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A NEW PROOF OF A THEOREM CONCERNING DECOMPOSABLE GROUPS

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Abstract. We give an elementary proof of the following result: If G is a compact non-zero Abelian group with dual isomorphic to a subgroup of $\mathbb{Q}$, such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$ for some open subset $U \subset G$, where $G_{(2)} = \{a \in G: 2a = 0\}$, then G is topologically isomorphic with $\mathbb{T}$.

1. Introduction

Let G be a locally compact Abelian group with dual $\hat{G}$. Denote by $G^{(2)}$ and $G_{(2)}$ the image and kernel of the homomorphism $G \ni a \mapsto 2a \in G$, respectively. Given a subset $X \subset G$, let

$$-X = \{a \in G: -a \in X\}.$$

In agreement with the terminology introduced in [1], G will be said to be *decomposable* if there exists an open subset $U \subset G$ such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$.

Let $\mathbb{T}$ be the multiplicative group of complex numbers with unit modulus, endowed with the usual topology. Let $\mathbb{Q}$ be the additive group of rational numbers, equipped with the discrete topology. For each $n \in \mathbb{N}$, let $\mathbb{Z}(n)$ be the cyclic group with n elements. Assume that the $\mathbb{Z}(n)$ are endowed with the discrete topology. Given Abelian groups G_i $(i = 1, \dots, n)$, denote by $G_1 \times \cdots \times G_n$ the direct product of the G_i. For a cardinal number $\mathfrak{m}$ and a compact Abelian group H, designate by $H^{\mathfrak{m}}$ the direct product of $\mathfrak{m}$ copies of H, enriched with the product topology (under which $H^{\mathfrak{m}}$ is compact).

In [1] the following characterisation of decomposable compact Abelian groups is given:

THEOREM 1. *Let G be a compact Abelian group. Then G is decomposable if and only if either $(\hat{G})^{(2)}$ is a countable torsion group or G is topologically isomorphic with $\mathbb{T} \times \mathbb{Z}(2)^{\mathfrak{m}} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group.*

The above theorem is a consequence of a number of results describing certain subclasses of the class of all decomposable compact Abelian groups. One of these results reads as follows:

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THEOREM 2. *Any decomposable compact connected Abelian group different from a singleton is topologically isomorphic with* $\mathbb{T}$.

The main part of the proof to Theorem 2 is embodied by the following result:

THEOREM 3. *Suppose that* G *is a decomposable compact Abelian group different from a singleton. Suppose, moreover, that* $\hat{G}$ *is isomorphic with a subgroup of* $\mathbb{Q}$. *Then* G *is topologically isomorphic with* $\mathbb{T}$.

The proof of Theorem 3 given in [1] (as part of the proof to Proposition 4.1) is short but quite involved. This note offers a longer but more elementary proof. While the first of these proofs utilises a rather special result concerning compact cancellative semigroups, the second uses only standard tools from general topology. This notwithstanding, both proofs invoke freely a basic lore on locally compact Abelian groups.

2. Proof of the main result

This section gives the proof of Theorem 3 alluded to above.

Proof of Theorem 3. We commence by showing that, for every $a \in G$, $G \setminus \{a\}$ is connected. Since $\hat{G}$ is isomorphic with a subgroup of $\mathbb{Q}$, it is torsion free. Hence, being compact, G is connected (cf. [2, Thm. 24.25]). We see that G is a continuum with more than one element. By a theorem of Moore-Wallace [4, 6] (see also [3, §47, Sec. IV, Thm. 5]), any continuum different from a singleton contains at least two elements, each of which has a connected complement. Therefore there exists $b \in G$ for which $G \setminus \{b\}$ is connected. Now, to conclude that $G \setminus \{a\}$ is connected for each $a \in G$, it suffices to observe that $G \setminus \{a\}$ is the image of $G \setminus \{b\}$ via the translation by $a - b$ (defined as $G \ni h \mapsto a - b + h \in G$), which is a homeomorphism.

Denote by 0 the neutral element of G. Let U be an open subset of G such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$. It is clear that $G \setminus G_{(2)}$ is disconnected. In view of the assertion established in the preceding paragraph, $G \setminus \{0\}$ is connected. Therefore $G_{(2)} \setminus \{0\}$ is non-empty.

Let ρ be a monomorphism mapping $\hat{G}$ into $\mathbb{Q}$. For each $n \in \mathbb{N}$, let K_n be the cyclic subgroup of $\mathbb{Q}$ given by

$$K_n = \{k/n! \mid k \in \mathbb{Z}\}$$

and let Γ_n be the subgroup of $\hat{G}$ given by

$$\Gamma_n = \rho^{-1}(K_n \cap \rho(\hat{G})).$$

It is clear that, for each $n \in \mathbb{N}$, Γ_n is cyclic and $\Gamma_n \subset \Gamma_{n+1}$. Furthermore, $\hat{G} = \bigcup_{n=1}^{\infty} \Gamma_n$.

We now prove that $G_{(2)} \setminus \{0\}$ has precisely one element. Select $g \in G_{(2)} \setminus \{0\}$ arbitrarily. Given $n \in \mathbb{N}$, let χ_n be a generator of Γ_n. Since $2g = 0$, we have $(g, \chi_n) = \pm 1$ for each $n \in \mathbb{N}$. Here $(\cdot, \cdot)$ represents the pairing between elements of G and $\hat{G}$. Now either there is a sequence $\{n_k\}_{k \in \mathbb{N}}$ in $\mathbb{N}$ diverging to infinity such that $(g, \chi_{n_k}) = 1$ for each $k \in \mathbb{N}$, or $(g, \chi_n) = -1$ for all but finitely many $n \in \mathbb{N}$. Suppose that the first possibility holds. Since $\{\Gamma_n\}_{n \in \mathbb{N}}$ is an increasing sequence of subgroups eventually exhausting all of $\hat{G}$, any given $\gamma \in \hat{G}$ can be written as $\gamma = l\chi_{n_k}$ for some $k, l \in \mathbb{N}$. It then follows that $(g, \gamma) = 1$, which, in view of the arbitrariness of γ, implies that $g = 0$, a contradiction. The first possibility being excluded, let $n_0 \in \mathbb{N}$ be such that $(g, \chi_n) = -1$ for each integer n greater than n_0. Given $\gamma \in \hat{G}$, choose $l \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n > n_0$ such that $\gamma = l\chi_n$. Then, clearly, $(g, \gamma) = (-1)^l$, which shows that (g, γ) does not depend on the particular choice of g. Consequently, g is uniquely determined, and so $G_{(2)} \setminus \{0\}$ is a singleton.

Denote by g the unique element of $G_{(2)} \setminus \{0\}$. We clearly have $G_{(2)} = \{0, g\}$. For each subset $X \subset G$, denote by ∂X the boundary of X relative to G. We now show that

$$\partial U = \{0, g\}. \tag{1}$$

It is evident that $\partial U \subset \{0, g\}$. Since the inversion $G \ni a \mapsto -a \in G$ is a homeomorphism, we have $\partial(-U) = -\partial U$. Taking into account that $g = -g$, we see that $\partial(-U) = \partial U$. Now $\partial U \subset \partial(U \cup (-U))$, since U is open. Moreover,

$$\partial(U \cup (-U)) \subset \partial U \cup \partial(-U) = \partial U.$$

It follows that $\partial U = \partial(U \cup (-U))$. In particular, the set $U \cup (-U) \cup \partial U$ is closed. Suppose that $\{0, g\} \setminus \partial U \neq \emptyset$. Being a finite set, $\{0, g\} \setminus \partial U$ is closed. Since G is the union of $\{0, g\} \setminus \partial U$ and $U \cup (-U) \cup \partial U$, we arrive at a contradiction with G being connected. Thus $\{0, g\} \setminus \partial U = \emptyset$, establishing (1).

We contend that $U \cup \{0\}$ is connected. Suppose, on the contrary, that $U \cup \{0\} = A \cup B$, where A and B are non-empty disjoint closed subsets of $U \cup \{0\}$. In view of (1), $U \cup \{0\}$ is closed in $G \setminus \{g\}$. Correspondingly, A and B are closed in $G \setminus \{g\}$. It is now clear that $A \cup (-A)$ and $B \cup (-B)$ are non-empty disjoint closed subsets of $G \setminus \{g\}$, whose union is the whole of $G \setminus \{g\}$. But this contradicts the connectedness of $G \setminus \{g\}$ (which follows from the assertion from the first paragraph) and establishes the contention.

Let

$$V_1 = (U + g) \cap (-U) \quad \text{and} \quad V_2 = (U + g) \cap U.$$

We claim that V_1 is not empty. Since G is compact and connected, it is also divisible (cf. [2, Thm. 24.25]). In particular, $g = 2h$ for some $h \in G$. Since g is non-zero, we see that h is a member of $G \setminus G_{(2)}$, and so either h or $-h$ falls into U. If $h \in U$, then, taking into account that $2g = 0$, we have $-h = h + g$, whence $h \in V_1$. If $-h \in U$, then, in view of $h = -h + g$, we have $-h \in V_1$. In either case, V_1 is non-empty, as claimed.

We shall now focus our attention on the set $V_1 \cup \{0, g\}$. We first show that it is closed in G.

Given a subset $X \subset G$ and $a \in G$, let

$$X + a = \{b \in G \colon b - a \in X\}.$$

Clearly, since G is connected, ∂V_1 is non-void. We have

$$\partial V_1 \subset \partial(U+g) \cup \partial(-U) = \{0, g\}.$$

Since V_1 is invariant under the composition of the inversion and the translation by g, so too is ∂V_1. It is easily seen that any non-empty subset of ∂V_1 invariant under the same composition coincides with ∂V_1. Therefore $\partial V_1 = \{0, g\}$, implying that $V_1 \cup \{0, g\}$ is closed.

We now show that $V_1 \cup \{0, g\}$ is connected. Suppose, on the contrary, that $V_1 \cup \{0, g\} = A \cup B$, where A and B are non-empty disjoint closed subsets of $V_1 \cup \{0, g\}$. Since $V_1 \cup \{0, g\}$ is closed in G, it follows that A and B are closed in G too. With no loss of generality, we may assume that $0 \in A$. Then, necessarily, $g \in B$. For otherwise B would be an open subset of V_1 and, since V_1 is open in G, B would be open in G; as B is also closed in G, we would thus arrive at a contradiction with G being connected. Now $A \setminus \{0\}$ is non-empty for otherwise $\{0\}$ would be an open subset of $V_1 \cup \{0, g\}$ contrary to the fact that $0 \in \partial V_1 \setminus V_1$. Since

$$A \setminus \{0\} \subset (U+g) \cup \{g\} \subset G \setminus \{0\}$$

and since A is closed in G, it follows that $A \setminus \{0\}$ is a closed subset of $(U+g) \cup \{g\}$. On the other hand, since ∂V_2 is contained in $\partial(U+g) \cup \partial U = \{0, g\}$, we find that $V_2 \cup \{g\}$ is a closed subset of $(U+g) \cup \{g\}$. As B is closed in G, $B \cup V_2 = B \cup V_2 \cup \{g\}$ is closed in $(U+g) \cup \{g\}$. We thus see that $A \setminus \{0\}$ and $B \cup V_2$ are closed non-empty subsets of $(U+g) \cup \{g\}$. Clearly, $A \setminus \{0\}$ and $B \cup V_2$ are disjoint and their union is all of $(U+g) \cup \{g\}$. This is, however, incompatible with the fact that, being the translate by g of the connected set $U \cup \{0\}$ (recall that the connectedness of $U \cup \{0\}$ was already shown earlier), $(U+g) \cup \{g\}$ is connected. The connectedness of $V_1 \cup \{0, g\}$ is thus established.

In preparation for the next step, we show now that if V_2 is non-void, then both $V_2 \cup \{0, g\}$ and $(-V_2) \cup \{0, g\}$ are connected. Assume then that $V_2 \neq \emptyset$. Since G is connected, ∂V_2 is not empty. V_2 is invariant under the translation by g, and so too is ∂V_2. Since $\partial V_2 \subset \{0, g\}$ and since any non-empty subset of $\{0, g\}$ invariant under the translation by g coincides with $\{0, g\}$, it follows that ∂V_2 is all of $\{0, g\}$. Repeating the argument employed in the proof of the connectedness of $V_1 \cup \{0, g\}$, we conclude that $V_2 \cup \{0, g\}$ is connected. Now $(-V_2) \cup \{0, g\}$ is connected too for it is the inverse of $V_2 \cup \{0, g\}$.

At this stage, we are in position to show that $V_1 \cup \{0, g\}$ is an arc with endpoints 0 and g. Note first that, since G is compact and $\hat{G}$ is countable, G is metrisable (cf. [2, Thm. 24.15]). In particular, $V_1 \cup \{0, g\}$ is a metrisable continuum. By a theorem of Moore [4] (see also [3, §47, Sec. V, Thm. 1]), if every point in a metrisable continuum with the exception of two points a and b has a disconnected complement, then the continuum is an arc with endpoints a and b. Thus to prove that $V_1 \cup \{0, g\}$ is an arc with endpoints 0 and g, it suffices to show that, for each $a \in V_1$, $(V_1 \cup \{0, g\}) \setminus \{a\}$ is disconnected. Suppose that $(V_1 \cup \{0, g\}) \setminus \{a\}$ is connected for some $a \in V_1$. Noting that $(V_1 \cup \{0, g\}) \setminus \{a\}$ coincides with

$(V_1 \setminus \{a\}) \cup \{0, g\}$ and that the translate of $(V_1 \setminus \{a\}) \cup \{0, g\}$ by g coincides with $((-V_1) \setminus \{a+g\}) \cup \{0, g\}$, we see that $((-V_1) \setminus \{a+g\}) \cup \{0, g\}$ is connected. Now both $(V_1 \setminus \{a\}) \cup \{0, g\}$ and $((-V_1) \setminus \{a+g\}) \cup \{0, g\}$ are connected and contain 0 and g, so their union C is connected. If V_2 is empty, then, as is easily seen, C coincides with $G \setminus \{a, a+g\}$, and in particular $G \setminus \{a, a+g\}$ is connected. If V_2 is not empty, then both $V_2 \cup \{0, g\}$ and $(-V_2) \cup \{0, g\}$ are connected and contain 0 and g, and so $(V_2 \cup \{0, g\}) \cup ((-V_2) \cup \{0, g\}) \cup C$ is connected. It is straightforwardly verified that the latter set coincides with $G \setminus \{a, a+g\}$. Thus, independently of whether or not V_2 is empty, $G \setminus \{a, a+g\}$ is connected. But $G \setminus \{a, a+g\}$ is disconnected, since it is the translate by a of the disconnected set $G \setminus \{0, g\}$ $(= G \setminus G_{(2)})$. This contradiction proves that $V_1 \cup \{0, g\}$ is an arc with endpoints 0 and g.

Now that V_2 is an open subset of G homeomorphic with the real line $\mathbb{R}$, G is locally connected. According to a theorem of Pontryagin [5, Thm. 42], any compact, metrisable, connected and locally connected group is the direct product of a finite or countable number of subgroups, each isomorphic with $\mathbb{T}$. Applying this theorem and taking into account that G contains an open subset homeomorphic with $\mathbb{R}$ (namely V_2), we find that G is topologically isomorphic with $\mathbb{T}$.

□

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