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THE MINIMAL 0-DIMENSIONAL OVERRINGS OF COMMUTATIVE RINGS

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Abstract. Throughout this paper rings are understood to be commutative with unity, and subrings are understood to have same identity as their overrings. In this paper the following results are given: *i*) if S is a 0-dimensional overring of a commutative ring R then there exists exactly one minimal 0-dimensional overring R_m of R contained in S and R_m is the total quotient ring of an integral overring of R; *ii*) some facts on the structure of minimal 0-dimensional overrings of a commutative ring R; and *iii*) a construction of the minimal regular overring of a semiprime ring R that is also a quotient ring of R.

A commutative ring with unity will be denoted by R. We will also use the symbol $T(R)$ when we want to emphasize the fact that $T(R)$ is a total quotient ring, i. e. a ring in which every regular element is invertible.

We next recall the following definitions.

Definition 1. A prime ideal P of R is called regular if P contains a regular element of R.

Definition 2. A commutative ring R is called semiprime if R contains no nonzero nilpotent elements.

Definition 3. A commutative ring R is called regular if for $a \in R$ there exists $a' \in R$ such that $a^2 a' = a$.

Definition 4. A commutative ring R is called π-regular if for $a \in R$ there exists $a' \in R$ and a positive integer n such that $a^n = (a^n)^2 a'$.

Definition 5. Let R_m be a 0-dimensional overring of R. We say that R_m is a minimal 0-dimensional overring of R if it has the property: If S is a 0-dimensional overring of R such that $R \subseteq S \subseteq R_m$, then $S = R_m$.

Definition 6. Let S be an overring of R. $s^{-1} R = \{r \in R \mid sr \in R\}$ is an ideal of R, for every $s \in S$. S is called the quotient ring of R if and only if $t(s^{-1} R) \neq 0$ for every $s \in S$, $t \in S$, $t \neq 0$. This definition is due to Utumi (see [4]).

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It is well known that R is a regular ring if and only if R is a semiprime 0-dimensional ring and that R is a π-regular ring if and only if R is a 0-dimensional ring.

Let R be a semiprime ring. Sometimes it is useful to embed R into a regular overring. [2]. It is known that the complete quotient ring $Q(R)$ of R is a regular ring. [4]. It is easy to construct a regular overring of R. Namely, let $\{P_\lambda\}$ be the set of the prime ideals of R. R/P_λ is a domain, for every λ, furthermore, $\cap P_\lambda = (0)$ and we can realize R as a subdirect product of the domains $\{R/P_\lambda\}$. Let $K_\lambda = Q(R/P_\lambda)$ be the quotient field of R/P_λ, for every λ. Then $R \subseteq \prod_\lambda K_\lambda$ and $\prod_\lambda K_\lambda$ is, as a direct product of fields, a regular ring.

THEOREM 1. *Let R be a semiprime ring and let $\overline{R}$ be its regular overring. Then $\overline{R}$ contains exactly one minimal regular overring R_m of R and R_m is the total quotient ring of an integral overring of R.*

Proof. Since $\overline{R}$ is a regular ring, by Theorem 2 of [1] for $a \in \overline{R}$ there exists an idempotent $e \in \overline{R}$ such that $a = ae$ and $a + (1 - e)$ is a regular element of $\overline{R}$. Let E denote the set of idempotent element of $\overline{R}$ formed in the following way: $e \in E$ if and only if there exists $a \in R$ such that $a = ae$ and $a + 1 - e$ is a regular element of $\overline{R}$. Let R_1 denote the subring of $\overline{R}$ generated by R and E. Consider the total quotient ring $T(R_1)$ of R_1. Clearly, every regular overring of R contained in $\overline{R}$ contains $T(R_1)$. We will show that $T(R_1)$ is a regular overring of R. It is sufficient to show that every prime ideal of $T(R_1)$ is maximal, i. e. that $T(R_1)$ is a 0-dimensional ring. Let P, P_1 be prime ideals of $T(R_1)$, $P_1 \subseteq P$. Then $P_1 \cap R$ and $P \cap R$ are prime ideals of R and $P_1 \cap R \subseteq P \cap R$. Let $a \in P \cap R$ and let e be an idempotent element of $T(R_1)$ such that $ae = a$ and $a + 1 - e$ is a regular element of $T(R_1)$. Since $a + 1 - e$ is a regular element, $1 - e \notin P_1$ and since $e(1 - e) = 0$, $e \in P_1$. It follows that $a \in P_1$. Hence $P_1 \cap R = P \cap R$ and since $T(R)$ is a quotient ring of an integral overring of R it follows that $P_1 = P$.

Let R be a semiprime commutative ring and let R_m be its minimal regular overring. For $r \in R$ let e_r be the idempotent element of R_m such that $r = re_r$ and $r + (1 - e_r)$ is a regular element of R_m. Let R_1 denote the subring of R_m generated by R and $\{e_r \mid r \in R\}$. If $x \in R_1$, then x has the form $x = \sum_{k=1}^{n} a_k e_{1_k} e_{2_k} \ldots e_{m_k}$; $a_1, a_2, \ldots, a_n \in R$; $e_{1_1}, e_{2_1}, \ldots, e_{m_1}, e_{1_2}, \ldots, e_{m_2}, \ldots, e_{1_n}, \ldots, e_{m_n} \in \{e_r \mid r \in R\}$. We will show that we can write x in the form $x = \sum_{k=1}^{n} a_k e_k$; $a_1, a_2, \ldots, a_n \in R$; $e_1, e_2, \ldots, e_n \in \{e_r \mid r \in R\}$. To see this it is sufficient to consider the following. Let $e_1, e_2, \ldots, e_n \in \{e_r \mid r \in R\}$ and let $r_1, r_2 \ldots, r_n$ be elements of R such that $r_i = r_i e_i$ and $r_i + (1 - e_i)$ is a regular element of R_1 ($i = 1, 2, \ldots, n$). Then for the idempotent element $e_1 e_2 \ldots e_n$ and $r_1 r_2 \ldots r_n \in R$ we have $r_1 r_2 \ldots r_n = (r_1 r_2 \ldots \ldots r_n)(e_1 e_2 \ldots e_n)$ and $r_1 r_2 \ldots r_n + (1 - e_1 e_2 \ldots e_n)$ is a regular element of R_1.

Let R be a semiprime ring and let R_m be its minimal regular overring. Let P be a prime ideal of R_m and $\overline{P} = P \cap R$. For $r \in R$ let e_r be the idempotent element of R_m such that $r = re_r$ and $r + (1 - e_r)$ is a regular element of R_m. Consider the subset of R_m given by $Q = \overline{P} \cup \{1 - e_r \mid r \in R \setminus \overline{P}\} \cup \{e_r \mid r \in \overline{P}\}$. Clearly, P contains Q. Let A be the ideal of R_m generated by Q.

PROPOSITION 2. $P = A$.

Proof. Let $r \in R$ and let e_r be the idempotent element of R_m such that $r = re_r$ and $r + (1 - e_r)$ is a regular element of R_m. Let R_1 be the overring of R generated by R and $\{e_r \mid r \in R\}$. It is sufficient to show that $P \cap R_1 = A \cap R_1$. Let $x \in P \cap R_1$. x has the form $x = \sum_{k=1}^{n} a_k e_k$; $a_1, a_2, \ldots, a_n \in R$; $e_1, e_2, \ldots, e_n \in \{e_r \mid r \in R\}$. It is sufficient to consider the case when no summand in the above sum is an element of A. Then $1 - e_1, 1 - e_2, \ldots, 1 - e_n \in A$ and $x = \sum_{k=1}^{n} a_k e_k = -\sum_{k=1}^{n} a_k (1 - e_k) + \sum_{k=1}^{n} a_k \in A$.

COROLLARY. *If P is a prime ideal of R then at most one prime ideal $\overline{P}$ of R_m lies over P.*

THEOREM 3. *Let R_m be a minimal regular overring of the semiprime ring R and let $\{\overline{P}_\lambda\}$ denote the set of prime ideals of R_m. Let $P_\lambda = \overline{P}_\lambda \cap R$, for $\lambda \in \Lambda$ (clearly these are prime ideals of R). For every λ, let $Q(R/P_\lambda)$ be the quotient field of the domain R/P_λ. Then R_m can be expressed as a subdirect product of the fields $\{Q(R/P_\lambda)\}$.*

Proof. It is easy to conclude that R_m can be expressed as a subdirect product of the fields $\{Q(R/P_\lambda)\}$. Therefore, R_m can be identified with the minimal regular overring of R that is contained in $\prod_\lambda Q(R/P_\lambda)$.

A minimal regular overring R_m of R does not in general contain the total quotient ring $T(R)$. For example, let R be a semiprime ring that is not a total quotient ring. Let $\{P_\lambda\}$ be the set of prime ideals of R. For every λ, let $Q(R/P_\lambda)$ be the quotient field of the domain R/P_λ. Let R_m be the minimal regular overring of R contained in $\prod_\lambda Q(R/P_\lambda)$. It is easy to see that R_m does not contain the total quotient ring $T(R)$ of R.

Let R be a commutative ring. It is well known that R, the total quotient ring $T(R)$ and the complete quotient ring $Q(R)$ of R are quotient rings of R. Also, it is known that every quotient ring S of R can be embedded into $Q(R)$ using the morphism that is the extension of the canonical morphism $R \to Q(R)$ [4]. In the following theorem we construct the minimal regular overring of R that is also a quotient ring of R.

THEOREM 4. *Let R be a semiprime ring and let $\{P_\lambda\}$ be the set of minimal prime ideals of R. Let R_{m_0} be the minimal regular overring of R contained in $\prod_\lambda Q(R/P_\lambda)$. Then R_{m_0} is the unique (up to an isomorphism) minimal regular overring of R that is also a quotient ring of R.*

Proof. There exists a minimal regular overring of R that is simultaneously a quotient ring of R. Namely, it is known that the complete quotient ring $Q(R)$ of R is a regular overring of R; consequently, the minimal regular overring of R contained in $Q(R)$ is the minimal regular overring of R that is simultaneously a quotient ring of R. Let R_m be a minimal regular overring of R that is simultaneously a quotient ring of R. Let $\{\overline{P}_\lambda\}$ be the set of minimal prime ideals of R. There exists a prime ideal P_λ of R_m lying over $\overline{P}_\lambda$, for every λ. Nemely, R_m is the total quotient ring of an integral overring R_1 of R, i. e. $R_m = T(R_1)$. Since R_1 is an integral overring of R, there exists a prime ideal P'_λ lying over P_λ for every λ. It is obligatory that P'_λ is a minimal prime ideal of R_1, and therefore P'_λ is also a non regular prime ideal of R_1, hence P'_λ is preserved in $R_m = T(R_1)$. Consider the ideal $\bigcap_\lambda P_\lambda$ of R_m. $(\bigcap_\lambda P_\lambda) \cap R = (0)$. Therefore if $x \in \bigcap_\lambda P_\lambda$ and $r \in R$ such that $xr \in R$ then it necessarily follows that $xr = 0$. Since R_m is a quotient ring of R it follows that $\bigcap_\lambda P_\lambda = (0)$. Hence we can realize R_m as a subdirect product of the fields $\{Q(R/\overline{P}_\lambda)\}$ and so we can identify R_m with R_{m_0}.

THEOREM 5. *Let R_m be a minimal regular overring of R and let R_{m_0} be the minimal regular overring of R that is simultaneuosly a quotient ring of R. Then there exists an epimorphism $\varphi : R_m \to R_{m_0}$ with the following property: $\varphi | R$ (the restriction of φ on R) is the identity mapping on R.*

Proof. Let $\{\overline{P}_\lambda\}$ be the set of minimal prime ideals of R. There exists a prime ideal P_λ of R_m lying over $\overline{P}_\lambda$ for every λ. $R_m/\cap P_\lambda$ is, as an epimorphic image of a regular ring, a regular ring. Furthermore, since $(\cap P_\lambda) \cap R = (0)$, $R_m/\cap P_\lambda$ is a regular overring of R, and it is easy to see that it is the minimal regular overring of R that is also a quotient ring of R. The natural epimorphism $\varphi : R_m \to R_m/\cap P_\lambda$ is the required epimorphism.

LEMMA 6. *Let R be a commutative ring. Let e be an idempotent element and let n be a nilpotent element of R. If $e+n$ is also an idempotent element of R, then $n=0$.*

Proof. Let $e+n$ be an idempotent element of R. Then $e+n = (e+n)^2 = e+2en+n^2$. Multiplying by $1-e$ we obtain $(1-e)n = (1-e)n^2$ from which it follows that $(1-e)n = (1-e)n^2 = (1-e)n^3 = \ldots = 0$. Hence $n = ne$ and therefore $e+n = e+en = e(1+n) = e(1+n)^2$ and since $1+n$ is a regular element of R, $e = e+en$, i. e. $n = en = 0$.

THEOREM 7. *Let R be a subring of a 0-dimensional commutative ring S. Then there exists exactly one minimal 0-dimensional overring R_m of R contained in S and R_m is the total quotient ring of an integral overring of R.*

Proof. Let $r \in R$. Then there exists an idempotent $e_r \in S$ such that $r(1-e_r)$ is a nilpotent element and $r+(1-e_r)$ is a regular element of S. Let N be the nilradical of S. $\bar{S} = S/N$ is a regular ring and $\bar{e}_r = e_r + N$ is the unique idempotent element of $\bar{S}$ such that $\bar{r}(\bar{1}-\bar{e}_r) = 0$ and $\bar{r}+(\bar{1}-\bar{e}_r)$ is a regular element of $\bar{S}$, where $\bar{r} = r+N \in \bar{S}$. If there exists another idempotent $e'_r \in S$ such that $r(1-e'_r)$ is a nilpotent element and $r+(1-e'_r)$ is a regular element of S, then $e'_r = e_r + n$, where n is a nilpotent element of S, and Lemma 6 shows that $n=0$. Therefore, there exists a unique idempotent element e_r of S such that $r(1-e_r)$ is a nilpotent element and $r+(1-e_r)$ is a regular element of S. Let R_1 be the ring generated with R and $\{e_r \mid r \in R\}$. Certainly R_1 is an integral overring of R. Let $T(R_1)$ be its total quotient ring. Clearly, every 0-dimensional overring of R, that is contained in S, contains $T(R_1)$. We will show that $T(R_1)$ is a 0-dimensional ring. Let P, P_1 be the prime ideals of $T(R_1)$ (they are, certainly, non-regular) such that $P \subsetneq P_1$. Since $T(R_1)$ is a quotient ring of an integral overring of R, $P \cap R \subsetneq P_1 \cap R$. Let $r \in (P_1 \cap R) \setminus (P \cap R)$ and let e_r be the idempotent element of $T(R_1)$ such that $r(1-e_r)$ is a nilpotent element and $r+(1-e_r)$ is a regular element of $T(R_1)$. Since $r(1-e_r)$ is a nilpotent element and $r \notin P$, it follows that $1-e_r \in P$. Therefore $r+(1-e_r) \in P_1$ and $r+(1-e_r)$ is a regular element of $T(R_1)$. This is a contradiction and it follows that $T(R_1)$ is a 0-dimensional ring.

PROPOSITION 8. *Let R be a subring of the 0-dimensional ring S and let R_m be the minimal 0-dimensional overring of R contained in S. If A is a primary ideal of R then at most one primary ideal $\bar{A}$ of R_m lies over A.*

Proof. The proof is similar to the proof of Proposition 2.

THEOREM 9. *Let R be a subring of the* 0*-dimensional ring* $\bar{R}$. *Then there exists a minimal* 0*-dimensional overring* R_m *of* R *with the property that if* P *is a prime ideal of* R *then there exists a prime ideal* $\bar{P}$ *of* R_m *lying over* P.

Proof. Let $\{P_\lambda\}$ be the set of prime ideals of R and let $Q(R/P_\lambda)$ be the quotient field of R/P_λ for every λ. $\bar{R} \oplus (\prod_\lambda Q(R/P_\lambda))$ is a 0-dimensional ring. Let $r \in R$ and let $\bar{r}$ be the element of $\prod_\lambda Q(R/P_\lambda)$ having $r + P_\lambda \in Q(R/P_\lambda)$ as its component in the λ-place, for every λ. The mapping $\varphi(r) = r \oplus \bar{r}$, $r \in R$, is a morphism of R into $\bar{R} \oplus (\prod_\lambda Q(R/P_\lambda))$. If we identify R with $\varphi(R)$, then $\bar{R} \oplus (\prod_\lambda Q(R/P_\lambda))$ is a 0-dimensional overring of R. The minimal 0-dimensional overring of R contained in $\bar{R} \oplus (\prod_\lambda Q(R/P_\lambda))$ has the property that if P is a prime ideal of R then there exists a prime ideal $\bar{P}$ of R_m lying over P.

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O MINIMALNIM 0-DIMENZIONALNIM NADPRSTENIMA KOMUTATIVNIH PRSTENOVA

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Sadržaj

U ovom radu se posmatraju komutativni prstenovi sa jedinicom. Dati su slijedeći rezultati: *i*) ako je S 0-dimenzionalan nadprsten komutativnog prstena R, tada postoji točno jedan minimalni 0-dimenzionalni nadprsten R_m od R sadržan u S i R_m je totalni prsten razlomaka cijelog nadprstena od R; *ii*) neke činjenice o strukturi minimalnih 0-dimenzionalnih nadprstena komutativnog prstena R; i *iii*) konstrukcija minimalnog regularnog nadprstena poluprostog prstena R koji je također i prsten razlomaka od R.