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THE GENERAL SOLUTION OF TWO FUNCTIONAL EQUATIONS BY REDUCTION TO FUNCTIONS ADDITIVE IN TWO VARIABLES AND WITH THE AID OF HAMEL BASES

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In this paper we give the general solutions of two important functional equations with the aid of general *antisymmetric* resp. *symmetric additive functions of two variables,* wich are then represented with the aid of *Hamel bases.* It is remarkable, that though the second equation is one for an unknown function of *one* variable, still it leads to a functional equation for functions of *two* variables. This reflects the connection between norms and inner products.

1. S. Kurepa [15] (cf. also [22], [23], [24], [10]) has solved the functional equation for real functions

$$F(x+y, z) + F(x, y) = F(y, z) + F(x, y+z), \tag{1}$$

which is closely connected with the notion of the second homology group (cf. e. g. [21]) under *differentiability* suppositions. J. Erdős has proved in [5] that the general *symmetric* solution of (1) among functions with variables in an arbitrary abelian group and values in an arbitrary divisible abelian group is of the form

$$F(x, y) = f(x+y) - f(x) - f(y) \tag{2}$$

and he reproduced in the same paper an argument of the present author which proves that, under *continuity* or weaker (say, boundedness) conditions, (2) is the most general solution of (1). Recently M. Hosszú [11] has proved, relying upon the above mentioned result of J. Erdős, that *without any suppositions (except the above restrictions on domain and range) the general solution of* (1) *is*

$$F(x, y) = f(x+y) - f(x) - f(y) + G(x, y), \tag{3}$$

where G is an arbitrary *antisymmetric* function

$$G(x, y) = -G(y, x), \tag{4}$$

which is *additive* in its single variables:

$$G(x+z, y) = G(x, y) + G(z, y) \tag{5}$$

(the additivity in the other variable is a consequence of (4) and (5)).

Now we show that this last result follows also from our considerations published in [5] (and give then also explicitly, in the case of real functions, the general solution of (1)).

For this purpose we repeat that argument: Define

$$G(x, y) = \frac{F(x, y) - F(y, x)}{2}. \tag{6}$$

This is evidently an antisymmetric function (cf. (4)). In order to prove (5) we change the role of x and y in (1) and get

$$F(x + y, z) + F(y, x) = F(x, z) + F(y, x + z). \tag{7}$$

Similarly we change z and y in (1) and get

$$F(x + z, y) + F(x, z) = F(z, y) + F(x, y + z). \tag{8}$$

By subtracting (1) from the sum of (7) and (8) we obtain

$$F(x + z, y) - F(y, x + z) = F(x, y) - F(y, x) + F(z, y) - F(y, z),$$

which by (6) yields the equation (5) to be proved. On the other hand with $F(x, y)$ also

$$H(x, y) = \frac{F(x, y) + F(y, x)}{2} \tag{9}$$

satisfies equation (1) and is symmetric, thus by the quoted result of J. Erdős [5] is of the form

$$H(x, y) = f(x + y) - f(x) - f(y). \tag{10}$$

(6), (9) and (10) show that, as asserted, every solution of (1) is of the form (3) while we have proved that also (4) and (5) hold.

On the other hand, (3) with (4) and (5) evidently always satisfies (1) ((3) *always satisfies* (1) even *if* G *is* only *additive in both variables*), which proves our above statement.

The statement just proved reduces the solution of the functional equation (1) to the pair of functional equations (4), (5). The question arises how to represent the general solution of the latter, at least for real functions. As G. Hamel [6] has proved, there exist subsets B of real numbers such that every real number x can be represented in a unique way as

$$x = \sum_{k=1}^{n} r_k b_k \tag{11}$$

with $b_k \in B$ and with rational coefficients r_k. Let the similar representation of y be

$$y = \sum_{j=1}^{m} s_j b_j \tag{12}$$

($b_j \in B$, s_j rational) and take into consideration that (4) and (5) imply, as already mentioned, also

$$G(x, y_1 + y_2) = G(x, y_1) + G(x, y_2). \tag{13}$$

From (5) and (13) by induction

$$G\left(\sum_{k=1}^{n} x_k, y\right) = \sum_{k=1}^{n} G(x_k, y) \tag{14}$$

and

$$G\left(x, \sum_{k=1}^{n} y_k\right) = \sum_{k=1}^{n} G(x, y_k) \tag{15}$$

follow. Furthermore, with $x_1 = x_2 = \cdots = x_n = x, y_1 = y_2 = \cdots = y_n = y$

$$G(nx, y) = nG(x, y) = G(x, ny)$$

and for $t = \frac{m}{n} x$, that is $nt = mx$,

$$nG(t, y) = G(nt, y) = G(mx, y) = mG(x, y)$$

or

$$G\left(\frac{m}{n} x, y\right) = \frac{m}{n} G(x, y). \tag{16}$$

As (5) also implies (with $z = 0$)

$$G(0, y) = 0$$

and (with $z = -x$)

$$G(-x, y) = -G(x, y),$$

the equation (16) remains also valid for nonpositive integers m. The same argument applies to the second variable and so we have *for all rational* r and for all real x, y

$$G(rx, y) = rG(x, y) = G(x, ry). \tag{17}$$

Now by (11), (14), (15) and (17), we have with

$$G(b_k, b_j) = a_{kj}: \tag{18}$$

$$G(x, y) = G\left(\sum_{k=1}^{n} r_k b_k, \sum_{j=1}^{m} s_j b_j\right) = \sum_{k=1}^{n} r_k G\left(b_k, \sum_{j=1}^{m} s_j b_j\right) =$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} r_k s_j G(b_k, b_j) = \sum_{k=1}^{n} \sum_{j=1}^{m} a_{kj} r_k s_j.$$

So we have proved the following

Lemma. *The general real solution of the pair* (5), (13) *of functional equations is*

$$G(x, y) = \sum_{k,j} a_{kj} r_k s_j, \tag{19}$$

where

$$x = \sum_k r_k b_k, \quad y = \sum_j s_j b_j,$$

the r_k, s_j being rational while the b_j are elements of a Hamel basis B and the a_{kj} arbitrarily depending upon b_k and b_j. In any of these sums only finite number of terms may be different from zero.

(The matrix a_{kj} is infinite, even not-countable, but in (19) only a finite segment of it figures.)

If we also take (4) into consideration, we get from (18)

$$a_{kj} = G(b_k, b_j) = -G(b_j, b_k) = -a_{jk}$$

and have thus the

Corollary 1. *The general real solution of the pair* (4), (5) *of functional equations is*

$$G(x, y) = \sum_{k,j} a_{kj} r_k s_j$$

where r_k, s_j are the rational coefficients figuring in (11), (12), *while the a_{kj} are elements of an arbitrary antisymmetric matrix, depending upon the basis elements b_k, b_j figuring there.*[1]

If we compare this with (3), (4) and (5) we get the following

Theorem 1. *The general solution of*

$$F(x+y, z) + F(x, y) = F(y, z) + F(x, y+z) \tag{1}$$

among functions defined on arbitrary abelian groups and with values in an arbitrary abelian group, where every equation $nx = c$ has a unique solution x (n being a positive integer), is of the form

$$F(x, y) = f(x+y) - f(x) - f(y) + G(x, y) \tag{3}$$

with arbitrary f and with G fulfilling (4) *and* (5) *or for real numbers explicitly with*

$$G(x, y) = \sum_{k,j} a_{kj} r_k s_j ,$$

where

$$x = \sum_k r_k b_k , \quad y = \sum_j s_j b_j$$

and the a_{kj} are elements of an arbitrary (antisymmetric) matrix, depending upon b_k and b_j (r_k, s_j rational, b_k, b_j elements of a Hamel basis).

(The theorem remains true both if we leave the matrix $\| a_{kj} \|$ arbitrary and if we state that it is antisymmetric.)

As there is no continuous function satisfying (4) *and* (5) *except that identically* 0 (as — for G continuous in x, (5) implies $G(x, y) = c(y)\,x$ see [1] and cf. (17)), while (4) gives $c(y)\,x = -c(x)\,y$, thus $\frac{c(x)}{x} = -\frac{c(y)}{y} = \text{constant} = 0$), so we have also (cf. [5]) the

[1] In this sum as well as in other sums of this paper only finite number of terms may be $\neq 0$.

Corollary 2. *The general continuous real solution of* (1) *is*

$$F(x, y) = f(x + y) - f(x) - f(y)$$

with arbitrary continuous f.

The continuity condition can be considerably weakened (to boundedness, measurability, etc.).

2. The functional equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y) \tag{20}$$

is very important as it serves in certain abstract spaces for the definition of the norm (resp. of the square of the norm). It was repeatedly examined ([12], [13], [26], [3], [7], [14], [8], [25], [16], [17], [9], [1], [18], [2], [4], [19], [20], [27]) and we will also make use here of some of these results.

In order to solve or reduce (20), we define a function H of two variables (this corresponds to the inner product) by

$$4H(x, y) = g(x + y) - g(x - y)\,. \tag{20}$$

We first prove, that *if g satisfies* (20), *then H is symmetric*

$$H(x, y) = H(y, x)\,, \tag{22}$$

additive

$$H(x, y + z) = H(x, y) + H(x, z) \tag{23}$$

and

$$g(x) = H(x, x)\,. \tag{24}$$

We register some consequences of (20). By substituting $y = 0$, we get

$$g(0) = 0 \tag{25}$$

and, for $x = 0$,

$$g(-y) = g(y)\,, \tag{26}$$

that is, g is *even*. Finally, with $y = x$, (20) gives

$$g(2x) = 4g(x)\,. \tag{27}$$

Now, from (26) we have (22); (24) follows from (27) and (25), while from (20) and (21) we have (23):

$$\begin{aligned}4H(x, y + z) = g(x + y + z) - g(x - y - z) = g(x + y + z) +\\ + g(x + y - z) - g(x - z + y) - g(x - z - y) =\\ = 2g(x + y) + 2g(z) - 2g(x - z) - 2g(y) =\\ = g(x + y) + g(x + y) - 2g(y) + 2g(z) -\\ - g(x - z) - g(x - z) = g(x + y) + 2g(x) -\\ - g(x - y) + g(x + z) - 2g(x) - g(x - z) =\\ = 4H(x, y) + 4H(x, z)\,,\end{aligned}$$

as asserted.

On the other hand, (24) *always satisfies* (20), *if* H *is additive in both variables:*

$$\begin{aligned} &g(x+y)+g(x-y)=H(x+y,x+y)+H(x-y,x-y)= \\ &=H(x,x+y)+H(y,x+y)+H(x,x-y)-H(y,x-y)= \\ &=H(x,x)+H(x,y)+H(y,x)+H(y,y)+H(x,x)- \\ &-H(x,y)-H(y,x)+H(y,y)=2H(x,x)+2H(y,y)= \\ &=2g(x)+2g(y)\,. \end{aligned}$$

Thus, every function of the form (24) *with* $H(x,y)$ *additive in both variables satisfies the functional equation* (20), *but already* (24) *with symmetric additive* H *gives the general solution of* (20) *among the functions defined on abelian groups with values in abelian groups in which equations of the form* $4x=c$ *have unique solutions* x.

Answering a problem raised by S. Kurepa at the Second Oberwolfach Symposium on Functional Equations [4], we give here the general real solution of (20). By the above Lemma, H being additive in both variables, has to be of the form

$$H(x,y)=\sum_{k=1}^{n}\sum_{j=1}^{m} a_{kj}\, r_k\, s_j \quad \left(x=\sum_{k=1}^{n} r_k\, b_k,\ y=\sum_{j=1}^{m} s_j\, b_j\right),$$

where the r_k, s_j are rational while the b_j are elements of a Hamel basis B and the $a_{kj}=H(b_k,b_j)$ *arbitrarily depending upon* b_k *and* b_j. If H is *symmetric*, then

$$a_{kj}=H(b_k,b_j)=H(b_j,b_k)=a_{jk}$$

and we have similary to Corollary 1 the

Corollary 3. *The most general real symmetric additive functions are of the form*

$$H(x,y)=H\left(\sum_k r_k\, b_k,\ \sum_j s_j\, b_j\right)=\sum_{k,j} a_{kj}\, r_k\, s_j\,,$$

where the a_{kj}, *are elements of an arbitrary symmetric matrix, depending upon the elements* b_k, b_j *of the Hamel basis* B.

Summarizing, and taking (24) into consideration, we have the

Theorem 2. *The general solution of*

$$g(x+y)+g(x-y)=2g(x)+2g(y) \tag{20}$$

among functions defined on arbitrary abelian groups and with values in an arbitrary abelian group, where every equation $4x=c$ *has a unique solution* x, *is of the form*

$$g(x)=H(x,x)\,, \tag{24}$$

where H fulfills (22) *and* (23) *or for real numbers explicitly*

$$g(x) = \sum_{k,j} a_{kj} r_k r_j , \tag{28}$$

where

$$x = \sum_k r_k b_k$$

and the a_{kj} *are elements of an arbitrary (symmetric) matrix, depending upon* b_k *and* b_j *(*r_k, r_j *rational,* b_k, b_j *elements of a Hamel basis).*

(By methods of [18] the suppositions concerning the range of g might still be somewhat reduced.)

The general continuous real solution of (22) and (23) evidently is (cf. [17] or [1])

$$H(xy) = cxy$$

and so, by (24), we have the following

Corollary 4. *The general continuous real solution of* (20) *is*

$$g(x) = cx^2 ,$$

where c is an arbitrary constant.

Here again the continuity supposition can be considerably weakened. It is so much the more supprising, that although $cx^2 = a(bx)^2$, $h(x) = bx$ being the general continuous real solution of the additive functional equation $h(x+y) = h(x) + h(y)$, whose general real solution ([6]) is $h(x) = h(\sum_{k=1}^{n} r_k b_k) = \sum_{k=1}^{n} r_k h(b_k)$, the general real solution of (20) is not

$$g(x) = a(\sum_{k=1}^{n} r_k h(b_k))^2 = \sum_{k,j=1}^{n} a h(b_k) h(b_j) r_k r_j ,$$

but (cf. (28)):

$$g(x) = \sum_{k,j} H(b_k, b_j) r_k r_j .$$

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OPĆE RJEŠENJE DVIJU FUNKCIONALNIH JEDNADŽBI REDUKCIJOM NA FUNKCIJE ADITIVNE U DVIJE VARIJABLE I POMOĆU HAMELOVE BAZE

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Sadržaj

Autor dokazuje slijedeća dva teorema:

Teorem 1. *Opće rješenje funkcionalne jednadžbe*

$$F(x+y,z)+F(x,y)=F(y,z)+F(x,y+z) \tag{1}$$

u skupu funkcija definiranih na proizvoljnoj Abelovoj grupi i s vrijednostima u Abelovoj grupi u kojoj je djeljenje s prirodnim brojevima definirano ima oblik

$$F(x,y)=f(x+y)-f(x)-f(y)+G(x,y)\,, \tag{3}$$

gdje je f proizvoljna funkcija, a G je antisimetrična i aditivna u obadvije varijable, tj. G zadovoljava (4), (5).

U slučaju da su spomenute grupe realni brojevi, G ima oblik (19), *gdje je* (a_{kj}) *proizvoljna antisimetrična matrica s konačno elemenata različitih od nule u svakom retku i svakom stupcu, a* a_{kj} *je vrijednost funkcije G na paru elemenata Hamelove baze.*

Teorem 2. *Opće rješenje funkcionalne jednadžbe* (20), *gdje je g definirano na proizvoljnoj Abelovoj grupi s vrijednostima u Abelovoj grupi u kojoj je djeljenje s 2 izvedivo ima oblik* $g(x)=H(x,x)$, *gdje je* $2H(x,y)=g(x+y)-g(x)-g(y)$ *simetrična i u obadvije varijable aditivna funkcija. U slučaju da su spomenute grupe realni brojevi, g je dano s* (28) *pomoću svojih vrijednosti* a_{kj} *na Hamelovoj bazi.*

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