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Stable Algorithm for Calculating with q–Splines*

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Abstract. We are using a technique to calculate with Chebyshevian splines of order ≤ 4, based on the known derivative formula for Chebyshevian splines and an Oslo type algorithm, to produce simple formulæ for qB-splines developed by Kulkarni and Laurent. Starting with the known fact that local basis for q-splines of order 3 and 4 can be evaluated by making positive linear combinations of less smooth, one order higher polynomial B-splines, we deduce a simple and stable algorithm for such splines.

It is an interesting fact in itself, that the coefficients in such linear combinations are discrete Chebyshevian splines, and therefore make a partition of unity. The same is true for qB-splines themselves.

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1. Introduction

The notion of q-spline has an origin in the beam theory. Consider a simply supported beam with supports $\{(x_i, f_i)\}_{i=0}^{k+1}$; then the deflection of the beam between successive supports is the solution $s(x)$ to the differential equation $[E \cdot I \cdot D^2]s = M$. Here E denotes Young's modulus of elasticity, I is the cross-sectional moment of inertia, and M is the bending moment. We suppose that $E \cdot I = 1/q$, $q > 0$, where q and, under assumption of weightlessness, M, are piecewise linear continuous functions with break points at the supports. Differentiating the above equation twice, we arrive at the two-point boundary value problem on $[x_i, x_{i+1}]$, for $i = 0, \ldots, k$:

$$D^2 \frac{1}{q} D^2 s = 0, \quad s(x_i) = f_i, \quad s(x_{i+1}) = f_{i+1}, \quad s''(x_i) = s''_i, \quad s(x_{i+1}) = s''_{i+1},$$

where s''_i and s''_{i+1} are chosen so as to ensure that $s \in C^2[x_0, x_{k+1}]$. Such a function s is called a *q-spline*.

The aim is to construct a stable algorithm for calculating with *q-splines*, based on the known derivative formula for Chebyshevian splines and an Oslo type algorithm. To

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this end, we will use one special *Canonical Complete Chebyshev* (CCT)–system, and some general techniques from the *Chebyshevian Spline Theory*. Instead of calculating directly with q-splines, we propose to write such splines as linear combinations of locally supported ones, which can be expressed as linear combinations of ordinary polynomial B-splines.

Since their introduction by Kulkarni and Laurent [2], q-splines have been used in various applications in computer aided geometric design.

2. Chebyshev theory preliminaries

Let $t_1 \le t_2 \le t_3 \le a = t_4 < t_5 < \cdots < t_{k+5} = b \le t_{k+6} \le t_{k+7} \le t_{k+8}$ be an extended partition of the interval $[a,b]$, and let q be a continuous, piecewise linear function defined by

$$q(x)\big|_{[t_i,t_{i+1}]} = \frac{q_{i+1}-q_i}{h_i}(x-t_i)+q_i,$$

where $h_i = t_{i+1}-t_i$, and $q_i > 0$. Consider the CCT–system $\{u_1,u_2,u_3,u_4\}$:

$$u_1(x) = 1, \qquad u_2(x) = \int_a^x ds_2,$$

$$u_3(x) = \int_a^x ds_2 \int_a^{s_2} q(s_3)\,ds_3, \qquad u_4(x) = \int_a^x ds_2 \int_a^{s_2} q(s_3)\,ds_3 \int_a^{s_3} ds_4.$$

We wish to construct a local basis for the spline space spanned piecewisely by these functions, that is, B-splines in $\mathcal{S}(4,\mathbf{m},d\boldsymbol{\sigma},\Delta)$, where $\mathbf{m}$ is the *multiplicity vector*, $\mathbf{m} = (1,\ldots,1)^T$, $d\boldsymbol{\sigma} := (ds_2, q(s_3)\,ds_3, ds_4)^T$ is the *measure vector*, and $\Delta = \{t_i\}_{i=1}^{k+8}$ (see [5] for details of the notation). An important role is played by the associated generalized derivatives:

$$L_{1,d\boldsymbol{\sigma}} = D, \quad L_{2,d\boldsymbol{\sigma}} = \frac{1}{q}D^2, \quad L_{3,d\boldsymbol{\sigma}} = D\,\frac{1}{q}D^2, \quad L_{4,d\boldsymbol{\sigma}} = D^2\frac{1}{q}D^2.$$

To begin with, we focus on the *reduced system* $\{u_{1,1},u_{1,2},u_{1,3}\}$, spanning the space $\mathcal{S}(3,\mathbf{m},d\boldsymbol{\sigma}^{(1)},\Delta)$, $d\boldsymbol{\sigma}^{(1)} = (q(s_3)\,ds_3, ds_4)^T$, on each interval. The CCT–system is:

$$u_{1,1}(x) = 1, \quad u_{1,2}(x) = \int_a^x q(s_3)\,ds_3, \quad u_{1,3}(x) = \int_a^x q(s_3)\,ds_3 \int_a^{s_3} ds_4. \tag{1}$$

Next consider less smooth B-splines $\widetilde{T}_j^3$ from the space $\mathcal{S}(3,\widetilde{\mathbf{m}},d\boldsymbol{\sigma}^{(1)},\Delta)$, with the multiplicity vector $\widetilde{\mathbf{m}} = (2,\ldots,2)^T$ on the same knot sequence. For the fixed index i, we denote the points in the new extended partition as $t_i = \tilde{t}_{r-1} = \tilde{t}_r < \tilde{t}_{r+1}$, and polynomial B-splines on this partition simply as $\widetilde{B}_j^n$. It is easily seen from the definition of the basis (1) that we can write $\widetilde{T}_j^3$ as

$$\widetilde{T}_{r-1}^3(x) = \sum_{j=s-3}^{s} a_{r-1,j}\widehat{B}_j^4(x), \tag{2}$$

$$\widetilde{T}_r^3(x) = \sum_{j=s-3}^{s+3} a_{r,j} \widehat{B}_j^4(x), \tag{3}$$

with $\widehat{B}_j^4 \in \mathcal{S}(4, \widehat{\mathbf{m}}, d\boldsymbol{\lambda}, \Delta)$, where $\widehat{\mathbf{m}} = (3, \dots, 3)^T$ on the same knot sequence Δ, and $d\boldsymbol{\lambda}$ is the measure vector determined by Lebesgue measures only. Points in this partition will be denoted as $t_i = \hat{t}_{s-2} = \hat{t}_{s-1} = \hat{t}_s < \hat{t}_{s+1}$.

We will use the following general theorem, which is a generalization to Chebyshevian splines of the derivative formula for polynomial B-splines [1, 4]:

Theorem 1. *Let $L_{1,d\boldsymbol{\sigma}}$ be the first generalized derivative with respect to CCT–system $\mathcal{S}(n, d\boldsymbol{\sigma})$, and let the multiplicity vector $\mathbf{m} = (m_1, \dots, m_k)^T$ satisfy $m_i < n-1$ for $i = 1, \dots, k$. Then for $x \in [a, b]$ and $i = 1, \dots, n + \sum_{i=1}^k m_i$, the following derivative formula holds:*

$$L_{1,d\boldsymbol{\sigma}} T_{i,d\boldsymbol{\sigma}}^n(x) = \frac{T_{i,d\boldsymbol{\sigma}^{(1)}}^{n-1}(x)}{C_{n-1}(i)} - \frac{T_{i+1,d\boldsymbol{\sigma}^{(1)}}^{n-1}(x)}{C_{n-1}(i+1)}, \tag{4}$$

where

$$C_{n-1}(i) := \int_{t_i}^{t_{i+n-1}} T_{i,d\boldsymbol{\sigma}^{(1)}}^{n-1} \, d\sigma_2, \tag{5}$$

with measure vectors

$$d\boldsymbol{\sigma} = (d\sigma_2(\delta), \dots, d\sigma_n(\delta))^T \in \mathbb{R}^{n-1}, \quad d\boldsymbol{\sigma}^{(1)} := (d\sigma_3(\delta), \dots, d\sigma_n(\delta))^T \in \mathbb{R}^{n-2},$$

for all measurable δ.

3. Construction of the local basis for q-spline spaces

It is obvious from (1) and Theorem 1 that

$$\bar{L}_1 \widetilde{T}_{r-1}^3(x) = \frac{\widetilde{B}_{r-1}^2(x)}{\widetilde{C}_2(r-1)} - \frac{\widetilde{B}_r^2(x)}{\widetilde{C}_2(r)},$$

where $\bar{L}_1 := L_{1,d\boldsymbol{\sigma}^{(1)}} = \frac{1}{q} D$ is the generalized derivative for the reduced CCT–system, and

$$\widetilde{C}_2(j) = \int_{\tilde{t}_j}^{\tilde{t}_{j+2}} \widetilde{B}_j^2(t) \, q(t) \, dt.$$

In particular,

$$\widetilde{C}_2(r-1) = \frac{(2q_i + q_{i+1})h_i}{6}, \quad \widetilde{C}_2(r) = \frac{(q_i + 2q_{i+1})h_i}{6}.$$

From the simple properties of B-splines:

$$\widetilde{T}_{r-1}^3(t_i) = 0, \qquad \bar{L}_1 \widetilde{T}_{r-1}^3(t_i^+) = \frac{1}{\widetilde{C}_2(r-1)},$$

$$\widetilde{T}_{r-1}^3(t_{i+1}) = 0, \qquad \bar{L}_1 \widetilde{T}_{r-1}^3(t_{i+1}^-) = -\frac{1}{\widetilde{C}_2(r)},$$

we get the coefficients in (2):

$$a_{r-1,s-3} = a_{r-1,s} = 0,$$

$$a_{r-1,s-2} = \frac{2q_i}{2q_i + q_{i+1}}, \quad a_{r-1,s-1} = \frac{2q_{i+1}}{q_i + 2q_{i+1}},$$

whence

$$\widetilde{T}^3_{r-1}(x) = \frac{2q_i}{2q_i + q_{i+1}} \widehat{B}^4_{s-2}(x) + \frac{2q_{i+1}}{q_i + 2q_{i+1}} \widehat{B}^4_{s-1}(x). \tag{6}$$

To calculate $\widetilde{T}^3_r$, we use the equations

$$\widetilde{T}^3_r(t_i) = \bar{L}_1 \widetilde{T}^3_r(t_i) = \widetilde{T}^3_r(t_{i+2}) = \bar{L}_1 \widetilde{T}^3_r(t_{i+2}) = 0,$$

$$\widetilde{T}^3_r(t_{i+1}) = 1, \quad \bar{L}_1 \widetilde{T}^3_r(t^-_{i+1}) = \frac{1}{\widetilde{C}_2(r)}, \quad \bar{L}_1 \widetilde{T}^3_r(t^+_{i+1}) = -\frac{1}{\widetilde{C}_2(r+1)},$$

to get the coefficients in (3):

$$a_{r,s-3} = a_{r,s-2} = a_{r,s+2} = a_{r,s+3} = 0,$$

$$a_{r,s} = 1, \quad a_{r,s-1} = \frac{q_i}{2q_{i+1} + q_i}, \quad a_{r,s+1} = \frac{q_{i+2}}{2q_{i+1} + q_{i+2}},$$

and, finally

$$\widetilde{T}^3_r(x) = \frac{q_i}{q_i + 2q_{i+1}} \widehat{B}^4_{s-1}(x) + \widehat{B}^4_s(x) + \frac{q_{i+2}}{2q_{i+1} + q_{i+2}} \widehat{B}^4_{s+1}(x). \tag{7}$$

By integrating (4) in Theorem 1, we can further calculate splines of higher order. We start with the equation

$$\widetilde{T}^4_{r-1}(x) = \frac{1}{\widetilde{C}_3(r-1)} \int_{\tilde{t}_{r-1}}^x \widetilde{T}^3_{r-1}(t)\,dt - \frac{1}{\widetilde{C}_3(r)} \int_{\tilde{t}_r}^x \widetilde{T}^3_r(t)\,dt. \tag{8}$$

It is easy to see from (6) and (7) that

$$\widetilde{C}_3(r-1) = \frac{h_i}{4}\left[\frac{2q_i}{2q_i + q_{i+1}} + \frac{2q_{i+1}}{q_i + 2q_{i+1}}\right],$$

$$\widetilde{C}_3(r) = \frac{1}{4}\left[\frac{q_i h_i}{q_i + 2q_{i+1}} + h_i + h_{i+1} + \frac{q_{i+2} h_{i+1}}{2q_{i+1} + q_{i+2}}\right].$$

From (8), by using (6), (7), and the well known recurrence for integrals of the polynomial B-splines

$$\int_{-\infty}^x B^n_i(t)\,dt = \frac{t_{i+n} - t_i}{n} \sum_{j=i}^{i+n-1} B^{n+1}_j(x),$$

where $\{t_i\}$ is now any extended partition, we obtain (by looking separately at x from each of the subintervals $[t_i, t_{i+1}]$ and $[t_{i+1}, t_{i+2}]$), that

$$\begin{aligned}\widetilde{T}^4_{r-1}(x) = {}& \frac{1}{\widetilde{C}_3(r-1)} \frac{2q_i}{2q_i + q_{i+1}} \frac{h_i}{4} \widehat{B}^5_{s-2}(x) \\ &+ \frac{1}{\widetilde{C}_3(r)} \left(\frac{h_i + h_{i+1}}{4} + \frac{q_{i+2}}{2q_{i+1} + q_{i+2}} \frac{h_{i+1}}{4} \right) \widehat{B}^5_{s-1}(x) \\ &+ \frac{1}{\widetilde{C}_3(r)} \frac{q_{i+2}}{2q_{i+1} + q_{i+2}} \frac{h_{i+1}}{4} \widehat{B}^5_s(x).\end{aligned}$$

In the same way,

$$\begin{aligned}\widetilde{T}^4_r(x) = {}& \frac{1}{\widetilde{C}_3(r)} \frac{q_i}{q_i + 2q_{i+1}} \frac{h_i}{4} \widehat{B}^5_{s-1}(x) + \frac{1}{\widetilde{C}_3(r)} \left(\frac{q_i}{q_i + 2q_{i+1}} \frac{h_i}{4} + \frac{h_i + h_{i+1}}{4} \right) \widehat{B}^5_s(x) \\ &+ \frac{1}{\widetilde{C}_3(r+1)} \frac{2q_{i+2}}{q_{i+1} + 2q_{i+2}} \frac{h_{i+1}}{4} \widehat{B}^5_{s+1}(x).\end{aligned}$$

The following lemma and theorem are connecting general T-splines of orders 3 and 4 with less smooth ones, which are simpler to calculate, and (in the case of q-splines) have already been constructed by the explicit formulæ. Proofs are omitted and may be found in [4].

Lemma 1. *Let $T^3_{i,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3, \mathbf{m}, d\boldsymbol{\sigma}^{(1)}, \Delta)$ be a Chebyshevian B-spline of order 3 associated with the multiplicity vector $\mathbf{m} = (1, \ldots, 1)^T$, and let us assume that $\widetilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3, \widetilde{\mathbf{m}}, d\boldsymbol{\sigma}^{(1)}, \Delta)$ are B-splines associated with multiplicity vector $\widetilde{\mathbf{m}} = (2, \ldots, 2)^T$ on the same knot sequence. If $\{t_1, \ldots, t_{k+6}\}$ and $\{\tilde{t}_1, \ldots, \tilde{t}_{2k+6}\}$ are the associated extended partitions, and r an index such that $t_i = \tilde{t}_r < \tilde{t}_{r+1}$, then for $i = 1, \ldots, k+3$:*

$$T^3_{i,d\boldsymbol{\sigma}^{(1)}} = T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\, \widetilde{T}^3_{r,d\boldsymbol{\sigma}^{(1)}} + \widetilde{T}^3_{r+1,d\boldsymbol{\sigma}^{(1)}} + T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\, \widetilde{T}^3_{r+2,d\boldsymbol{\sigma}^{(1)}}.$$

Theorem 2. *Let $T^4_{i,d\boldsymbol{\sigma}} \in \mathcal{S}(4, \mathbf{m}, d\boldsymbol{\sigma}, \Delta)$, $\widetilde{T}^4_{i,d\boldsymbol{\sigma}} \in \mathcal{S}(4, \widetilde{\mathbf{m}}, d\boldsymbol{\sigma}, \Delta)$, the multiplicity vectors $\mathbf{m}$, $\widetilde{\mathbf{m}}$ being as in Lemma 1. Then positive $\delta^4_i(j)$ exist such that*

$$T^4_{i,d\boldsymbol{\sigma}} = \sum_{j=r}^{r+3} \delta^4_i(j)\, \widetilde{T}^4_{j,d\boldsymbol{\sigma}},$$

where $r = r_i$ satisfies $t_i = \tilde{t}_{r_i} < \tilde{t}_{r_i+1}$. Let the extended partitions be $\{t_1, \ldots, t_{k+8}\}$ and $\{\tilde{t}_1, \ldots, \tilde{t}_{2k+8}\}$. Then $\delta^4_i(j)$, $j = r, \ldots, r+3$, are determined by the formulæ:

$$\delta^4_i(r) = \frac{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\, \widetilde{C}(r)}{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\, \widetilde{C}(r) + \widetilde{C}(r+1) + T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\, \widetilde{C}(r+2)},$$

$$\delta^4_i(r+1) = \frac{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\, \widetilde{C}(r) + \widetilde{C}(r+1)}{T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+1})\, \widetilde{C}(r) + \widetilde{C}(r+1) + T^3_{i,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\, \widetilde{C}(r+2)},$$

$$\delta_i^4(r+2) = \frac{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\,\widetilde{C}(r+4) + \widetilde{C}(r+3)}{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\,\widetilde{C}(r+2) + \widetilde{C}(r+3) + T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\,\widetilde{C}(r+4)},$$

$$\delta_i^4(r+3) = \frac{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\,\widetilde{C}(r+4)}{T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+2})\,\widetilde{C}(r+2) + \widetilde{C}(r+3) + T^3_{i+1,d\boldsymbol{\sigma}^{(1)}}(t_{i+3})\,\widetilde{C}(r+4)},$$

where, as in (5)

$$\widetilde{C}(i) = \int_{\text{supp}} \widetilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}}\, d\sigma_2.$$

To use Lemma 1 and Theorem 2, it remains to calculate $T_i^3(t_{i+1})$ and $T_i^3(t_{i+2})$. The derivative formula in Theorem 1 implies

$$T_i^3(x) = \frac{1}{C_2(i)} \int_{t_i}^x B_i^2(t)\,q(t)\,dt - \frac{1}{C_2(i+1)} \int_{t_{i+1}}^x B_{i+1}^2(t)\,q(t)\,dt,$$

where

$$C_2(i) = \int_{t_i}^{t_{i+2}} B_i^2(t)\,q(t)\,dt = \frac{1}{6}\big[(q_i + 2q_{i+1})h_i + (2q_{i+1} + q_{i+2})h_{i+1}\big].$$

One finds easily that

$$T_i^3(t_{i+1}) = \frac{h_i(q_i + 2q_{i+1})}{6C_2(i)}, \quad T_i^3(t_{i+2}) = \frac{h_{i+2}(2q_{i+2} + q_{i+3})}{6C_2(i+1)},$$

and we have everything that is needed for the evaluation of T_i^4 by means of Theorem 2.

4. Conclusion

We have constructed formulæ for calculating with q-splines as linear combinations of polynomial B-splines. Moreover, all the coefficients involved are positive, and thus we have to calculate scalar products of positive quantities only, guaranteeing numerical stability of such an algorithm.

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