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Computation of Power–Series Expansions in Homogenization of Nonlinear Equations*

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Abstract. In the theory of homogenization it is of particular interest to determine the classes of problems which are stable on taking the homogenization limit. A notable situation where the limit enlarges the class of original problems is known as memory (nonlocal) effect. A number of results in that direction has been obtained for linear problems.

Tartar initiated the study of effective equation corresponding to nonlinear equation:

$$\partial_t u_n + a_n u_n^2 = f.$$

Significant progress has been hampered by the complexity of required computations needed in order to obtain the terms in power-series expansion. We propose a method which overcomes that difficulty by introducing graphs representing the domain of integration of the integrals in each term. The graphs are relatively simple, it is easy to calculate with them and they give us a clear image of the form of each term. The method allows us to discuss the form of the effective equation and the convergence of power-series expansions.

AMS subject classification: 35B27, 05C90

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1. Introduction

In the theory of non-periodic deterministic homogenization, as developed by F. Murat and L. Tartar, one important goal is to determine classes of differential (or more generally pseudo–differential) operators which are closed on taking the homogenization limit (more precisely, the limit in the sense of H-convergence, see [4]). The first precise mathematical result in that direction was obtained by S. Spagnolo [5] for stationary heat conduction with symmetric conductivity tensor.

In the quest for such classes of operators one phenomenon is ubiquitous: the nonlocal effects — the homogenization limit of a sequence of partial differential operators is an integro–differential operator.

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In abstract terms, for linear differential operators which commute with translations, the limit has the same property. Thus, by the L. Schwartz theorem, it can be expressed as a convolution operator.

Various results for particular linear equations, including the precise description of the kernels of convolution operators, have been obtained by various authors [1, 2, 3, 6, 7].

Of particular interest for us is the only such result for the so-called time-dependent coefficients [8], where one considers a sequence of initial value problems of the form:

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(x,t) + a_n(x,t)\, u_n(x,t) = f(x,t) \\ \hfill u_n(x,0) = v(x), \end{cases} \tag{1}$$

with $t \in \mathbb{R}^+$ and $x \in \Omega$, a set equipped by a non-atomic measure (for simplicity, one can take a measurable set $\Omega \subseteq \mathbb{R}^d$ with the Lebesgue measure).

It is assumed that the sequence (a_n) satisfies:

$$\alpha \le a_n(x,t) \le \beta \quad \text{(a.e.)}$$
$$|a_n(x,t) - a_n(x,s)| \le \varepsilon(|t-s|) \quad \text{(a.e. } x \in \Omega), \quad t, s \in \mathbb{R}^+,$$

where $\varepsilon(\sigma) \to 0$ as $\sigma \to 0$.

The effective (homogenized) equation is then of the form:

$$\frac{\partial u_\infty}{\partial t} + a_\infty(\cdot,t)\, u_\infty = f(\cdot,t) + \int_0^t K(\cdot,t,s)\, u_\infty(\cdot,s)\, ds, \tag{2}$$

with the same initial condition, where a_∞ is L^∞ weak $*$ limit of a_n.

After defining a new sequence of functions $b_n := a_n - a_\infty$, one considers an equation with parameter γ:

$$\begin{cases} \dfrac{\partial U_n(\cdot,\cdot,\gamma)}{\partial t} + (a_\infty + \gamma b_n)\, U_n(\cdot,\cdot,\gamma) = f \\ \hfill U_n(\cdot,0,\gamma) = v, \end{cases}$$

so that (1) is obtained by taking $\gamma = 1$. Tartar based his analysis on analytic dependence on γ. In the first step, a subsequence is extracted such that for each $k \ge 1$ and $s_1, \ldots, s_k \in \mathbb{R}^+$

$$b_n(\cdot,s_1)\cdots b_n(\cdot,s_k) \rightharpoonup M_k(\cdot,s_1,\ldots,s_k) \quad \text{in } L^\infty(\Omega) \text{ weak } *. \tag{3}$$

It is assumed that U_n admits the expansion

$$U_n(x,t;\gamma) = \sum_{k=0}^{\infty} \gamma^k U_{n,k}(x,t) \tag{4}$$

in which $U_{n,0} = U_*$ is independent of n, and is taken as the unique solution of

$$\begin{cases} \dfrac{\partial U_*}{\partial t} + a_\infty U_* = f \\ \qquad U_*(\cdot, 0) = v, \end{cases} \tag{5}$$

while the functions $U_{n,k}$ are defined by induction for $k \geq 1$ as solutions of

$$\begin{cases} \dfrac{\partial U_{n,k}}{\partial t} + a_\infty U_{n,k} + b_n U_{n,k-1} = 0 \\ \qquad U_{n,k}(\cdot, 0) = 0. \end{cases} \tag{6}$$

The functions $U_{n,k}$ could be written explicitly, as well as their weak $*$ limits $U_{\infty,k}$, after passing to an appropriate subsequence. The weak $*$ limit $W_{\infty,k}$ of $b_n U_{n,k}$ is computed in the same way.

Passing to the limit in (6) we get

$$\begin{cases} \dfrac{\partial U_{\infty,k}}{\partial t} + a_\infty U_{\infty,k} + W_{\infty,k-1} = 0 \\ \qquad U_{\infty,k}(\cdot, 0) = 0 \end{cases}$$

for $k \geq 1$, so that, after taking (5) into account, one can deduce that

$$\frac{\partial \left(\sum_{k=0}^{\infty} \gamma^k U_{\infty,k} \right)}{\partial t} + a_\infty \left(\sum_{k=0}^{\infty} \gamma^k U_{\infty,k} \right) + \sum_{k=0}^{\infty} \gamma^{k+1} W_{\infty,k} = f. \tag{7}$$

Next step is to identify the terms in the following power-series expansion:

$$K(x, s, t; \gamma) = \sum_{k=2}^{\infty} \gamma^k K_k(x, s, t).$$

After inserting it in (2), comparing with (7) and equating the terms with the same power of γ, an explicit expression for K_k is obtained. Each K_k is represented by sums and integrals of functions M and a_∞.

As an academic example, Tartar attempted to apply the above procedure to a nonlinear ordinary differential equation depending on a parameter x:

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(x,t) + a_n(x,t)\, u_n^2(x,t) = f(x,t) \\ \qquad u_n(x, 0) = v(x). \end{cases}$$

In order to avoid questions of blow-up, which would force us to work on a finite interval $\langle 0, T \rangle$ adapted to the data, it is reasonable to assume that $\alpha \geq 0$, $0 \leq v \leq M_0$, and $0 \leq f \leq F$ (a.e.) in $\Omega \times \mathbb{R}^+$ for some constants M_0 and F.

Again, one looks for an expansion (4) in which $U_{n,0} = U_*$ is now taken as the unique solution of

$$\begin{cases} \dfrac{\partial U_*}{\partial t} + a_\infty U_*^2 = f \\ \qquad U_*(\cdot, 0) = v, \end{cases}$$

while the functions $U_{n,k}$ are defined by induction for $k \geq 1$ as solutions of

$$\begin{cases} \dfrac{\partial U_{n,k}}{\partial t} + a_\infty V_{n,k} + b_n V_{n,k-1} = 0 \\ \qquad U_{n,k}(\cdot, 0) = 0, \end{cases} \tag{8}$$

where

$$V_{n,k} = \sum_{j=0}^{k} U_{n,j}\, U_{n,k-j}.$$

The weak $*$ limits $U_{\infty,k}$ of $U_{n,k}$, $V_{\infty,k}$ of $V_{n,k}$ and $W_{\infty,k}$ of $b_n V_{n,k}$ (on a subsequence, of course) are then related by

$$\begin{cases} \dfrac{\partial U_{\infty,k}}{\partial t} + a_\infty V_{\infty,k} + W_{\infty,k-1} = 0 \\ \qquad U_{\infty,k}(\cdot, 0) = 0. \end{cases} \tag{9}$$

Tartar outlined two tasks which should be accomplished in order to apply the above method successfully:

- Find an efficient representation for the integrals, which allows easy manipulation.
- Determine an adequate summation procedure for the above power-series.

In this paper we propose an answer to the first question.

2. Graph representation of integrals

The unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial U(t)}{\partial t} + g(t)\, U(t) + f(t) = 0 \\ \qquad U(0) = 0 \end{cases}$$

is given by the formula

$$U(t) = -\int_0^t \exp\left(-\int_s^t g(\sigma)\, d\sigma\right) f(s)\, ds.$$

Accordingly, by using

$$R(s,t) = \exp\left(-\int_s^t 2a_\infty(x,\sigma)\, U_*(x,\sigma)\, d\sigma\right),$$

the solutions of (8) can be expressed by the recursive formula

$$U_{n,k}(t) = -\int_0^t R(s,t)\Bigg(a_\infty(s)\sum_{j=1}^{k-1} U_{n,j}(s)\,U_{n,k-j}(s) + b_n(s)\sum_{j=0}^{k-1} U_{n,j}(s)\,U_{n,k-1-j}(s)\Bigg)\,ds. \tag{10}$$

It follows that the explicit expression for $U_{n,k}$ consists of integrals, sums and products of four *basic* functions R, b_n, a_∞ and U_*. We are interested in $U_{\infty,k}$, the weak $*$ limit of $(U_{n,k})$ in $L^\infty(\Omega)$. When taking the limit, the powers of function b_n change according to relation (3), while the other factors remain the same. Accordingly, we will make no explicit difference between functions b_n and M, writing the first in expressions for $U_{n,k}$ and the second in the corresponding expressions for $U_{\infty,k}$. Therefore, the expression for $U_{\infty,k}$ is of the form

$$U_{\infty,k}(t) = \sum_{j=1}^{N_k} a_j^k\, S_j^k(t), \tag{11}$$

where a_j^k is an integer and S_j^k denotes the multiple integral

$$S_j^k(t) = \int_{s_1}^{t_1}\int_{s_2}^{t_2}\cdots\int_{s_{M_j^k}}^{t_{M_j^k}} f_j^k(s_1,\ldots,s_{M_j^k},t_1,\ldots,t_{M_j^k},z_1,\ldots,z_{M_j^k},t)\, dz_1\cdots dz_{M_j^k}, \tag{12}$$

with f_j^k being the product of basic functions. It can easily be shown that $\max_j M_j^k = 2k-1$.

In the analysis of (10) we encountered the following problem. With increasing k, the number N_k of terms in the expression for $U_{\infty,k}$ grows, as well as the number of both the integrals and factors in each term. Hence the computation of $U_{\infty,k}$ becomes very complicated, almost impossible. Our goal is to find a simpler way of presentation of these expressions.

We start with a few lemmas.

Lemma 1. *Each integral in the expression for $U_{\infty,k}$ has zero as a lower limit of integration; in other words, each s_j in* (12) *is equal to* 0.

Proof. Let us look at the explicit expression for $U_{\infty,1}$:

$$U_{\infty,1} = -\int_0^t R(s,t)\,M(s)\,U_*^2(s)\,ds.$$

In this case the statement is valid. By using the induction on k and (10) it can easily be shown that it is valid in general. ∎

Lemma 2. *In each integrand f_j^k in (12) the function R appears exactly M_j^k times as a factor, with variables as follows*

$$R(z_1, t_1)\, R(z_2, t_2) \cdots R(z_{M_j^k}, t_{M_j^k}).$$

Lemma 3. *In each term S_j^k in the expression for $U_{\infty,k}$ the basic function M appears as a factor in the integrand f_j^k exactly once, as a function of k variables. This means that the factor is of the form*

$$M(\sigma_1, \sigma_2, \dots, \sigma_k), \quad \text{with} \quad \sigma_i \in \{t_1, t_2, \dots, t_{M_j^k}, z_1, z_2, \dots, z_{M_j^k}, t\},$$

for $i = 1, \dots, k$.

Proof. The statement is valid for $k = 1$. Let us assume that it is valid for $j < k$ and prove that it is valid for $j = k$, as well. In the first term of the integrand in (10) we have terms including products $U_{n,j}U_{n,k-j}$. According to the assumption of induction, in each term S_i^j in the expression for $U_{n,j}$ the basic function b_n appears as a factor j times. Similarly, in each term in the expression for $U_{n,k-j}$ the basic function b_n appears as a factor $k - j$ times. Accordingly, in each term of the product $U_{n,j}U_{n,k-j}$ the basic function b_n appears as a factor k times. The same proof can easily be applied to the second part of expression (10). ■

Lemma 4. *The upper limits of integration $t_1, t_2, \dots, t_{M_j^k}$ in expression (12) belong to the set $\{t, z_1, z_2, \dots, z_{M_j^k}\}$, and the domains of integration are described by the following sequence of inequalities*

$$\begin{cases} 0 \le z_{m_1^1} \le z_{m_2^1} \le \cdots \le z_{m_{n_1}^1} \le t \\ 0 \le z_{m_1^2} \le z_{m_2^2} \le \cdots \le z_{m_{n_2}^2} \le u_2 \\ \qquad\qquad \cdots\cdots\cdots \\ 0 \le z_{m_1^l} \le z_{m_2^l} \le \cdots \le z_{m_{n_l}^l} \le u_l. \end{cases} \tag{13}$$

In these inequalities every member of the set $\{z_1, z_2, \dots, z_{M_j^k}\}$ appears exactly once. Furthermore, let us introduce the following sets

$$\begin{aligned} A_1 &= \{z_{m_1^1}, z_{m_2^1}, \dots, z_{m_{n_1}^1}\}, \\ A_2 &= \{z_{m_1^2}, z_{m_2^2}, \dots, z_{m_{n_2}^2}\}, \\ &\vdots \\ A_l &= \{z_{m_1^l}, z_{m_2^l}, \dots, z_{m_{n_l}^l}\}. \end{aligned}$$

Then we have

$$u_2 \in A_1, \quad u_3 \in A_1 \cup A_2, \quad \dots, \quad u_l \in A_1 \cup A_2 \cup \cdots \cup A_{l-1}. \tag{14}$$

Now we are ready to associate graphs with integrals. With graphs we want primarily to show clearly the domain of integration in each term S_j^k. Each graph will contain a finite number of vertices and edges, two of the vertices being emphasized:

- bottom or starting point 0,
- top or final point t.

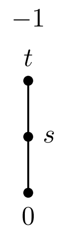
All other vertices will be called *inner vertices*. Each inner vertex denotes a variable of integration. From every inner vertex exactly one edge goes upward. The vertex at the top end of that edge will represent the upper limit of integration concerning the variable below. The graph will contain some kind of orientation. We go from the starting point 0 upward by one or more edges, which continue to go upwards through one or more vertices, and at the end they all come to the final point t. The exact algorithm follows.

We go from the starting point 0 upward drawing an edge having the variable $z_{m_1^1}$ from (13) as the upper vertex. From that vertex we draw the next edge, having the variable $z_{m_2^1}$ as the other end, and we continue such a procedure by passing through the variables from the first sequence in (13), till the final point t is reached. After that, we repeat the procedure for the second sequence. We again start from 0 drawing upward a new edge having the variable $z_{m_1^2}$ as the upper vertex. From that vertex we draw the next edge, having the variable $z_{m_2^2}$ as the other end, and continue such a procedure by passing through the variables from the second sequence, till the variable $z_{m_{n_2}^2}$. We join that variable by an edge with the vertex u_2 already existing, because of (14) (we draw only one vertex for each variable). We repeat this procedure for all sequences from (13). In the end, we attach to each graph an integer denoting how many times (and with which sign) that graph (summand) appears in the expression of $U_{\infty,k}$. Of course, this integer corresponds to a_j^k in (11).

Let us now draw a graph for $k = 1$, namely

$$U_{\infty,1} = -\int_0^t R(s,t)\, M(s)\, U_*^2(s)\, ds.$$

Its graph is



This is the simplest graph, which tells us that the function $f(s,t)$ is integrated from 0 to t. We do not know exactly the structure of f, but we know that it is a product of basic functions. Our next goal is to determine the exact structure of the integrand f from the graph.

3. Generation of graphs

In this section we describe a procedure to draw graphs describing the terms in the expression for $U_{\infty,k}$ without knowing the expression itself, but only by looking at the graphs for $U_{\infty,j}$, $j = 1, \dots, k-1$. In fact, we want to express (10) in terms of graphs.

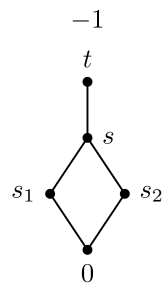
Let us first consider the explicit formula for $U_{n,2}$:

$$U_{n,2}(t) = -\int_0^t R(s,t)\big(a_\infty(s)\, U_{n,1}^2(s) + 2b_n(s)\, U_*(s)\, U_{n,1}(s)\big)\, ds.$$

After taking the limit, the only difference is that the product $b_n(s_1)\cdots b_n(s_j)$ changes into $M(s_1, \dots, s_j)$. The first term on the right-hand side reads:

$$-\int_0^t \int_0^s \int_0^s R(s,t)\, a_\infty(s)\, R(s_1,s)\, R(s_2,s)\, M(s_1,s_2)\, U_*^2(s_1)\, U_*^2(s_2)\, ds_1\, ds_2\, ds.$$

It can be represented by the graph:



By looking at the graph we come to an idea — taking products of summands S_j^k from $U_{n,k}$ and S_i^l from $U_{n,l}$ and integrating them from 0 to t corresponds to joining of the graphs in the following way: draw two graphs, each representing one of the terms with the common starting point 0, and the common endpoint, which we will denote by s. Then connect the vertex s with t, the endpoint of the new graph. This last step corresponds to the integral $\int_0^t$ in (10).

We want our graphs to give us all the information about the terms. For this purpose let us complicate our graphs a bit. We will introduce two kinds of inner vertices, which we will denote by $\circ$ and $\bullet$. The first sign will denote a vertex in which the function a_∞ appears as a factor.

Lemma 5. *The function M depends exactly on those variables corresponding to the vertices denoted by $\bullet$.*

The following lemma gives us information about the fourth, until now unmentioned basic function U_*.

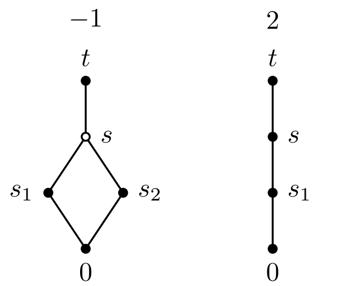
Lemma 6. *The function U_* does not depend on the variables corresponding to the vertices with two entering edges. For other vertices, the following rule is valid: if we have a sequence of vertices $0, s_1, s_2, \ldots, s_n$ connected by the edges $0 — s_1$, $s_1 — s_2$, ..., $s_{n-1} — s_n$, where s_n is the first vertex in the sequence in which two edges enter, or s_n is the endpoint t, then U_* appears as a function of these variables in the following way*

$$U_*^2(s_1)\,U_*(s_2)\,U_*(s_3)\cdots U_*(s_{n-1}).$$

If we have a sequence as above, where only the point 0 is replaced by a vertex s_0 in which two edges enter, then U_* appears in the following way

$$U_*(s_1)\,U_*(s_2)\,U_*(s_3)\cdots U_*(s_{n-1}).$$

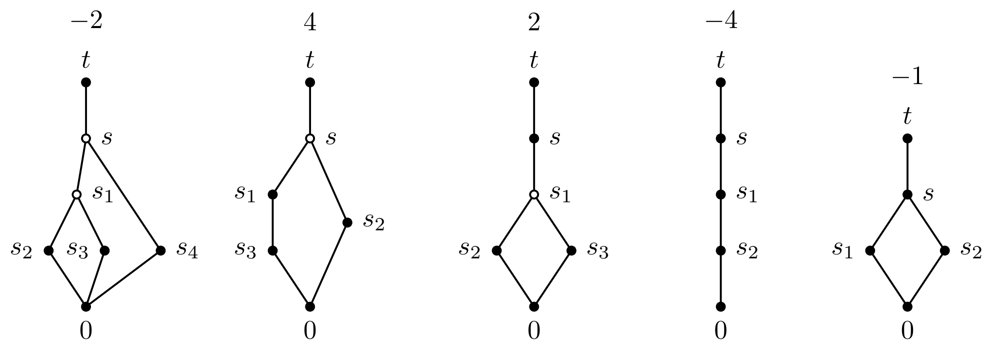
Let us see how the graphs look like. For $U_{\infty,2}$ the complete graph is:



From the previous graphs it is now easy to draw the graph for $U_{\infty,3}$ by using the particular case of (10):

$$U_{\infty,3}(t) = -\int_0^t R(s,t)\big(2a_\infty(s)\,U_{\infty,1}(s)\,U_{\infty,2}(s) \\ + 2b_n(s)\,U_*(s)\,U_{\infty,2}(s) + b_n\,U_{\infty,1}^2(s)\big)\,ds.$$

The corresponding graph is

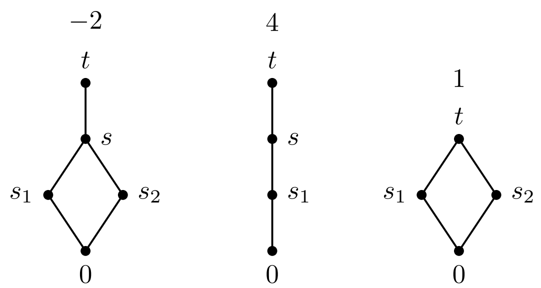


By using previous lemmas we can easily write down the explicit expression for $U_{\infty,3}$:

$$U_{\infty,3} = -2 \int_0^t \int_0^s \int_0^s \int_0^{s_1} \int_0^{s_1} a_\infty(s)\, a_\infty(s_1)\, R(s,t)\, R(s_1,s)\, R(s_2,s_1)\, R(s_3,s_1)\, R(s_4,s) \cdot M(s_2,s_3,s_4)\, U_*^2(s_2)\, U_*^2(s_3)\, U_*^2(s_4)\, ds_2\, ds_3\, ds_4\, ds_1\, ds$$
$$+ 4 \int_0^t \int_0^s \int_0^s \int_0^{s_1} a_\infty(s)\, R(s,t)\, R(s_1,s)\, R(s_3,s_1)\, R(s_2,s) \cdot M(s_1,s_2,s_3)\, U_*(s_1)\, U_*^2(s_2)\, U_*^2(s_3)\, ds_3\, ds_1\, ds_2\, ds$$
$$+ 2 \int_0^t \int_0^s \int_0^{s_1} \int_0^{s_1} a_\infty(s_1)\, R(s,t)\, R(s_1,s)\, R(s_2,s_1)\, R(s_3,s_1) \cdot M(s,s_2,s_3)\, U_*(s)\, U_*^2(s_2)\, U_*^2(s_3)\, ds_3\, ds_2\, ds_1\, ds$$
$$- 4 \int_0^t \int_0^s \int_0^{s_1} R(s,t)\, R(s_1,s)\, R(s_2,s_1) \cdot M(s,s_1,s_2)\, U_*(s)\, U_*(s_1)\, U_*^2(s_2)\, ds_2\, ds_1\, ds$$
$$- \int_0^t \int_0^s \int_0^s R(s,t)\, R(s_1,s)\, R(s_2,s) \cdot M(s,s_1,s_2)\, U_*(s)\, U_*^2(s_1)\, U_*^2(s_2)\, ds_2\, ds_1\, ds.$$

4. Formal expansion

In the same way we can compute other functions appearing in (9): $V_{\infty,k}$ and $W_{\infty,k}$. They can also be represented by graphs, constructed in the same way as graphs for $U_{\infty,k}$. These graphs allow us to write easily the explicit expressions for $V_{\infty,k}$ and $W_{\infty,k}$. For example, the graph for $V_{\infty,1}$ is same as the graph for $U_{\infty,1}$, only the corresponding integer is -2, and the graph for $V_{\infty,2}$ is



By our method we can compute the first dozen terms in the expansion of U, with the aid of a personal computer. Of course, the formulæ involving integrals are cumbersome; thus we explicitly write down only the expansion up to terms involving γ^3.

Theorem 1. *The function*

$$\widetilde{U}(x,t) = U_*(x,t) + \gamma^2 U_{\infty,2}(x,t) + \gamma^3 U_{\infty,3}(x,t)$$

satisfies the equation (where we omit the explicit writing of the variable x):

$$\begin{aligned}
&\frac{\partial \widetilde{U}}{\partial t}(t) + a_\infty(t)\,\widetilde{U}^2(t) \\
&\quad + \gamma^2\, a_\infty(t) \int_0^t\!\!\int_0^t R^*(s_1,t)\,R^*(s_2,t)\,M(s_1,s_2)\,\widetilde{U}^2(s_1)\,\widetilde{U}^2(s_2)\,ds_1\,ds_2 \\
&\quad - 2\gamma^2\,\widetilde{U}(t) \int_0^t R^*(s,t)\,M(s,t)\,\widetilde{U}^2(s)\,ds \\
&\quad + 2\gamma^3\, a_\infty(t) \int_0^t\!\!\int_0^t\!\!\int_0^{s_1}\!\!\int_0^{s_1} a_\infty(s_1)\,R^*(s_1,t)\,R^*(s_2,s_1)\,R^*(s_3,s_1)\,R^*(s_4,t) \\
&\qquad\qquad \cdot M(s_2,s_3,s_4)\,\widetilde{U}^2(s_2)\,\widetilde{U}^2(s_3)\,\widetilde{U}^2(s_4)\,ds_1\,ds_2\,ds_3\,ds_4 \\
&\quad - 4\gamma^3\, a_\infty(t) \int_0^t\!\!\int_0^t\!\!\int_0^{s_1} R^*(s_1,t)\,R^*(s_3,s_1)\,R^*(s_2,t) \\
&\qquad\qquad \cdot M(s_1,s_2,s_3)\,\widetilde{U}(s_1)\,\widetilde{U}^2(s_2)\,\widetilde{U}^2(s_3)\,ds_1\,ds_2\,ds_3 \\
&\quad - 2\gamma^3\,\widetilde{U}(t) \int_0^t\!\!\int_0^{s_1}\!\!\int_0^{s_1} a_\infty(s_1)\,R^*(s_1,t)\,R^*(s_2,s_1)\,R^*(s_3,s_1) \\
&\qquad\qquad \cdot M(t,s_2,s_3)\,\widetilde{U}^2(s_2)\,\widetilde{U}^2(s_3)\,ds_3\,ds_2\,ds_1 \\
&\quad + 4\gamma^3\,\widetilde{U}(t) \int_0^t\!\!\int_0^{s_1} R^*(s_1,t)\,R^*(s_2,s_1)\,M(t,s_1,s_2)\,\widetilde{U}(s_1)\,\widetilde{U}^2(s_2)\,ds_2\,ds_1 \\
&\quad + \gamma^3 \int_0^t\!\!\int_0^t R^*(\cdot,s_1,t)\,R^*(\cdot,s_2,t)\,M(\cdot,s_1,s_2,t)\,\widetilde{U}^2(s_1)\,\widetilde{U}^2(s_2)\,ds_1\,ds_2 \\
&\quad = f + O(\gamma^4),
\end{aligned}$$

with initial condition

$$\widetilde{U}(\cdot,0) = v,$$

where

$$R^*(x,s,t) = \exp\left(-\int_s^t 2a_\infty(x,\sigma)\,\widetilde{U}(x,\sigma)\,d\sigma\right).$$

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