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ACCELERATING ORBITS OF TWIST DIFFEOMORPHISMS ON A TORUS

S.SLIJEPČEVIĆ

ABSTRACT. Given an area-preserving twist diffeomorphism on a 2D torus, we prove existence of orbits asymptotic to arbitrary periodic or quasiperiodic Aubry-Mather minimising set and with arbitrary shear rotation number from the shear rotation interval; and orbits whose ends have two arbitrary shear rotation numbers from the shear rotation interval. As a corollary, we construct infinitely many ergodic measures with positive metric entropy supported on the set of accelerating orbits, and therefore mutually singular with the invariant measures constructed in [MF].

1. INTRODUCTION

Area-preserving twist diffeomorphisms on a cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ are studied as the simplest example of a Hamiltonian dynamical system. The condition of area-preserving is the standard condition of preserving a symplectic form, and the twist condition enables one to set up a discrete variational principle (i.e. provides existence of a discrete Legendre transform).

The important results on area-preserving twist diffeomorphisms are concerned with the existence of different types of orbits. The existence of periodic and quasiperiodic (Birkhoff) orbits of arbitrary period have been proved independently in the late seventies by Aubry ([ALD], generalised by [BAN]) and Mather (see [MF]) using different variational techniques. If the map is given by $(x,p) \mapsto (x',p')$, and we interprete the coordinate p as momentum, Aubry and Mather proved existence of orbits with arbitrary uniform average velocity. More recently Mather in [MAT] proved existence of 'connecting' orbits that have two almost arbitrary average 'velocities' (i.e. rotation numbers) forward and backward in time.

In this paper we prove results in the same spirit, proving existence of orbits with different asymptotic 'accelerations' forward and backward in time; and orbits having arbitrary asymptotic acceleration forward in time, and arbitrary asymptotic velocity backwards in time. The technique used in [MAT]

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is variational, while in this paper we rely on the monotonicity property of the gradient dynamics of the action functional (i.e. Frenkel-Kontorova model). An alternative proof of the results from [MAT] using a similar technique is reported in [SL2].

Since we analyse only the diffeomorphisms with well-defined asymptotic 'acceleration', we restrict our analysis to those periodic in 'vertical' direction, or equivalently on a torus. Acceleration is defined as the shear-rotation number of an orbit, recently studied in [DOE].

All our results are interesting in the case when the shear-rotation interval (the set of all possible shear-rotation numbers) is non-trivial. The shear-rotation interval is not a single point if and only if there are no invariant circles non-homotopic to a point (called shortly invariant circles in the following); we discuss the result in section 4.

One of important standing conjectures of Hamiltonian dynamics, due to Arnold, is that near-integrable Hamiltonian systems are generically unstable, and as a method of proof Arnold suggested construction of orbits with unbounded action variable ([ARN]). The problem of existence of such orbits became known as the problem of *Arnold diffusion* (for recent results see [XIA]). In the case of twist maps it has been completely solved by Mather [MAT]: such orbits exist if and only if there are no invariant circles.

Our result is a contribution towards understanding of Arnold diffusion phenomena in the case of twist maps; in particular, we show that when the invariant set of accelerating orbits is non-empty, it has positive topological entropy.

In section 2 we give all the definitions and formulate known properties of twist maps in our setting. In section 3 we construct the connecting orbits. In section 4, as a corollary, we construct ergodic measures with positive metric entropy supported on the set of accelerating orbits and discuss and announce related results.

2. The twist diffeomorphisms

Let $\Omega = dp \wedge dx$ be the canonical 2-form on the cylinder $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where x is the angular variable and p the vertical variable. Then $\Omega = d\alpha$, where $\alpha = pdx$. We say that a diffeomorphism F of $\mathbb{A}$ is area-preserving if

$$F^*\Omega - \Omega = 0,$$

(i.e. F is symplectic). We say that F is exact symplectic, if there exists a function $\tilde{V} : \mathbb{A} \mapsto \mathbb{R}$ such that

$$F^*\alpha - \alpha = d\tilde{V}$$

(exact symplectic implies area-preserving). We write $F(x,p) = (x',p')$.

Definition 2.1. *A diffeomorphism $F : \mathbb{A} \mapsto \mathbb{A}$ is a twist diffeomorphism if it is orientation preserving and:*

(i) F is exact symplectic: $F^*\alpha - \alpha = d\tilde{V}$, $\tilde{V} : \mathbb{A} \mapsto \mathbb{R}$,

(ii) F is a twist map: $\frac{\partial x'}{\partial p} \geq \delta > 0$, *where $\delta > 0$ is a fixed constant.*

A diffeomorphism $f : \mathbb{T} \mapsto \mathbb{T}$ on a torus $\mathbb{T} = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ is a twist diffeomorphism on a torus, if its lift $F : \mathbb{A} \mapsto \mathbb{A}$ is a twist diffeomorphism.

For a given twist diffeomorphism $F : \mathbb{A} \mapsto \mathbb{A}$, (ii) implies that we can define (working in the covering space $\mathbb{R}^2$) the diffeomorphism $\psi : (x, p) \mapsto (x, x')$ of $\mathbb{R}^2$ (the discrete Legendre transform), and the Lagrange transform V of $\tilde{V}$, $V \circ \psi = \tilde{V}$, $V : \mathbb{R}^2 \mapsto \mathbb{R}^2$. We call V a generating function of F, and it is easy to show (see [KAT]) that it is a C^2 function satisfying the following properties:

(i) **Periodicity:** for each $x, y \in \mathbb{R}^2$, $V(x+1, y+1) = V(x, y)$,

(ii) **Twist condition:** for each $x, y \in \mathbb{R}^2$, $V_{12}(x, y) \leq -\delta < 0$ (where indices denote partial derivatives),

(iii) **Uniform divergence at infinity:** $\lim_{|\eta| \to \infty} V(x, x + \eta) = +\infty$, uniformly in x.

(iv) The lift of the twist diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, p) \mapsto (x', p')$ satisfies the relations:

$$\begin{aligned} p &= -V_1(x, x') \\ p' &= V_2(x, x'); \end{aligned} \tag{2.1}$$

where indices denote partial derivatives.

The conditions (i)-(iii) are sometimes the axioms of the Aubry-Mather theory (i.e. in [BAN]).

Example 2.1. *The family of standard (Chirikov) maps $(x, p) \mapsto (x + p + k\sin(2\pi x), p + k\sin(2\pi x))$ is a family of twist diffeomorphisms on a torus (k is a parameter), with generating functions $V(x, y) = \frac{1}{2}(x - y)^2 + \frac{k}{2\pi}\cos(2\pi x)$.*

We assume in the following that f is a twist diffeomorphism on a torus, and that $V : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is its generating function. We denote by F its lift $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

Let $L = \pi_1(f) : \mathbb{Z}^2 \to \mathbb{Z}^2$ be the induced map on the fundamental group $\mathbb{Z}^2$ of torus.

Proposition 2.1. *If $L = \pi_1(f)$, there exists $N \in \mathbb{N}$, $N > 0$, such that the representation of L is*

$$\pi_1(f) = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Proof. Since f is a torus homeomorphism, the induced map on the fundamental group $L = \pi_1(f)$ is an isomorphism, and therefore $\det(L) = \pm 1$. Since f is orientation-preserving, $\det(L) = 1$. We represent $L \in GL(2, \mathbb{Z})$ as a matrix

$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that for each $u \in \mathbb{R}^2$, $z \in \mathbb{Z}^2$,

$$F(u+z) = F(u) + Lz. \tag{2.3}$$

Periodicity implies that if $F(x,p) = (x',p')$, then $F(x+1,p) = (x'+1,p')$. Relation (2.3) now implies

$$F\begin{pmatrix} x^* \\ p \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = F\begin{pmatrix} x^*+1 \\ p \end{pmatrix} = F\begin{pmatrix} x^* \\ p \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and we conclude that $a = 1$, $c = 0$. Since $\det(L) = 1$, $d = 0$. The twist condition implies that $b > 0$, and $N = b$.□

We call N constructed above the twist number of a twist map on a torus f.

From the classification of homeomorphisms on a torus now follows that f is homotopic to a Dehn twist (see [THU], [DOE]). Therefore we can apply both the results of Aubry-Mather theory of area-preserving twist diffeomorphisms ([BAN]), and the theory of shear rotation numbers of homeomorphisms of the torus homotopic to a Dehn twist ([DOE]).

The variational approach ([ALD],[BAN],[MAT]) minimises the formal sum $\sum_{i=-\infty}^{\infty} V(x_i, x_{i+1})$ (energy, or action functional), and uses the fact that the stationary configurations of the functional correspond $1-1$ to the orbits of the twist diffeomorphism (a discrete Lagrange principle). In detail:

Definition 2.2. *The configuration* $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ *is stationary, if for each* i,

$$V_2(x_{i-1}, x_i) + V_1(x_i, x_{i+1}) = 0.$$

The set of all stationary configurations is denoted by $\mathcal{S}$.

The topology on $\mathcal{S}$ is the induced product topology from $\mathbb{R}^{\mathbb{Z}}$. It is easy to see that the mapping $\iota : \mathcal{S} \mapsto \mathbb{R}^2$; $\mathbf{x} \mapsto (x_0, -V_1(x_0, x_1))$ is a homeomorphism, and that $(x_n, -V_1(x_n, x_{n+1}))_{n \in \mathbb{Z}}$ is the F-orbit of $(x_0, -V_1(x_0, x_1))$.

We will work mainly in the space of configurations $\mathbb{R}^{\mathbb{Z}}$, and look for stationary configurations with desired properties.

We now define the rotation numbers. The rotation number measures average change in the first coordinate; and shear rotation number of the second coordinate of the F-iterated point.

In the following definition, we denote by (x_n, p_n) the F-orbit of (x,p).

Definition 2.3. *Given a twist diffeomorphism on a torus* f, *the rotation a point* $(x,p) \in \mathbb{T}$ *is (if the expression is convergent):*

$$\rho((x,p), f) = \lim_{|n| \to \infty} \frac{x_n - x_0}{n}. \tag{2.4}$$

The α-shear rotation number $\sigma_\alpha((x,p),f)$ and ω-shear rotation number $\sigma_\omega((x,p),f)$ are defined (if the expressions are convergent):

$$\sigma_\alpha((x,p),f) = \lim_{n\to-\infty} \frac{p_n - p_0}{n}. \tag{2.5}$$

$$\sigma_\omega((x,p),f) = \lim_{n\to\infty} \frac{p_n - p_0}{n}. \tag{2.6}$$

The α-shear (ω-shear) rotation sets are

$$\begin{aligned} \sigma_\alpha(f) &= \{\sigma_\alpha((x,p),f),\ (x,p)\in\mathbb{T}\}, \\ \sigma_\omega(f) &= \{\sigma_\omega((x,p),f),\ (x,p)\in\mathbb{T}\}. \end{aligned}$$

Since all lifts $F:\mathbb{R}^2\to\mathbb{R}^2$ of the map f are the same up to a constant vector, the definition above is independent of the choice of a lift F. Given a stationary configuration $\mathbf{x}$, relation (2.1) implies that we can write

$$\sigma_\alpha(\mathbf{x},f) = \lim_{n\to-\infty} \frac{-V_1(x_n,x_{n+1})+V_1(x_0,x_1)}{n}, \tag{2.7}$$

$$\sigma_\omega(\mathbf{x},f) = \lim_{n\to\infty} \frac{-V_1(x_n,x_{n+1})+V_1(x_0,x_1)}{n}, \tag{2.8}$$

and then $\sigma(\mathbf{x},F)=\sigma(\iota(\mathbf{x}),f)$.

Proposition 2.2. *For each $\rho\in\mathbb{R}$, there exists $\mathbf{x}\in S$, such that $\rho(\mathbf{x},f)=\rho$. The α and ω-shear rotation sets coincide $\sigma_\alpha(f)=\sigma_\omega(f)$. They are a closed interval containing 0.*

Proof. The fact that the shear rotation sets are closed follows from [DOE], Theorem 6.7; and that they are equal follows from the construction in the proof. Construction of orbits with arbitrary rotation number is a standard result of Aubry-Mather theory (see [BAN]), and every orbit (stationary configuration) with well defined rotation number has shear rotation numbers 0. □

We write $\sigma(f)=\sigma_\alpha(f)=\sigma_\omega(f)$.

3. Construction of connecting orbits

Instead of variational methods, we will construct the stationary configurations (and corresponding orbits of the twist diffeomorphism) as critical points of the 'energy flow' (or gradient dynamics of the Frenkel-Kontorova model) below. The main tool will be the monotonicity of the dynamics.

We use the following equations:

$$\begin{aligned} u_i'(t) &= -V_1(u_{i-1},u_i)-V_2(u_i,u_{i+1}) \\ u_i(0) &= u_i^0; \end{aligned} \tag{3.1}$$

where $\mathbf{u}^0\in\mathbb{R}^{\mathbb{Z}}$ is the initial condition.

Definition 3.1. *The set $\mathcal{S}^+$ is the set of all (supercritical) configurations* $\mathbf{u}$, *such that for each $i \in \mathbb{Z}$, $V_2(u_{i-1}, u_i) + V_1(u_i, u_{i+1}) \geq 0$. The set $\mathcal{S}^-$ is the set of all (subcritical) configurations* $\mathbf{u}$, *such that for each $i \in \mathbb{Z}$, $V_2(u_{i-1}, u_i) + V_1(u_i, u_{i+1}) \leq 0$.*

We define the ordering on $\mathbb{R}^{\mathbb{Z}}$ with

$$\mathbf{u} \geq \mathbf{v} \iff \forall i \in \mathbb{Z},\ u_i \geq v_i;$$
$$\mathbf{u} \gg \mathbf{v} \iff \forall i \in \mathbb{Z},\ u_i > v_i.$$

Two configurations $\mathbf{u}$ *and* $\mathbf{v}$ *intersect if for some i, $(u_{i-1} - v_{i-1})(u_{i+1} - v_{i+1}) < 0$ and $u_i = v_i$; or $(u_i - v_i)(u_{i+1} - v_{i+1}) < 0$.*

The following properties of the gradient dynamics are well known (it is a simple consequence of the fact that the off-diagonal elements of the linearised equation (3.1) are positive, for a proof see e.g. [GOL]).

Lemma 3.1. *(i)* **No passing property:** *if* $\mathbf{u}(0) \geq \mathbf{v}(0)$, *then for each $t \geq 0$,* $\mathbf{u}(t) \geq \mathbf{v}(t)$.

(ii) If $\mathbf{u}^0 \in \mathcal{S}^+$, then the solution $\mathbf{u}(t)$ is decreasing in t. If $\mathbf{u}^0 \in \mathcal{S}^-$, then the solution $\mathbf{u}(t)$ is increasing in t.

The No-passing property in Lemma 3.1 is a discrete analogue of the comparison principle for parabolic differential equations.

The following Lemma will be the main tool in the construction of different orbits. We prove it using the monotonicity of the gradient dynamics (9). An alternative, more topological proof can be found in [ANA], Theorem 4.2.

Lemma 3.2. *Assume that there are two configurations $\mathbf{u}^- \leq \mathbf{u}^+$ such that $\mathbf{u}^- \in \mathcal{S}^-$, $\mathbf{u}^+ \in \mathcal{S}^+$. Then there exists a configuration $\mathbf{x} \in \mathcal{S}$, $\mathbf{u}^- \leq \mathbf{x} \leq \mathbf{u}^+$.*

Proof. Assume that $\mathbf{v}(t), \mathbf{w}(t)$ are solutions of (3.1) with the initial conditions $\mathbf{u}^-$, $\mathbf{u}^+$. Lemma 3.1 implies that $\mathbf{v}(t)$ is increasing, $\mathbf{w}(t)$ is decreasing, and that for each t:

$$\mathbf{v}(t) \leq \mathbf{w}(t) \leq \mathbf{u}^+.$$

Therefore $\mathbf{v}(t)$ is increasing and bounded, and hence it converges in the product topology. It is easy to check that its limit $\mathbf{x} = \lim_{t\to\infty} \mathbf{v}(t)$ must be an equilibrium of (3.1), and therefore $\mathbf{x} \in \mathcal{S}$. □

Lemma 3.3. *Assume that two stationary configurations* $\mathbf{u}$, $\mathbf{v}$ *intersect so that for some j, $u_{j-1} < v_{j-1}$, $u_j \geq v_j$. If the configuration* $\mathbf{z}$ *is defined with:*

(i) $z_i = u_i$ for $i < j$, $z_i = v_i$ for $i \geq j$; then $\mathbf{z} \in \mathcal{S}^+$;

(ii) $z_i = v_i$ for $i < j$, $z_i = u_i$ for $i \geq j$; then $\mathbf{z} \in \mathcal{S}^-$.

Proof. We prove the case (i). There are two possibilities; in the calculations we apply the mean value theorem and the twist condition $V_{12}(x, y) \leq -\delta$.

a) Assume that $u_j > v_j$. Then

$$\begin{aligned}
V_2(z_{j-2}, z_{j-1}) + V_1(z_{j-1}, z_j) &= V_2(u_{j-2}, u_{j-1}) + V_1(u_{j-1}, v_j) \\
&= V_2(u_{j-2}, u_{j-1}) + V_1(u_{j-1}, u_j) \\
&\quad + V_{12}(u_{j-1}, \eta)(v_j - u_j) > 0, \\
V_2(z_{j-1}, z_j) + V_1(z_j, z_{j+1}) &= V_2(u_{j-1}, v_j) + V_1(v_j, v_{j+1}) \\
&= V_{12}(\eta', v_j)(u_{j-1} - v_{j-1}) + V_2(u_{v-1}, v_j) \\
&\quad + V_1(v_j, v_{j+1}) > 0, \\
V_2(z_{i-1}, z_i) + V_1(z_i, z_{i+1}) &= 0 \quad \forall i \neq j-1, j.
\end{aligned}$$

b) Assume that $u_j = v_j$. We will first show that $u_{j+1} \geq v_{j+1}$. Assume the contrary; now we get a contradiction:

$$\begin{aligned}
0 &= V_2(u_{j-1}, u_j) + V_1(u_j, u_{j+1}) - V_2(v_{j-1}, v_j) - V_1(v_j, v_{j+1}) \\
&= V_{12}(\eta, u_j)(u_{j-1} - v_{j-1}) + V_{12}(u_j, \eta')(u_{j+1} - v_{j+1}) < 0.
\end{aligned}$$

Now we deduce that for each i, $V_2(z_{i-1}, z_i) + V_1(z_i, z_{i+1}) \geq 0$ in the same way as in a). □

We recall from (2.3) that the twist number N is the constant such that for each $x, p \in \mathbb{R}$

$$\begin{aligned}
F(x+1, p) &= F(x, p) + (1, 0) \\
F(x, p+1) &= F(x, p) + (N, 1).
\end{aligned} \tag{3.2}$$

Lemma 3.4. *There exist constants A, B such that for each $(x, y) \in \mathbb{R}^2$,*

$$V_1(x, y) \in [\frac{x-y}{N} + A, \frac{x-y}{N} + B].$$

Proof. Let C_1 and C_2 be minimum and maximum of $V_1(x, y)$ on the compact set of all x, y such that $x \in [0, 1]$, and $x - y \in [0, N]$. Then for each (x, y) such that $x \in [0, 1]$, $x - y \in [0, N]$,

$$V_1(x, y) \in [\frac{x-y}{N} + C_1 - 1, \frac{x-y}{N} + C_2]. \tag{3.3}$$

Because of periodicity $V_1(x+1, y+1) = V_1(x, y)$, (3.3) is valid for each x, y such that $x - y \in [0, N]$. Substituting relation (3.2) into (2.1) one gets

$$V_1(x, x' + N) = V_1(x, x') - 1. \tag{3.4}$$

The claim now follows inductively from (3.3), (3.4), setting $A = C_1 - 1$, $B = C_2$. □

We define a translation $\Theta_{a,b} : \mathcal{S} \to \mathcal{S}$, $(x_i)_{i \in \mathbb{Z}} \mapsto (x_i + a + bNi)_{i \in \mathbb{Z}}$; where $(a, b) \in \mathbb{Z}^2$. The translation $\Theta_{a,b}$ adds a straight line with the slope bN to the stationary configuration. Its image is in $\mathcal{S}$ because of (3.2). The translation $\Theta_{a,b}$ leaves shear rotation numbers invariant.

The following lemma shows that the translation $\Theta_{a,b}$ plays the same role for shear rotation numbers, as the translation $\tau_{a,b}$ for rotation numbers (see [BAN]).

Lemma 3.5. *Assume that* $\mathbf{u} \in \mathcal{S}$ *has* ω*-shear rotation number* σ*, and that* $\mathbf{v} \in \mathcal{S}$*.*

(i) If for some $a \in \mathbb{Z}$*,* $\mathbf{u}' = \Theta_{a,0}\mathbf{u}$ *and there is* j *such that for each* $i \geq j$*,* $u_i \leq v_i \leq u_i'$*, then* $\mathbf{v}$ *has* ω*-shear rotation number* σ*.*

(ii) If for each $(a, b) \in \mathbb{Z}^2$*,* $\Theta_{a,b}\mathbf{u}$ *and* $\mathbf{v}$ *intersect at most once, then* $\mathbf{v}$ *has* ω*-shear rotation number* σ*.*

Analogous statements are true for α*-shear rotation numbers.*

Proof. (i) The conditions imply that for all $i \geq j$, $u_i \leq v_i \leq u_i + a$ and then

$$u_i - u_{i+1} - a \leq v_i - v_{i+1} \leq u_i - u_{i+1} + a.$$

Applying Lemma 3.4 we get that for each $i \geq j$,

$$|V_1(v_i, v_{i+1}) - V_1(u_i, u_{i+1})| \leq$$

$$\frac{v_i - v_{i+1} + B}{N} - \frac{u_i - u_{i+1} - A}{N} \leq B - A + \frac{a}{N}. \tag{3.5}$$

The claim now follows from the definition of the ω-shear rotation number.

(ii) First choose $(a, b) \in \mathbb{R}^2$ such that $\mathbf{u}' = T_{a,b}\mathbf{u}$, $u_0' < v_0$, $u_1' > v_1$. Because $\mathbf{u}'$ and $\mathbf{v}$ intersect at most once, for each $i > 0$, $u_i' > v_i$. Now choose $(c, d) \in \mathbb{R}^2$ such that $\mathbf{u}'' = T_{a,b}\mathbf{u}$, $u_0'' > v_0$, $u_1'' < v_1$. Because $\mathbf{u}''$ and $\mathbf{v}$ intersect at most once, for each $i > 0$, $u_i'' < v_i$. We distinguish three cases:

a) There exists $j > 0$ such that $v_{j+1} - v_j < u_{j+1}'' - u_j'' - 1$. Then we can find $n \in \mathbb{Z}$ such that $u_j'' + n \in [v_j - 1, v_j)$, and then $u_j'' + n < v_j$, $u_{j+1}'' + n \geq v_{j+1}$. Define $\mathbf{u}''' = \Theta_{n,0}\mathbf{u}'' = \Theta_{c+n,d}\mathbf{u}$. Now for each $i > j$, $u_j''' \geq v_j$, and (i) implies that $\mathbf{v}$ has the same ω-shear rotation number as $\mathbf{u}''$, and it is σ.

b) There exists $j > 0$ such that $v_{j+1} - v_j > u_{j+1}' - u_j' + 1$. Analogously as in a) we conclude that $\mathbf{v}$ has the same ω-shear rotation number as $\mathbf{u}'$, and it is σ.

c) Assume that it is neither a) nor b). Then for each $j > 0$,

$$u_{j+1}'' - u_j'' - 1 \leq v_{j+1} - v_j \leq u_{j+1}' - u_j' + 1.$$

Applying Lemma 3.4 similarly as in (3.5) one gets

$$V_1(u_j'', u_{j+1}'') - (B - A + \frac{1}{N}) \leq V_1(v_j, v_{j+1}) \leq V_1(u_j', u_{j+1}') + (B - A + \frac{1}{N}).$$

Since $\mathbf{u}'$, $\mathbf{u}''$ have the same ω-shear rotation number σ, so does $\mathbf{v}$. □

Now we are ready for the construction of connecting orbits. If we plot a stationary configuration $\mathbf{x}$ in the Aubry diagram (graph (i, x_i)), one should think about the configurations with well defined rotation number as approximately straight lines (the rotation number is the slope), and the configurations with well defined shear-rotation number as approximately parabolas.

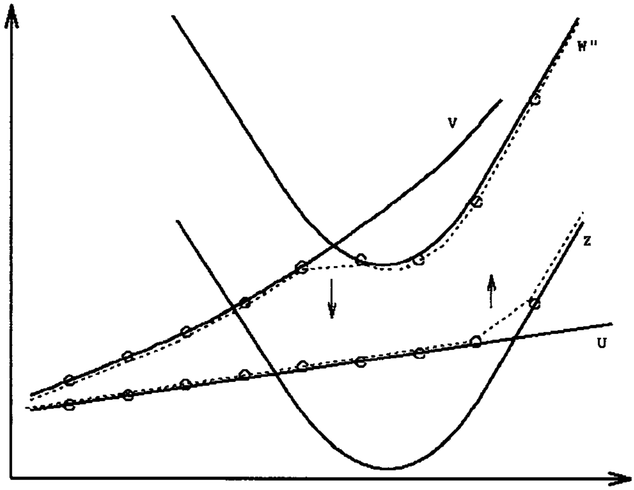

FIGURE 1. Proof of Theorem 3.6, configurations are plotted in the Aubry diagram. Dashed lines: configurations $\mathbf{u}^-$ and $\mathbf{u}^+$; arrows: direction of their dynamics.

We need the basic definition of the Aubry-Mather theory.

Definition 3.2. *A configuration* $\mathbf{u} \in \mathcal{S}$ *is minimising if for each* $i < j$ *and for each segment* $(v_i, v_{i+1}, ..., v_{j-1}, v_j) \in \mathbb{R}^{j-i-1}$ *with* $u_i = v_i$, $u_j = v_j$,

$$V(u_i, u_{i+1}, ..., u_{j-1}, u_j) \leq V(v_i, v_{i+1}, ..., v_{j-1}, v_j).$$

Aubry and Mather proved (see [BAN]) that for each $\rho \in \mathbb{R}$, there exists a minimising configuration with rotation number ρ (i.e. periodic or quasiperiodic orbit of F).

We say that two configurations $\mathbf{u}$, $\mathbf{v}$ are α-asymptotic (ω-asymptotic) if $\lim_{i\to-\infty}(u_i - v_i) = 0$ ($\lim_{i\to\infty}(u_i - v_i) = 0$ respectively).

Theorem 3.6. *Given a twist diffeomorphism on a torus* f, *choose* $\rho \in \mathbb{R}$, *and* $\sigma \in \sigma(f)$. *Then there exists a stationary configuration* $\mathbf{x}$ *(i.e. an orbit of* F*) with* ω*-shear rotation number* σ, *and* α*-asymptotic to a minimising configuration* $\mathbf{u}$ *with rotation number* ρ.

Proof. Assume first that the set of all minimising configurations with rotation number ρ is an invariant circle (see [BAN] for details). Since invariant circles exist, each Birkhoff region of instability on a cylinder is bounded (see [MAT]); hence for each stationary configuration the set $\{-V_1(x_n, x_{n+1}),\ n \in \mathbb{Z}\}$ is bounded. The definition of the shear rotation numbers now implies that the shear rotation set is $\sigma(F) = \{0\}$ and the claim is trivial.

Assume now the set of all minimising configurations with the rotation number ρ is not an invariant circle, and that $\sigma > 0$ (the case $\sigma < 0$ is analogous). In [MAT] it was proved that there exists a connecting orbit $\mathbf{v}$ α-asymptotic to a minimising orbit $\mathbf{u}$ with rotation number ρ, and ω-asymptotic to a minimising configuration with rotation number $\rho + \epsilon$, for some $\epsilon > 0$. Furthermore, we can find $\mathbf{v}$ such that $\mathbf{v} \gg \mathbf{u}$, and such that

$$v_0 \in (u_0, u_0 + 1]. \tag{3.6}$$

We can find m such that

$$\forall i \geq m, \;\; v_m > u_m + 1. \tag{3.7}$$

Choose $\mathbf{w}$ with ω-shear rotation number σ.

Construction: (Figure 1)

(i) $\mathbf{u}^+$: Choose b such that $\mathbf{w}' = \Theta_{0,b}\mathbf{w}$ and such that for all i, $0 \leq i \leq m$ implies

$$w'_{i+1} - w'_i < u_{i+1} - u_i - 1. \tag{3.8}$$

Since $\sigma > 0$, Lemma 3.4 implies that $\lim_{i\to\infty} w'_{i+1} - w'_i = \infty$, and Aubry-Mather theory ([BAN]) implies that for each i, $|u_{i+1} - u_i| \in [\rho - 1, \rho + 1]$. Therefore we can find a such that $\mathbf{w}'' = \Theta_{a,b}\mathbf{w}$, and for all $i \geq 0$, $w''_i \geq u_i$. We can choose the smallest such a, such that for some $k \geq 0$,

$$w''_k \leq u_k + 1 \tag{3.9}$$

(and (3.8) then implies that $k \geq m$). Relations (3.6), (3.7) and (3.9) imply that $w''_0 > v_0$ and $w''_k < v_k$, and therefore $\mathbf{w}''$ and $\mathbf{v}$ intersect somewhere between 0 and k, say in such a way that for some $0 < j \leq k$, $v_{j-1} < w''_{j-1}$, $v_j \geq w''_j$. We define $u^+_i = v_i$ for $i \leq j-1$, and $u^+_i = w''_i$ for $i \geq j$. Lemma 3.3 implies that $\mathbf{u}^+ \in \mathcal{S}^+$.

(ii) $\mathbf{u}^-$: Define $\mathbf{z} = \Theta_{-1,0}\mathbf{w}''$, and then $z_k \leq u_k$. Since z_i grows faster then u_i (faster than linearly in i, see Lemma 3.4), we can find the last intersection of $\mathbf{u}$ and $\mathbf{z}$, say in such a way that for some $j \geq m$, $z_{j-1} < u_{j-1}$, $z_j \geq u_j$. We define $u^-_i = u_i$ for $i < j$, and $u^-_i = z_i$ for $i \geq j$. Lemma 3.3 implies that $\mathbf{u}^- \in \mathcal{S}^-$; it is easy to see that $\mathbf{u}^- < \mathbf{u}^+$.

(iii) $\mathbf{x}$: Lemma 3.2 implies that there exists a stationary configuration $\mathbf{x}$, $\mathbf{u}^- \leq \mathbf{x} \leq \mathbf{u}^+$. Since $\mathbf{u}^-$, $\mathbf{u}^+$ and $\mathbf{u}$ are α asymptotic, so is $\mathbf{x}$ α-asymptotic to $\mathbf{u}$; from the construction and Lemma 3.5, (i) it follows that the ω-shear rotation number of $\mathbf{x}$ is σ. □

Theorem 3.7. *Choose σ_1, σ_2 from $\sigma(F)$. Then there exists a stationary configuration (i.e. an orbit of F) with α-shear rotation number σ_1 and ω-shear rotation number σ_2.*

Proof. We can find stationary configurations $\mathbf{u}$ with α-shear rotation number σ_1 and $\mathbf{v}$ with ω-shear rotation number σ_2. If for each $(a, b) \in \mathbb{Z}$, $\mathbf{u}$ and

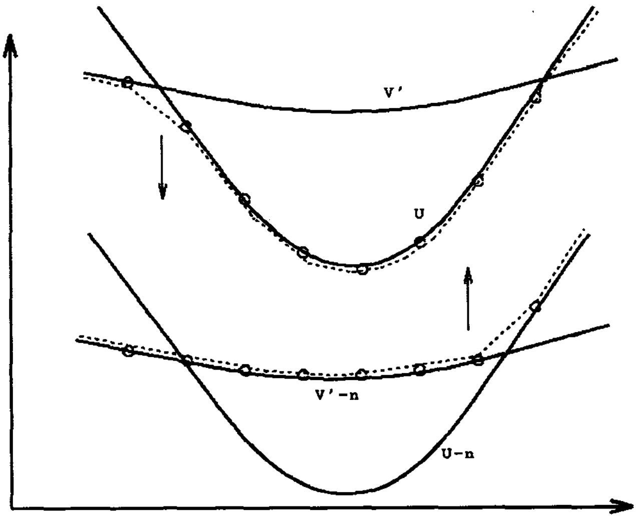

FIGURE 2. Proof of Theorem 3.7. Dashed lines: configurations $\mathbf{u}^-$ and $\mathbf{u}^+$; arrows: direction of their dynamics.

$\Theta_{a,b}\mathbf{v}$ intersect at most once, Lemma 3.5, (ii) implies that $\mathbf{u}$ and $\mathbf{v}$ have the same ω-shear rotation numbers, and the claim is proved. Assume that for some $(a,b) \in \mathbb{Z}$, $\mathbf{u}$ and $\mathbf{v}' = \Theta_{a,b}\mathbf{v}$ intersect twice, say in such a way that $u_{j-1} < v'_{j-1}$, $u_j \geq v'_j$; and $u_{k-1} > v'_{k-1}$, $u_k \leq v'_k$, with $k > j$.

Construction: (Figure 2)

(i) $\mathbf{u}^+$: Define $u_i^+ = u_i$ for $i < j$; $u_i^+ = v'_i$ for $i \geq j$. Lemma 3.3 implies that $\mathbf{u}^+ \in \mathcal{S}^+$.

(ii) $\mathbf{u}^-$: We can find $n \in \mathbb{N}$ in such a way that for each $j \leq i \leq k$, $u_i - n < v_i$. We define $u_i^- = u_i - n$ for $i < k$, $u_i^- = v'_i - n$ for $i \geq k$. The construction and Lemma 3.3 imply that $\mathbf{u}^- \in \mathcal{S}^-$ and that $\mathbf{u}^- \leq \mathbf{u}^+$.

(iii) $\mathbf{x}$: Lemma 3.2 implies that there exists a stationary configuration $\mathbf{x}$, $\mathbf{u}^- \leq \mathbf{x} \leq \mathbf{u}^+$. Lemma 3.5, (i) implies that $\mathbf{x}$ has α-shear rotation number σ_1 and ω-shear rotation number σ_2. □

Remark 3.1. *generalisation of all the results to finite compositions of twist diffeomorphisms (setting as in [MAT]) on a torus is straightforward, using sums of generating functions.*

4. DISCUSSION

Here we discuss corollaries of Theorem 3.7 and announce related results. For detailed proofs of Corollary 4.1, Proposition 4.3 and Theorem 4.4 we refer the reader to [SL3].

Topological entropy and accelerating invariant measures. Angenent in [ANA] proved that non-integrable twist maps on a cylinder (non-integrable means that there exists an orbit not on an invariant circle) have positive topological entropy. In detail, Angenent assumed the existence of two stationary configurations $\mathbf{x}$, $\mathbf{y}$ which 'exchange rotation numbers', in the sense that

$$\begin{aligned} \omega_0 = \rho_\alpha(\mathbf{x}) &= \lim_{n\to-\infty} \frac{x_n - x_0}{n} = \lim_{n\to\infty} \frac{y_n - y_0}{n} = \rho_\omega(\mathbf{y}), \\ \omega_1 = \rho_\alpha(\mathbf{y}) &= \lim_{n\to-\infty} \frac{y_n - y_0}{n} = \lim_{n\to\infty} \frac{x_n - x_0}{n} = \rho_\omega(\mathbf{x}), \end{aligned} \tag{4.1}$$

$\omega_0 < \omega_1$. Under this hypothesis, Angenent constructed an invariant set $\mathcal{A}$, such that $f|_{\mathcal{A}}$ has a finite shift as a factor and therefore positive topological entropy. The set $\mathcal{A}$ is such that for each $(x,p) \in \mathcal{A}$,

$$\omega_1 \le \liminf_{|n|\to\infty} \frac{x_n - x_0}{n} \le \limsup_{|n|\to\infty} \frac{x_n - x_0}{n} \le \omega_2,$$

where (x_n, p_n) is the F-orbit of (x,p), and $F : \mathbb{R}^2 \to \mathbb{R}^2$ the lift of f.

Given a twist diffeomorphism on a torus f, and $\alpha_1 < \alpha_2$ such that $[\alpha_1, \alpha_2] \subseteq \sigma(f)$, we define the invariant set $\mathcal{A}_{\alpha_1,\alpha_2}$ as the set of all $(x,p) \in \mathbb{T}$ such that

$$\alpha_1 \le \liminf_{|n|\to\infty} \frac{p_n - p_0}{n} \le \limsup_{|n|\to\infty} \frac{p_n - p_0}{n} \le \alpha_2, \tag{4.2}$$

where (x_n, p_n) is the F-orbit of (x,p). The definition of shear-rotation numbers (2.7), (2.8) implies that $\mathcal{A}_{\alpha_1,\alpha_2}$ contains all orbits with both α and ω shear rotation number in the interval $[\alpha_1, \alpha_2]$.

The proof of the following corollary is analogous to Angenent's construction, but instead of orbits with property (4.1) one uses connecting accelerating orbits with exchanging shear rotation numbers, constructed in Theorem 3.7. The proof does not contain any new ideas, and is omitted.

Corollary 4.1. *Given a twist diffeomorphism f on a torus with the shear-rotation interval $[a,b]$, for each α_1, α_2 such that $a \le \alpha_1 < \alpha_2 \le b$, the map $f|_{\mathcal{A}_{\alpha_1,\alpha_2}}$ has positive topological entropy.*

We now construct infinitely many ergodic measures with positive metric entropy, supported on the set of accelerating orbits.

Corollary 4.2. *Assume that the shear rotation interval $[a,b]$ of a twist diffeomorphism on a torus is non-trivial, $a < b$. Then there exists infinitely many $\sigma \in [a,b]$, and ergodic measures μ_σ with positive metric entropy. For μ_σ-almost every point $(x,p) \in \mathbb{T}$, the shear-rotation number of (x,p) is σ.*

Proof. Choose $a \le \alpha_1 < \alpha_2 \le b$. The well-known variational principle for topological and metric entropy (see e.g. [POL], Prop. 3.1 and Theorem 3.2, (i)) and Corollary 4.1 imply existence of an ergodic measure μ supported on $\mathcal{A}_{\alpha_1,\alpha_2}$, with positive metric entropy. Since all lifts F of the map f are the same up to a constant vector, the function $(x,p) \mapsto p' - p$ is well defined on $\mathbb{T}$, where $p' = F(x,p)$. We define σ as

$$\sigma = \int_{\mathbb{T}} (p' - p) d\mu.$$

Now Birkhoff ergodic theorem implies that for μ-almost each $(x,p) \in \mathbb{T}$, the following sum is convergent,

$$\lim_{|n|\to\infty} \frac{p_n - p_0}{n} = \lim_{|n|\to\infty} \frac{1}{n} \sum_{k=1}^{n} (p_k - p_{k-1}) = \sigma, \tag{4.3}$$

where (x_n, p_n) is the F-orbit of (x,p), and we see that $\sigma \in [\alpha_1, \alpha_2]$. But the lefthand side of (4.3) is shear-rotation number $\sigma(x,p)$, and we write $\mu_\sigma = \mu$.

Since the construction was independent of the interval $[\alpha_1, \alpha_2]$, inductively we deduce existence of infinitely many invariant measures μ_σ with positive metric entropy. □

Note that all the invariant measures of twist diffeomorphisms with positive metric entropy constructed by Mather and Forni (see [MF]) have well defined rotation number, and hence have shear-rotation number 0. We conclude that all ergodic measures μ_σ constructed above, $\sigma \neq 0$, are mutually singular with Mather's invariant measures and therefore describe complementary portions of the phase space.

We conjecture that Corollary 4.2 could be strengthened, namely that for each $\sigma \in [a,b]$ there exists an ergodic measure μ_σ supported on the set of orbits with shear-rotation number σ, and with positive metric entropy.

Arbitrary large shear-rotation intervals. In a typical one-parameter family of twist maps on a torus, one can choose values of parameters such that shear rotation interval can be arbitrary large. Precisely, define one parameter family of generating functions

$$H_k(x,y) = kP(x) + V(x,y),$$

where V is a generating function of a twist diffeomorphism on a torus, and P is a C^2, 1-periodic function. Furthermore, we assume that P is non-degenerate, i.e. that for each $x \in \mathbb{R}$, $P'(x) = 0$ implies $P''(x) \neq 0$. Note that the family of standard maps (Example 2.1) is such a family.

Proposition 4.3. *Given the one-parameter family of generating functions H_k as above, and the corresponding family of twist maps on a torus f_k, for a given $n \in \mathbb{N}$ there exists k_0 such that for each $k > k_0$, $[-n,n] \subset \sigma(f_k)$.*

The proof is an application of Implicit Function Theorem, and follows Aubry's idea of approximation by 'anti-integrable limit' (see [AUB]; in the case of standard family, Proposition 4.3 is a direct corollary of [AUB], Theorem 1).

An intuitive conclusion is that, for large values of parameter k, the invariant measures μ_σ describe significant portion of the phase portrait. Their further study could provide some information about such important open problems as whether there exists a Lebesgue absolute-continuous invariant measure with positive metric entropy for large values of k in the standard family ([MK2]).

Non-triviality of the shear-rotation interval. We already showed in the proof of Theorem 3.6 that existence of invariant circles is a sufficient condition for shear-rotation interval to be trivial, $\sigma(f) = \{0\}$. The converse is also true:

Theorem 4.4. *Given a twist diffeomorphism on a torus f, the shear rotation interval is not reduced to a point if and only if invariant circles do not exist, and then $\sigma(f) = [a, b]$, $a < 0 < b$.*

The proof is rather technical, depends on careful modification of construction of Mather's shadowing orbits reported in [SL2], and will be published elsewhere.

We conclude that break-up of the last invariant circle (studied e.g. in [MK1]) is a global bifurcation where an invariant set of accelerating orbits with positive topological entropy is created.

Numerical determination of the shear-rotation interval. Pulling (and pushing) of the Frenkel-Kontorova model defined as the action functional of a given twist map is an algorithm to determine the shear-rotation interval (see [SL1], Theorem 5.6). Its efficiency and numerical accuracy remains to be tested in practice.

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Address: Department of Mathematics, Bijenička 30, 10000 Zagreb, Croatia
Current address: The Nonlinear Centre, DAMTP, Silver Street, Cambridge CB3 9EW, UK
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