theorem 2.

§ 5. *The increasing of local weight under continuous mappings*

The weight w, degree of separability s and degree of cellularity c cannot increase under a continuous mapping. In other words, if $f: X \to Y$ is a mapping of a compactum X onto Y, then $w(Y) \leq w(X)$, $s(Y) \leq s(X)$ and $c(Y) \leq c(X)$.

On the contrary, we have

Theorem 4. *A mapping* f *can increase the local weight of a compactum. Moreover, there exist ordered continua* C *and such quasi-open light mappings* $f: C \to X$ *onto* X *that* $\mathrm{l\,w}(C) < \mathrm{l\,w}(X)$.

This answers a question raised by Đ. Kurepa several years ago (unpublished).

An example proving Theorem 4 is provided by the square in lexicographic order (see § 1), which we have denoted by Q. Q is a continuum and $l\,w(Q) = \aleph_0$. Y is defined as follows. Let

$$P = \prod_{t \in I} I_t\,,\ I_t = I = [0, 1]\,, \tag{1}$$

be the direct product of $2^{\aleph_0}$ copies of $I = [0, 1]$. Let $Y_{t_0} \subset P$ be the set of all $p \in P$ having all coordinates $p_t = 0$, for $t \neq t_0$.

Then we set

$$Y = \bigcup_{t \in I} Y_t\,. \tag{2}$$

Clearly, Y is a continuum. The point $O \in P$, having all coordinates zero, belongs to Y and it is readily seen, that Y does not admit of a countable basis of neighbourhoods at O. Actually, $l\,w(Y) = 2^{\aleph_0} > \aleph_0 = l\,w(Q)$.

However, there exists a mapping $f: Q \to Y$ onto Y. f is defined as follows. It maps the segment $[t \times 0, t \times \frac{1}{2}]$ of $(Q, <)$ linearly onto Y_t (recall that $Y_t = I_t \times 0 \times 0 \times \ldots = [0, 1] \times 0 \times 0 \times \ldots$) in such a way that $f(t \times 0) = O$, and f maps the segment $[t \times \frac{1}{2}, t \times 1]$ of $(Q, <)$ linearly onto Y_t in such a way that $f(t \times 1) = O$. It is readily seen, that f is continuous and that $f(Q) = Y$. Moreover, f is light and quasi-open. Of course, Y has not the Suslin property. This completes the proof of Theorem 4.

If $f: K \to X$ is any continuous map of the ordered compactum K onto X, we introduce the cardinal

$$\varkappa(f) = \operatorname*{Sup}_{x \in X} k(f^{-1}(x))\,. \tag{3}$$

$\varkappa(f)$ exists and, clearly, $\varkappa(f) \leq k(K)$.

Theorem 5. *The cardinal* $\varkappa(f)$ *given by* (3) *satisfies the inequality*

$$l\,w(X) \leq \varkappa(f) \cdot l\,w(K)\,. \tag{4}$$

Proof. If $l\,w(K)$ is finite, then K and X are finite sets and (4) is trivially true. We assume henceforth that $l\,w(K) \geq \aleph_0$. Given any $x \in X$, consider $f^{-1}(x)$ and for any $t \in f^{-1}(x)$ choose such a basis $\mathfrak{U}(t)$ of neighbourhoods that $k(\mathfrak{U}(t)) \leq l w(K)$. Let T be the set of all finite subsets of $f^{-1}(x)$.

Since $k(f^{-1}(x)) \leq \varkappa(f)$, clearly, $k(T) \leq \varkappa(f)$ provided $\varkappa(f)$ is infinite. If $\varkappa(f)$ is finite, then $k(T)$ is finite too. Let $\mathfrak{U}$ be the family of all sets $U = U(t_1) \cup \ldots \cup U(t_n)$, where $\{t_1, \ldots, t_n\} \in T$ and $U(t_i) \in \mathfrak{U}(t_i)$. Since $lw(K)$ is infinite ,it follows that

$$k(\mathfrak{U}) \leq \varkappa(f) \cdot lw(K). \tag{5}$$

Now we shall prove that $\{\mathrm{Int}\, f(U)\}$, $U \in \mathfrak{U}$, is a basis of neighbourhoods of x. Then (5) shall imply (4).

Let V be an open set in X about $x \in X$. Choose for any $t \in f^{-1}(x)$ a set $U(t) \in \mathfrak{U}(t)$ such that $U(t) \subset f^{-1}(V)$. By compactness of $f^{-1}(x)$, there is a finite set $\{t_1, \ldots, t_n\} \in T$, such that

$$f^{-1}(x) \subset U(t_1) \cup \ldots \cup U(t_n) = U. \tag{6}$$

Thus $U \in \mathfrak{U}$ and, clearly, $x \in \mathrm{Int}\, f(U) \subset V$, which completes our proof.

C o r o l l a r y 4. *If* $f: K \to X$ *maps* K *onto* X *and* $\varkappa(f) < lw(X)$ *then the local weight cannot increase, i. e. we have*

$$lw(X) \leq lw(K). \tag{7}$$

Indeed, this is trivial if $lw(X)$ is finite, because then $lw(X) = 1$. Thus assume that $lw(X)$ is infinite and (7) false. Then we would have $lw(K) < lw(X)$ beside the assumed inequality $\varkappa(f) < lw(X)$. Multiplying these two inequalities, we would obtain

$$\varkappa(f) \cdot lw(K) < lw(X), \tag{8}$$

which, however, contradicts (4).

§ 6. *Light mappings and the decreasing of weight and local weight*

In this section we consider the question of the decreasing of numbers $w(K)$ and $lw(K)$ under continuous mappings. Clearly, these numbers, as well as $s(K)$ and $c(K)$, can always decrease. This occurs e. g. if we map an ordered compactum K with $\aleph_0 < lw(K)$ onto a point. However, the question becomes interesting if we restrict ourselves to mappings $f: K \to X$ which are light in the sense of ordering.

L e m m a 7. *Let* $f: K \to X$ *be a mapping, light in the sense of ordering, and let* $x_0 \in X$ *and* $t_0 \in f^{-1}(x_0)$ *be two points. Furthermore, let* $\mathfrak{B} = \{V_a\}$, $a \in A$, *be a basis of neighbourhoods at* x_0 *and* U_a, $a \in A$, *the order component of* $f^{-1}(V_a)$ *containing* t_0. *Then* $\mathfrak{U} = \{U_a\}$, $a \in A$, *is a basis of neighbourhoods at* t_0.

Proof. Let (a, b) be any interval of K containing t_0. Then there exists an a', $a \leq a' < t_0$ with $f(a') \neq f(t_0) = x_0$, for otherwise we would have $f([a, t_0]) = \{x_0\}$, $a < t_0$, contradicting the lightness of f. Similarly, there is a b', $t_0 < b' \leq b$ with $f(b') \neq x_0$. Let $V_a \in \mathfrak{B}$ be such that $x_0 \in V_a \subset X \setminus \{f(a'), f(b')\}$. Then $f^{-1}(V_a) \cap \{a', b'\} = 0$ and, therefore, the component U_a of $f^{-1}(V_a)$, which contains t_0, is itself contained in (a', b'). This proves that $\mathfrak{U} = \{U_a\}$, $a \in A$, is indeed a basis of neighbourhoods at t_0.

Theorem 6. *Let* $f: K \to X$ *be a map of the ordered compactum* K *onto* X. *If* f *is light in the sense of ordering, then the weight* $w(K) \leq w(X)$, *whenever* $w(X)$ *is infinite. If* $w(X)$ *is finite, then* $w(K)$ *is finite too.*

Proof. Let $\mathfrak{V} = \{V\}$ be such an open basis for the topology of X, that $k(\mathfrak{V}) \leq w(X)$. For any $V \in \mathfrak{V}$, $f^{-1}(V)$ is an open set of K.

Let $\mathfrak{U}$ be the family of all the order components of $f^{-1}(V)$, when V runs through $\mathfrak{V}$. Given any $t_0 \in K$, consider $x_0 = f(t_0)$ and let $\mathfrak{V}' \subset \mathfrak{V}$ consist of all $V \in \mathfrak{V}$, which contain x_0. Then $\mathfrak{V}'$ is a basis of neighbourhoods at x_0 and, by Lemma 7, there is a subset $\mathfrak{U}' \subset \mathfrak{U}$, constituting a basis of neighbourhoods at t_0. This proves that $\mathfrak{U}$ is a basis for the topology of K.

Now, observe that every open set $V \subset X$ is the union of at most $w(X)$ closed sets F. It suffices, to consider all $W \in \mathfrak{V}$ such that $Cl\, W \subset V$ and recall that X is regular. However, if $F \subset V$ is a closed subset, then due to compactness $f^{-1}(F)$ is contained in finitely many components of $f^{-1}(V)$. Since V is a union of at most $w(X)$ closed sets, it follows that $f^{-1}(V)$ has at most $w(X)$ components if $w(X)$ is infinite and has finitely many components if $w(X)$ is finite. This and $k(\mathfrak{V}) \leq w(X)$ proves that $k(\mathfrak{U}) \leq w(X)$ if $w(X)$ is infinite and $k(\mathfrak{U})$ is finite if such is $w(X)$. This completes our proof.

Remark. Theorem 6 is the order analogue of Theorem 1 of [6].

Now we apply Theorem 6 to obtain a strengthening of the main result of [8] (Theorem 1)[5]

Theorem 7. *Let* X *be the continuous image of an ordered compactum and* $p: X \to Y$ *a mapping of* X *onto* Y *such that* $\operatorname{Int} p^{-1}(y) = 0$, *for each* $y \in Y$. *If* X *is locally connected, then* $w(X) \leq w(Y)$.

The proof follows the same plan as in [8]. By Lemma 1 we can assume that $X = f(K)$, where K is an ordered compactum and f is quasi-open. Then, for any $y \in Y$ the set $(p\,f)^{-1}(y) = f^{-1}(p^{-1}(y))$ cannot contain a non-empty interval U, because $f(U)$ would be part of $p^{-1}(y)$ and thus would have an empty interior.

Now consider all pairs of points $t, t' \in K$, $t < t'$, such that the interval $(t, t') = 0$ and $(pf)(t) = (pf)(t')$. Identifying the points in each such pair, we obtain a new ordered compactum K_1. Observe that any two such pairs $\{t, t'\}$ and $\{s, s'\}$ are disjoint, since otherwise we would have, say $t' = s$, and thus $\{s\} = (t, s') \neq 0$ would be a non-empty interval contained in $(pf)^{-1}(y)$, $y = (pf)(s)$.

[5] Theorem 7 is not used in proving other tneorems of this paper.

Denoting the identification map by $m: K \to K_1$, there exists a uniquely defined map $g: K_1 \to Y$ such that $gm = pf$. The map g is light in the sense of ordering. Therefore, by Theorem 6,

$$w(K_1) \leq w(Y) \tag{1}$$

($w(Y)$ is infinite, for otherwise Y would be finite and for any $y \in Y$ the set $p^{-1}(y)$ would be open contrary to the assumptions). (1) implies $s(K_1) \leq w(Y)$. Let $R_1 \subset K_1$ be a set dense in K_1 and such that $k(R_1) \leq w(Y)$.

Clearly, the set $R = m^{-1}(R_1)$ is then dense in K and $k(R) \leq \leq 2k(R_1) \leq w(Y)$. Hence

$$s(K) \leq w(Y). \tag{2}$$

We conclude the proof by combining (2) and a proposition from [8] (Lemma 3), which reads as follows:

Let K *be an ordered compactum and* $\mathrm{f}: \mathrm{K} \to \mathrm{X}$ *a mapping onto* X. *If* X *is locally connected, then* $\mathrm{w}(\mathrm{X}) \leq \mathrm{s}(\mathrm{K})$.

Now, following the same plan as in [8], we can prove[6]

Theorem 8. *If* X *is the continuous image of an ordered compactum and is locally connected, then* $\mathrm{w}(\mathrm{X}) = \mathrm{s}(\mathrm{X})$.

This improves Theorem 4 of [8].

Local connectedness is not a redundant condition in Theorems 7 and 8, since one can have, even for ordered compacta K (without isolated points) $s(K) < w(K)$ (see Q_1 in § 1).

We conclude this section by proving

Theorem 9. *Let* $\mathrm{f}: \mathrm{K} \to \mathrm{X}$ *be the mapping of an ordered compactum* K *onto* X. *If* f *is light in the sense of ordering, then the local weight* $\mathrm{lw}(\mathrm{K}) \leq \mathrm{lw}(\mathrm{X})$.

Proof. Given $t_0 \in K$, consider $x_0 = f(t_0) \in X$ and choose such a basis $\mathfrak{V}$ of neighbourhoods at x_0 that $k(\mathfrak{V}) \leq lw(X)$. Then, by Lemma 7, there is a basis $\mathfrak{U}$ of neighbourhoods at t_0 such that $k(\mathfrak{U}) = k(\mathfrak{V}) \leq lw(X)$, which proves that $lw(K) \leq lw(X)$.

§ 7. *Light quasi-open mappings and the decreasing of the degree of separability*

Theorem 10. *Let* K *be an ordered compactum and* $\mathrm{f}: \mathrm{K} \to \mathrm{X}$ *a mapping onto* X. *If* f *is quasi-open and light in the sense of ordering, then the degree of separability* $\mathrm{s}(\mathrm{K}) = \mathrm{s}(\mathrm{X})$, *whenever* $\mathrm{s}(\mathrm{X})$ *is infinite; if* $\mathrm{s}(\mathrm{X})$ *is finite, then* $\mathrm{s}(\mathrm{K})$ *is finite too.*

This theorem is an analogue of Theorem 1.

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[6] For an alternate proof see § 7.

Proof. $s(X) \leq s(K)$ is fulfilled for any continuous map f. In order to establish the reversed inequality, let f be quasi-open and light. By Theorem 2 we have

$$l\,w(X) \leq c(X) \leq s(X)\,. \tag{1}$$

Thus, for any given $x \in X$, there is a basis of neighbourhoods $\{V_a\}$, $a \in A$, where $k(A) \leq l\,w(X) \leq s(X)$.

Consider $f^{-1}(V_a)$ and its order components $U_{a\beta}$, $\beta \in B(a)$. Since $f^{-1}(x)$ is compact, there is a finite subset $B'(a) \subset B(a)$ such that $\{U_{a\beta}\}$, $\beta \in B'(a)$, covers $f^{-1}(x)$. Clearly, the sets $U_{a\beta}$, $\beta \in B'(a)$, $a \in A$, form a family $\mathfrak{U}$ of at most $s(X)$ intervals if $s(X)$ is infinite; if $s(X)$ is finite, $\mathfrak{U}$ is finite too.

It follows readily from Lemma 7, that $\mathfrak{U}$ is a basis for the topology of $f^{-1}(x)$. Thus, for any $x \in X$, the weight

$$w(f^{-1}(x)) \leq s(X)\,, \tag{2}$$

if $s(X)$ is infinite, and is finite if $s(X)$ is finite. Since, we always have $s \leq w$, we obtain

$$s(f^{-1}(x)) \leq s(X)\,, \tag{3}$$

for $s(X)$ infinite and $s(f^{-1}(x))$ finite, for finite $s(X)$.

Now, let R be a dense subset of X with $k(R) \leq s(X)$, and consider

$$f^{-1}(R) = \bigcup_{x \in R} f^{-1}(x)\,. \tag{4}$$

It follows from (3), that

$$s(f^{-1}(R)) \leq s(X)\,, \tag{5}$$

if $s(X)$ is infinite, and $s(f^{-1}(R))$ is finite if so is $s(X)$. Thus our proof will be completed, if we can show that $f^{-1}(R)$ is dense on K. However, for any open set $U \subset K$, $U \neq 0$, we have $\operatorname{Int} f(U) \neq 0$, f being quasi-open. Therefore,

$$f(U) \cap R \supset [\operatorname{Int} f(U)] \cap R \neq 0\,, \tag{6}$$

and thus

$$U \cap f^{-1}(R) \neq 0\,. \tag{7}$$

P r o b l e m 2. *Does Theorem 10 remain true if one only assumes that* f *is light in the sense of ordering and do not require that* f *be quasi-open?*

Lemma 3. and Theorems 6, 9, 10 and 1 yield

C o r o l l a r y 5. *Let* X *be the continuous image of an ordered compactum. Then there exists an ordered compactum* K *and a map* f : K → X *onto* X *such that*

$$w(K) \leq w(X)\,,\ l\,w(K) \leq l\,w(X)\,,\qquad s(K) \leq s(X) \text{ and } c(K) \leq c(X)\,.$$

Now, by means of Theorem 10, we can prove Theorem 8 without recourse to Theorem 7. Indeed, let X be locally connected and the image of an ordered compactum K under a map f. By Lemma 3 we can always assume that f is quasi-open and light. Then Theorem 10 yields

$$s(K) = s(X), \tag{8}$$

if $s(X)$ is infinite.

By Lemma 3 of [8] (quoted in § 6 of the present paper) we know that

$$w(X) \leq s(K). \tag{9}$$

Thus, for infinite $s(X)$, we have

$$w(X) \leq s(X). \tag{10}$$

If $s(X)$ is finite, then X is finite too, and $w(X) = s(X)$. This proves Theorem 8.

By means of Theorem 8, we can give an affirmative answer to Problem 2, in the case of locally connected X. Indeed, we obtain

Theorem 11. *Let* K *be an ordered compactum and* $f: K \to X$ *a mapping onto* X *which is light in the sense of ordering. If* X *is locally connected, then* $s(K) = s(X)$, *whenever* X *is infinite. If* $s(X)$ *is finite, then so is* $s(K)$.

Proof. $s(X) \leq s(K)$ is obvious. In order to prove the reversed inequality, first observe that $s(K) \leq w(K)$ (true for all spaces). Furthermore, if $s(X)$ is infinite, then so is $w(X) \geq s(X)$, and thus $w(K) \leq w(X)$ (Theorem 6). If $s(X)$ is finite, then so is $w(X) = s(X)$ and thus also $w(K) = s(K)$ is finite (Theorem 6). Now, by Theorem 8, we have $w(X) = s(X)$. Combining these facts, we readily obtain our assertion.

Concluding this section, notice that a compactum X can contain subcompacta $X' \subset X$ with $s(X') > s(X)$ and $c(X') > c(X)$. E. g. if $X = \prod_{\alpha \in A} I_\alpha$, $I_\alpha = I$, and $k(A) = 2^{\aleph_0}$, then $c(X) = \aleph_0$ and $s(X) = \aleph_0$ (see § 1).

On the other hand, for the square in the lexicographic order Q (see § 1) we have $w(Q) = c(Q) = s(Q) = 2^{\aleph_0}$. Therefore, by a well-known theorem, Q can be topologically imbedded in X.

For continuous images of ordered compacta, c and s are always monotone and we have

Theorem 12. *Let* X *be the continuous image of an ordered compactum. Then for any pair of closed subsets* $X' \subset X''$ *of* X *we have* $s(X') \leq s(X'')$ *and* $c(X') \leq c(X'')$.

Proof. Let $X = f(K)$, where K is an ordered compactum and let $K' = f^{-1}(X')$, $K'' = f^{-1}(X'')$, $K' \subset K''$. By Lemma 3, we can always assume that $f'' = f \mid K''$ is quasi-open and light. Assuming that $s(X'')$ is infinite, we have, by Theorem 10,

$$s(K'') = s(X''). \tag{11}$$

Furthermore, $K' \subset K''$ implies readily

$$s(K') \leq s(K''). \tag{12}$$

Indeed, let $R'' \subset K''$ be a set dense in K'' with $k(R'') \leq s(K'')$. For each $r \in R''$ consider

$$r_0 = \mathrm{Sup}\,\{(\cdot, r] \cap K'\} \tag{13}$$

and

$$r_1 = \mathrm{Inf}\,\{[r, \cdot) \cap K'\}, \tag{14}$$

where $(\cdot, r] = \{t \in K'' \mid t \leq r\}$ and $[r, \cdot) = \{t \in K'' \mid t \geq r\} \cdot r_0$ and r_1 are well-defined elements of K' and $r_0 \leq r \leq r_1$.
Let

$$R' = \bigcup_{r \in R''} \{r_0, r_1\}. \tag{15}$$

Since $s(K'')$ is infinite, we have $k(R') \leq 2k(R'') \leq s(K'')$.

However, it is readily seen that R' is dense in K', which establishes (12).

Finally, we have

$$s(X') \leq s(K') \tag{16}$$

because X' is the image of K' under $f \mid K'$. Composing (16), (12) and (11) we obtain

$$s(X') \leq s(X''), \tag{17}$$

for $s(X'')$ infinite.

If $s(X'')$ is finite, then X'' and $X' \subset X''$ are finite, and therefore, $s(X') = k(X') \leq k(X'') = s(X'')$, and we obtain again (17).

The proof in the case of the degree of cellularity follows the same scheme and is based on Theorem 1.

Remark. w and lw are always monotone.

§ 8. *The Suslin problem and continuous images of ordered compacta*

In this section we compare the degree of cellularity $c(X)$ with the degree of separability $s(X)$, for spaces X which are continuous images of ordered compacta. We state three hypotheses:

H$_1$. *If* K *is an ordered compactum, then* c (K) = s (K).

H$_2$. *If* X *is the continuous image of an ordered compactum* K, *then* c (X) = s (X).

H$_3$. *If* X *is the continuous image of an ordered continuum* C, *then* c (X) = w (X).

The hypothesis **H$_1$** has been conjectured and much studied by Đ. Kurepa (cf. [3], [4] and [5]). If K is a Suslin compactum, then **H$_1$** implies, that K is separable, and thus answers the Suslin problem (stated in § 1) in the affirmative.

T h e o r e m 13. *The hypotheses* $\mathbf{H}_1$, $\mathbf{H}_2$ *and* $\mathbf{H}_3$ *are equivalent.*

$\mathbf{H}_1 \Rightarrow \mathbf{H}_2$. $c(X) \leq s(X)$ is always true. In order to prove the reversed inequality, choose an ordered compactum K and a map $f: K \to X$ onto X such that $c(K) \leq c(X)$ (apply Corollary 5). Then, by $\mathbf{H}_1$, $s(K) = c(K)$ and thus $s(K) \leq c(X)$. However, $X = f(K)$ implies that $s(X) \leq s(K)$ and we obtain $s(X) \leq c(X)$.

$\mathbf{H}_2 \Rightarrow \mathbf{H}_3$. Let X be the continuous image of an ordered continuum C. Then, X is locally connected, and by Theorem 8 we have $w(X) = s(X)$. However, by $\mathbf{H}_2$, we also have $s(X) = c(X)$.

$\mathbf{H}_3 \Rightarrow \mathbf{H}_1$. Let K be an infinite ordered compactum. We have to prove that $s(K) \leq c(K)$. Denote by $Z \subset K$ the set of all isolated points $t \in K$. Replacing each $t \in Z$ by a copy of the real line segment I, we obtain an ordered compactum K' without isolated points. Since $k(Z) \leq c(K)$, we infer that $c(K') \leq c(K)$. Thus there is no loss of generality in assuming that K itself has no isolated points.

Now consider all empty intervals (a_α, b_α), $a_\alpha < b_\alpha$, $a_\alpha, b_\alpha \in K$, and identify all pairs $\{a_\alpha, b_\alpha\}$. (Observe that these pairs are disjoint, because K has no isolated points). One obtains an ordered continuum C. Let $p: K \to C$ be the identification map. p being continuous, we have $c(C) \leq c(K)$. By $\mathbf{H}_3$, we conclude that $s(C) = c(C) \leq c(K)$. Let $R \subset C$ be a set, dense in C and such that $k(R) \leq s(C) \leq c(K)$. Then $p^{-1}(R) \subset K$ is dense in K and since $k(p^{-1}(R)) \leq 2k(R) = k(R) \leq$ $\leq c(K)$, we conclude that $s(K) \leq c(K)$. This completes the proof of Theorem 3.

C o r o l l a r y 6. *The affirmative answer to the Suslin problem is equivalent to each of these two propositions:*

A compactum X, *which is the continuous image of an ordered compactum, has the Suslin property if and only if it is separable.*

A continuum X, *which is the continuous image of an ordered continuum, has the Suslin property if and only if it is metrizable.*

C o r o l l a r y 7. *The hypothesis* $\mathbf{H}_1$ *implies an affirmative answer to Problem* 1, *for the case of locally connected* X.

The proof is immediate from $\mathbf{H}_1$, the implication $\mathbf{H}_1 \Rightarrow \mathbf{H}_2$ (Theorem 13) and Theorem 11.

C o r o l l a r y 8. *The hypothesis* $\mathbf{H}_1$ *and an affirmative answer to Problem* 2 *imply an affirmative answer to Problem* 1.

The proof is immediate by applying the implication $\mathbf{H}_1 \Rightarrow \mathbf{H}_2$ (Theorem 13).

§ 9. *Continuous images of ordered compacta and diadic compacta*[7]

Let D denote a discrete space consisting of just two points. *A diadic compactum* is any space X which is obtainable as the continuous image of a direct product $\Pi_\alpha D_\alpha$, $\alpha \in A$, where $D_\alpha = D$,

[7] This section depends only on §§ 1—4.

for each $a \leqslant A$. There are no restrictions on $k(A)$ (P. S. Aleksandrov). If $k(A) = \aleph_0$, we obtain continuous images of the Cantor triadic set, namely all metrizable compacta.

N. A. Šanin has shown that each diadic ordered compactum is necessarily metrizable (Theorem 51, p. 92 of [11]). Strengthening this theorem we prove

Theorem 14. *A diadic compactum* X *is the continuous image of an ordered compactum if and only if* X *is metrizable.*

This has been recently[8] conjectured by P. S. Aleksandrov.

Proof. Let X be a diadic compactum and the continuous image of an ordered compactum. Then, by Szpilrajn's theorem (quoted in § 1) X has the Suslin property. Hence, by Corollary 2, X satisfies the first axiom of countability, i. e. $\mathrm{lw}(X) \leq \aleph_0$. However, A. S. Esenin-Vol'pin [1] has proved that a diadic compactum X, satisfying the first axiom of countability, is metrizable.

The converse is trivial, the Cantor triadic set being at the same time diadic and an ordered compactum. This completes the proof of Theorem 14.

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NEPREKIDNE SLIKE UREĐENIH KOMPAKATA, SUSLINOVO SVOJSTVO I DIJADSKI KOMPAKTI

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Sadržaj

U ovom se radu ispituju Hausdorffovi prostori X, koji se mogu dobiti kao slike barem jednog uređenog kompakta K, pri neprekidnom preslikavanju $f: K \to X$ na čitav X. Ovakve prostore zvat ćemo kraće *neprekidnim slikama uređenih kompakata.* Pri tome se u ovom članku pod kompaktom razumijeva Hausdorffov kompaktni prostor, koji ne mora biti metrizabilan.

Promatraju se i neke specijalne klase neprekidnih preslikavanja uređenih kompakata na Hausdorffove prostore koje se definiraju ovako:

Definicija 2. Preslikavanje $f: K \to X$ uređena kompakta $(K, <)$ na X se naziva *laganim u uređajnom smislu,* ako za svaki $x \in X$, skup $f^{-1}(x)$ ne sadrži niti jedan zatvoreni interval koji ima više od jedne tačke.

Definicija 3. Neka su X i Y topološki prostori i neka je $f: X \to Y$ neprekidno preslikavanje. Preslikavanje f se naziva *kvazi-otvorenim,* ako za svaki neprazni otvoren skup $U \subset X$, skup $f(U)$ ima nepraznu nutrinu, Int $f(U) \neq 0$.

Važnost ovih klasa preslikavanja izlazi iz ove leme:

Lema 3. Neka je X neprekidna slika uređenog kompakta. Tada postoji uređeni kompakt K i preslikavanje $f: K \to X$ na X, koje je i lagano u uređajnom smislu i kvazi-otvoreno.

Posebna je pažnja u ovom radu obraćena vezama između *stepena celularnosti* c(X) i *lokalne težine* lw(X) neprekidnih slika uređenih kompakata (§ 4).

Stepen celularnosti $c(X)$ prostora X je Sup $k(\mathfrak{U})$, gdje $\mathfrak{U}$ prolazi svim familijama $\mathfrak{U} = \{U_a\}$ disjunktnih nepraznih otvorenih skupova[1] $U_a \subset X$. Ovaj je pojam uveo Đ. Kurepa ([3], str. 131). Kaže se da prostor X ima Suslinovo svojstvo ako je $c(X) \leq \aleph_0$. Težina $w(x, X)$ prostora X u tački $x \in X$ je najmanji kardinalni broj k sa svojstvom da tačka x ima bazu okolina kardinalnog broja $\leq k$. Lokalna težina $lw(X)$ se definira kao $\operatorname{Sup}_{x \in X} w(x, X)$. Jasno je da je $lw(X) \leq w(X)$, gdje je $w(X)$ *težina* prostora X.

Sa $s(X)$ se označava *stepen separabilnosti* prostora X, tj. najmanji kardinalni broj k, za koji postoji podskup $R \subset X$, $k(R) \leq k$, koji je svuda gust na X.

Osnovni rezultat rada može se izreći u ovom obliku:

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[1] k(A) označava kardinalni broj skupa A.

T e o r e m 1. *Neka je* K *uređen kompakt, a* $f : K \to X$ *preslikavanje na* X, *koje je kvazi-otvoreno i lagano u uređajnom smislu. Tada za stepene celularnosti od* K *i* X *vrijedi relacija* $c(K) = c(X)$, *ako je* $c(X)$ *beskonačan; ako je* $c(X)$ *konačan, onda je konačan i* $c(K)$.

K o r o l a r 1. *Neka je* K *uređen kompakt, a* $f : K \to X$ *neprekidno preslikavanje na* X, *koje je kvazi-otvoreno i lagano u uređajnom smislu. Tada* X *ima Suslinovo svojstvo onda i samo onda kada* K *ima Suslinovo svojstvo.*

T e o r e m 2. *Ako je* X *neprekidna slika uređena kompakta, tada njegova lokalna težina* $\mathrm{lw}(X)$ *ne može biti veća od njegovog stepena celularnosti, tj.* $\mathrm{lw}(X) \leq c(X)$.

K o r o l a r 2. *Ako je Suslinov kompakt* X *neprekidna slika uređena kompakta, tada* X *zadovoljava prvi aksiom prebrojivosti, tj.* $\mathrm{lw}(X) \leq \aleph_0$.

Korolar 2, zajedno s nekim poznatim rezultatima o dijadskim kompaktima (vidi § 9), dokazuje ovu slutnju P. S. A l e k s a n d r o v a[2]:

T e o r e m 14. *Dijadski kompakt* X *je neprekidna slika uređena kompakta, onda i samo onda, ako je* X *metrizabilan.*

Pri tome se dijadski kompakti definiraju ovako:

Neka je D diskretan prostor sastavljen od tačno dvije tačke. Dijadskim kompaktom se naziva svaki kompakt X koji se može dobiti kao neprekidna slika direktnog produkta $\prod_{\alpha \in A} D_\alpha$, gdje je $D_\alpha = D$ za svaki $\alpha \leqslant A$, a $k(A)$ može biti bilo koji kardinalni broj (P. S. Aleksandrov).

U §§ 3, 5, 6 i 7 izučava se vladanje stepena celularnosti c, stepena separabilnosti s, lokalne težine lw i težine w pri neprekidnim preslikavanjima $f : K \to X$ na X, posebno, ako je f kvazi-otvoreno i lagano u uređajnom smislu. Evo nekih rezultata te vrste:

K o r o l a r 5. Neka je X neprekidna slika uređena kompakta. Tada postoji uređen kompakt K i neprekidno preslikavanje $f : K \to X$ na X tako, da bude:

$$w(K) \leq w(X), \quad lw(K) \leq lw(X), \quad s(K) \leq s(X) \ \text{ i } \ c(K) \leq c(X).$$

T e o r e m 6. *Neka je* $f : K \to X$ *neprekidno preslikavanje na* X. *Ako je* f *lagano u uređajnom smislu, a težina* $w(X)$ *prostora* X *beskonačna, onda je* $w(K) \leq w(X)$. *Ako je pak težina* $w(X)$ *konačna, onda je konačna i težina* $w(K)$.

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[2] Slutnja je izrečena u diskusiji na IV Svesaveznom matematičkom kongresu, Lenjingrad, 3.—12. VII 1961.

Teoremom 7 je pooštren jedan od ranijih rezultata autorâ iz [8].

T e o r e m 12. *Neka je* X *neprekidna slika uređenog kompakta. Tada, za svaki par zatvorenih podskupova* $X' \subset X''$ *iz* X, *vrijedi*

$$s(X') \leq s(X'') \text{ i } c(X') \leq c(X'').$$

Pitanje jednakosti stepena celularnosti $c(X)$ i stepena separabilnosti $s(X)$ za prostore X, koji su neprekidne slike uređenih kompakata, svedeno je u § 8 na Suslinov problem.

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