GLASNIK MAT. - FIZ. I ASTR.
Tom 18. — No. 1—2 — 1963.

ON A FUNCTIONAL EQUATION TREATED BY S. KUREPA

Miklós Hosszú, Miskolc

S. Kurepa [2] has treated the functional equation

$$f(x+y,z)+f(x,y)=f(x,y+z)+f(y,z) \tag{1}$$

defined on reals. J. Erdős [1] has proved that every symmetric solution of (1) is of the form

$$f(x,y)=F(x+y)-F(x)-F(y)\,. \tag{2}$$

Here, without any additional conditions, we prove the following

Theorem. *The most general form of the solutions of* (1) *is*

$$f(x,y)=S(x,y)+A(x,y) \tag{3}$$

where S is a symmetric solution of (1) *and A is an antisymmetric one, additive in both variables.*

Proof. (1) is a linear functional equation invariant with respect to the changing of the variables of $f(x,y)$, therefore, together with f both the functions

$$S(x,y)=\frac{1}{2}[f(x,y)+f(y,x)],\ A(x,y)=\frac{1}{2}[f(x,y)-f(y,x)]$$

are its solutions. One sees that S is a symmetric function, while A is antisymmetric one,

$$S(x,y)=S(y,x)\,,\ A(x,y)=-A(y,x)\,.$$

Thus in order to prove the Theorem it is enough to show that $A(x,y)$ is additive, e. g. with respect to its first variable (in fact, then the antisymmetricity implies the additivity with respect to the second variable too). But this follows immediately by the repeated application of (1)

$$\begin{aligned}2A(x+y,z)&=f(x+y,z)-f(z,x+y)=f(x,y+z)+f(y,z)-f(x,y)-\\&-[f(z+x,y)+f(z,x)-f(x,y)]=f(x,y+z)+f(y,z)-\\&-[f(x,z+y)+f(z,y)-f(x,z)+f(z,x)]=\\&=2A(y,z)+2A(x,z)\,.\end{aligned}$$

Thus, our Theorem is proved.

C o r o l l a r y. *Every continuous, or measurable, or bounded solution of* (1) *has the form* (2).

In fact, every continuous or measurable or bounded function $A(x, y)$, additive in both variables, is of the form

$$A(x, y) = c\,x\,y\,,$$

which is antisymmetric only if $c = 0$. Thus we have the solution (2) due to J. Erdös.

Observe that in the general case, where x, y, z are elements of an Abelian group and the value of the function belongs to another Abelian group, then by a similar method we get the result

$$2f(x, y) = S^*(x, y) + A^*(x, y)\,, \qquad (3')$$

where S^*, A^* are solutions of (1), symmetric respectively antisymmetric, and A^* is additive.

R E F E R E N C E S :

[1] *J. Erdős,* A remark on the paper »On some functional equations« by S. Kurepa, Glasnik Mat.-Fiz. Astr. **14** (1959), 3—5,

[2] *S. Kurepa,* On some functional equations, Glasnik Mat.-Fiz. Astr. **11** (1956), 3—5.

O FUNKCIONALNOJ JEDNADŽBI KOJU JE PROMATRAO S. KUREPA

M i k l ó s H o s s z ú, M i s k o l c

Sadržaj

U članku je dokazan slijedeći

T e o r e m. *Opće rješenje funkcionalne jednadžbe* (1) *je zbroj simetričnog i antisimetričnog rješenja te jednadžbe. Pri tome je antisimetrično rješenje aditivna funkcija u svakoj varijabli posebno.*

Napomenimo da je J. E r d ő s [1] dokazao da je svako simetrično rješenje od (1) oblika (2).

(Primljeno 6. V 1963.)