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Croatian Quants Day

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Multidefaults

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This presentation is devoted to modeling multiple default times in presence of some extra information.

Marked Point Processes

We recall some results on Marked Point Processes.

A MPP $\mathbb{M}$ is a sequence $(\sigma_k, Y_k)_{k\geq 1}$ where

1. The random variables σ_k satisfy $0 \leq \sigma_k < \sigma_{k+1}$
2. The r.vs Y_k (the marks) are valued in $\mathbb{R}^d$

We note $(\mathcal{M}_t, t \geq 0)$ the history of $\mathbb{M}$ (the marked point process filtration generated by $\mathbb{M}$) so that $\mathcal{M}_{\sigma_k} = \sigma\{(\sigma_1, Y_1), \ldots, (\sigma_k, Y_k)\}$.

To any MPP, we associate the random measure μ defined as

$$\mu(]0,t] \times C) = \sum_k \mathbb{1}_{\{(\sigma_k, Y_k) \in]0,t] \times C\}}$$

for $C \in \mathcal{B}(\mathbb{R}^d - 0)$

For any integrable r.v. U, setting $\sigma_0 = 0$, one has

$$\mathbb{E}(U|\mathcal{M}_t) = \sum_{k\geq 0} \mathbb{1}_{\{\sigma_k < t \leq \sigma_{k+1}\}} \frac{\mathbb{E}(\mathbb{1}_{\{t<\sigma_{k+1}\}} U|\mathcal{M}_{\sigma_k})}{\mathbb{P}(t < \sigma_{k+1}|\mathcal{M}_{\sigma_k})}$$

An important tool is $\eta^{k+1|k}(dt, dy)$, the regular version of the conditional distribution of (σ_{k+1}, Y_{k+1}) w.r.t. $\mathcal{M}_{\sigma_k}$.

The compensator of the point process $\mathbb{M}$ is the (unique) random measure $\nu(dt, dy)$ such that for any (bounded) predictable function K, the process $K \star (\mu - \nu)$ is a local martingale, where

$$(K \star (\mu - \nu))_t(\omega) = \int_{]0,t] \times \mathbb{R}^d} K(\omega; s, y)(\mu(\omega; ds, dy) - \nu(\omega; ds, dy))$$

given by

$$\begin{aligned} \nu(dt, dy) &= \sum_{k \geq 0} \mathbb{1}_{\{\sigma_k \leq t < \sigma_{k+1}\}} \frac{\eta^{k+1|k}(dt, dy)}{\eta^{k+1|k}([t, \infty[\times \mathbb{R}^d)} \\ &= \sum_{k \geq 0} \mathbb{1}_{\{\sigma_k \leq t < \sigma_{k+1}\}} \frac{\mathbb{P}((\sigma_{k+1}, Y_{k+1}) \in (dt, dx) | \mathcal{M}_{\sigma_k})}{\mathbb{P}(\sigma_{k+1} \geq t | \mathcal{M}_{\sigma_k})} \end{aligned}$$

Ranked Default Times

We restrict our attention to a finite number of ranked default times $(\sigma_k, k \le n)$. We set $\sigma_0 = 0, \sigma_{n+1} = \infty$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$. This is a MPP (without marks!)

We assume that the vector $\boldsymbol{\sigma}$ has a density $\eta(\boldsymbol{u})$, i.e.,

$$\mathbb{E}[f(\boldsymbol{\sigma})] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\eta(\boldsymbol{u})d\boldsymbol{u},$$

Here, we make use of the following notation

- $\boldsymbol{u} = (u_1, \cdots, u_n), \quad \boldsymbol{u}_{(k:p)} = (u_k, \cdots, u_p), \quad \boldsymbol{u}_{(p)} = \boldsymbol{u}_{(1:p)}$
- $d\boldsymbol{u} = du_1 \cdots du_n, \quad d\boldsymbol{u}_{(k:p)} = du_k \dots du_p$
- $\boldsymbol{u} > \boldsymbol{\theta}$ stands for $u_i > \theta_i$ for all $i \in \{1, \cdots, n\}$
- $\int_{]\boldsymbol{t},+\infty[} f(\boldsymbol{u}_{(k:n)})d\boldsymbol{u}_{(k:n)} := \int_{]t,+\infty[} du_k \cdots \int_{]t,+\infty[} du_n\, f(u_k, \dots, u_n)$.

The (marginal) density of $\boldsymbol{\sigma}_{(k)}$ is

$$\eta^{(k)}(\boldsymbol{u}_{(k)}) = \int_{\mathbb{R}_+^{n-k}} \eta(\boldsymbol{u})d\boldsymbol{u}_{(k+1:n)}$$

Furthermore, on $\sigma_k \leq t < \sigma_{k+1}$

$$\mathbb{P}(\sigma_{k+1} > \theta | \mathcal{M}_t) = \int_\theta^\infty \eta^{k+1|k}(s) ds$$

where

$$\eta^{k+1|k}(s) = \frac{1}{\eta^{(k)}(\boldsymbol{\sigma}_{(k)})} \int_{\mathbb{R}_+^{n-(k+2)}} d\boldsymbol{u}_{(k+2:n)} \eta(\boldsymbol{\sigma}_{(k)}, s, \boldsymbol{u}_{(k+2:n)})$$

It follows that

$$\mathbb{E}(f(\boldsymbol{\sigma}) | \mathcal{M}_t) = \int_{\mathbb{R}_+^n} f(\boldsymbol{u}) \eta_t^{\mathcal{M}}(d\boldsymbol{u})$$

where, on the set $\sigma_k \leq t < \sigma_{k+1}$

$$\eta_t^{\mathcal{M}}(d\boldsymbol{u}) = \frac{\mathbb{1}_{\{\boldsymbol{t} < \boldsymbol{u}_{(k+1:n)}\}}}{\int_t^\infty \eta^{k+1|k}(s) ds} \delta_{\boldsymbol{\sigma}_{(k)}}(d\boldsymbol{u}_{(k)}) \, \eta(\boldsymbol{u}_{(k)}, \boldsymbol{u}_{(k+1:n)}) d\boldsymbol{u}_{(k+1:n)}$$

Let $N_t = \sum_{k=1}^{n} 1\!\!1_{\{\sigma_k \leq t\}}$. The compensator of N is

$$\Lambda_t = \int_0^{t \wedge \sigma_n} \lambda_s ds = \sum_{k=0}^{n-1} \int_{\sigma_k}^{\sigma_{k+1} \wedge t} \frac{1}{\int_s^\infty \eta^{k+1|k}(y)dy} \eta^{k+1|k}(s)ds$$

Remark: it is useful to remember that the support of η is contained in $\{u_1 < u_2 < \cdots < u_n\}$.

Ranked Default Times with Reference Filtration

We assume now that a reference filtration $\mathbb{F}$ is given and that there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable functions $(\omega, \boldsymbol{u}) \to \alpha_t(\omega, \boldsymbol{u})$ such that

$$\mathbb{E}[f(\boldsymbol{\sigma})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\alpha_t(\boldsymbol{u})d\boldsymbol{u},$$

We call the family $\alpha(\boldsymbol{u})$ the $\mathbb{F}$-*conditional density* of $\boldsymbol{\sigma}$. Note that α_0 is the unconditional law of $\boldsymbol{\sigma}$.

We denote by $\mathbb{G}$ the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{M}_t$.

It can be useful to keep in mind that, if one defines

$$d\mathbb{Q}|_{\mathcal{F}_t \vee \sigma(\boldsymbol{\sigma})} = \frac{1}{\alpha_t(\boldsymbol{\sigma})} d\mathbb{P}|_{\mathcal{F}_t \vee \sigma(\boldsymbol{\sigma})}$$

then, $\mathbb{F}$ and $\boldsymbol{\sigma}$ are independent under $\mathbb{Q}$, and $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$.

For any fixed $\boldsymbol{u} \in \mathbb{R}_+^n$, the process $(\alpha_t(\boldsymbol{u}), t \geq 0)$ is an $\mathbb{F}$-martingale. The joint conditional survival law is given, for any $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n) \in \mathbb{R}_+^n$, by

$$S_t(\boldsymbol{\theta}) := \mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta} | \mathcal{F}_t) = \int_{\theta_1}^{\infty} du_1 \cdots \int_{\theta_n}^{\infty} du_n \, \alpha_t(\boldsymbol{u}) = \int_{\boldsymbol{\theta}}^{\infty} \alpha_t(\boldsymbol{u}) d\boldsymbol{u}$$

The marginal density of $\boldsymbol{\sigma}_{(k)}$ with respect to $\mathcal{F}_t$ is the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^k)$-measurable function $\alpha_t^{(k)}(\boldsymbol{u}_{(k)})$ given by

$$\alpha_t^{(k)}(\boldsymbol{u}_{(k)}) = \int_{\mathbb{R}_+^{n-k}} \alpha_t(\boldsymbol{u}) d\boldsymbol{u}_{(k+1:n)}$$

and, on $\sigma_k \leq t < \sigma_{k+1}$,

$$\mathbb{P}(\sigma_{k+1} > \theta | \mathcal{G}_t) = \int_{\theta}^{\infty} \alpha_t^{k+1|k}(s) ds$$

where

$$\alpha_t^{k+1|k}(s) = \frac{1}{\alpha_t^{(k)}(\boldsymbol{\sigma}_{(k)})} \int_{\mathbb{R}^{n-(k+2)}} d\boldsymbol{u}_{(k+2,n)} \alpha_t(\boldsymbol{\sigma}_{(k)}, s, \boldsymbol{u}_{(k+2,n)})$$

It follows that

$$\mathbb{E}(f(\boldsymbol{\sigma})|\mathcal{G}_t) = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\mu_t(d\boldsymbol{u})$$

where, on the set $\sigma_k \leq t < \sigma_{k+1}$

$$\mu_t(d\boldsymbol{u}) = \frac{\mathbb{1}_{\{t<\boldsymbol{u}_{(k+1,n)}\}}}{\int_t^\infty \alpha_t^{k+1|k}(s)ds}\delta_{\boldsymbol{\sigma}_{(k)}}(d\boldsymbol{u}_{(k)})\ \alpha_t(\boldsymbol{u})\ d\boldsymbol{u}_{(k+1,n)}$$

Furthermore, for $Y_T(\boldsymbol{u})$ a family of positive $\mathcal{F}_T$ adapted random variables,

$$\begin{aligned}
\mathbb{E}(Y_T(\boldsymbol{\sigma})|\mathcal{G}_t) &= \int_{\mathbb{R}_+^n} \frac{1}{\alpha_t(\boldsymbol{u})}\mathbb{E}(Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})|\mathcal{F}_t)\mu_t(d\boldsymbol{u}) \\
&= \sum_{k=0}^{n-1} \frac{\mathbb{1}_{\{\sigma_k \leq t < \sigma_{k+1}\}}}{\int_t^\infty \alpha_t^{k+1|k}(s)ds} \int_{\boldsymbol{t}}^\infty \mathbb{E}(Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})|\mathcal{F}_t)|_{\boldsymbol{u}_{(k)}=\boldsymbol{\sigma}_{(k)}} d\boldsymbol{u}_{(k+1,n)}
\end{aligned}$$

Let $N_t = \sum_{k=1}^{n} 1\!\!1_{\{\sigma_k \le t\}}$. The compensator of N in the filtration $\mathbb{G}$ is

$$\Lambda_t = \int_0^{t\wedge\sigma_n} \lambda_s ds = \sum_{k=0}^{n-1} \int_{\sigma_k}^{\sigma_{k+1}\wedge t} \frac{\alpha_s^{k+1|k}(s)}{\int_s^\infty \alpha_s^{k+1|k}(u)du}\, ds = \sum_{k=0}^{n-1} \int_{\sigma_k}^{\sigma_{k+1}\wedge t} \lambda_s^{k+1}\, ds$$

where $\lambda_s^{k+1|k} = \frac{\alpha_s^{k+1|k}(s)}{\int_s^\infty \alpha_s^{k+1|k}(u)du}$. Note that $\lambda_s^{k+1|k}$ depends on $\boldsymbol{\sigma}_{(k)}$.

General Construction

The random variable Ξ is a random variable of law η taking values in a complete metric space E with countable base and equipped with Borel σ-algebra $\mathcal{B}(E)$. The main example is $\Xi = (\tau_k, Y_k)_{1\le k\le n}$ where $\boldsymbol{\tau}$ is a sequence (not necessarily ranked) of random times and Y_k some marks.

Without loss of generality, we assume that Ξ is the canonical map from E in E, defined as $\Xi(\chi) = \chi$ so that $\mathbb{E}(f(\Xi)) = \int_E f(\chi)\eta(d\chi)$ where η is the law of Ξ.

The "default-free" market risk is represented by a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.

We denote be $\boldsymbol{\sigma}$ the ranked sequence of times, the filtration $\mathcal{M}_t$ is the one of the associated MMP $\mathbb{M} = (\sigma_k, Y_{\sigma_k})_k$.
The filtration $\mathbb{F}$ is considered as well on Ω or on the product space.
The filtration $\mathbb{H}$ is defined as $\mathcal{H}_t = \mathcal{F}_t \otimes \mathcal{B}(E) = \mathcal{F}_t \otimes \sigma(\Xi)$.
The filtration $\mathbb{G}$ is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{M}_t$.
All the filtrations are defined in such a way that they satisfy usual conditions.

We start with the fundamental case where the two sources of risks are independent (i.e., the random variable Ξ is independent from $\mathcal{F}_\infty$), the probability measure is the product measure $\overline{\mathbb{P}}^0(d\omega, d\chi) = \mathbb{P}(d\omega) \otimes \eta(d\chi)$.

The conditional law of Ξ given $\mathcal{M}_t$ is denoted by $\eta_t^{\mathcal{M}}$.

Given a non-negative measurable function Y on $\Omega \times E$, we define

$$\eta_t^{\mathcal{M}}(Y) = \int_E Y(.,\chi)\eta_t^{\mathcal{M}}(d\chi) = \overline{\mathbb{E}}^0[Y(.,\Xi)|\mathcal{F}_\infty \vee \mathcal{M}_t]$$

which is $\mathcal{F}_\infty \vee \mathcal{M}_t$-measurable.

One should take care about the notation: $\eta^{\mathcal{M}}$ refers to the filtration $\mathcal{F}_\infty \vee \mathcal{M}_t$ and not to $\mathcal{M}_t$.

Note that, from the independence assumption, $\overline{\mathbb{E}}^0(f(\Xi)|\mathcal{M}_t) = \overline{\mathbb{E}}^0(f(\Xi)|\mathcal{M}_t \vee \mathcal{F}_\infty)$.

As an exemple, let us study the case where $\Xi = (\tau_1, \tau_2)$, with law η and define $G(t,s) = \overline{\mathbb{P}}^0(\tau_1 > t, \tau_2 > s)$. Then, denoting by $\mathcal{M}_t$ the filtration generated by $1\!\!1_{\tau^i \le t}$,

$$\mathbb{E}(f(\tau_1,\tau_2)|\mathcal{M}_t) = I_t(1,1)f(\tau_1,\tau_2) + I_t(1,0)\Psi_{1,0}(\tau_1,t) + I_t(0,1)\Psi_{0,1}(t,\tau_2) + I_t(0,0)\Psi_{0,0}(t)$$

where

$$\Psi_{1,0}(t,u) = -\frac{1}{\partial_1 G(u,t)}\int_t^\infty f(u,v)\partial_1 G(u,dv)$$

$$\Psi_{0,1}(t,v) = -\frac{1}{\partial_2 G(t,v)}\int_t^\infty f(u,v)\partial_2 G(du,v)$$

$$\Psi_{0,0}(t) = \frac{1}{G(t,t)}\int_t^\infty \int_t^\infty f(u,v)G(du,dv)$$

$$I_t(1,1) = 1\!\!1_{\{\tau_1 \le t, \tau_2 \le t\}}\ , \qquad I_t(0,0) = 1\!\!1_{\{\tau_1 > t, \tau_2 > t\}}$$

$$I_t(1,0) = 1\!\!1_{\{\tau_1 \le t, \tau_2 > t\}}\ , \qquad I_t(0,1) = 1\!\!1_{\{\tau_1 > t, \tau_2 \le t\}}$$

Given a non-negative measurable function Y on $\Omega \times E$ (that is $(\omega, \chi) \to Y(\omega, \chi)$), there exists a family of $\mathbb{F}$-adapted processes, parametrized by χ, say $Y^{\mathcal{F}}(\chi)$, such that $\mathbb{P}$-a.s, for any $\chi \in E$, and for any $t \geq 0$, $Y_t^{\mathcal{F}}(\chi) = \mathbb{E}[Y(\cdot, \chi)|\mathcal{F}_t]$.

An useful example is $Y = Xh(\Xi)$ where $X \in \mathcal{F}_\infty$.

We shall call $Y^{\mathcal{F}}$ the universal version of conditional expectation.

One has $\overline{\mathbb{E}}^0(Y|\mathcal{H}_t) = Y_t^{\mathcal{F}}(\Xi)$ and, for any $\mathcal{H}_t$-measurable r.v. Y_t

$$\overline{\mathbb{E}}^0(Y_t|\mathcal{G}_t) = \int_E Y_t(\chi)\eta_t^{\mathcal{M}}(d\chi) =: \eta_t^{\mathcal{M}}(Y_t)$$

In the same way, if $K = K(X, M_t)$ where $X \in \mathcal{F}_\infty$ and $M_t \in \mathcal{M}_t$, one has

$$\overline{\mathbb{E}}^0(K|\mathcal{G}_t) = \overline{\mathbb{E}}^0(K(X, m)|\mathcal{F}_t)_{m=M_t} =: K_t^{\mathcal{F}}$$

$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\Xi)$

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$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\Xi)$

Consider now a non-negative measurable random variable Y on $\Omega \times E$. The calculation of its conditional expectation w.r.t. $\mathcal{G}_t$ can be done in two different ways as shown below:

On the one hand, using the notation of the universal martingale

$$\overline{\mathbb{E}}^0[Y|\mathcal{G}_t] = \overline{\mathbb{E}}^0[\overline{\mathbb{E}}^0[Y|\mathcal{H}_t]\,|\,\mathcal{G}_t] = \overline{\mathbb{E}}^0[Y_t^{\mathcal{F}}\,|\,\mathcal{G}_t] = \eta_t^{\mathcal{M}}(Y_t^{\mathcal{F}})$$

On the other hand, using the intermediary σ-algebra $\mathcal{F}_\infty \vee \mathcal{M}_t$

$$\overline{\mathbb{E}}^0[Y|\mathcal{G}_t] = \overline{\mathbb{E}}^0[\overline{\mathbb{E}}^0[Y|\mathcal{F}_\infty \vee \mathcal{M}_t]\,|\,\mathcal{G}_t] = \overline{\mathbb{E}}^0[\eta_t^{\mathcal{M}}(Y)|\mathcal{G}_t] = (\eta_t^{\mathcal{M}}(Y))_t^{\mathcal{F}}$$

In the general case, we characterize the dependence between Ξ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by a change of probability w.r.t. the probability measure $\overline{\mathbb{P}}^0$.

We suppose that there exist an $\mathbb{F}$-stopping time T and a strictly positive $\mathcal{F}_T \otimes \mathcal{B}(E)$-measurable random variable $\beta_{T(\omega)}(\omega, \chi)$ with expectation under $\overline{\mathbb{P}}^0$ equal to 1 and we define the probability measure $\overline{\mathbb{P}}$ on the product space by

$$\overline{\mathbb{P}}(d\omega, d\chi) = \beta_T(\omega, \chi)\overline{\mathbb{P}}^0(d\omega, d\chi)$$

In the following, we suppose the process $\beta^{\mathcal{F}} > 0$ where $\beta_t^{\mathcal{F}}(\chi) = \overline{\mathbb{E}}^0(\beta_T(\cdot, \chi)|\mathcal{H}_t)$.

We can generate different types of density processes depending on the structure information:

$$\begin{aligned}
\beta_t^{\mathcal{H}} &= \overline{\mathbb{E}}^0[\beta_T|\mathcal{H}_t] = \beta_t^{\mathcal{F}}(\Xi) \\
\beta_t^{\mathcal{M}} &= \overline{\mathbb{E}}^0[\beta_T|\mathcal{F}_\infty \vee \mathcal{M}_t] = \eta_t^{\mathcal{M}}(\beta_T) \\
\beta_t^{\mathcal{G}} &= \overline{\mathbb{E}}^0[\beta_T|\mathcal{G}_t] = (\beta_t^{\mathcal{M}})_t^{\mathcal{F}} = \eta_t^{\mathcal{M}}(\beta_t^{\mathcal{F}})
\end{aligned}$$

Then

$$\begin{aligned}
\overline{\mathbb{E}}(f(\Xi)|\mathcal{M}_t \vee \mathcal{F}_\infty) &= \int_E f(\chi)\overline{\eta}_t^{\mathcal{M}}(d\chi) = \int_E f(\chi)\frac{\beta_T(\chi)\eta_t^{\mathcal{M}}(d\chi)}{\beta_t^{\mathcal{M}}} \\
\overline{\mathbb{E}}(f(\Xi)|\mathcal{G}_t) &= \int_E f(\chi)\overline{\eta}_t^{\mathcal{G}}(d\chi) = \int_E f(\chi)\frac{\beta_t^{\mathcal{F}}(\chi)\eta_t^{\mathcal{M}}(d\chi)}{\beta_t^{\mathcal{G}}}
\end{aligned}$$

and, for any integrable $\mathcal{G}_T$ measurable random variable Y_T

$$\overline{\mathbb{E}}[Y_T \mid \mathcal{G}_t] = \frac{\overline{\mathbb{E}}^0[Y_T\beta_T|\mathcal{G}_t]}{\overline{\mathbb{E}}^0[\beta_T|\mathcal{G}_t]} = \frac{\eta_t^{\mathcal{M}}((Y_T\beta_T)_t^{\mathcal{F}})}{\eta_t^{\mathcal{M}}(\beta_t^{\mathcal{F}})}$$

$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\Xi)$

The density of τ under $\mathbb{Q}$ is

$$\alpha_t^{\mathbb{Q}}(\boldsymbol{u}) = \frac{\beta_t(\boldsymbol{u})}{\int_{\mathbb{R}_+^n} \beta_t(\boldsymbol{v}) d\boldsymbol{v}}$$

The computation in a closed form is not difficult, even if tedious. Let us present the case where $\Xi = \tau$ (unidimensional case), with law with density η, and $\mathbb{F}$ is a Brownian filtration. Furthermore, assume that β so that

$$\beta_t(\chi) = \exp\left(\int_0^t \Psi_s(\chi) dB_s - \frac{1}{2}\int_0^t (\Psi_s(\chi))^2 ds\right)$$

Then, $\mathbb{E}(\beta_T(\Xi)|\mathcal{G}_t) = L_t$ where

$$dL_t = L_{t-}(\psi_t dB_t + \gamma_t dM_t)$$

$$\psi_t = 1\!\!1_{\{t \le \tau\}} \frac{\int_t^\infty \Psi_t(u)\beta_t(u)\eta(u)du}{\int_t^\infty \beta_t(u)\eta(u)du} + 1\!\!1_{\{\tau < t\}} \Psi_t(\tau)$$

and $\gamma_t = \frac{\beta_t(t)G(t)}{\int_t^\infty \beta_t(u)\eta(u)du} - 1$.

Let us mention that one can obtain a characterization of martingales in the large filtration, in terms of martingales in the reference filtration. Let us reduce our attention, for simplicity, to the case where $\Xi = \tau$. Then, a process $Y_t = y_t \mathbb{1}_{t<\tau} + \mathbb{1}_{\tau \le t} y_t(\tau)$ is a $\mathbb{G}$ martingale if and only if

1. For any u, the process $y_t(u)\alpha_t(u), t \ge u$ is an $\mathbb{F}$-martingale
2. The process $E(Y_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale

In the multidimensional case, for a ranked sequence, the process

$$Y_t = \sum_{k=0}^{n-1} y_t^k(\boldsymbol{\sigma}_{(k)}) \mathbb{1}_{\{\sigma_k \le t < \sigma_{k+1})\}}$$

is a $\mathbb{G}$ martingale if and only if

$$y_t^k(\boldsymbol{\theta}_{(k)}) \mathbb{P}(\sigma_{k+1} > t | \mathcal{G}_t^{(k)} + \int \theta_k^t d\boldsymbol{\theta}_{(k+1:n)} \mathbb{1}_{\{\theta_{k+1} < t\}} y_{\theta_{k+1}}^{k+1} \alpha_{\theta_{k+1}}(\boldsymbol{\theta})$$

are martingales

Examples

Gaussian model

Let $f_i, i = 1, \ldots, n$ be a family of functions with L^2 norm equal to 1 and $X_i = \int_0^\infty f_i(s) dB_s^i$ where B^i are $\mathbb{F}$-BMs with correlation $\rho^{i,j}$.

Then

$$\mathbb{P}(X_i > \theta_i, \forall i = 1, \ldots, n | \mathcal{F}_t)$$
$$= \Phi_n^*\left(\frac{\theta_1}{\sqrt{1-\rho_1^2}} - m_t^1, \ldots, \frac{\theta_n}{\sqrt{1-\rho_n^2}} - m_t^n; \gamma(t)\right)$$

where

- $m_t^i = \int_0^t f_i(s) dB_s^i$
- $\Phi_n^*(x_1, \ldots, x_n; \gamma(t)) = \mathbb{P}(G_i^{(t)} > x_i, \forall i = 1, \ldots, n)$

where $G^{(t)} = (G_i^{(t)}, i = 1, \ldots, n)$ is a Gaussian vector, centered, with covariance matrix $\gamma(t)$ with

$$\gamma_{i,j}(t) = \int_t^\infty f_i(s) f_j(s) \rho^{i,j} ds \, .$$

Let H_i be an increasing function from $\mathbb{R}$ to $\mathbb{R}^+$ with inverse h_i and $\tau_i = H_i(X_i)$.

Then

$$\mathbb{P}(\tau_i > t_i, \forall i = 1, \ldots, n | \mathcal{F}_t)$$
$$= \Phi_n^*(\frac{h_1(t_1)}{\sqrt{1-\rho_1^2}} - m_t^1, \ldots, \frac{h_n(t_n)}{\sqrt{1-\rho_n^2}} - m_t^n; \gamma(t), t)$$

In particular,

$$\mathbb{P}(\tau_i > t_i) = \Phi^*(\frac{h_i(t_i)}{\sqrt{1-\rho_i^2}})$$

where $\Phi^*(x)$ is the survival function of a standard Gaussian law.

Uniform law (From Kchia and Larson)

On start with r.v. U_i, with exponential law, independent from $\mathbb{F}$ and R a r.v. with given conditional density $p_t(r)$. Set $\tau_i = RU_i$. Then

$$\mathbb{P}(\tau_i > t_i, i = 1, \ldots, n|\mathcal{F}_t) = \int p_t(r) \prod_{i=1}^{n} (1 - \frac{t_i}{r})^+$$

Some references

El Karoui, N., J.M., Jiao, Y. : Modeling defaults events, preprint

Fermanian, J.-D. and Vigneron, O.: Pricing and hedging basket credit derivatives in the Gaussian copula. *Risk Magazine*, February 2010.

Kchia, Y., Larsson, M. : Credit contagion and risk management with multiple non-ordered defaults, preprint

Crépey, S., J.M., Wu, D .: Informationally Dynamized Gaussian Copula, preprint

Thank you for your attention