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MAPPINGS ON WEAKLY ARCWISE OPEN DENDROIDS

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Abstract. A dendroid X is said to be *weakly arcwise open* (WAO) at $p \in X$ if each arc component of $X \setminus \{p\}$ is either open or has empty interior. The dendroid X is WAO if it is WAO at p for each $p \in X$. This paper deals with mappings which preserve the WAO property. In particular we prove that the WAO property is preserved by monotone mappings on dendroids and by locally monotone mappings on fans.

Introduction

A variety of classes of mappings between continua have revealed to be important in Continuum Theory. In [7] T. Mackowiack presents a wide study of most of these mappings. In particular J. J. Charatonik and Wardle [2 and 10] studied mappings which preserve Kelley's property in continua. In this paper we are interested in mappings which preserve the WAO property in dendroids. This property is a generalization of Kelley's property [8, Lemma 1.3, p.940] and has been studied in [9]. A dendroid X has the WAO property if for every $p \in X$ and every arc-component α of $X \setminus \{p\}$, $\mathrm{Int}(\alpha)$ is either α or the empty set. The mappings considered here are mainly those mappings studied by T. Mackowiack in [7]. The following results will be proved in this paper: Monotone mappings preserve the WAO property on dendroids (Theorem 2.3) and locally monotone mappings preserve WAO property on fans (Theorem 2.7).

1. Preliminary results and definitions

A continuum means a compact, connected, metric space. A dendroid is a hereditarily unicoherent and path connected continuum. Given a dendroid X and a subset W of X, we denote by $A(W)$ the set of arc components of W. Let K be a closed subset of X. We say that X is *weakly arcwise open* (shortly WAO) at K if every $\alpha \in A(X \setminus K)$ is either open or has empty interior; X is WAO (or X has WAO property) provided X is WAO at $\{p\}$ for every $p \in X$.

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We recall that any subcontinuum of a dendroid X is a dendroid and for every two points $x, y \in X$ there is a unique arc $[x, y]$ joining x and y. A ramification point (resp. end point) p of X is a point for which the set $A(X \setminus \{p\})$ has more than two elements (resp. has exactly one element). The dendroid X is a fan if it has exactly one ramification point called the top of X and denoted by $\mathrm{top}(X)$. The dendroid X is smooth if there exists a point $p \in X$ with the following property: the sequence $\{[p, a_n]\}_{n \in \mathbb{N}}$ convergs to $[p, a]$ in the Hausdorff metric whenever the sequence $\{a_n\}_{n \in \mathbb{N}}$ converges to a in X.

We recall that a continuous map f from a topological space X onto a topological space Y is

a) *monotone* if the inverse image of every point of Y is a connected subset of X.

b) *locally monotone* if given any point $p \in X$ there exists a closed neighborhood U of p such that $f(p) \in \mathrm{Int}(f(U))$ and $f|U : U \to f(U)$ is a monotone mapping.

c) *quasi-monotone* if for each subcontinuum Q of Y with nonempty interior, the set $f^{-1}(Q)$ has a finite number of components and each of them is mapped onto Q.

d) *Confluent* if every component of $f^{-1}(K)$ is mapped onto K whenever K is a subcontinuuum of Y, and

e) *semiconfluent* if every for every subcontinuum K of Y and every two components C and C^* of $f^{-1}(K)$ either $f(C) \subseteq f(C^*)$ or $f(C^*) \subseteq f(C)$.

f) *atomic* if for every subcontinuum K of X such that $f(K)$ is nondegenerate, $f^{-1}(f(K)) = K$.

g) *light* if $f^{-1}(q)$ is totally disconnected for every $q \in Y$.

h) We say that f is *hereditarily* (monotone, confluent, etc.) if restricted to any subcontinuum of X, it is monotone, confluent, etc.

The following results will be used in this paper:

1.1 The inverse image of a subcontinuum of Y under a monotone mapping is a subcontinuum of X [11, VIII, 2.2, p.138].

1.2 The semiconfluent images of dendroids are dendroids [7, 7.30, p. 66].

1.3 Every locally monotone mapping is confluent [7, 4.32, 4.30 and 4.9 p. 16–22] and quasi-monotone [7, 4.43 p.25].

1.4 The image of a fan under a confluent mapping is a fan (or an arc) and the top of the model is mapped into the top of the image [1, Theorem 12, p.32]. This result remains true for semiconfluent mappings [5, Theorem 5.6, p.263] and locally confluent mappings [7, 7.34, p.67].

Now we prove results that will be used in Section 2.

1.5 LEMMA. *Let $f : X \to Y$ be a mapping from a dendroid X onto a dendroid Y and let W be a closed subset of Y.*

a) Assume there is $\beta \in A(Y \setminus W)$ such that $\beta \setminus \mathrm{Int}(\beta) \neq \emptyset$. Then for some $\alpha \in A(X \setminus f^{-1}(W))$, $f^{-1}(\beta \setminus \mathrm{Int}(\beta)) \cap (\alpha \setminus \mathrm{Int}(\alpha)) \neq \emptyset$. Moreover, for each $v \in \beta \setminus \mathrm{Int}(\beta)$ there is a point $u \in f^{-1}(v)$ such that $u \in \alpha_u \setminus \mathrm{Int}(\alpha_u)$ where $\alpha_u \in A(X \setminus f^{-1}(W))$.

b) Assume f is monotone. Let $\beta \in A(Y \setminus W)$ and $\alpha \in A(X \setminus f^{-1}(W))$ such that $f^{-1}(\mathrm{Int}(\beta)) \cap \alpha \neq \emptyset$. Then $f^{-1}(\mathrm{Int}(\beta)) \subseteq \mathrm{Int}(\alpha)$.

c) Assume that for every subcontinuum M of Y with nonempty interior $f^{-1}(M)$ has only a finite number of components. If the subcontinuum M of Y does not intersect W, then $f^{-1}(\mathrm{Int}(M)) \cap \alpha \subseteq \mathrm{Int}(\alpha)$ for any $\alpha \in A(X \setminus f^{-1}(W))$.

Proof. Let $v \in \beta \setminus \mathrm{Int}(\beta)$ and $\{v_n\}_{n \in \mathbf{N}}$ a sequence contained in $Y \setminus (W \cup \beta)$ which converges to v. We may assume that there is a convergent sequence $\{u_n\}_{n \in \mathbf{N}} \subseteq X$ such that $f(u_n) = v_n$. If $u = \lim_{n \to \infty} u_n$ then $f(u) = v$. Let $\alpha \in A(X \setminus f^{-1}(W))$ such that $u \in \alpha$. Then $f(\alpha) \subseteq \beta$ and hence $u_n \notin \alpha$, since $u_n \in \alpha$ implies $f(u_n) = v_n \in \beta$. Therefore $u \in \alpha \setminus \mathrm{Int}(\alpha)$. This proves a).

Let α and β be as in the hypothesis of b) and suppose there is a point $u \in f^{-1}(\mathrm{Int}(\beta)) \setminus \mathrm{Int}(\alpha)$. We consider two cases: i) $u \in \alpha$. Since $f^{-1}(\mathrm{Int}(\beta))$ is a neighborhood of u, there is a point $s \in f^{-1}(\mathrm{Int}(\beta)) \cap (X \setminus \alpha)$, thus the set $K = [f(s), f(u)]$ is a connected subset of β and therefore $f^{-1}(K)$ is connected. Since $u, s \in f^{-1}(K)$ then $[u, s] \subseteq f^{-1}(K)$. But u and s are different arc-components of $X \setminus f^{-1}(W)$ implies $[u, s] \cap f^{-1}(W) \neq \emptyset$ and then $K \cap W \neq \emptyset$, a contradiction since $K \subseteq \beta \subseteq Y \setminus W$.

ii) If $u \notin \alpha$ then, by the hypothesis, we may choose a point $x \in f^{-1}(\mathrm{Int}(\beta)) \cap \alpha$ so that $f(x)$ and $f(u)$ are both in β, while u and s are in different arc-components of $X \setminus f^{-1}(W)$ and we proceed as in the preceding case.

To prove c) we may assume that $f^{-1}(\mathrm{Int}(M)) \cap \alpha \neq \emptyset$ and therefore $f^{-1}(M)$ has a finite number of components. Suppose there is a point $u \in f^{-1}(\mathrm{Int}(M)) \cap (\alpha \setminus \mathrm{Int}(\alpha))$. Since $f^{-1}(\mathrm{Int}(M))$ is a neighborhood of u, it contains an infinite sequence of points $\{u_n\}_{n \in \mathbf{N}} \subseteq X \setminus \alpha$ converging to u. Therefore one component L of $f^{-1}(M)$ contains an infinite subsequence of $\{u_n\}_{n \in \mathbf{N}}$ so that L contains u. Since L is a subdendroid of X and contains points in α and points not in α, then $L \cap f^{-1}(W) \neq \emptyset$, a contradiction to the hypothesis.

1.6 PROPOSITION. *Let X be dendroid and K a closed subset of X which has a finite number of components. If X is WAO at $\{p\}$ for every $p \in K$, then X is WAO at K.*

Proof. Suppose that there is a nonopen $\alpha \in A(X \setminus K)$. Then, some $u \in \alpha$ is the limit of a sequence $\{u_n\}_{n \in \mathbf{N}} \subseteq X \setminus (K \cup \alpha)$ and therefore there exists a point $p \in K$ such that $p \in [u_n, u]$ for infinitely many indices n. If $u \in \alpha^* \in A(X \setminus \{p\})$ then $u \notin \mathrm{Int}(\alpha^*)$, so that *Int*$(\alpha^*) = \emptyset$ since X is WAO at $\{p\}$. But $\alpha \subseteq \alpha^*$ implies $\mathrm{Int}(\alpha) = \emptyset$ as desired.

1.7 PROPOSITION. *If the dendroid X does not contain a dendroid of type 1, K is a closed subset of X and X is WAO at $\{q\}$ for every $q \in K$, then X is WAO at K.*

Proof. We recall that a dendroid Y is said to be of type 1, [4, p.192] if $Y = \mathrm{Cl}(\bigcup_{n=1}^{\infty} [p, a_n])$ where $\lim_{n \to \infty} a_n = a$, $L = \lim_{n \to \infty} [p, a_n]$ and the following holds:

There is an end point $s \in L$, $s \neq a$ and an open set D containing s such that $C \cap \bigcup_{n=1}^{\infty} [p, a_n] = \emptyset$ where C is the component of $D \cap L$ containing s. Suppose that there is a nonopen $\alpha \in A(X \setminus K)$. Then some $u \in \alpha$ is the limit of a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq X \setminus (K \cup \alpha)$. Let $q_n \in K \cap [u, u_n]$. Without losing generality, the sequence $\{q_n\}_{n \in \mathbb{N}}$ converges to a point $q \in K$. Let $u \in \beta \in A(X \setminus \{q\})$. Then $\alpha \subseteq \beta$. On the other hand, supose that $q \notin [u, u_{n_k}]$ for infinitely many indices n_k. Then it is easy to verify that $Y = \mathrm{Cl}\left(\bigcup_{k=1}^{\infty} [u, u_{n_k}]\right)$ is a dendroid of type 1. This implies that $q \in [u, u_n]$ for almost every n and hence $u \in \beta \setminus \mathrm{Int}(\beta)$ so that $\mathrm{Int}(\beta) = \emptyset$ since X is WAO at $\{q\}$. Therefore $\mathrm{Int}(\alpha) = \emptyset$.

Containing no subdendroid of type 1 is essential in Proposition 1.7. In fact, let X be the WAO fan with $\mathrm{top}(X) = q$ and $K = \{p_1, p_2, \ldots, p\}$ (Figure 1). Let $\alpha \in A(X \setminus K)$ such that $q \in \alpha$. Then $x \in \alpha \setminus \mathrm{Int}(\alpha)$ and $y \in \mathrm{Int}(\alpha)$.

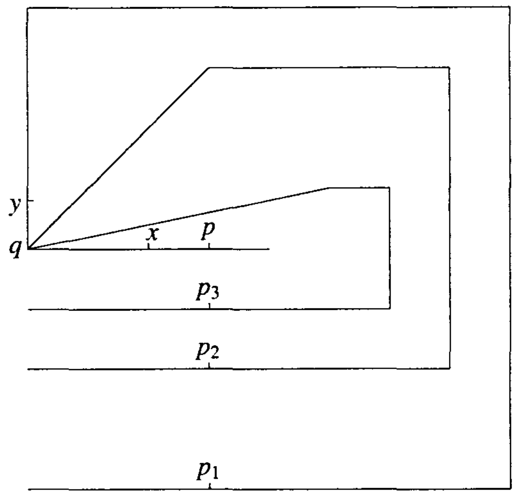

Figure 1.

Since smooth dendroids do not contain subdendroids of type 1 [4, Theorem 1, p.194], we have the following

1.8 COROLLARY. *A smooth dendroid X is WAO at a closed subset K of X provided X is WAO at $\{q\}$ for every $q \in K$.*

1.9 PROPOSITION. *A fan Y is WAO iff it is WAO at* $\mathrm{top}(Y)$.

Proof. The necessity of condition is immediate. It is also immediate that Y is WAO at every one of its end points. Let w be a nonend point of Y, $w \neq \mathrm{top}(Y) = q$.

There are exactly two elements $\beta_1, \beta_2 \in A(Y \setminus \{w\})$, one of them, say β_1, is an arc. Then β_2 is open. Since β_1 is an arc, $q \in \beta_2$, so that $\beta_1 \subseteq \beta \in A(Y \setminus \{q\})$ and thus $\mathrm{Int}(\beta_1)$ is β_1 or the empty set depending $\mathrm{Int}(\beta)$ is β or the empty set respectively.

2. Mappings which preserve WAO property

In this section we will prove that monotone mappings preserve WAO property on dendroids. With respect to the problem of determining whether or not locally monotone and quasi-monotone mappings preserve the WAO property, we present some partial results. In particular we prove that locally monotone mappings preserve WAO property on fans.

2.1 PROPOSITION. *Let X, Y be continua and $f : X \to Y$ a monotone mapping. Let $q \in Y$ and assume that X is a dendroid which is WAO at $f^{-1}(q)$. Then Y is a dendroid which is WAO at q.*

Proof. It follows from 1.2 that the monotone image of a dendroid is a dendroid, so Y is a dendroid. To prove that Y is WAO at q, suppose there is a $\beta \in A(Y \setminus \{q\})$ for which the following two conditions hold:

$$\mathrm{Int}(\beta) \neq \emptyset \tag{1}$$

and

$$\beta \setminus \mathrm{Int}(\beta) \neq \emptyset. \tag{2}$$

Then by (1) $f^{-1}(\mathrm{Int}(\beta)) \cap \alpha \neq \emptyset$ for some $\alpha \in A(X \setminus f^{-1}(q))$. It follows from Lemma 1.5 b) that $f^{-1}(\mathrm{Int}(\beta)) \subseteq \mathrm{Int}(\alpha)$ so that $\mathrm{Int}(\alpha) \neq \emptyset$. Since $f^{-1}(q)$ is a continuum, Proposition 1.4 implies that α is open.

Assertion (2) and Lemma 1.5 a) imply that $f^{-1}(\beta \setminus \mathrm{Int}(\beta)) \cap \sigma \setminus \mathrm{Int}(\sigma) \neq \emptyset$ for some $\sigma \in A(X \setminus f^{-1}(q))$. In particular $\sigma \setminus \mathrm{Int}(\sigma) \neq \emptyset$ so that $\mathrm{Int}(\sigma) = \emptyset$. This proves that $\sigma \neq \alpha$. Take now $v \in \mathrm{Int}(\beta)$ and $v^* \in \beta \setminus \mathrm{Int}(\beta)$. Therefore $A = [v, v^*] \subseteq \beta$ and $f^{-1}(A)$ is a connected subset of $X \setminus f^{-1}(q)$. But $A \cap \alpha \neq \emptyset$ and $A \cap \sigma \neq \emptyset$. This is a contradiction which proves the theorem.

In the preceeding proposition the hypothesis of WAO at $f^{-1}(q)$ cannot be substituted by WAO at some point of $f^{-1}(q)$ as it can seen by the following example.

For $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n$, we denote by $[\mathbf{p}, \mathbf{q}]$ the linear segment from $\mathbf{p}$ to $\mathbf{q}$ contained in $\mathbf{R}^n$. Let $Y = [(0,0),(2,0)] \cup \bigcup_{n=1}^{\infty} [(0,0),(1,\frac{1}{n})]$. We obtain Z from Y by removing the set $A = [(0,0),(1,1)] \setminus \{(0,0)\}$ so that Z and Y are homeomorphic dendroids which are not WAO at $(0,0)$. Notice that Y is WAO at $(1,1)$. Let $f : Y \to Z$ be the map that shrinks A into $(0,0)$ and is the identity map on $Y \setminus A$. Then f is monotone and $f(1,1) = (0,0)$.

The following theorem follows immediately from Proposition 2.1.

2.2 THEOREM. *Let X be a WAO dendroid, Y a continuum and $f : X \to Y$ a monotone mapping from X onto Y. Then Y is WAO dendroid.*

2.3 COROLLARY. *Let X be a WAO dendroid, and $f : X \to Y$ a mapping from X onto the dendroid Y. If f is either atomic or hereditarily monotone or hereditarily confluent, then Y is a WAO dendroid.*

Proof. It follows from the definition that a hereditarily monotone mapping is monotone. By [3, Theorem 1, p.49] an atomic mapping is monotone. A hereditarily confluent map onto a dendroid is monotone by [7, Theorem 6.8, p.52].

Every local homeomorphism from a dendroid X onto a dendroid Y is a homeomorphism [6, p.64], thus local homeomorphisms preserve the WAO property in dendroids.

2.4 LEMMA. *Let X and Y be dendroids, $f : X \to Y$ a quasimonotone mapping and $q \in Y$. Suppose that X is WAO at $f^{-1}(q)$. Then Y is WAO at q provided the following property is satisfied:*

(A) *If $\beta \in A(Y \setminus \{q\})$ and $\mathrm{Int}(\beta) \neq \emptyset$, then there exists a subcontinuum K of Y contained in β such that $\mathrm{Int}(K) \neq \emptyset$.*

Proof. Let $\beta \in A(Y \setminus \{q\})$ and suppose that $\beta \setminus \mathrm{Int}(\beta) \neq \emptyset$ and $\mathrm{Int}(\beta) \neq \emptyset$. Let $v \in \beta \setminus \mathrm{Int}(\beta)$. By Lemma 1.5 a) there is $u \in f^{-1}(v)$ such that if $u \in \alpha \in A(X \setminus f^{-1}(q))$ then $u \notin \mathrm{Int}(\alpha)$ so that, $\mathrm{Int}(\alpha) = \emptyset$. In the other hand, by (A), there exists a subcontinuum K contained in β such that $\mathrm{Int}(K) \neq \emptyset$. Let $v^* \in \mathrm{Int}(K)$ and $M = K \cup [v, v^*]$. Then M is a subcontinuum of Y, $M \subseteq \beta$ and $\mathrm{Int}(M) \neq \emptyset$. Since f is quasi-monotone the component L of $f^{-1}(M)$ containing u, contains a point u^* such that $f(u^*) = v^*$ and since $L \subseteq \alpha$, $u^* \in f^{-1}(\mathrm{Int}(M)) \cap \alpha$. But f is quasi-monotone implies that $f^{-1}(M)$ has a finite number of components, therefore, by Lemma 1.5 c) $\mathrm{Int}(\alpha) \neq \emptyset$, a contradiction.

2.5 PROPOSITION. *Let $f : X \to Y$ be a quasi-monotone mapping from the WAO fan X onto the fan Y which maps* $\mathrm{top}(X)$ *into* $\mathrm{top}(Y)$. *Then Y is WAO.*

Proof. Let $p = \mathrm{top}(X)$ and $q = \mathrm{top}(Y)$. By Proposition 1.9 it is enough to prove that Y is WAO at q. Let $\alpha \in A(X \setminus f^{-1}(q))$ and suppose $u \in \alpha \setminus \mathrm{Int}(\alpha)$. Let $\alpha^* \in A(X \setminus \{p\})$ such that $\alpha \subseteq \alpha^*$. Then $u \in \alpha^* \setminus \mathrm{Int}(\alpha^*)$ and so, by hypothesis, $\mathrm{Int}(\alpha^*) = \emptyset$, which implies $\mathrm{Int}(\alpha) = \emptyset$. This proves that X is WAO at $f^{-1}(q)$. On the other hand, let $\beta \in A(Y \setminus \{q\})$ such that $\mathrm{Int}(\beta) \neq \emptyset$ and let $K = [v, v^*]$ where $v^* \in \mathrm{Int}(\beta)$ and $v \in (q, v^*)$. It is clear that $K \subseteq \beta$ and $\mathrm{Int}(K) \neq \emptyset$. This proves (A) of Lemma 2.4. Then Y is WAO at q.

2.6 THEOREM. *Let X be WAO fan, Y a continuum and $f : X \to Y$ a locally monotone mapping from X onto Y. Then Y is a WAO fan.*

Proof. A locally monotone mapping is confluent (1.3). Then, by 1.4, f maps a fan onto a fan (or an arc) and the top of the model is mapped into the top of the image. Again, by 1.3, f is quasi-monotone, so that the theorem follows from Theorem 2.5.

2.7 COROLLARY. *Let X and Y be dendroids and $f : X \to Y$ a quasi-monotone mapping. Suppose X is a smooth dendroid which satisfies WAO property and suppose that condition* (A) *of Lemma 2.4 is satisfied for every $q \in Y$. Then Y is WAO.*

Proof. Since the dendroid X is smooth and WAO, it follows from Corollary 1.8 that X is WAO at $f^{-1}(q)$ for every $q \in Y$. Therefore by Lemma 2.4 Y is WAO.

2.8 EXAMPLE Let $Y = [(0,0,0),(2,0,0)] \cup \bigcup_{n=1}^{\infty} [(0,0,0),(1,\frac{1}{n},0)]$ and C the Cantor set containd in $\{0\} \times \{0\} \times [0,1]$. We define X as the dendroid obtained from $Y \times C$ by identifying C into one point. Now define $f : X \to Y$ as $f(x,y,z) = (x,y,0)$. Then f is an open, light retract which maps a WAO dendroid onto a dendroid which is not WAO.

So open mappings do not preserve the WAO property and obviously classes of mappings containing the class of oppen mappings (see [7, Table II, p. 28]) do not preserve WAO property. In particular confluent mappings do not preserve WAO property.

We end the paper with the following open problems

Problem 1. Is the locally monotone image of a WAO dendroid a WAO dendroid?

Problem 2. Is the quasi-monotone image of a WAO dendroid a WAO dendroid?

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