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ON A SYMMETRIC DESIGN (133,33,8) AND THE GROUP $e_8 \cdot f_{21}$ AS ITS AUTOMORPHISM GROUP

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ABSTRACT. This article presents the examination of the possibility that group $E_8 \cdot F_{21}$ operates on a symmetric design with parameters $(133, 33, 8)$ as its automorphism group. The method based on coset enumeration in group is used.

1. INTRODUCTION

So far, only one symmetric design with parameters (133,33,8) is known [1, p. 625], on which operates Singer group of order 133. Its full automorphism group is $F_{18\cdot 19} \cdot Z_7$ of order 2394.

In this paper we examine the possibility that group $E_8 \cdot F_{21}$ would operate on a design with these parameters as its automorphism group. Namely, using the operation of this group, already several designs have been obtained [2], [5]. We use well known method based on coset enumeration in groups; see [2], [4]. By additional conditions on group operating the research is directed towards cases where the existence of design would be more likely, at least according to update knowledge and experience.

2. OPERATION OF THE GROUP

Group $G = E_8 \cdot F_{21}$ of order $168 = 2^3 \cdot 3 \cdot 7$ is a faithful extension of elementary abelian group of order 8 with Frobenius group F_{21}. In terms of generators and relations it is given as follows:

$$\begin{aligned} G = \; & \langle a, b, c, d, e | a^7 = 1, b^3 = 1, c^2 = d^2 = e^2 = 1, \\ & (cd)^2 = (ce)^2 = (de)^2 = 1, b^{-1}ab = a^2, a^{-1}ca = d, \\ & a^{-1}da = e, a^{-1}ea = cd, b^{-1}cb = c, b^{-1}db = e, b^{-1}eb = de \rangle . \end{aligned}$$

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We presume that G is an automorphism group of a symmetric design D with parameters (133,33,8), that $Z_7 = \langle a \rangle$ operates fixed point free and that $Z_3 = \langle b \rangle$ stabilizes all G-orbits on design D. In that case 7 is a divisor of all G-orbit lengths on D, while 3 can be divisor of none. This leads us to a conclusion that lengths of G-orbits on D must be indices of the subgroups of G listed in Table 1.

Table 1

Subgroup	Degree of repres. (Index)	Number of fixed points of Z_2	Z_3
$\langle b,c,d,e \rangle \approx E_8 \cdot Z_3$	7	7	1
$\langle b,d,e \rangle \approx A_4$	14	6	2
$\langle b,c \rangle \approx Z_6$	28	4	1
$\langle b \rangle \approx Z_3$	56	0	2

Thus we shall need only the permutation representation of G-generators of degrees 7,14,28 and 56 provided by the corresponding computer program (Hrabe de Angelis). From the permutation representation of the generators we also determine the number of fixed points of prime-order automorphisms for all the necessary degrees (Table 1).

Let $f(Z_2)$ and $f(Z_3)$ denote respectively the number of fixed points of automorphisms of order 2 and 3 on design D. Using their well known upper and lower bounds

$$f(Z_2) \geq 1 + \frac{k-1}{\lambda} \quad , \quad f(Z_{2,3}) \leq k + \sqrt{k-\lambda} \quad ,$$

as well as the fact $f(Z_2) \equiv 1 \pmod 2$ and $f(Z_3) \equiv 1 \pmod 3$, we obtain $f(Z_2) \in \{5,7,9,11,\dots,37\}$ and $f(Z_3) \in \{1,4,7,\dots,37\}$.

Our additional assumption is that Z_3 has at most 7 fixed points on D. A motivation for this is the manner of acting of the automorphism of order 3 on Hall's design [3], the only so far known one with these parameters. Considering all this, we finally get possible lengths of G-orbits of points (and blocks) on design D as given in Table 2.

Table 2

Fixed points schedule			Orbit lengths						
$f(Z_2)$	$f(Z_3)$	$f(Z_7)$							
13	7	0	7	14	56	56			*
21	7	0	7	14	28	28	56		*
			7	7	7	56	56		
29	7	0	7	7	7	28	28	56	
			7	14	28	28	28	28	*
37	7	0	7	7	7	28	28	28	28

The task of finding orbit structures is performed by computer. They are obtained in three cases only, marked "*" in Table 2. After bringing certain number of them to contradiction, it's left to do the indexing of orbit structures given in Fig. 1 by means of computer. Here we make use of the algorithm presented in the next section.

7	14	56	56	
1	0	24	8	7
0	1	12	20	14
3	3	13	14	56
1	5	14	13	56

$O4$

7	14	28	28	56	
1	8	12	4	8	7
4	1	10	6	12	14
3	5	4	7	14	28
1	3	7	4	18	28
1	3	7	9	13	56

$(O5)_1$

7	14	28	28	56	
1	8	12	4	8	7
0	1	10	10	12	14
1	5	4	9	14	28
1	3	7	4	18	28
3	3	7	7	13	56

$(O5)_2$

7	14	28	28	56	
1	0	12	12	8	7
0	1	6	6	20	14
3	3	9	4	14	28
3	3	4	9	14	28
1	5	7	7	13	56

$(O5)_3$

7	14	28	28	56	
1	0	4	4	24	7
0	1	10	10	12	14
1	5	9	4	14	28
1	5	4	9	14	28
3	3	7	7	13	56

$(O5)_4$

7	14	28	28	28	28	
1	8	12	4	4	4	7
4	1	10	6	6	6	14
3	5	4	7	7	7	28
1	3	7	9	9	4	28
1	3	7	9	4	9	28
1	3	7	4	9	9	28

$(O6)_1$

7	14	28	28	28	28	
1	8	12	4	4	4	7
0	1	10	10	6	6	14
3	3	7	7	9	4	28
3	3	7	7	4	9	28
1	5	4	9	7	7	28
1	3	7	4	9	9	28

$(O6)_2$

7	14	28	28	28	28	
1	0	12	12	4	4	7
0	1	6	6	10	10	14
3	3	9	4	7	7	28
3	3	4	9	7	7	28
1	5	7	7	9	4	28
1	5	7	7	4	9	28

$(O6)_3$

Fig. 1

3. Indexing the orbit structures and corresponding algorithm

Let $B_1, B_2, \cdots, B_t$ be G-orbits of blocks on design D. To index an orbit structure, that is to index a representative of each block orbit, means to find all points from point orbit I, $I \in \{1, 2, \cdots, t\}$, which lie on that block.

As a representative of the block orbit B_i, $i \in \{1, 2, \cdots, t\}$, we take block p stabilized by the subgroup $H_i < G$ for which $[G : H_i] = |B_i|$. Such a block can contain only H_i-orbits, on G-orbits of points, in whole. One possible choice of points of block p, regarding this criterion, will be denoted *compose_block_p* hereafter.

For the composed block p we have to check the number of points that are common to it and each block of its own orbit B_i, that is, we check upon the accuracy of the relation $|p \cap p^\alpha| = \lambda = 8$ for all $\alpha \in G$. In fact, it is enough to check the intersection of p and the representatives of H_i-orbits on B_i because H_i stabilizes p. The procedure of these consecutive checkings we shall call *check_inprod_p*.

Next, p must be submitted to checking upon its intersection with the blocks from all the other orbits. Let block q belong to B_j, $j \in \{1, \cdots, t\}$, $j \neq i$. Checking the criterion $|p \cap q^\alpha| = \lambda$ for all $\alpha \in G$ will be called *check_outprod_*(p,q). Blocks $p_1 \in B_1, p_2 \in B_2, \cdots, p_t \in B_t$, that would satisfy all the cited conditions, completely determine design we are searching for. $\{p_1^\alpha, p_2^\alpha, \cdots, p_t^\alpha \mid \alpha \in G\}$ is the set of all blocks of D.

The procedure of indexing, which we accomplish iteratively in t steps, is presented by algorithm in pseudocode, Fig. 2.

Algorithm for step 1:

```
while input matrices permit do
/input matrices are composed of
  corresponding H-orbits on
  G-orbits of points, H¡G/
    begin
      compose_block_p
      check_inprod_p
      if p satisfactory then
      save p to 'RES 1'
    end
```

Algorithm for step i, $i = 2, 3, \ldots, t$:

```
while input matrices permit do
  begin
    compose_block_p
    check_inprod_p
    if p satisfactory then
    begin
      while not eof 'RES i-1' do
      begin
        read (p1,p2, ... ,pi-1)
          from 'RES i-1'
        w = 1
        repeat
          check_outprod_(p,pw)
          if p satisfactory then
            w = w + 1
        until p not satisfactory or w = i
        if w = i then
        save (p1, ... ,pi-1,p) to 'RES i'
      end
    end
  end
```

Fig. 2

Statistics of the number of solutions obtained by indexing cited orbit structures, step by step, is given in Table 3.

Table 3

orbit structure	step 1	step 2	step 3	step 4	step 5	step 6
$O4$	2	16	96	0	—	—
$(O5)_1$	4	16	0	0	0	—
$(O5)_2$	4	0	0	0	0	—
$(O5)_3$	4	16	0	0	0	—
$(O5)_4$	4	16	0	0	0	—
$(O6)_1$	4	32	0	0	0	0
$(O6)_2$	4	0	0	0	0	0
$(O6)_3$	4	32	0	0	0	0

That result proves the following conclusion.

Theorem. *There is no symmetric design with parameters (133,33,8) on which group $G = E_8 \cdot F_{21}$ would operate so that Z_7 acts fixed point free and Z_3 stabilizes G-orbits having at most 7 fixed points.*

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