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SYMMETRIC (70,24,8) DESIGNS HAVING $Frob_{21} \times Z_2$ AS AN AUTOMORPHISM GROUP

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Abstract. Up to isomorphism there are twenty-two symmetric (70,24,8) designs having automorphism group isomorphic to $Frob_{21} \times Z_2$. Among them there are four self-dual, and nine pairs of dual designs. Full automorphism groups of those designs are isomorphic to $Frob_{21} \times Z_2$. Designs are constructed by means of tactical decomposition, using a principal series of the group $Frob_{21} \times Z_2$.

1. Introduction and preliminaries

A symmetric (v, k, λ) design is a finite incidence structure $(\mathcal{P}, B, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = |\mathcal{B}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every pair of elements of $\mathcal{P}$ is incident with exactly λ elements of $\mathcal{B}$.

Let $\mathcal{D} = (P, B, I)$ be a symmetric (v, k, λ) design and $G \leq Aut\mathcal{D}$. Group G has the same number of point and block orbits. Let us denote the number of G-orbits by t, point orbits by $\mathcal{P}_1, \dots, \mathcal{P}_t$, block orbits by $\mathcal{B}_1, \dots, \mathcal{B}_t$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$. We shall denote points of the orbit $\mathcal{P}_r$ by $\mathcal{P}_r = \{r_0, \dots, r_{\omega_r - 1}\}$. Further, denote by γ_{ir} the number of points of $\mathcal{P}_r$ which are incident with the representative of the block orbit $\mathcal{B}_i$. For those numbers the

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following equalities hold:

$$\sum_{r=1}^{t} \gamma_{ir} = k, \tag{1}$$

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij} \cdot (k - \lambda). \tag{2}$$

DEFINITION 2. *The* $(t \times t)$*-matrix* (γ_{ir}) *with entries satisfying properties* (1) *and* (2) *is called the orbit structure for parameters* (v, k, λ) *and orbit distribution* $(\omega_1, \ldots, \omega_t)$, $(\Omega_1, \ldots, \Omega_t)$.

The first step of the construction of designs is to find all orbit structures (γ_{ir}) for some parameters and orbit distribution. The next step, called indexing, is to determine for each number γ_{ir} exactly which points from the point orbit $\mathcal{P}_r$ are incident with representative of the block orbit $\mathcal{B}_i$. Because of the large number of possibilities, it is often necessary to involve a computer in both steps of the construction.

DEFINITION 3. *The set of indices of points of the orbit* $\mathcal{P}_r$ *indicating which points of* $\mathcal{P}_r$ *are incident with the representative of the block orbit* $\mathcal{B}_i$ *is called the index set for the position* (i, r) *of the orbit structure.*

First symmetric $(70, 24, 8)$ design is constructed by Z. Janko and Tran van Trung (see [6]). Full automorphism group of that design is isomorphic to $Frob_{21} \times Z_2$. Later on, A. Golemac has proved that there are up to isomorphism and duality 5 symmetric $(70, 24, 8)$ designs having automorphism group isomorphic to $E_8 \cdot Frob_{21}$ (see [4]).

2. $Frob_{21}$ ACTING ON A SYMMETRIC (70,24,8) DESIGN

Let H be the Frobenius group of order 21. Since there is only one isomorphism class of nonabelian groups of order 21, we may write

$$H = \langle \rho, \sigma \mid \rho^7 = 1,\, \sigma^3 = 1,\, \rho^\sigma = \rho^2 \rangle.$$

We shall need the permutation representations of ρ and σ on H-orbits consisting of 7 and 21 points, which can be notated as $0, 1, \ldots, 6$, or $0, 1, \ldots, 20$ respectively. Without losing generality we write

$$\rho = (0,1,2,3,4,5,6) \quad \text{and} \quad \sigma = (0)\,(1,2,4)(3,6,5)$$

in the first, and

$$\rho = (0,1,2,3,4,5,6)(7,8,9,10,11,12,13)(14,15,16,17,18,19,20),$$

$$\sigma = (0,7,14)(1,9,18)(2,11,15)(3,13,19)(4,8,16)(5,10,20)(6,12,17)$$

in the second case.

Let α be an automorphism of a symmetric design. We shall denote by $F(\alpha)$ the number of points fixed by α. In that case, the number of blocks fixed by α is also $F(\alpha)$.

LEMMA 2. *Let ρ be an automorphism of a symmetric* $(70, 24, 8)$ *design. If* $|\rho| = 7$, *then* $F(\rho) = 0$.

PROOF. It is known that $F(\rho) < k + \sqrt{k - \lambda}$ and $F(\rho) \equiv v(mod\ |\rho|)$. Therefore, $F(\rho) \in \{0, 7, 14, 21\}$. One can not construct orbit structures for $F(\rho) \in \{7, 14, 21\}$. □

LEMMA 3. *Let H be a Frobenius automorphism group of order* 21 *of a symmetric* $(70, 24, 8)$ *design $\mathcal{D}$. H acts semistandardly on $\mathcal{D}$ with orbit distribution* $(7, 7, 7, 7, 21, 21)$.

PROOF. Frobenius kernel $\langle\rho\rangle$ acts on $\mathcal{D}$ with orbit distribution $(7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$. Since $\langle\rho\rangle H$, σ maps $\langle\rho\rangle$-orbits on $\langle\rho\rangle$-orbits. Therefore, only possibilities for orbit distributions are $(7, 21, 21, 21)$, $(7, 7, 7, 7, 21, 21)$, $(7, 7, 7, 7, 7, 7, 7, 21)$ and $(7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$. Since automorphism group of a symmetric design has the same number of orbits on sets of blocks and points, H acts semistandardly on $\mathcal{D}$.

Stabilizer of each block from the block orbit of length 7 is conjugated to $\langle\sigma\rangle$. Therefore, entries of orbit structures corresponding to point and block orbits of length 7 must satisfy the condition $\gamma_{ir} \equiv 0, 1\ (mod\ 3)$.

With the help of the computer program by V. Ćepulić we got 11 orbit structures for orbit distribution $(7, 7, 7, 7, 21, 21)$. Some of them will produce designs.

There are 7 orbit structures for orbit distribution $(7, 7, 7, 7, 7, 7, 7, 21)$, but none of them gives rise to designs. We shall not describe that unsuccessful attempt of indexing.

There are no orbit structures for orbit distributions $(7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$ and $(7, 21, 21, 21)$. □

3. $Frob_{21} \times Z_2$ ACTING ON A SYMMETRIC (70,24,8) DESIGN

Let G be direct product of the Frobenius group of order 21 with an involution. We may write

$$G = \langle \rho, \sigma, \tau \mid \rho^7 = 1,\ \sigma^3 = 1,\ \tau^2 = 1,\ \rho^\sigma = \rho^2,\ \rho^\tau = \rho,\ \sigma^\tau = \sigma \rangle.$$

Obviously, only possibilities for orbit distributions of the automorphism group $Frob_{21} \times Z_2$ acting on a symmetric (70,24,8) design are $(14, 14, 42)$, $(7, 7, 14, 42)$ and $(7, 7, 7, 7, 42)$. Using the computer program by V. Ćepulić, we got the following results:

LEMMA 4. *Up to isomorphism there are exactly three orbit structures for symmetric* $(70, 24, 8)$ *designs and the automorphism group* $Frob_{21} \times Z_2$ *acting with orbit distribution* $(14, 14, 42)$. *Those structures are:*

OS1	14	14	42
14	8	4	12
14	4	8	12
42	4	4	16

OS2	14	14	42
14	8	4	12
14	4	2	18
42	4	6	14

OS3	14	14	42
14	7	2	15
14	2	4	18
42	5	6	13

LEMMA 5. *Up to isomorphism there are exactly four orbit structures for symmetric* $(70, 24, 8)$ *designs and the automorphism group* $Frob_{21} \times Z_2$ *acting with orbit distribution* $(7, 7, 14, 42)$. *Those structures are:*

OS4	7	7	14	42
7	4	0	8	12
7	0	4	8	12
14	4	4	4	12
42	2	2	4	16

OS5	7	7	14	42
7	4	0	8	12
7	0	4	8	12
14	1	1	4	18
42	3	3	4	14

OS6	7	7	14	42
7	4	0	2	18
7	0	4	2	18
14	4	4	4	12
42	2	2	6	14

OS7	7	7	14	42
7	4	0	2	18
7	0	4	2	18
14	1	1	7	15
42	3	3	5	13

LEMMA 6. *Up to isomorphism there are exactly two orbit structures for symmetric* $(70, 24, 8)$ *designs and the automorphism group* $Frob_{21} \times Z_2$ *acting with orbit distribution* $(7, 7, 7, 7, 42)$. *Those structures are:*

OS8	7	7	7	7	42
7	4	4	4	0	12
7	4	4	0	4	12
7	4	0	4	4	12
7	0	4	4	4	12
42	2	2	2	2	16

OS9	7	7	7	7	42
7	4	4	4	0	12
7	4	4	0	4	12
7	4	0	1	1	18
7	0	4	1	1	18
42	2	2	3	3	14

4. Z_{14} ACTING ON A SYMMETRIC (70,24,8) DESIGN

Let K be the cyclic group of order 14, namely

$$K = \langle \rho, \tau \mid \rho^7 = 1,\ \tau^2 = 1,\ \rho^\tau = \rho \rangle.$$

In the process of indexing we shall need the permutation representations of ρ and τ on K-orbits consisting of 7 and 14 points, which can be notated as $0, 1, \ldots, 6$, or $0, 1, \ldots, 13$ respectively. Without losing generality we write

$$\rho = (0, 1, 2, 3, 4, 5, 6) \quad \text{and} \quad \tau = (0)(1)(2)(3)(4)(5)(6)$$

in the first, and

$$\rho = (0, 1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12, 13),$$

$$\tau = (0, 7)(1, 8)(2, 9)(3, 10)(4, 11)(5, 12)(6, 13)$$

in the second case.

Since we shall consider the Z_{14} a subgroup of the $Frob_{21} \times Z_2$, we require that in orbit structures for Z_{14} and symmetric $(70, 24, 8)$ designs entries γ_{ir} corresponding to point and block orbits of length 7 must satisfy the condition $\gamma_{ir} \equiv 0, 1 \ (mod\ 3)$. Also, parts of those orbit structures corresponding to point and block orbits of length 14 must admit the automorphism of order 3.

With the help of the computer program by V. Čepulić, we got the following results:

LEMMA 7. *Up to isomorphism there are three orbit structures for symmetric* $(70, 24, 8)$ *designs and the automorphism group* Z_{14} *acting with orbit distribution* $(14, 14, 14, 14, 14)$. *All of them admit an automorphism of order 3. Those structures are:*

OS1'	14	14	14	14	14
14	8	4	4	4	4
14	4	8	4	4	4
14	4	4	8	4	4
14	4	4	4	8	4
14	4	4	4	4	8

OS2'	14	14	14	14	14
14	8	4	4	4	4
14	4	2	6	6	6
14	4	6	6	6	2
14	4	6	2	6	6
14	4	6	6	2	6

OS3'	14	14	14	14	14
14	7	2	5	5	5
14	2	4	6	6	6
14	5	6	7	3	3
14	5	6	3	7	3
14	5	6	3	3	7

LEMMA 8. *Up to isomorphism there are eight orbit structures for symmetric* $(70, 24, 8)$ *designs and the automorphism group* Z_{14} *acting with orbit distribution* $(7, 7, 14, 14, 14, 14)$. *Four of them are orbit structures for the group* Z_{14} *as a subgroup of the* $Frob_{21} \times Z_2$. *Those structures are:*

OS4'	7	7	14	14	14	14
7	4	0	8	4	4	4
7	0	4	8	4	4	4
14	4	4	4	4	4	4
14	2	2	4	8	4	4
14	2	2	4	4	8	4
14	2	2	4	4	4	8

OS5'	7	7	14	14	14	14
7	4	0	8	4	4	4
7	0	4	8	4	4	4
14	1	1	4	6	6	6
14	3	3	4	6	6	2
14	3	3	4	2	6	6
14	3	3	4	6	2	6

OS6'	7	7	14	14	14	14
7	4	0	2	6	6	6
7	0	4	2	6	6	6
14	4	4	4	4	4	4
14	2	2	6	6	6	2
14	2	2	6	2	6	6
14	2	2	6	6	2	6

OS7'	7	7	14	14	14	14
7	4	0	2	6	6	6
7	0	4	2	6	6	6
14	1	1	7	5	5	5
14	3	3	5	7	3	3
14	3	3	5	3	7	3
14	3	3	5	3	3	7

In the case of orbit distribution $(7,7,7,7,14,14,14)$ involution acts with 28 fixed points. Following theorem gives the additional condition for orbit structures:

THEOREM 6. *Suppose that a nonidentity automorphism σ of a nontrivial symmetric (v,k,λ) design fixes F points. Then $F \leq v - 2(k-\lambda)$ and $F \leq \frac{\lambda}{k-\sqrt{k-\lambda}} \cdot v$. Moreover, if equality holds in either inequality, σ must be an involution and every non-fixed block contains exactly λ fixed points.*

PROOF. Lander [7]. □

LEMMA 9. *Up to isomorphism there are forty-five orbit structures for symmetric $(70,24,8)$ designs and the automorphism group Z_{14} acting with orbit distribution $(7,7,7,7,14,14,14)$. Only one of them is orbit structure for the group Z_{14} as a subgroup of the $Frob_{21} \times Z_2$. That structure is:*

OS8'	7	7	7	7	14	14	14
7	4	4	4	0	4	4	4
7	4	4	0	4	4	4	4
7	4	0	4	4	4	4	4
7	0	4	4	4	4	4	4
14	2	2	2	2	8	4	4
14	2	2	2	2	4	8	4
14	2	2	2	2	4	4	8

5. Orbit distribution (14,14,42)

It would be very difficult to proceed with indexing of orbit structures OS1, OS2 and OS3. For example, there are $\binom{42}{12}$ possibilities for index sets for the position $(1,3)$ in the OS1. Therefore, we shall use the principal series $\langle 1\rangle\langle\rho\rangle\langle\rho,\sigma\rangle G$ of the automorphism group $G = \langle \rho,\sigma,\tau \mid \rho^7 = 1,\ \sigma^3 = 1,\ \tau^2 = 1,\ \rho^\sigma = \rho^2,\ \rho^\tau = \rho,\ \sigma^\tau = \sigma\rangle$. Our aim is to find all orbit structures for the group $\langle\rho\rangle$ corresponding to structures OS1, OS2 and OS3. We shall construct designs from those orbit structures for $\langle\rho\rangle$, having in mind the action of permutations σ and τ on $\langle\rho\rangle$-orbits.

Theorem 7. *Up to isomorphism and duality there are five symmetric* $(70,24,8)$ *designs with automorphism group* $Frob_{21} \times Z_2$ *acting with orbit distribution* $(14,14,42)$. *Only one of them is self-dual. Full automorphism groups of those designs are isomorphic to* $Frob_{21} \times Z_2$.

Proof. In order to find orbit structures for the group $\langle\rho\rangle$, we shall determine orbit structures for the group $\langle\rho,\sigma\rangle \cong Frob_{21}$ corresponding to OS1, OS2 and OS3. Those structures are:

OS1"	7	7	7	7	21	21
7	4	0	4	4	6	6
7	0	4	4	4	6	6
7	4	4	4	0	6	6
7	4	4	0	4	6	6
21	2	2	2	2	10	6
21	2	2	2	2	6	10

OS2"	7	7	7	7	21	21
7	4	0	4	4	6	6
7	0	4	4	4	6	6
7	4	4	3	1	9	3
7	4	4	1	3	3	9
21	2	2	3	1	7	9
21	2	2	1	3	9	7

OS3"	7	7	7	7	21	21
7	4	0	4	4	6	6
7	0	4	4	4	6	6
7	1	1	4	0	9	9
7	1	1	0	4	9	9
21	3	3	2	2	9	5
21	3	3	2	2	5	9

OS4"	7	7	7	7	21	21
7	4	0	4	4	6	6
7	0	4	4	4	6	6
7	1	1	3	1	12	6
7	1	1	1	3	6	12
21	3	3	3	1	6	8
21	3	3	1	3	8	6

OS5"	7	7	7	7	21	21
7	4	0	1	1	9	9
7	0	4	1	1	9	9
7	4	4	3	1	9	3
7	4	4	1	3	3	9
21	2	2	4	2	6	8
21	2	2	2	4	8	6

Taking into consideration orbit structures for Z_{14} we shall determine which orbit structures for Z_7 correspond to orbit structures OS1", ...,OS5". Up to isomorphism and duality, those decomposed structures are:

OS1'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	4	4	4	0	2	2	2	2	2	2
7	4	4	0	4	2	2	2	2	2	2
7	2	2	2	2	4	4	2	4	0	2
7	2	2	2	2	2	4	4	2	4	0
7	2	2	2	2	4	2	4	0	2	4
7	2	2	2	2	4	0	2	4	4	2
7	2	2	2	2	2	4	0	2	4	4
7	2	2	2	2	0	2	4	4	2	4

OS2'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	4	4	4	0	2	2	2	2	2	2
7	4	4	0	4	2	2	2	2	2	2
7	2	2	2	2	6	2	2	2	2	2
7	2	2	2	2	2	6	2	2	2	2
7	2	2	2	2	2	2	6	2	2	2
7	2	2	2	2	2	2	2	6	2	2
7	2	2	2	2	2	2	2	2	6	2
7	2	2	2	2	2	2	2	2	2	6

OS3'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	4	4	3	1	3	3	3	1	1	1
7	4	4	1	3	1	1	1	3	3	3
7	2	2	3	1	5	1	1	3	3	3
7	2	2	3	1	1	5	1	3	3	3
7	2	2	3	1	1	1	5	3	3	3
7	2	2	1	3	3	3	3	5	1	1
7	2	2	1	3	3	3	3	1	5	1
7	2	2	1	3	3	3	3	1	1	5

OS4'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	4	4	3	1	3	3	3	1	1	1
7	4	4	1	3	1	1	1	3	3	3
7	2	2	3	1	3	3	1	5	1	3
7	2	2	3	1	1	3	3	3	5	1
7	2	2	3	1	3	1	3	1	3	5
7	2	2	1	3	5	1	3	3	3	1
7	2	2	1	3	3	5	1	1	3	3
7	2	2	1	3	1	3	5	3	1	3

OS5'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	1	1	4	0	3	3	3	3	3	3
7	1	1	0	4	3	3	3	3	3	3
7	3	3	2	2	5	3	1	1	3	1
7	3	3	2	2	1	5	3	1	1	3
7	3	3	2	2	3	1	5	3	1	1
7	3	3	2	2	1	3	1	5	3	1
7	3	3	2	2	1	1	3	1	5	3
7	3	3	2	2	3	1	1	3	1	5

OS6'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	1	1	3	1	4	4	4	2	2	2
7	1	1	1	3	2	2	2	4	4	4
7	3	3	3	1	4	2	0	2	4	2
7	3	3	3	1	0	4	2	2	2	4
7	3	3	3	1	2	0	4	4	2	2
7	3	3	1	3	2	4	2	4	2	0
7	3	3	1	3	2	2	4	0	4	2
7	3	3	1	3	4	2	2	2	0	4

OS7'''	7	7	7	7	7	7	7	7	7	7
7	4	0	4	4	2	2	2	2	2	2
7	0	4	4	4	2	2	2	2	2	2
7	1	1	3	1	4	4	4	2	2	2
7	1	1	1	3	2	2	2	4	4	4
7	3	3	3	1	2	2	2	4	4	0
7	3	3	3	1	2	2	2	0	4	4
7	3	3	3	1	2	2	2	4	0	4
7	3	3	1	3	4	4	0	2	2	2
7	3	3	1	3	0	4	4	2	2	2
7	3	3	1	3	4	0	4	2	2	2

OS8'''	7	7	7	7	7	7	7	7	7	7
7	4	0	1	1	3	3	3	3	3	3
7	0	4	1	1	3	3	3	3	3	3
7	4	4	3	1	3	3	3	1	1	1
7	4	4	1	3	1	1	1	3	3	3
7	2	2	4	2	4	2	0	2	4	2
7	2	2	4	2	0	4	2	2	2	4
7	2	2	4	2	2	0	4	4	2	2
7	2	2	2	4	2	4	2	4	2	0
7	2	2	2	4	2	2	4	0	4	2
7	2	2	2	4	4	2	2	2	0	4

We shall proceed with indexing of orbit structures OS1''',...,OS8''', knowing that σ and τ act on the set of ten $\langle\rho\rangle$-orbits of points and blocks as $\sigma = (5,6,7)(8,9,10)$, $\tau = (1,2)(3,4)(5,8)(6,9)(7,10)$. Obviously, it is sufficient to determine index sets for the first, third and fifth row of orbit structures OS1''',...,OS8'''. Also, in the first and third row we have to determine index sets for positions (i,r) only for $r \in \{1,2,3,4,5,8\}$. We shall denote points by $1_i, \ldots, 10_i$, $i = 0,1,\ldots,6$, and assume that automorphisms ρ, σ and τ act on the set of points as follows:

$$\begin{aligned}
\rho &= (I_0, I_1, \ldots, I_6),\ I = 1,2,\ldots,10,\\
\sigma &= (K_0)(K_1,K_2,K_4)(K_3,K_6,K_5)(5_i,6_{2i},7_{4i})(8_i,9_{2i},10_{4i}),\\
&\qquad\qquad K = 1,2,3,4, \quad i = 0,1,\ldots,6,\\
\tau &= (1_i,2_i)(3_i,4_i)(5_i,8_i)(6_i,9_i)(7_i,10_i),\ i = 0,1,\ldots,6.
\end{aligned}$$

Of course, operation with indices is multiplication modulo seven. As representatives of block orbits 1,2,3 and 4, we shall choose blocks fixed by $\langle\sigma\rangle$. Therefore, index sets for positions (i,r), $1 \le i,r \le 4$, have to be unions of sets $\emptyset, \{0\}, \{1,2,4\}, \{3,5,6\}$. To eliminate isomorphic structures during the indexing (see [3]), we have been using the permutation which on each $\langle\rho\rangle$-orbit acts as $x \mapsto 3x$ (*mod* 7), and automorphisms of orbit structures OS1''',...,OS8''' which commute with σ and τ. Constructed designs presented by their base blocks are:

$\mathcal{D}_1$:	$1_0 1_1 1_2 1_4 3_0 3_3 3_5 3_6 4_0 4_1 4_2 4_4 5_0 5_4 6_0 6_1 7_0 7_2 8_1 8_4 9_1 9_2 10_2 10_4$
	$1_0 1_3 1_5 1_6 2_0 2_1 2_2 2_4 3_0 3_1 3_2 3_4 5_0 5_6 6_0 6_5 7_0 7_3 8_1 8_6 9_2 9_5 10_3 10_4$
	$1_0 1_1 2_0 2_2 3_1 3_5 4_2 4_5 5_0 5_1 5_2 5_3 5_4 5_5 6_3 6_4 7_2 7_6 8_3 8_5 9_0 9_5 10_3 10_4$
$\mathcal{D}_2$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_3 4_5 4_6 5_2 5_3 6_4 6_6 7_1 7_5 8_0 8_3 9_0 9_6 10_0 10_5$
	$1_0 2_0 3_1 3_2 3_4 4_0 5_0 5_1 5_5 5_6 6_0 6_2 6_3 6_5 7_0 7_3 7_4 7_6 8_3 8_5 9_3 9_6 10_5 10_6$
	$1_1 1_2 1_4 2_1 2_5 2_6 3_0 3_1 3_2 4_0 5_0 5_1 5_2 5_4 6_3 6_6 8_0 8_3 9_1 9_2 9_3 9_5 10_1 10_3$
$\mathcal{D}_3$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_3 4_5 4_6 5_2 5_5 6_3 6_4 7_1 7_6 8_1 8_2 9_2 9_4 10_1 10_4$
	$1_0 2_0 3_1 3_2 3_4 4_0 5_1 5_3 5_5 5_6 6_2 6_3 6_5 6_6 7_3 7_4 7_5 7_6 8_0 8_5 9_0 9_3 10_0 10_6$
	$1_0 1_3 1_5 2_2 2_4 2_5 3_0 3_1 3_3 4_0 5_0 5_1 5_2 5_4 6_0 6_6 8_0 8_6 9_0 9_3 9_4 9_5 10_2 10_5$
$\mathcal{D}_4$:	$1_0 1_1 1_2 1_4 3_0 4_0 5_3 5_5 5_6 6_3 6_5 6_6 7_3 7_5 7_6 8_1 8_3 8_4 9_1 9_2 9_6 10_2 10_4 10_5$
	$1_0 1_3 1_5 1_6 2_0 2_1 2_2 2_4 3_3 3_5 3_6 4_0 5_0 5_1 5_5 6_0 6_2 6_3 7_0 7_4 7_6 8_1 9_2 10_4$
	$1_0 1_1 2_0 2_2 3_1 3_2 3_4 3_5 4_1 4_6 5_0 5_1 5_2 5_5 6_4 6_6 8_5 8_6 9_1 9_2 9_5 9_6 10_0 10_2$
$\mathcal{D}_5$:	$1_0 1_1 1_2 1_4 3_0 4_0 5_2 5_4 5_6 6_1 6_4 6_5 7_1 7_2 7_3 8_0 8_1 8_6 9_0 9_2 9_5 10_0 10_3 10_4$
	$1_0 1_3 1_5 1_6 2_0 2_1 2_2 2_4 3_3 3_5 3_6 4_0 5_2 5_3 5_6 6_4 6_5 6_6 7_1 7_3 7_5 8_3 9_6 10_5$
	$1_0 1_1 2_0 2_3 3_0 3_2 3_3 3_5 4_4 4_5 5_0 5_1 5_2 5_5 6_2 6_5 8_1 8_3 9_0 9_1 9_2 9_6 10_2 10_6$

One can get whole designs by applying permutations ρ, σ and τ on base blocks. Design $\mathcal{D}_1$ is constructed from the orbit structure OS2"', designs $\mathcal{D}_2$ and $\mathcal{D}_3$ from OS6"', and designs $\mathcal{D}_4$ and $\mathcal{D}_5$ from OS8"'.

The statistics of intersection of any three blocks proves that those designs are mutually non-isomorphic. With the help of the computer program by V. Tonchev we have computed that orders of full automorphism groups of those five designs are 42. Using the computer program by V. Ćepulić it was determined that $\mathcal{D}_1$ is the only self-dual design among them. □

REMARK With the help of the computer program by V. Tonchev we have determined that 2-rank of $\mathcal{D}_1$ is 27, 2-ranks of $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_5$ are 22, and 2-rank of $\mathcal{D}_4$ is 28.

6. Orbit distribution (7,7,14,42)

THEOREM 8. *Up to isomorphism and duality there are eight symmetric* $(70, 24, 8)$ *designs with automorphism group* $Frob_{21} \times Z_2$ *acting with orbit distribution* $(7, 7, 14, 42)$. *Three of them are self-dual. Full automorphism groups of those designs are isomorphic to* $Frob_{21} \times Z_2$.

PROOF. Orbit structures for the group $\langle\rho\rangle \cong Z_7$ corresponding to OS4, OS5, OS6 and OS7 are structures OS1"', OS2"',..., OS8"', as in the case of orbit distribution $(14, 14, 42)$. Automorphisms ρ, σ and τ act on the set of points as follows:

$\rho = (I_0, I_1, \dots, I_6),\ I = 1, 2, \dots, 10,$

$$\sigma = (K_0)(K_1, K_2, K_4)(K_3, K_6, K_5)(5_i, 6_{2i}, 7_{4i})(8_i, 9_{2i}, 10_{4i}),$$
$$K = 1, 2, 3, 4, \quad i = 0, 1, \dots, 6,$$
$$\tau = (1_i)(2_i)(3_i, 4_i)(5_i, 8_i)(6_i, 9_i)(7_i, 10_i),\ i = 0, 1, \dots, 6.$$

It is sufficient to determine index sets for the first, second, third and fifth row of orbit structures. In the similar way as in the case of the orbit distribution (14, 14, 42), following designs are constructed:

$\mathcal{D}_6$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_2 5_4 6_1 6_4 7_1 7_2 8_2 8_4 9_1 9_4 10_1 10_2$
	$2_0 2_3 2_5 2_6 3_0 3_3 3_5 3_6 4_0 4_3 4_5 4_6 5_3 5_4 6_1 6_6 7_2 7_5 8_3 8_4 9_1 9_6 10_2 10_5$
	$1_0 1_1 1_2 1_4 2_0 2_3 2_5 2_6 3_0 3_1 3_2 3_4 5_1 5_5 6_2 6_3 7_4 7_6 8_2 8_3 9_4 9_6 10_1 10_5$
	$1_4 1_5 2_4 2_6 3_0 3_1 4_2 4_4 5_0 5_1 5_3 5_5 6_1 6_4 6_5 6_6 7_4 7_5 8_0 8_3 8_4 8_6 10_0 10_3$
$\mathcal{D}_7$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_3 5_5 6_3 6_6 7_5 7_6 8_3 8_5 9_3 9_6 10_5 10_6$
	$2_0 2_3 2_5 2_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_2 5_6 6_4 6_5 7_1 7_3 8_2 8_6 9_4 9_5 10_1 10_3$
	$1_0 1_1 1_2 1_4 2_0 2_1 2_2 2_4 3_0 3_3 3_5 3_6 5_2 5_3 6_4 6_6 7_1 7_5 8_5 8_6 9_3 9_5 10_3 10_6$
	$1_0 1_4 2_1 2_3 3_0 3_1 4_3 4_4 5_0 5_1 5_2 5_3 5_4 5_5 6_1 6_4 7_1 7_3 8_3 8_4 9_2 9_6 10_0 10_2$
$\mathcal{D}_8$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_4 5_5 6_1 6_3 7_2 7_6 8_4 8_5 9_1 9_3 10_2 10_6$
	$2_0 2_3 2_5 2_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_3 5_6 6_5 6_6 7_3 7_5 8_3 8_6 9_5 9_6 10_3 10_5$
	$1_0 1_1 1_2 1_4 2_0 2_1 2_2 2_4 3_3 3_5 3_6 4_0 5_3 5_4 5_5 6_1 6_3 6_6 7_2 7_5 7_6 8_6 9_5 10_3$
	$1_0 1_3 2_1 2_2 3_0 3_1 3_2 4_0 5_0 5_1 5_2 5_3 5_4 6_3 7_0 8_0 8_2 8_5 9_2 9_5 9_6 10_1 10_3 10_5$
$\mathcal{D}_9$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_3 5_4 6_1 6_6 7_2 7_5 8_3 8_4 9_1 9_6 10_2 10_5$
	$2_0 2_1 2_2 2_4 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_1 5_6 6_2 6_5 7_3 7_4 8_1 8_6 9_2 9_5 10_3 10_4$
	$1_0 2_0 3_0 3_1 3_2 3_4 5_0 5_3 5_6 6_0 6_5 6_6 7_0 7_3 7_5 8_0 8_2 8_5 9_0 9_3 9_4 10_0 10_1 10_6$
	$1_1 1_4 1_6 2_2 2_3 2_6 3_0 3_1 4_2 4_4 5_0 5_1 5_2 5_3 5_4 6_1 6_3 6_4 7_0 8_5 9_2 9_3 9_5 10_1$
$\mathcal{D}_{10}$:	$1_0 1_3 1_5 1_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_3 5_5 6_3 6_6 7_1 7_6 8_3 8_5 9_3 9_6 10_1 10_6$
	$2_0 2_1 2_2 2_4 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_2 5_6 6_4 6_5 7_1 7_3 8_2 8_6 9_4 9_5 10_1 10_3$
	$1_0 2_0 3_1 3_2 3_4 4_0 5_0 5_1 5_2 5_4 6_0 6_1 6_2 6_4 7_0 7_1 7_2 7_4 8_3 8_4 9_1 9_6 10_2 10_5$
	$1_1 1_2 1_5 2_4 2_5 2_6 3_0 3_1 3_2 4_4 5_0 5_3 6_0 6_4 7_2 7_4 8_1 8_4 8_5 8_6 9_0 9_2 9_3 9_5$
$\mathcal{D}_{11}$:	$1_0 1_1 1_2 1_4 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_2 5_3 6_4 6_6 7_1 7_5 8_2 8_3 9_4 9_6 10_1 10_5$
	$2_0 2_3 2_5 2_6 3_0 3_1 3_2 3_4 4_0 4_1 4_2 4_4 5_1 5_4 6_1 6_2 7_2 7_4 8_1 8_4 9_1 9_2 10_2 10_4$
	$1_0 2_0 3_1 3_2 3_4 4_0 5_0 5_2 5_5 5_6 6_0 6_3 6_4 6_5 7_0 7_1 7_3 7_6 8_4 8_6 9_1 9_5 10_2 10_3$
	$1_1 1_3 1_5 2_2 2_3 2_4 3_0 3_1 3_2 4_4 5_0 5_3 6_3 6_6 7_0 7_3 8_1 8_4 8_5 8_6 9_0 9_2 9_3 9_4$
$\mathcal{D}_{12}$:	$1_0 1_1 1_2 1_4 3_0 4_0 5_4 5_5 5_6 6_1 6_3 6_5 7_2 7_3 7_6 8_4 8_5 8_6 9_1 9_3 9_5 10_2 10_3 10_6$
	$2_0 2_3 2_5 2_6 3_0 4_0 5_1 5_2 5_5 6_2 6_3 6_4 7_1 7_4 7_6 8_1 8_2 8_5 9_2 9_3 9_4 10_1 10_4 10_6$
	$1_0 1_1 1_2 1_4 2_0 2_1 2_2 2_4 3_1 3_2 3_4 4_0 5_1 5_2 5_4 6_1 6_2 6_4 7_1 7_2 7_4 8_0 9_0 10_0$
	$1_0 1_1 2_2 2_6 3_2 3_3 3_4 3_5 4_0 4_4 5_0 5_1 5_3 5_5 6_4 6_6 8_3 8_6 9_1 9_2 9_3 9_5 10_4 10_5$
$\mathcal{D}_{13}$:	$1_0 1_1 1_2 1_4 3_0 4_0 5_4 5_5 5_6 6_1 6_3 6_5 7_2 7_3 7_6 8_4 8_5 8_6 9_1 9_3 9_5 10_2 10_3 10_6$
	$2_0 2_3 2_5 2_6 3_0 4_0 5_1 5_2 5_5 6_2 6_3 6_4 7_1 7_4 7_6 8_1 8_2 8_5 9_2 9_3 9_4 10_1 10_4 10_6$
	$1_0 1_3 1_5 1_6 2_0 2_3 2_5 2_6 3_3 3_5 3_6 4_0 5_3 5_5 5_6 6_3 6_5 6_6 7_3 7_5 7_6 8_0 9_0 10_0$
	$1_0 1_1 2_2 2_6 3_2 3_3 3_4 3_5 4_0 4_4 5_0 5_1 5_3 5_5 6_4 6_6 8_3 8_6 9_1 9_2 9_3 9_5 10_4 10_5$

Design $\mathcal{D}_6$ is constructed from the orbit structure OS1"', design $\mathcal{D}_7$ from the orbit structure OS2"', $\mathcal{D}_8$ from OS3"', $\mathcal{D}_9$ from OS5"', designs $\mathcal{D}_{10}$ and

$\mathcal{D}_{11}$ from OS7''', and designs $\mathcal{D}_{12}$ and $\mathcal{D}_{13}$ from OS8'''. Orders of full automorphism groups of all constructed designs are 42. Designs $\mathcal{D}_6$, $\mathcal{D}_7$ and $\mathcal{D}_8$ are self-dual. □

REMARK Design $\mathcal{D}_6$ is isomorphic to the one constructed by Z. Janko and Tran van Trung. Using the computer program by V. Tonchev it was computed that 2-rank of $\mathcal{D}_6$ is 24, 2-rank of $\mathcal{D}_7$ is 30, 2-rank of $\mathcal{D}_8$ is 31, 2-ranks of $\mathcal{D}_9$, $\mathcal{D}_{10}$, $\mathcal{D}_{11}$ and $\mathcal{D}_{13}$ are 28, and 2-rank of $\mathcal{D}_{12}$ is 25.

LEMMA 10. *Group $Frob_{21} \times Z_2$ can not act as an automorphism group of a symmetric $(70, 24, 8)$ design with orbit distribution $(7, 7, 7, 7, 42)$.*

PROOF. In this case involution τ acts on the set of points as

$$\tau = (1_i)(2_i)(3_i)(4_i)(5_i, 8_i)(6_i, 9_i)(7_i, 10_i), \ i = 0, 1, \dots, 6.$$

Only orbit structures for Z_7 corresponding to orbit structures OS8 and OS9 are OS1''' and OS2'''. Indexing of these structures doesn't lead to designs. □

Thereby we have proved the following theorem:

THEOREM 9. *Up to isomorphism, there are 22 symmetric $(70, 24, 8)$ designs with automorphism group isomorphic to $Frob_{21} \times Z_2$. Among them there are four self-dual and nine pairs of dual designs. Full automorphism groups of those designs are isomorphic to $Frob_{21} \times Z_2$.*

Since there was only one known design with automorphism group isomorphic to $Frob_{21} \times Z_2$, twenty-one of constructed designs are new.

REFERENCES

[1] M. Aschbacher, *On Collineation Groups of Symmetric Block Designs*, J. Combin. Theory 11 (1971), 272–281.

[2] C. J. Colbourn, J. H. Dinitz, *The CRC Handbook of Combinatorial Designs*, CRC press, Boca Raton–New York–London–Tokyo, 1996.

[3] V. Ćepulić, *On symmetric block designs (40,13,4) with automorphisms of order 5*, Discrete Math. 128 (1994) no. 1–3, 45–60.

[4] A. Golemac, *Construction of new symmetric designs with parameters* $(70, 24, 8)$, Discrete Math. 120 (1993), 51–58.

[5] Z. Janko, *Coset Enumeration in Groups and Constructions of Symmetric Designs*, Combinatorics '90 (1992), 275–277.

[6] Z. Janko and Tran v. Trung, *The existence of a symmetric block design for* $(70, 24, 8)$, Mitt. Math. Sem. Gießen, vol. 165 (1984), 17–18.

[7] E. Lander, *Symmetric Designs: An Algebraic Approach*, Cambridge University Press (1983).

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