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ON FUNCTIONS WITH α-CLOSED GRAPHS

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Abstract. The concept of α-closed graph has been introduced by S. Kasahara [9]. In this paper functions with α-closed graphs are further investigated. Also, several sufficient conditions for a function to be continuous are established.

1. Introduction

In 1979, S. Kasahara [9] introduced the concept of α-closed graph of a function, which generalizes the concepts of closed, strongly-closed, and almost-strongly-closed graph of a function, with the help of a certain operation of topology τ into the power set of $\cup\,\tau$. By using the notion of functions with α-closed graphs S. Kasahara unified several known characterizations of compact spaces, nearly-compact spaces, and H-closed spaces.

In the present paper we further investigate functions with α-closed graphs, and, particularly, functions with strongly-closed graphs. We generalize the notion of locally closed functions due to R. Fuller [2] and generalize some earlier results for locally closed functions. We also give some sufficient conditions for a function to be continuous.

We point out that all the consequences of theorems that follow are not cited.

2. Preliminary definitions and theorems

Throughout, X and Y denote topological spaces (X, τ) and (Y, τ'), respectively, and $f : X \to Y$ denote a function from X into Y. By $\mathrm{Cl}\,(A)$ and $\mathrm{Int}\,(A)$ we denote the closure and the interior of a subset A of a topological space, respectively.

Definition 2.1 [9]. An *operation* α on τ is a function from τ into the power set of $\cup\,\tau$ such that $U \subset U^{\alpha}$ for each $U \in \tau$, where U^{α} denotes the value of α at U. An operation α on τ is *regular* if for each $x \in X$ and for each $U, V \in \tau$ such that $x \in U \cap V$, there exists a $W \in \in \tau$ such that $W^{\alpha} \subset U^{\alpha} \cap V^{\alpha}$.

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The function α defined by $U^\alpha = U$ (resp. $U^\alpha = \mathrm{Cl}(U)$, $U^\alpha = \mathrm{Int}(\mathrm{Cl}(U))$) for each $U \in \tau$ is an operation on τ and it is called the identity (resp. closure, interior-closure) operation on τ [9].

Let α be an operation on the topology τ of X.

Definition 2.2. A point $x \in X$ is in the *α-closure* of a set $A \subset X$ ($x \in \mathrm{Cl}_\alpha(A)$) if $U^\alpha \cap A \neq \emptyset$ for each open neighborhood U of x. A set $A \subset X$ is *α-closed* if $\mathrm{Cl}_\alpha(A) \subset A$.

If α is the identity operation on τ, then the α-closure coincides with the closure in the usual sense. In the case where α is the closure (resp. interior-closure) operation on τ, the α-closure is identical with the ϑ-closure [15] (resp. δ-closure [15]).

Let α be an operation on the topology τ' of Y.

Definition 2.3 [9]. The graph $G(f)$ of a function $f: X \to Y$ is *α-closed* if for each $(x, y) \in (X \times Y) - G(f)$ there exist open sets U and V containing x and y, respectively, such that $(U \times V^\alpha) \cap G(f) = \emptyset$.

Evidently, if α is the identity (resp. closure, interior-closure) operation on τ', then the α-closedness of a graph is identical with the closedness (resp. strong-closedness [7], almost-strong-closedness [8]) of the graph.

The following lemma will be used in the sequel.

LEMMA 2.4. *A function $f: X \to Y$ has an α-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V^\alpha = \emptyset$.*

Let α be an operation on a topology τ of X.

Definition 2.5 [9]. A subset A of X is *α-compact* if for every open cover $\mathscr{C}$ of A there exists a finite subfamily $\{U_1, \ldots, U_n\}$ of $\mathscr{C}$ such that $A \subset \cup_{i=1}^n U_i^\alpha$.

If α is the identity (resp. closure, interior-closure) operation on τ, then an α-compact set is compact (resp. $H(i)$ set [5], N-closed relative to X [1]). An $H(i)$ space [12] is α-compact space for the closure operation α on τ. A space is H-closed if it is $H(i)$ and Hausdorff. If α is the interior-closure operation on τ, then an α-compact space is nearly-compact [13].

Definition 2.6 [16]. A space X is *C-compact* if every closed subset of X is an $H(i)$ set.

Definition 2.7 [17]. A function $f: X \to Y$ is *almost-open* if for each open set V in Y, $f^{-1}(\mathrm{Cl}(V)) \subset \mathrm{Cl}(f^{-1}(V))$.

Every open function is almost-open, but the converse is not true [17].

Let α be an operation on the topology τ' of Y.

Definition 2.8 [9]. A function $f : X \to Y$ is *α-continuous* if for each $x \in X$ and for each open neighborhood V of $f(x)$ in Y there exists an open neighborhood U of x in X such that $f(U) \subset V^{\alpha}$.

If α is the identity operation on τ', then the α-continuity concides with the continuity. The weak-continuity [11] (resp. almost-continuity [14]) is the α-continuity for the closure (resp. interior-closure) operation on τ'.

Other terms and notations not explained herein are those of Kelley [10].

3. Functions with α-closed graphs

In the remainder of this paper α will be an operation on the topology τ' of Y.

THEOREM 3.1. *Let α be a regular operation on τ', and let $f : X \to Y$ be a function with an α-closed graph. If A is a compact subset of X, then $f(A)$ is an α-closed subset of Y.*

Proof. Let A be a compact subset of X. Suppose that $f(A)$ is not α-closed in Y. Then, there exists a point $y \in \mathrm{Cl}_{\alpha} f(A) - f(A)$. Therefore, $y \neq f(x)$ for each $x \in A$. Since f has an α-closed graph, by Lemma 2.4 it follows that for each $x \in A$, there exist open sets $U(x)$ and $V(x)$ containing x and y, respectively, such that $f(U(x)) \cap (V(x))^{\alpha} = \emptyset$. Now $\{U(x) : x \in A\}$ is an open cover of A and, since A is compact, there is a finite subset $\{x_1, \ldots, x_n\}$ of A such that $A \subset \subset \cup_{i=1}^{n} U(x_i)$. The regularity of α implies that there is an open neighborhood V of y such that $V^{\alpha} \subset \cap_{i=1}^{n} (V(x_i))^{\alpha}$. Then $f(A) \cap V^{\alpha} \subset \subset \cup_{i=1}^{n} (f(U(x_i)) \cap V^{\alpha}) \subset \cup_{i=1}^{n} (f(U(x_i)) \cap (V(x_i))^{\alpha}) = \emptyset$. Therefore $y \notin \mathrm{Cl}_{\alpha} f(A)$. This contradiction completes the proof.

The proof of the following theorem is omitted since it is similar to that of Theorem 3.1.

THEOREM 3.2. *Let $f : X \to Y$ be a function with an α-closed graph. If B is an α-compact subset of Y, then $f^{-1}(B)$ is a closed subset of X.*

Definition 3.3. A function $f : X \to Y$ is *locally α-closed* if for each neighborhood U of x there is a neighborhood V of x such that $V \subset U$ and $f(V)$ is α-closed in Y.

If α is the identity operation on τ', then the locally α-closedness of a function coincides with the locally closedness [2]. A function f will be called locally ϑ-closed (resp. locally δ-closed) if it is locally α-closed and α is the closure (resp. interior-closure) operation on τ'.

Definition 3.4. A function $f : X \to Y$ is *α-closed* (resp. *almost α-closed*) if f maps closed (resp. regularly-closed) subsets of X onto α-closed subsets of Y.

Clearly, if α is the identity operation on τ', then the α-closedness (resp. almost α-closedness) of a function coincides with the closedness (resp. almost-closedness [14]). In the case when α is the closure operation on τ', the α-closedness (resp. almost α-closedness) of a function will be called ϑ-closedness (resp. almost ϑ-closedness). We shall say that a function is δ-closed (resp. almost δ-closed) if it is α-closed (resp. almost α-closed) and α is the interior-closure operation on τ'.

It is obvious that the class of α-closed functions is contained in the class of almost α-closed functions. Also, if the domain of an almost α-closed function is a regular space, then the function is locally α-closed.

LEMMA 3.5. *If a function $f: X \to Y$ is locally α-closed and has closed point inverses, then f has an α-closed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $x \notin f^{-1}(y)$ and since $f^{-1}(y)$ is closed, there exists an open neighborhood U of x such that $U \cap f^{-1}(y) = \emptyset$. The locally α-closedness of f implies that there is a neighborhood V of x such that $V \subset U$ and $f(V)$ is α-closed in Y. Since $y \notin f(V)$, there exists an open neighborhood W of y such that $f(V) \cap W^\alpha = \emptyset$. Let V_0 be an open neighborhood of x such that $V_0 \subset V$. Then $f(V_0) \cap W^\alpha = \emptyset$ and hence, by Lemma 2.4 it follows that f has an α-closed graph.

The following theorem is an immediate consequence of Lemma 3.5.

THEOREM 3.6. *If a function $f: X \to Y$ is almost α-closed with closed point inverses and X is a regular space, then f has an α-closed graph.*

If α is the identity operation on τ', then Lemma 3.5 becomes Corollary 3.8 of [2], and Theorem 3.6 is an improvement of Corollary 3.9 of [2].

The following theorem shows that the converse of Theorem 3.1 holds if X is locally compact and regular.

THEOREM 3.7. *Let α be a regular operation on τ'. If $f: X \to Y$ is a function where X is locally compact and regular, the following conditions are equivalent.*

(*a*) *f maps compact sets onto α-closed sets and has closed point inverses.*

(*b*) *f is locally α-closed and has closed point inverses.*

(*c*) *f has an α-closed graph.*

Proof. Assuming (*a*), let U be a neighborhood of some $x \in X$. Since X is locally compact and regular, X has a compact basis of each point and hence, there is a compact neighborhood V of x such that $V \subset U$. Therefore, $f(V)$ is α-closed and (*b*) is verified. By Lemma

3.5, (*b*) implies (*c*). Thus it remains to show that (*c*) implies (*a*). If f has an α-closed graph, then Theorem 3.1 gives that f maps compact sets onto α-closed sets. Since points are α-compact, Theorem 3.2 establishes that f has closed point inverses.

If α is the identity operation on τ', then Theorem 3.7 becomes Theorem 3.11 of [2].

LEMMA 3.8. *If a function $f : X \to Y$ is almost α-closed and has ϑ-closed point inverses, then f has an α-closed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $x \notin f^{-1}(y)$ and since $f^{-1}(y)$ is ϑ-closed, there is an open neighborhood U of x such that $\mathrm{Cl}(U) \cap f^{-1}(y) = \emptyset$. Since $\mathrm{Cl}(U)$ is regularly-closed, the almost α-closedness of f implies that $f(\mathrm{Cl}(U))$ is α-closed in Y. Therefore, there is an open neighborhood V of y such that $f(\mathrm{Cl}(U)) \cap V^{\alpha} = \emptyset$. By Lemma 2.4 it follows that f has an α-closed graph.

Since ϑ-closure and closure coincide for subsets of a regular space, Theorem 3.6 follows from Lemma 3.8.

Now, we utilize Lemma 3.8 (resp. Lemma 3.5) to obtain a sufficient condition for a function to be α-continuous.

THEOREM 3.9. *If $f : X \to Y$ is an almost α-closed* (resp. *a locally α-closed*) *function with ϑ-closed* (resp. *closed*) *point inverses and Y is α-compact, then f is α-continuous.*

Proof. It follows from Lemma 3.8 (resp. Lemma 3.5) and Theorem 11 of [9].

COROLLARY 3.10. *Let $f : X \to Y$ be an almost α-closed function from a regular space X into an α-compact space Y such that $f^{-1}(y)$ is closed for every $y \in Y$. Then f is α-continuous.*

If α is the identity operation on τ', then Corollary 3.10 gives us an improvement of Theorem 4.9 of [4].

Utilizing the fact that a space is Hausdorff if and only if its points are ϑ-closed we obtain the next consequence of Lemma 3.8.

LEMMA 3.11. *If $f : X \to Y$ is an almost α-closed injection where X is Hausdorff, then f has an α-closed graph.*

Combining Lemma 3.11 and Theorem 11 of [9] we have the following theorem.

THEOREM 3.12. *If $f : X \to Y$ is an almost α-closed injection from a Hausdorff space X into an α-compact space Y, then f is α-continuous.*

For the identity operation on τ', Theorem 3.12 becomes Theorem 4.12 of [4].

In the case where α is the closure operation on τ', we extend and improve Theorem 3.9, Corollary 3.10 and Theorem 3.12.

THEOREM 3.13. *Let $f : X \to Y$ be an almost ϑ-closed function where Y is minimal Hausdorff* (resp. *C-compact, $H(i)$*) *space.*

(*a*) *If $f^{-1}(y)$ is ϑ-closed for every $y \in Y$, then f is continuous* (resp. *continuous, almost-continuous*).

(*b*) *If X is regular, $f^{-1}(y)$ is closed for every $y \in Y$, then f is continuous* (resp. *continuous, almost-continuous*).

(*c*) *If f is an injection and X is Hausdorff, then f is continuous* (resp. *continuous, almost-continuous*).

Proof. In all cases f has a strongly-closed graph. If Y is a C-compact (resp. $H(i)$) space, then every closed (resp. regularly--closed) set in Y is an $H(i)$ set. Hence, by Theorem 3.2 it follows that the inverse image under f of every closed (resp. regularly-closed) set is closed. This shows that f is continuous (resp. almost-continuous). In fact, we utilized Theorem 3.2 to prove Theorem 6 of [6] and a slightly improvement of Theorem 9 of [7]. Finally, if Y is minimal Hausdorff, then by Theorem 7 of [6] it follows that f is continuous.

Now, we give some sufficient conditions for a function to have a strongly-closed graph.

LEMMA 3.14. *If $f : X \to Y$ is an almost-open function with a closed graph, then f has a strongly-closed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Since f has a closed graph, there exist open sets U and V containing x and y, respectively, such that $f(U) \cap V = \emptyset$. This implies that $U \cap f^{-1}(V) = \emptyset$. Therefore, $U \cap \mathrm{Cl}(f^{-1}(V)) = \emptyset$. Since f is almost-open, $U \cap f^{-1}(\mathrm{Cl}(V)) = \emptyset$. Hence, $f(U) \cap \mathrm{Cl}(V) = \emptyset$. So, f has a strongly-closed graph.

The following theorem improves Theorem 3.1 and Theorem 3.3 of [3].

THEOREM 3.15. *Let f be an almost-open function from a space X into a minimal Hausdorff* (resp. *H-closed*) *space Y. Then f is continuous* (resp. *almost-continuous*) *if and only if f has a closed graph.*

Proof. We have only to prove the sufficiency. By Lemma 3.14 f has a strongly-closed graph. Now, the proof is parallel to that of Theorem 3.13.

Combining Lemma 3.8 (resp. Lemma 3.5) with Lemma 3.14 we obtain the following theorem.

THEOREM 3.16. *If $f : X \to Y$ is an almost-open, almost-closed* (resp. *locally closed*) *function with ϑ-closed* (resp. *closed*) *point inverses, then f has a strongly-closed graph.*

COROLLARY 3.17. *If $f: X \to Y$ is an almost-open, almost--closed function with ϑ-closed point inverses and Y is a minimal Hausdorff* (resp. *C-compact, $H(i)$*) *space, then f is continuous* (resp. *continuous, almost-continuous*).

Clearly if we replace the expressions »almost-closed« and »ϑ--closed« in Corollary 3.17 with the expressions »locally closed« and »closed«, respectively, then Corollary 3.17 is still valid.

The following corollary improves Theorem 3.4 of [3] and Theorem 11 of [6].

COROLLARY 3.18. *Let f be an almost-open function from a space X into a minimal Hausdorff* (resp. *C-compact, $H(i)$*) *space Y.*

(*a*) *If X is regular, f is almost-closed, and $f^{-1}(y)$ is closed for every $y \in Y$, then f is continuous* (resp. *continuous, almost-continuous*).

(*b*) *If f is an almost-closed injection and X is Hausdorff, then f is continuous* (resp. *continuous, almost-continuous*).

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O FUNKCIJAMA SA α-ZATVORENIM GRAFOM

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Sadržaj

Pojam α-zatvorenog grafa je uveo S. Kasahara [9]. U ovom radu je nastavljeno ispitivanje funkcija sa α-zatvorenim grafom, te posebno funkcija sa jako-zatvorenim grafom. Važniji su rezultati:

Ako je α regularna operacija na topologiji τ' prostora (Y, τ') i ako je $f : X \to Y$ funkcija sa α-zatvorenim grafom, onda je slika svakog kompaktnog skupa α-zatvorena.

Ako je $f : X \to Y$ lokalno α-zatvorena funkcija i $f^{-1}(y)$ zatvoreno za svako $y \in Y$, onda f ima α-zatvoren graf.

Ako je $f : X \to Y$ gotovo α-zatvorena funkcija i $f^{-1}(y)$ ϑ-zatvoreno za svako $y \in Y$, onda f ima α-zatvoren graf.

Ako je $f : X \to Y$ gotovo-otvoreno gotovo-zatvoreno preslikavanje, $f^{-1}(y)$ ϑ-zatvoreni za svako $y \in Y$ i Y je minimalan Hausdorff-ov (resp. C-kompaktan, $H(i)$) prostor, onda je f neprekidno (resp. neprekidno, gotovo-neprekidno) preslikavanje.