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Homogenization of Nonlinear Elliptic Systems*

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Abstract. We consider nonperiodic homogenization of nonlinear elliptic equations of arbitrary order following the ideas of Murat and Tartar developed in the case of stationary diffusion equation. Although the notion of H-convergence is defined analogously, the main problem arises in the determination of stable classes of functions with respect to H-convergence, and in the proof of the compactness result. Some other properties of H-convergence, such as locality principle and independence of boundary conditions, are also obtained.

The same concept of H-convergence can be successfully applied to the case of linear and nonlinear elliptic systems of higher order. We present some properties of such a convergence.

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1. Linear elliptic systems of higher order

Following the ideas of Murat and Tartar developed in the case of nonperiodic homogenization of the stationary diffusion equation and monotone operators, we consider possible generalizations to higher order nonlinear elliptic equations and systems.

In adapting the theory to other types of equations, two choices appear to be crucial: the appropriate definition of H-convergence (which has to be adjusted to each type of equation) and the determination of a class of coefficients (functions) stable with respect to that notion of H-convergence.

In this section we shall consider linear elliptic systems with r equations and the same number of unknown functions. The order of each equation in the system will be $2m$. Let Ω be an open and bounded subset of $\mathbb{R}^d$, and suppose that the following functions are given:

$$A_{ij}^{\gamma\delta} \in L^\infty(\Omega) \quad \text{and} \quad f^i \in H^{-m}(\Omega),$$

for $i, j \in \{1, \dots, r\}$ and $\gamma, \delta \in \{1, \dots, d\}^m$, the set of all multiindices of order m.

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We can write our problem as follows:

$$\begin{cases} \text{Find } u \in H_0^m(\Omega;\mathbb{R}^r) \text{ such that} \\ (-1)^m \sum_{|\delta|=m} \partial_\delta \Big(\sum_{|\gamma|=m} \sum_{j=1}^{r} A_{ij}^{\gamma\delta} \partial_\gamma u^j \Big) = f_i, \quad i = 1, \dots, r. \end{cases} \tag{1}$$

Due to Schwarz' rule, it is natural to assume the following symmetry for coefficients:

$$A_{ij}^{\gamma\delta} = A_{ij}^{\gamma\sigma(\delta)} = A_{ij}^{\sigma(\gamma)\delta},$$

for any permutation σ.

Let us adjust the notation in order to write this problem in a more compact form. Firstly, all partial derivatives of order m of a vector function $u \in H_0^m(\Omega;\mathbb{R}^r)$ can be written as a tensor function $\nabla^m u$ with values in $\mathbf{T}_S^r$, the space of all symmetric tensors in $\mathcal{L}((\mathbb{R}^d)^m;\mathbb{R}^r)$. The coefficients $A_{ij}^{\gamma\delta}$ define $\mathbf{A} \in L^\infty(\Omega;\mathcal{L}(\mathbf{T}_S^r;\mathbf{T}_S^r))$ by

$$\big(\mathbf{A}(\mathbf{x})\boldsymbol{\Xi}\big)_i^\delta := \sum_{j=1}^{r} \sum_{|\gamma|=m} A_{ij}^{\gamma\delta}\, \Xi_\gamma^j,$$

for $i \in \{1,\dots,r\}$ and $\delta \in \{1,\dots,d\}^m$. With this notation, the equation in (1) can be written as

$$(-1)^m \operatorname{div}^m(\mathbf{A}\nabla^m u) = f.$$

Further assumptions on $\mathbf{A}$ are given by the following inequalities:

$$\mathbf{A}(\mathbf{x})\boldsymbol{\Xi}\cdot\boldsymbol{\Xi} \geq \alpha\,|\boldsymbol{\Xi}|^2, \quad \mathbf{A}(\mathbf{x})\boldsymbol{\Xi}\cdot\boldsymbol{\Xi} \geq \frac{1}{\beta}\,|\mathbf{A}(\mathbf{x})\boldsymbol{\Xi}|^2,$$

for every tensor $\boldsymbol{\Xi} \in \mathbf{T}_S^r$ and almost every $\mathbf{x} \in \Omega$. We shall denote the set of all such tensor functions $\mathbf{A}$ by $\mathcal{M}(\alpha,\beta;\Omega)$.

The variational formulation of our problem reads

$$\begin{cases} \text{Find } u \in H_0^m(\Omega;\mathbb{R}^r) \text{ such that} \\ (\forall v \in H_0^m(\Omega;\mathbb{R}^r)) \quad \int_\Omega \mathbf{A}\nabla^m u \cdot \nabla^m v \, d\mathbf{x} = \int_\Omega f \cdot v \, d\mathbf{x}. \end{cases} \tag{2}$$

It has the unique solution, as a direct consequence of the Lax–Milgram lemma.

Our main goal in this section is to define a topology on the set $\mathcal{M}(\alpha,\beta;\Omega)$ such that the mapping $\mathbf{A} \mapsto u$, determined by (2), is continuous.

Definition 1. *We say that a sequence* $(\mathbf{A}_n)$ *in* $\mathcal{M}(\alpha,\beta;\Omega)$ *H-converges to* $\mathbf{A}_\infty \in \mathcal{M}(\alpha,\beta;\Omega)$ *if for any* $f \in H^{-m}(\Omega;\mathbb{R}^r)$ *the corresponding sequence of solutions* (u_n) *of* (2) *satisfies the following weak convergences*

$$u_n \rightharpoonup u_\infty \quad \text{in } H_0^m(\Omega;\mathbb{R}^r),$$

$$\mathbf{A}_n\nabla^m u_n \rightharpoonup \mathbf{A}_\infty\nabla^m u_\infty \quad \text{in } L^2(\Omega;\mathbf{T}_S^r).$$

Remark 1. *The second convergence in particular implies that u_∞ is a solution of* (2) *with $\mathbf{A}_\infty$ instead of $\mathbf{A}$.*

Remark 2. *Using the integration by parts one can show that*

$$\mathbf{A}_n \nabla^m u_n \cdot \nabla^m u_n \xrightarrow{*} \mathbf{A}_\infty \nabla^m u_\infty \cdot \nabla^m u_\infty$$

in the weak-$*$ topology of Radon measures (vaguely).

The second remark can be easily improved (see [1], their proof can be easily adapted to this case).

Lemma 1. *Let (v_n) and $(\mathbf{D}_n)$ be weakly convergent sequences in $H^m_{\mathrm{loc}}(\Omega;\mathbb{R}^r)$ and $L^2_{\mathrm{loc}}(\Omega;\mathbf{T}^r_S)$, respectively, with limits v_∞ and $\mathbf{D}_\infty$, such that the sequence $(\operatorname{div}^m \mathbf{D}_n)$ belongs to a compact set in $H^{-m}_{\mathrm{loc}}(\Omega;\mathbb{R}^r)$. Then*

$$\mathbf{D}_n \cdot \nabla^m v_n \xrightarrow{*} \mathbf{D}_\infty \cdot \nabla^m v_\infty$$

vaguely.

It can be shown that this convergence comes from a weak topology (we shall call it H-topology) on $\mathcal{M}(\alpha,\beta;\Omega)$, which is metrizable. Some of the properties of this topology are presented in the following theorems.

Theorem 1. *The set $\mathcal{M}(\alpha,\beta;\Omega)$ is compact in the H-topology.*

Proof. For simplicity, we denote

$$V = H^m_0(\Omega;\mathbb{R}^r), \quad H = L^2(\Omega;\mathbb{R}^r), \quad V' = H^{-m}(\Omega;\mathbb{R}^r).$$

Let $(\mathbf{A}_n)$ be a sequence in $\mathcal{M}(\alpha,\beta;\Omega)$, and

$$\mathcal{A}_n u = (-1)^m \operatorname{div}^m(\mathbf{A}_n \nabla^m u)$$

the corresponding operators $\mathcal{A}_n : V \to V'$. By the Lax–Milgram lemma, these operators are invertible with uniformly bounded inverse operators:

$$(\forall f \in V') \quad \|\mathcal{A}_n^{-1} f\|_V \le \frac{1}{\alpha}\|f\|_{V'}.$$

Let $\mathcal{F} = \{f_n \mid n \in \mathbb{N}\}$ be a countable dense subset of V'. By using the above inequality and standard diagonal procedure, we obtain a subsequence $(\mathcal{A}_{n'}^{-1})$ such that

$$(\forall f \in \mathcal{F}) \quad \mathcal{A}_{n'}^{-1} f \rightharpoonup B(f) \quad \text{weakly in } V,$$

where the operator $B : \mathcal{F} \to V$ is linear and bounded. Therefore, it can be uniquely extended to the whole space V'.

We can reapply the above procedure to the sequence $(\mathbf{A}_{n'}\nabla^m u_{n'})$, and obtain a subsequence such that for each $f \in V'$, the corresponding sequence $(u_{n''})$ of solutions of our boundary value problems satisfies weak convergences

$$\begin{aligned} u_{n''} &\rightharpoonup u_\infty = B(f) \quad \text{in } V, \\ \mathbf{A}_{n''}\nabla^m u_{n''} &\rightharpoonup R(f) \quad \text{in } L^2(\Omega; \mathbf{T}^r_S), \end{aligned}$$

where in such a way defined operator $R : V' \to L^2(\Omega; \mathbf{T}^r_S)$ is linear and bounded. We have to show that $R(f) = \mathbf{A}_\infty \nabla^m u_\infty$ for some $\mathbf{A}_\infty \in \mathcal{M}(\alpha, \beta; \Omega)$ or, more precisely, the equality $Cv := RB^{-1}v = \mathbf{A}_\infty \nabla^m v$, for each $v \in V$ (notice that B is invertible, because every $\mathcal{A}_n$ is coercive with the same constant α).

For $v \in V$ we define $f = (-1)^m \operatorname{div}^m(Cv)$, so that the corresponding u_∞ is equal to v. Using the integration by parts we can show that for any $\varphi \in C^m(\overline{\Omega})$

$$\int_\Omega \mathbf{A}_{n''}\nabla^m u_{n''} \cdot \varphi \nabla^m u_{n''} \, d\mathbf{x} \longrightarrow \int_\Omega \varphi \, Cv \cdot \nabla^m v \, d\mathbf{x}.$$

This implies, due to uniform coercivity of coefficients $\mathbf{A}_{n''}$, the following inequality

$$Cv \cdot \nabla^m v \geq \alpha |\nabla^m v|^2 \quad \text{(a.e. on } \Omega).$$

Analogously, we can obtain the other inequality:

$$Cv \cdot \nabla^m v \geq \frac{1}{\beta} |Cv|^2 \quad \text{(a.e. on } \Omega). \tag{3}$$

If we prove that $Cv = \mathbf{A}\nabla^m v$ for some measurable function $\mathbf{A} : \Omega \to \mathcal{L}(\mathbf{T}^r_S; \mathbf{T}^r_S)$, then the previous two inequalities confirm that such $\mathbf{A}$ belongs to $\mathcal{M}(\alpha, \beta; \Omega)$. Inequality (3) and the linearity of mapping C imply the following:

$$\nabla^m v = \nabla^m w \quad \text{(a.e. on } \omega) \implies Cv = Cw \quad \text{(a.e. on } \omega),$$

for any open set ω.

This allows us to define $\mathbf{A}$ in the following way: for any point $\mathbf{x} \in \Omega$ and a tensor $\mathbf{\Xi} \in \mathbf{T}^r_S$, let $U \subseteq \Omega$ be an open neighbourhood of $\mathbf{x}$, and let $v \in V$ be such that $\nabla^m v = \mathbf{\Xi}$ on U. Now set $\mathbf{A}(\mathbf{x})\mathbf{\Xi} = Cv(\mathbf{x})$. It is easy to prove that for any $v \in V$ we have $Cv = \mathbf{A}\nabla^m v$ almost everywhere on Ω. ∎

The following theorem gives more precise bounds for the H-limit. The proof is based on Lemma 1.

Theorem 2. *Let $(\mathbf{A}_n)$ be a sequence of symmetric tensor functions in $\mathcal{M}(\alpha, \beta; \Omega)$ H-converging to $\mathbf{A}_\infty$. Then $\mathbf{A}_\infty$ is symmetric, as well. Moreover, if*

$$\begin{aligned} \mathbf{A}_n &\overset{*}{\rightharpoonup} \mathbf{A}_+ \\ (\mathbf{A}_n)^{-1} &\overset{*}{\rightharpoonup} (\mathbf{A}_-)^{-1} \end{aligned} \qquad \textit{weak} * \textit{ in } L^\infty(\Omega; \mathcal{L}(\mathbf{T}^r_S; \mathbf{T}^r_S)),$$

then $\mathbf{A}_- \leq \mathbf{A}_\infty \leq \mathbf{A}_+$ almost everywhere on Ω.

Theorem 3. *Let $\mathbf{A}_n$ and $\mathbf{B}_n$ be H-convergent sequences in $\mathcal{M}(\alpha, \beta; \Omega)$, with limits $\mathbf{A}_\infty$ and $\mathbf{B}_\infty$, respectively, such that*

$$\mathbf{A}_n = \mathbf{A}_n^\tau \quad \textit{and} \quad \mathbf{A}_n \leq \mathbf{B}_n, \quad n \in \mathbb{N} \quad (\textit{a.e. on } \Omega).$$

Then the inequality $\mathbf{A}_\infty \leq \mathbf{B}_\infty$ holds almost everywhere on Ω.

Proof. Let $g \in H^{-m}(\Omega; \mathbb{R}^r)$. For $n \in \mathbb{N}$ consider the boundary value problem

$$\begin{cases} (-1)^m \operatorname{div}^m(\mathbf{B}_n \nabla^m v_n) = g & \text{in } \Omega \\ v_n \in H_0^m(\Omega; \mathbb{R}^r). \end{cases}$$

According to the assumptions, for any nonnegative function $\varphi \in C_c(\Omega)$ we have

$$\int_\Omega \varphi\, \mathbf{B}_n \nabla^m v_n \cdot \nabla^m v_n \, d\mathbf{x} \geq \int_\Omega \varphi\, \mathbf{A}_n \nabla^m v_n \cdot \nabla^m v_n \, d\mathbf{x}. \tag{4}$$

We can pass to the limit at the left-hand side of this inequality by Remark 2. For the right-hand side, we define $f = (-1)^m \operatorname{div}^m(\mathbf{A}_\infty \nabla^m v_\infty)$, which then determines the sequence (u_n) of solutions of corresponding boundary value problems with coefficients $\mathbf{A}_n$. From the inequality

$$\int_\Omega \varphi\, \mathbf{A}_n(\nabla^m v_n - \nabla^m u_n) \cdot (\nabla^m v_n - \nabla^m u_n) \, d\mathbf{x} \geq 0, \tag{5}$$

we conclude that

$$\liminf_n \int_\Omega \varphi\, \mathbf{A}_n \nabla^m v_n \cdot \nabla^m v_n \, d\mathbf{x} \geq \int_\Omega \varphi\, \mathbf{A}_\infty \nabla^m v_\infty \cdot \nabla^m v_\infty \, d\mathbf{x},$$

(since all the other terms in (5) have limits by Lemma 1), so we can pass to the limit inferior on the right-hand side of (4):

$$\int_\Omega \varphi\, \mathbf{B}_\infty \nabla^m v_\infty \cdot \nabla^m v_\infty \, d\mathbf{x} \geq \int_\Omega \varphi\, \mathbf{A}_\infty \nabla^m v_\infty \cdot \nabla^m v_\infty \, d\mathbf{x}.$$

Since φ is arbitrary, we obtain

$$\mathbf{B}_\infty \nabla^m v_\infty \cdot \nabla^m v_\infty \geq \mathbf{A}_\infty \nabla^m v_\infty \cdot \nabla^m v_\infty \quad (\text{a.e. on } \Omega).$$

Due to the arbitrariness of function g, we can obtain any tensor in $\mathbf{T}_S^r$ in place of $\nabla^m v_\infty$, so $\mathbf{A}_\infty \leq \mathbf{B}_\infty$ holds almost everywhere on Ω. ■

2. Nonlinear scalar elliptic equations of higher order

In this section we turn our attention to the notion of H-convergence for nonlinear scalar elliptic equations. We study the case where the nonlinearity is of a particular Leray–Lions type. We begin by introducing a natural class of (nonlinear) coefficients for which we define H-convergence. Throughout this section $\mathbf{T}_S$ will refer to the space of tensors $\mathbf{T}_S^1$ defined as above.

Definition 2. *A Carathéodory function (measurable in the first, continuous in the second variable)* $\mathbf{A} : \Omega \times \mathbf{T}_S \to \mathbf{T}_S$ *belongs to a class* $\mathbf{Mon}(\alpha, \beta; \Omega)$ *if the following estimates hold for every* $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2 \in \mathbf{T}_S$ *and almost every* $\mathbf{x} \in \Omega$*:*

$$(\mathbf{A}(\mathbf{x}, \boldsymbol{\Xi}_1) - \mathbf{A}(\mathbf{x}, \boldsymbol{\Xi}_2)) \cdot (\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2) \geq \alpha \, |\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2|^2,$$
$$(\mathbf{A}(\mathbf{x}, \boldsymbol{\Xi}_1) - \mathbf{A}(\mathbf{x}, \boldsymbol{\Xi}_2)) \cdot (\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2) \geq \frac{1}{\beta} \, |\mathbf{A}(\mathbf{x}, \boldsymbol{\Xi}_1) - \mathbf{A}(\mathbf{x}, \boldsymbol{\Xi}_2)|^2.$$

Definition 3. *A sequence* $(\mathbf{A}_n)$ *in the class* $\mathbf{Mon}(\alpha, \beta; \Omega)$ *H-converges to* $\mathbf{A}_\infty \in \mathbf{Mon}(\alpha, \beta; \Omega)$ *if for any* $f \in H^{-m}(\Omega)$ *the sequence of solutions* $u_n \in H^m_0(\Omega)$ *to the equations*

$$(-1)^m \operatorname{div}^m(\mathbf{A}_n(\,\cdot\,, \nabla^m u_n)) = f$$

satisfies the following weak convergences

$$u_n \rightharpoonup u_\infty \quad \text{in } H^m_0(\Omega),$$
$$\mathbf{A}_n(\,\cdot\,, \nabla^m u_n) \rightharpoonup \mathbf{A}_\infty(\,\cdot\,, \nabla^m u_\infty) \quad \text{in } L^2(\Omega; \mathbf{T}_S),$$

where $u_\infty \in H^m_0(\Omega)$ *is the unique solution of*

$$(-1)^m \operatorname{div}^m(\mathbf{A}_\infty(\,\cdot\,, \nabla^m u_\infty)) = f.$$

Theorem 4. *Consider a sequence* $(\mathbf{A}_n)$ *in* $\mathbf{Mon}(\alpha, \beta; \Omega)$ *with the property*

$$(\exists C > 0)(\forall n \in \mathbb{N}) \quad \|\mathbf{A}_n(\,\cdot\,, 0)\|_{L^2(\Omega)} \leq C.$$

Then there exists a subsequence $(\mathbf{A}_{n_k})$ *and an* $\mathbf{A}_\infty \in \mathbf{Mon}(\alpha, \beta; \Omega)$ *such that* $(\mathbf{A}_{n_k})$ *H-converges to* $\mathbf{A}_\infty$*.*

Proof. First, we note that the operators $\mathcal{A}_n : H^m_0(\Omega) \to H^{-m}(\Omega)$ defined by

$$\mathcal{A}_n(u) := (-1)^m \operatorname{div}^m(\mathbf{A}_n(\,\cdot\,, \nabla^m u))$$

satisfy the conditions of Browder–Minty's theorem (see [4]). Since the properties of coefficients $\mathbf{A}_n$ imply the inequalities

$${}_{H^{-m}(\Omega)}\langle \mathcal{A}_n(u) - \mathcal{A}_n(v), u - v \rangle_{H^m_0(\Omega)} \geq \alpha \, \|u - v\|^2_{H^m_0(\Omega)},$$
$$\|\mathcal{A}_n(u) - \mathcal{A}_n(v)\|_{H^{-m}(\Omega)} \leq \beta \, \|u - v\|_{H^m_0(\Omega)},$$

we are able to deduce the estimates of the same type for operators $(\mathcal{A}_n)^{-1}$, as well. Furthermore, by the assumption we have

$$(\forall n \in \mathbb{N}) \quad \|\mathcal{A}_n(0)\|_{H^{-m}(\Omega)} \leq C,$$

and thus

$$\|(\mathcal{A}_n)^{-1}(0)\|_{H^m_0(\Omega)} \leq \frac{1}{\alpha} \, \|\mathcal{A}_n(0)\|_{H^{-m}(\Omega)} \leq C,$$

which means

$$\|(\mathcal{A}_n)^{-1}(f)\|_{H_0^m(\Omega)} \le \frac{1}{\alpha}\,\|f\|_{H^{-m}(\Omega)} + C.$$

Now we choose $\mathcal{F} = \{f_1, f_2, f_3, \ldots\}$ to be an arbitrary countable dense subset in $H^{-m}(\Omega)$. By using the same line of reasoning as in the linear case, we extract a subsequence (u_{n_k}) satisfying the following weak convergences

$$(\forall s \in \mathbb{N}) \quad (\mathcal{A}_{n_k})^{-1}(f_s) \rightharpoonup B_\infty(f_s) \quad \text{in } H_0^m(\Omega),$$

therefore defining a mapping $B_\infty : \mathcal{F} \to H_0^m(\Omega)$, and then using a similar construction for $R : \mathcal{F} \to L^2(\Omega; \mathbf{T}_S)$ such that (weakly)

$$\mathbf{A}_{n_k}(\,\cdot\,, \nabla^m u_{n_k}) \rightharpoonup R(f) \quad \text{in } L^2(\Omega; \mathbf{T}_S).$$

By the standard extension theorem we extend B_∞ and R to the Lipschitz mappings on $H^{-m}(\Omega)$ (which we still denote by B_∞ and R, respectively).

It is easy to verify that B_∞ satisfies the conditions of the Browder–Minty theorem, and that it has an inverse which is Lipschitz continuous, as well. In the next step we define $C := R \circ (B_\infty)^{-1}$, having in that way the weak convergence

$$\mathbf{A}_{n_k}(\,\cdot\,, \nabla^m u_{n_k}) \rightharpoonup C(u_\infty) \quad \text{in } L^2(\Omega).$$

Finally, we show that C is a local operator of the form $C(u_\infty) = \mathbf{A}_\infty(\,\cdot\,, \nabla^m u_\infty)$, for some $\mathbf{A}_\infty \in \mathbf{Mon}(\alpha, \beta; \Omega)$.

To this end, we choose an arbitrary open set $\omega \subset\subset \Omega$. It is easy to verify that for any tensor $\mathbf{\Xi} \in \mathbf{T}_S$ we can find $f \in H^{-m}(\Omega)$ such that

$$\nabla^m v_\infty(\mathbf{x}) = \mathbf{\Xi} \quad (\text{a.e. } \mathbf{x} \in \omega),$$

where $v_\infty \in H_0^m(\Omega)$ is the (unique) solution of the equation

$$(-1)^m \operatorname{div}^m(C(v_\infty)) = f.$$

For such $\omega \subset\subset \Omega$, $\mathbf{x} \in \omega$, $\mathbf{\Xi} \in \mathbf{T}_S$ and $f \in H^{-m}(\Omega)$ we define

$$\mathbf{A}_\infty\big|_\omega(\mathbf{x}, \mathbf{\Xi}) := R(f)(\mathbf{x}).$$

To check that $\mathbf{A}_\infty : \Omega \times \mathbf{T}_S \to \mathbf{T}_S$ is well-defined mapping, we choose $\omega_i \subset\subset \Omega$, $\mathbf{\Xi}_i \in \mathbf{T}_S$ and $f_i \in H^{-m}(\Omega)$ as above (where $i = 1, 2$), $(\mathbf{x}, \mathbf{\Xi}) \in \omega_i \times \mathbf{T}_S$, and we define $\mathbf{A}_{\infty,i}(\mathbf{x}, \mathbf{\Xi}_i) := \mathbf{A}_\infty(\mathbf{x}, \mathbf{\Xi}_i)$. Now it is enough to show that it holds

$$(\forall \mathbf{\Xi} \in \mathbf{T}_S) \quad \mathbf{A}_{\infty,1}\big|_{\omega_1\cap\omega_2}(\,\cdot\,, \mathbf{\Xi}) = \mathbf{A}_{\infty,2}\big|_{\omega_1\cap\omega_2}(\,\cdot\,, \mathbf{\Xi}).$$

For $j = 1, 2$, we choose $v_{n_k,j} \in H_0^m(\Omega)$ such that

$$v_{n_k,j} \rightharpoonup v_{\infty,j} \quad \text{weakly in } H_0^m(\Omega),$$

where $\nabla^m v_j\big|_\omega = \mathbf{\Xi}_j$.

By the construction, with $E_{n_k,j} := \nabla^m v_{n_k,j}$ and $D_{n_k,j} := \mathbf{A}_{n_k}(\,\cdot\,,\nabla^m v_{n_k,j})$, we have the weak convergence

$$E_{n_k,2} - E_{n_k,1} \rightharpoonup \mathbf{\Xi}_2 - \mathbf{\Xi}_1 \quad \text{in } L^2(\omega_2 \cap \omega_1; \mathbf{T}_S),$$

and, similarly,

$$D_{n_k,2} - D_{n_k,1} \rightharpoonup \mathbf{A}_\infty(\,\cdot\,,\mathbf{\Xi}_2) - \mathbf{A}_\infty(\,\cdot\,,\mathbf{\Xi}_1) \quad \text{in } L^2(\omega_2 \cap \omega_1; \mathbf{T}_S).$$

By Lemma 1 we conclude

$$(\forall \psi \in C_c(\omega_1 \cap \omega_2)) \qquad \int \psi\,(D_{n_k,2} - D_{n_k,1}) \cdot (E_{n_k,2} - E_{n_k,1}) \longrightarrow L,$$

where

$$L := \int \psi\,(\mathbf{A}_\infty(\,\cdot\,,\mathbf{\Xi}_2) - \mathbf{A}_\infty(\,\cdot\,,\mathbf{\Xi}_1)) \cdot (\mathbf{\Xi}_2 - \mathbf{\Xi}_1),$$

and, furthermore, obtain the inequalities

$$\lim_{k\to\infty} \int \psi\,(D_{n_k,2} - D_{n_k,1}) \cdot (E_{n_k,2} - E_{n_k,1}) \geq \alpha\,|\mathbf{\Xi}_1 - \mathbf{\Xi}_2|^2 \int_{\omega_1\cap\omega_2} \psi(\mathbf{x})\,d\mathbf{x},$$

$$L \geq \liminf_{k\to\infty} \int_\Omega \psi\,\frac{1}{\beta}\,|D_{n_k,2} - D_{n_k,1}|^2 \geq \int_\Omega \psi\,\frac{1}{\beta}\,(\mathbf{A}_\infty(\,\cdot\,,\mathbf{\Xi}_2) - \mathbf{A}_\infty(\,\cdot\,,\mathbf{\Xi}_1))^2.$$

Taking $\mathbf{\Xi}_1 = \mathbf{\Xi}_2$ we finally see that $\mathbf{A}_\infty$ is well-defined. To prove that $\mathbf{A}_\infty$ belongs to the class $\mathbf{Mon}(\alpha,\beta;\Omega)$, it suffices to consider the above inequalities for varying nonnegative functions ψ. At last, an application of Lemma 1 and the continuity of $\mathbf{A}_\infty$ yield that $C(u_\infty) = \mathbf{A}_\infty(\,\cdot\,,\nabla^m u_\infty)$. The weak convergence

$$\mathbf{A}_{n_k}(\,\cdot\,,\nabla^m u_{n_k}) \rightharpoonup \mathbf{A}_\infty(\,\cdot\,,\nabla^m u_\infty) \quad \text{in } L^2(\Omega; \mathbf{T}_S)$$

assures that u_∞ satisfies the equation

$$(-1)^m \operatorname{div}^m \mathbf{A}_\infty(\,\cdot\,,\nabla^m u_\infty) = f.$$

∎

Theorem 5 (Locality of H-limits). *Let $\omega \subseteq \Omega$ be an open set, and $(\mathbf{A}_n)$ and $(\mathbf{B}_n)$ sequences in $\mathbf{Mon}(\alpha,\beta;\Omega)$ such that $\mathbf{A}_n\big|_\omega = \mathbf{B}_n\big|_\omega$, for $n \in \mathbb{N}$, and H-converging to $\mathbf{A}_\infty$ and $\mathbf{B}_\infty$, respectively. Then*

$$\mathbf{A}_\infty\big|_\omega = \mathbf{B}_\infty\big|_\omega.$$

Proof. Without loss of generality we may assume $\omega \subset\subset \Omega$. Therefore we can find an open set ω' such that $\omega \subset\subset \omega' \subset\subset \Omega$. Let us fix $\mathbf{\Xi}_1$, $\mathbf{\Xi}_2 \in \mathbf{T}_S$ and choose functions $\overline{u}$, $\overline{v} \in C_c^\infty(\Omega)$ in such a way that

$$\nabla^m \overline{u}\big|_{\omega'} = \mathbf{\Xi}_1 \quad \text{and} \quad \nabla^m \overline{v}\big|_{\omega'} = \mathbf{\Xi}_2.$$

Next we define $f, g \in H^{-m}(\Omega)$ by

$$f := (-1)^m \operatorname{div}^m(\mathbf{A}_\infty(\,\cdot\,, \nabla^m \overline{u})), \quad g := (-1)^m \operatorname{div}^m(\mathbf{B}_\infty(\,\cdot\,, \nabla^m \overline{v})),$$

and set $u_n, v_n \in H_0^m(\Omega)$ to be solutions of the equations

$$(-1)^m \operatorname{div}^m(\mathbf{A}_n(\,\cdot\,, \nabla^m u_n)) = f \quad \text{and} \quad (-1)^m \operatorname{div}^m(\mathbf{B}_n(\,\cdot\,, \nabla^m v_n)) = g,$$

respectively.

By the assumptions, for any nonnegative $\varphi \in C_c(\omega)$ we obtain

$$\int_\Omega \varphi\,(\mathbf{A}_n(\,\cdot\,, \nabla^m u_n) - \mathbf{B}_n(\,\cdot\,, \nabla^m v_n)) \cdot (\nabla^m u_n - \nabla^m v_n) \geq \alpha \int_\Omega \varphi\,|\nabla^m u_n - \nabla^m v_n|^2,$$

and, quite analogously as in previous cases, also

$$\int_\Omega \varphi\,(\mathbf{A}_\infty(\,\cdot\,, \nabla^m \overline{u}) - \mathbf{B}_\infty(\,\cdot\,, \nabla^m \overline{v})) \cdot (\nabla^m \overline{u} - \nabla^m \overline{v}) \geq \alpha \int_\Omega \varphi\,|\nabla^m \overline{u} - \nabla^m \overline{v}|^2,$$

which means that for any $\mathbf{\Xi}_1, \mathbf{\Xi}_2 \in \mathbf{T}_S$ and $\varphi \in C_c(\omega)$

$$\int_\omega \varphi\,(\mathbf{A}_\infty(\,\cdot\,, \mathbf{\Xi}_1) - \mathbf{B}_\infty(\,\cdot\,, \mathbf{\Xi}_2)) \cdot (\mathbf{\Xi}_1 - \mathbf{\Xi}_2) \geq \alpha \int_\Omega \varphi\,|\mathbf{\Xi}_1 - \mathbf{\Xi}_2|^2.$$

Therefore,

$$(\forall\, \mathbf{\Xi}_1, \mathbf{\Xi}_2 \in \mathbf{T}_S) \quad (\mathbf{A}_\infty(\mathbf{x}, \mathbf{\Xi}_1) - \mathbf{B}_\infty(\mathbf{x}, \mathbf{\Xi}_2)) \cdot (\mathbf{\Xi}_1 - \mathbf{\Xi}_2) \geq \alpha\,|\mathbf{\Xi}_1 - \mathbf{\Xi}_2|^2 \quad (\text{a.e. } \mathbf{x} \in \omega).$$

If we take $\mathbf{\Xi}_1 = \mathbf{\Xi}_2 + t\mathbf{\Xi}$, where $t > 0$ and $\mathbf{\Xi} \in \mathbf{T}_S$, the continuity of $\mathbf{\Xi}_2 \mapsto \mathbf{A}_\infty(\mathbf{x}, \mathbf{\Xi}_2)$ implies that for arbitrary $\mathbf{\Xi}_2 \in \mathbf{T}_S$ and $\mathbf{\Xi} \in \mathbf{T}_S$ we have

$$(\mathbf{A}_\infty(\mathbf{x}, \mathbf{\Xi}_2) - \mathbf{B}_\infty(\mathbf{x}, \mathbf{\Xi}_2)) \cdot \mathbf{\Xi} \geq 0 \quad (\text{a.e. } \mathbf{x} \in \omega),$$

which yields the claim. ■

Theorem 6 (Independence of boundary conditions). *Consider weakly convergent sequences* $(\mathbf{A}_n)$ *in* $\mathbf{Mon}(\alpha, \beta; \Omega)$ *and* (u_n) *in* $H^m(\Omega)$*:*

$$\mathbf{A}_n \rightharpoonup \mathbf{A}_\infty \quad (H\text{-convergence})$$

and

$$u_n \rightharpoonup u_\infty \quad \text{in } H^m_{\mathrm{loc}}(\Omega)$$

such that $\operatorname{div}^m(\mathbf{A}_n(\,\cdot\,, \nabla^m u_n))$ *is pre-compact (strongly) in* $H^{-m}(\Omega)$*. Then we have the weak convergence*

$$\mathbf{A}_n(\,\cdot\,, \nabla^m u_n) \rightharpoonup \mathbf{A}_\infty(\,\cdot\,, \nabla^m u_\infty) \quad \text{in } L^2_{\mathrm{loc}}(\Omega; \mathbf{T}_S).$$

3. Generalization to nonlinear systems

In the last two sections we have defined the notion of H-convergence both for linear systems and for nonlinear scalar equations of higher order. These ideas can be carried over to nonlinear systems, as well. For simplicity, we shall make some notes regarding second order systems.

More precisely, we consider a boundary value problem

$$\begin{cases} -\operatorname{div} \mathbf{A}(\,\cdot\,, \nabla u) = f \\ u \in H_0^1(\Omega; \mathbb{R}^r), \end{cases}$$

where $f \in H^{-1}(\Omega; \mathbb{R}^r)$ and $\mathbf{A} : \Omega \times M_{r\times d} \to M_{r\times d}$ is a Carathéodory function which, for arbitrarily chosen $\mathbf{B}, \mathbf{C} \in M_{r\times d}$, satisfies the following estimates:

$$(\mathbf{A}(\mathbf{x}, \mathbf{B}) - \mathbf{A}(\mathbf{x}, \mathbf{C})) \cdot (\mathbf{B} - \mathbf{C}) \geq \alpha\, |\mathbf{B} - \mathbf{C}|^2,$$
$$(\mathbf{A}(\mathbf{x}, \mathbf{B}) - \mathbf{A}(\mathbf{x}, \mathbf{C})) \cdot (\mathbf{B} - \mathbf{C}) \geq \frac{1}{\beta}\, |\mathbf{A}(\mathbf{x}, \mathbf{B}) - \mathbf{A}(\mathbf{x}, \mathbf{C})|^2,$$

for almost every $\mathbf{x} \in \Omega$ and some $\beta \geq \alpha > 0$.

If the notion of H-convergence is defined as in Definition 3 (with appropriate modifications regarding the vector valued functions), it can be shown, using similar techniques, that previous results are still valid.

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